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A General Asymptotic Scheme for the Analysis of Partition Statistics

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Dedicated to the memory of Philippe Flajolet

We consider statistical properties of random integer partitions. In order to compute means, variances and higher moments of various partition statistics, one often has to study generating functions of the form $P(x)F(x)$, where $P(x)$ is the generating function for the number of partitions. In this paper, we show how asymptotic expansions can be obtained in a quasi-automatic way from expansions of $F(x)$ around $x = 1$, which parallels the classical singularity analysis of Flajolet and Odlyzko in many ways. Numerous examples from the literature, as well as some new statistics, are treated via this methodology. In addition, we show how to compute further terms in the asymptotic expansions of previously studied partition statistics.

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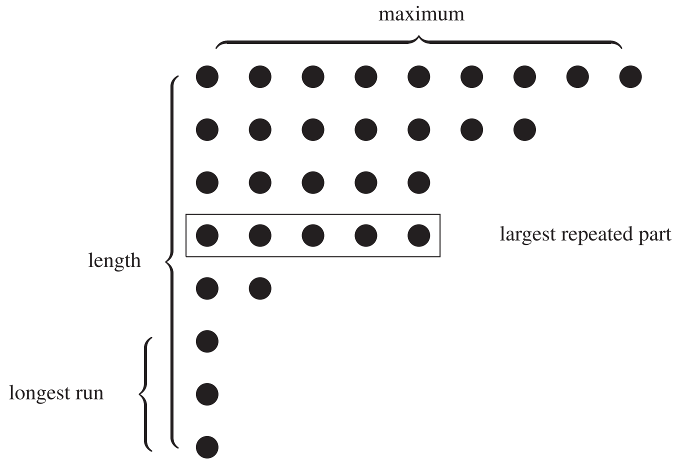

Figure 1. The Ferrers diagram of the partition $9+7+5+5+2+1+1+1$ and some partition statistics.

1. Introduction

The aim of this paper is to provide a tool for studying statistical properties of random integer partitions. Formally, a *partition* of a positive integer n is a representation as a sum of positive integers, where the order of summands is not taken into account. This means that one can assume, without loss of generality, that the summands are in decreasing order:

$$n = c_1 + c_2 + \cdots + c_\ell, \quad c_1 \geqslant c_2 \geqslant \cdots \geqslant c_\ell. \tag{1.1}$$

The numbers $c_1, c_2, \ldots$ are called the *parts*. For instance, the partitions of 5 are

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

Andrews' classical book [1] provides an introduction to the rich subject of integer partitions. Partitions are conveniently represented by means of *Ferrers diagrams*, such as in Figure 1, which shows the partition $9+7+5+5+2+1+1+1$ of 31: each summand is represented by a row of dots. This figure also exhibits a couple of partition statistics, which are the main subject of this paper. They capture different structural aspects of an integer partition.

The study of statistics of random partitions has a long and rich history; the earliest results actually appear in the physics literature [12], since partition statistics occur naturally in statistical physics. In the following, we will discuss various different instances of partition statistics that have been investigated in the literature. The first thing to look at is usually the mean of a certain parameter, followed by the variance and higher moments, and – if possible – the limiting distribution.

In the analysis of partition statistics, one often has to study generating functions of the form $P(x)F(x)$, where

$$P(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-1}$$

is the generating function for the number of partitions. In this paper, we develop a general asymptotic scheme that allows us to derive an asymptotic formula for the nth coefficient of $P(x)F(x)$ from the behaviour of $F(x)$ as $x \to 1$ in an almost automatic fashion. This scheme is then applied to a variety of examples of partition statistics: we re-prove (and extend) known results and also treat some new examples.

It is well known that $p(n) = [x^n]P(x)$ essentially behaves like

$$\frac{1}{4\sqrt{3}n}\exp\big(\pi\sqrt{2n/3}\big),$$

which is made much more precise by Rademacher's celebrated formula

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\,\frac{d}{dn}\,\frac{\sinh\Big(\frac{\pi}{k}\sqrt{\frac{2}{3}\big(n-\frac{1}{24}\big)}\Big)}{\sqrt{n-1/24}}, \tag{1.2}$$

a sum formula that is both exact and asymptotic (in the sense that the asymptotic order of the summands is decreasing). The coefficients $A_k(n)$ can be expressed in terms of Dedekind sums. See Rademacher's original book [18] or Apostol's book [2] for an excellent exposition. This result depends heavily on a functional equation for $P(x)$, a special case of which is given by

$$P(e^{-t}) = \sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{6t} - \frac{t}{24}\right)P(e^{-4\pi^2/t}). \tag{1.3}$$

We will make frequent use of this functional equation throughout the paper. Let us now consider a few examples that lead to generating functions of the form $P(x)F(x)$.

1.1. The length of a partition

The length (number of parts) of a partition has bivariate generating function

$$\prod_{j=1}^{\infty}(1-ux^j)^{-1},$$

where the exponent of u marks the length [8, p. 171, Example III.7]. Differentiating with respect to u and setting $u = 1$, we obtain the generating function

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{x^j}{1-x^j}$$

for the total length, summed over all partitions. The coefficient of x^n, divided by the number $p(n)$ of partitions of n, yields the average length of a random partition of n. This was used by Husimi [12] and Kessler and Livingston [14] to obtain rather precise asymptotics for the average length: it is given by

$$\frac{\sqrt{6n}}{2\pi}\cdot\big(\log n + 2\gamma - \log(\pi^2/6)\big) + O(\log n). \tag{1.4}$$

Husimi actually gives two more terms, for an error term of $O(n^{-1/2})$. In Section 4, we will use our results to compute arbitrarily many terms of such an expansion. Higher moments were studied by Richmond in [20]. The length asymptotically follows a Gumbel

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distribution, as was shown by Erdős and Lehner [5]. It is well known that the largest part follows the same distribution (as can be seen by conjugation of the Ferrers diagram), and so its mean is the same as well. These two parameters are asymptotically independent, as shown by Szekeres [22].

1.2. Number of distinct parts

The number of distinct parts is known to follow a normal distribution in the limit, as proved by Goh and Schmutz [9], with mean asymptotically equal to $\sqrt{6n}/\pi$. If we are interested in an asymptotic expansion for the mean, we again have to consider a bivariate generating function:

$$\prod_{j=1}^{\infty}\left(1+\frac{ux^j}{1-x^j}\right)$$

(see [8, p. 169, Note III.7]). Differentiating and setting $u=1$ yields

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\frac{x}{1-x},$$

which is of the form $P(x)F(x)$ as well. A generalization of this parameter, namely the sum of the mth powers of all distinct parts, was studied by Hwang and Yeh [13]. The associated generating function is

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}j^m x^j,$$

which also belongs to our scheme.

1.3. Moments of a partition

For a partition $(c_1, c_2, \ldots, c_\ell)$ of an integer n, consider the kth moment

$$\sum_{j=1}^{\ell}c_j^k.$$

The case $k=0$ clearly corresponds to the length, while the above sum is always equal to n for $k=1$ by definition. As before, differentiating the bivariate generating function

$$\prod_{j=1}^{\infty}(1-u^{j^k}x^j)^{-1}$$

with respect to u and setting $u=1$ yields

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{j^k x^j}{1-x^j},$$

see [11, Theorem 6.6]. [11, Corollary 6.5] gives an interesting connection to so-called hook lengths; see also [24] for an occurrence of the special case $k=2$ in graph theory.

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1.4. Number of parts of a given size

The bivariate generating function for the number of parts of size d (in terms of (1.1), the number of parts c_j that are equal to d) is given by

$$\frac{1-x^d}{1-ux^d}\cdot\prod_{j=1}^{\infty}(1-x^j)^{-1},$$

so we have to study

$$\frac{x^d}{1-x^d}\cdot\prod_{j=1}^{\infty}(1-x^j)^{-1}$$

for the average number of occurrences of the number d among the parts of a random partition of n. Note that in the case $d=1$, this coincides with the aforementioned generating function for the number of distinct parts, which is known as Stanley's theorem [21, p. 48, Exercise 1.26].

1.5. Number of parts with given multiplicity

The *multiplicity* of a positive integer d is the number of times it occurs as a summand c_j in a partition $(c_1, c_2, \dots)$ as in (1.1). If u marks all parts that occur with some prescribed multiplicity d, we obtain the bivariate generating function

$$\prod_{j=1}^{\infty}\left(\frac{1}{1-x^j}+(u-1)x^{dj}\right)$$

(see [4, Theorem 1]), so that the generating function for the total number of parts of multiplicity d is given by

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{k=1}^{\infty}x^{dk}(1-x^k)=\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\frac{(1-x)x^d}{(1-x^d)(1-x^{d+1})}.$$

The asymptotic behaviour of the mean was found by Corteel, Pittel, Savage and Wilf [4]: the average number of parts of multiplicity d is asymptotically $\sqrt{6n}/(\pi d(d+1))$. The limiting distribution is Gaussian for fixed d: see [19]. In Section 4, we will show how to determine a further asymptotic expansion for the mean. It is also worth mentioning that the number of parts of multiplicity d almost exactly follows the same distribution as the number of d-successions (*i.e.*, occurrences of two subsequent parts whose difference is d), which can be seen by conjugation of the Ferrers diagram. Successions in partitions were investigated in another recent paper [15]. Another related parameter that also falls under our scheme is the number of ascents of size d or more, which was studied in [3]. A special case is the number of gaps (ascents of size 2 or more); see [16].

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1.6. Largest repeated part

It was shown in [10, equation (6.4)] that the generating function for the sum of the largest repeated part sizes over all partitions of n is given by

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{x^{2j}}{1-x^{2j}},$$

and this can easily be generalized to yield the generating function for the sum of the largest part sizes that are repeated at least d times (if no such part exists, it is defined to be 0):

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{x^{dj}}{1-x^{dj}}.$$

Note that $d=1$ corresponds to the maximum of the parts in a partition. Results on the limiting distributions of the largest repeated part and related parameters can be found in [26].

1.7. Longest run

The longest run (maximum multiplicity, *i.e.*, the greatest number of times a part gets repeated in a partition) was shown to follow a rather unusual limit law in [17]; see also [25], where the mean of the longest run was found to be asymptotically equal to

$$\left(4\sqrt{2}-\frac{6\sqrt{6}}{\pi}\right)\sqrt{n}.$$

In Section 4 we will considerably improve on this. Since the generating function for the number of partitions whose longest run is $< k$ (k a positive integer) is easily found to be [1, p. 3, Theorem 1.1]

$$\prod_{j=1}^{\infty}\frac{1-x^{jk}}{1-x^j}=P(x)P(x^k)^{-1},$$

we have to study the generating function

$$P(x)\cdot\sum_{k=1}^{\infty}k\big(P(x^{k+1})^{-1}-P(x^k)^{-1}\big)=P(x)\cdot\sum_{k=1}^{\infty}\big(1-P(x^k)^{-1}\big),$$

or, more generally for the mth moment,

$$P(x)\cdot\sum_{k=1}^{\infty}(k^m-(k-1)^m)\big(1-P(x^k)^{-1}\big).$$

A related question concerns the probability that a certain integer k is one of the parts of largest multiplicity. The generating function for the number of partitions for which this is the case is

$$\sum_{\ell=1}^{\infty}\frac{x^{k\ell}(1-x^k)}{1-x^{k(\ell+1)}}\cdot\prod_{j=1}^{\infty}\frac{1-x^{(\ell+1)j}}{1-x^j}=P(x)\cdot\sum_{\ell=1}^{\infty}\frac{x^{k\ell}(1-x^k)}{(1-x^{k(\ell+1)})P(x^{\ell+1})},$$

which is once again of the form $P(x)F(x)$. In a similar manner, we obtain the following generating function for the number of partitions with the property that all parts have strictly smaller multiplicity than k:

$$\sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{1-x^{k\ell}} \cdot \prod_{j=1}^{\infty} \frac{1-x^{\ell j}}{1-x^j} = P(x) \cdot \sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{(1-x^{k\ell})P(x^\ell)}.$$

There are two main approaches to determining the asymptotic behaviour of the coefficients of generating functions of the form $P(x)F(x)$. The first is to write

$$[x^n]P(x)F(x) = \sum_{k=0}^{n} p(k)[x^{n-k}]F(x)$$

and to use Rademacher's formula (1.2) to approximate $p(k)$ in this sum, so that it can be determined by means of the Euler–Maclaurin formula or other techniques (see for instance [3] or [4]). The second possibility is to work directly with the generating function and to apply Cauchy's integral formula (as for instance in [13] or [14]) in combination with the saddle point method. We will adopt this approach here as well, but we also add some other ingredients: in particular, the functional equation of $P(x)$ allows us to transform some of the arising integrals to Bessel functions in the (common) situation that $F(e^{-t})$ has an asymptotic expansion into powers of t as $t \to 0^+$. This has several benefits.

- The asymptotic behaviour of the coefficients of $P(x)F(x)$ and even asymptotic expansions can be determined completely automatically (to the extent that the method can be implemented in a computer algebra system) from the behaviour of $F(x)$ around $x = 1$. Knowing further terms in an asymptotic expansion is often desirable, since cancellations frequently occur in higher moments, in particular the variance.
- For certain statistics (for example the longest run, see Section 4), one can obtain the asymptotic behaviour of all moments directly from our theorems and thus also the limiting distribution.
- In some situations (see the discussion of moments of partitions and longest runs in partitions in Section 4), very strong error estimates (superpolynomial decay) can be obtained, which lead to excellent approximations of the exact quantities.

The following section gives a more detailed description of our results.

2. A general asymptotic scheme

Let us now develop the general scheme that allows us to obtain asymptotic expansions for the coefficients of a generating function of the form $P(x)F(x)$. It is based on the classical saddle point method. The technical conditions we impose on $F(x)$ for this purpose are rather mild. First of all, we express the coefficient of x^n by means of Cauchy's integral formula,

$$[x^n]P(x)F(x) = \frac{1}{2\pi i} \oint_C z^{-n-1} P(z)F(z)\,dz \tag{2.1}$$

for a closed curve C around 0 inside the unit circle. If we take C to be the circle of radius e^{-r} around 0, where the choice of r will be specified later and is determined by the

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position of the saddle point, the change of variable $z = e^{-r+iu}$ yields

$$\frac{1}{2\pi i}\oint_C z^{-n-1}P(z)F(z)\,dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du. \tag{2.2}$$

As usual in applications of the saddle point method, only the central part of the integral is relevant, while the rest only contributes a very small error term. Therefore, the main asymptotic information can be determined from the behaviour as z goes to 1. Essentially,

$$\frac{1}{p(n)}\cdot [x^n]P(x)F(x)$$

is given by the value of $F(z)$ at the saddle point $z = e^{-\pi/\sqrt{6n}}$: see Theorem 2.2 below. For this purpose, it is necessary that $F(z)$ does not grow too quickly as $|z| \to 1$. Specifically, we assume that

$$|F(z)| \ll e^{C/(1-|z|)^\eta} \text{ as } |z| \to 1 \text{ for some } C > 0 \text{ and } \eta < 1. \tag{2.3}$$

Remark 1. It will be convenient for us throughout this paper to use the Vinogradov notation $f(z) \ll g(z)$ interchangeably with $f(z) = O(g(z))$.

A simple sufficient condition for (2.3) to hold (that will actually be satisfied in all our examples) is that the coefficients of $F(z)$ grow at most polynomially.

Our main results parallel the classical Flajolet–Odlyzko singularity analysis [7] in many ways. We first prove a O-result (see [8, p. 390, Theorem VI.3]), mostly as an auxiliary tool, that reads as follows.

Theorem 2.1. *Suppose that the function $F(z)$ satisfies (2.3) and $F(e^{-t}) = O(f(|t|))$ as $t \to 0$, $\operatorname{Re} t > 0$. Then we have*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = O\left(\exp\left(-n^{1/2-\epsilon}\right) + f\left(\frac{\pi}{\sqrt{6n}} + O\left(n^{-1/2-\epsilon}\right)\right)\right)$$

for any fixed $0 < \epsilon < (1-\eta)/2$ as $n \to \infty$.

With just a little further technical assumption, we get an asymptotic formula with a $o(1)$ error term, which is useful if we are only interested in first-order asymptotics (for an example, see the discussion of the position of the longest run at the end of this paper).

Theorem 2.2. *Suppose that the function $F(z)$ satisfies (2.3) and that*

$$\frac{F(e^{-t+iu})}{F(e^{-t})} \to 1$$

if $|u| \leqslant At^{1+\epsilon}$ for some $A > 0$ and for some $\epsilon < (1-\eta)/2$, uniformly in u as $t \to 0^+$. Then we have

$$\frac{1}{p(n)}[x^n]P(x)F(x) = F\left(e^{-\pi/\sqrt{6n}}\right)\left(1 + o(1)\right) + O\left(\exp\left(-Bn^{1/2-\epsilon}\right)\right)$$

as $n \to \infty$ for some $B > 0$.

Remark 2. The required technical condition on F avoids pathological examples (*e.g.*, zeros of $F(z)$ accumulating as $z \to 1$). It is easy to check them for all the examples described in the Introduction. A sufficient condition is that

$$\left| \frac{t^{1+\epsilon_0} F'(e^{-t})}{F(e^{-t})} \right|$$

remains bounded as $t \to 0$ ($\operatorname{Re} t > 0$) for some $\epsilon_0 < \epsilon$.

In many cases, however, much more is known than in those two theorems, *i.e.*, we know the asymptotic behaviour of the function F at $z = 1$ very precisely. Indeed, as will be shown in Section 4, we often have an asymptotic expansion for $F(e^{-t})$ into powers of t (and possibly logarithms) around $t = 0$. In this case, we can be much more precise by relating the central part of the saddle point integral (2.2) to classical integral representations of Bessel functions. The final result is stated in the following theorem.

Theorem 2.3. *Suppose that the function $F(z)$ satisfies* (2.3) *and $F(e^{-t}) = at^b + O(f(|t|))$ as $t \to 0$, $\operatorname{Re} t > 0$, for real numbers a, b. Then we have*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \cdot \frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + O\left(\exp\left(-n^{1/2-\epsilon}\right) + f\left(\frac{\pi}{\sqrt{6n}} + O\left(n^{-1/2-\epsilon}\right)\right)\right)$$

as $n \to \infty$ for any $0 < \epsilon < (1-\eta)/2$, where I_ν denotes a modified Bessel function of the first kind. Similarly, if $F(z)$ satisfies (2.3) *and*

$$F(e^{-t}) = at^b \log\frac{1}{t} + O(f(|t|)) \quad \text{as } t \to 0,$$

then

$$\frac{1}{p(n)}[x^n]P(x)F(x) = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \cdot \left(\log\left(\frac{\sqrt{24n-1}}{2\pi}\right)\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + \sum_{k=1}^{2K}\frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{I_{|b+j+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}\right) + O\left(n^{-(b+K+1)/2} + f\left(\frac{\pi}{\sqrt{6n}} + O\left(n^{-1/2-\epsilon}\right)\right)\right)$$

for any non-negative integer K.

Remark 3. Of course, this theorem generalizes to asymptotic expansions of the form

$$F(e^{-t}) = \sum_{j=1}^{J} a_j t^{b_j} + O(f(|t|)),$$

or even to mixed expansions involving logarithms.

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The absolute value in the index of the Bessel function can actually be dropped, since the difference $I_\nu(z) - I_{-\nu}(z)$ decreases exponentially as $z \to \infty$. Note further that the modified Bessel function $I_{|b+3/2|}$ can be written in terms of elementary functions for integer values of b (see [27, Section 6.22]). For non-negative integer h, we have

$$I_{h+1/2}(z) = \frac{1}{\sqrt{2\pi z}} \sum_{j=0}^{h} \left(-\frac{1}{2z}\right)^j \cdot \frac{(h+j)!}{j!(h-j)!}(e^z - (-1)^{h-j}e^{-z}),$$

to the effect that the quotient simplifies, with $h = |b+3/2| - 1/2$ and

$$m = \sqrt{\frac{2\pi^2}{3}\left(n - \frac{1}{24}\right)},$$

to

$$\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} = \frac{m}{m-1} \cdot \sum_{j=0}^{h} \frac{(h+j)!}{j!(h-j)!}\left(-\frac{1}{2m}\right)^j + O(e^{-2m}).$$

For non-integer values of b, this is at least asymptotically correct, in the sense that

$$\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} = \frac{m}{m-1} \cdot \sum_{j=0}^{J} \binom{h+j}{2j} \frac{(2j)!}{j!}\left(-\frac{1}{2m}\right)^j + O(m^{-J-1})$$

for any fixed J, with the same abbreviations as above. This also shows that in the sum

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{I_{|b+j+3/2|}(m)}{I_{3/2}(m)},$$

the power $m^{-\ell}$ in the asymptotic expansion vanishes for $\ell \leqslant \lfloor (k-1)/2 \rfloor$, since its coefficient is a polynomial of order 2ℓ, and the alternating sum above amounts to taking kth differences. Therefore, this sum is of asymptotic order $m^{-\lfloor (k+1)/2 \rfloor}$.

Equipped with these theorems, one can now analyse the asymptotic behaviour of many partition statistics following a fixed algorithm.

- Determine the generating function for the mean (higher moments, ...) and write it in the form $P(x)F(x)$.
- Make sure that $F(x)$ satisfies the technical conditions (trivial in most cases).
- Theorem 2.2 readily gives the main term, which may already be sufficient.
- If further terms are desired (*e.g.*, because one would like to obtain higher accuracy approximations, or because cancellations occur), determine an expansion of $F(e^{-t})$ into powers of t (and possibly logarithmic terms) if possible.
- Apply Theorem 2.3. Use the asymptotic expansions of the Bessel functions to obtain asymptotic formulae in terms of powers of n. Those last steps are fully automatic and can be performed by a computer algebra system.

This process will be illustrated in Section 4 by means of various examples.

3. Proofs of the main results

For the proofs we need the following technical lemma. Similar estimates are frequently used in the theory of partitions; see for instance [17].

Lemma 3.1. *Uniformly as $|u| \leqslant \pi$, we have*

$$\frac{|P(e^{-r+iu})|}{P(e^{-r})} \leqslant \exp\left(-\frac{u^2}{r(u^2+(\pi r/2)^2))} + O(r)\right)$$

as $r \to 0^+$.

Proof. The proof essentially follows the ideas of [17]. First we get

$$\begin{aligned}
|P(e^{-r+iu})| &= \left|\exp\left(-\sum_{j=1}^{\infty}\log(1-e^{-jr+iju})\right)\right| = \left|\exp\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-jkr+ijku}\right)\right| \\
&= \exp\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-jkr}\cos(jku)\right) \\
&= P(e^{-r})\exp\left(-\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-jkr}(1-\cos(jku))\right) \\
&\leqslant P(e^{-r})\exp\left(-\sum_{j=1}^{\infty}e^{-jr}(1-\cos(ju))\right) \\
&= P(e^{-r})\exp\left(-\frac{1}{2}\sum_{j=1}^{\infty}(2e^{-jr}-e^{-j(r+iu)}-e^{-j(r-iu)})\right) \\
&= P(e^{-r})\exp\left(-\frac{1}{e^r-1}+\frac{1}{2(e^{r+iu}-1)}+\frac{1}{2(e^{r-iu}-1)}\right) \\
&= P(e^{-r})\exp\left(-\coth\left(\frac{r}{2}\right)\cdot\frac{1-\cos u}{2(\cosh r-\cos u)}\right)
\end{aligned}$$

after a few simplifications. Now, making use of the inequality $1-\cos u \geqslant 2(u/\pi)^2$, the exponent can be estimated below by

$$\begin{aligned}
\coth\left(\frac{r}{2}\right)\cdot\frac{1-\cos u}{2(\cosh r-\cos u)} &= \frac{1}{2}\coth\left(\frac{r}{2}\right)\cdot\left(1+\frac{\cosh r-1}{1-\cos u}\right)^{-1} \\
&\geqslant \frac{1}{2}\coth\left(\frac{r}{2}\right)\cdot\left(1+\frac{\cosh r-1}{2u^2/\pi^2}\right)^{-1} \\
&= \left(\frac{1}{r}+O(r)\right)\cdot 2u^2(2u^2+\pi^2r^2/2+O(r^4))^{-1} \\
&= \frac{u^2}{r(u^2+\pi^2r^2/4)}(1+O(r^2)) \\
&= \frac{u^2}{r(u^2+\pi^2r^2/4)}+O(r).
\end{aligned}$$

This completes the proof of our lemma. □

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Proof of Theorem 2.1. We start with the integral representation (2.2), where we choose r to be the saddle point $r = \pi/\sqrt{6n}$. Now choose $\epsilon < (1-\eta)/2$ and split the integral into a central part for which $|u| \leqslant 2r^{1+\epsilon}$ and the rest. For sufficiently large n and $2r^{1+\epsilon} \leqslant |u| \leqslant \pi$, we have

$$\frac{u^2}{r(u^2+\pi^2 r^2/4)} = \frac{1}{r(1+\pi^2 r^2/(4u^2))} \geqslant \frac{1}{r(1+\pi^2 r^{-2\epsilon}/16)} \geqslant \frac{3}{2} r^{2\epsilon-1},$$

and thus, by Lemma 3.1 and the assumptions on F,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{2r^{1+\epsilon}\leqslant|u|\leqslant\pi} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du\right| \\
&\quad\leqslant \sup_{2r^{1+\epsilon}\leqslant|u|\leqslant\pi} |e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})| \\
&\quad\ll e^{rn}P(e^{-r})\exp\left(-\frac{3}{2}r^{2\epsilon-1} + C(1-e^{-r})^{-\eta} + O(r)\right) \\
&\quad= e^{rn}P(e^{-r})\exp\left(-\frac{3}{2}\left(\frac{\pi}{\sqrt{6n}}\right)^{2\epsilon-1} + O(n^{\eta/2})\right).
\end{aligned}$$

The functional equation (1.3) shows that

$$e^{nr}P(e^{-r}) = \sqrt{\frac{r}{2\pi}}\exp\left(nr+\frac{\pi^2}{6r}-\frac{r}{24}\right)P(e^{-4\pi^2/r}) \ll n^{3/4}p(n),$$

so that finally

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{2r^{1+\epsilon}\leqslant|u|\leqslant\pi} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du\right| \\
&\quad\ll p(n)\exp\left(-\frac{3}{2}\left(\frac{\pi}{\sqrt{6n}}\right)^{2\epsilon-1} + O(n^{\eta/2})\right) \\
&\quad\ll p(n)\exp(-n^{1/2-\epsilon}).
\end{aligned}$$

Hence it suffices to consider the remaining part of the integral. Since $|r-iu| = r + O(r^{1+2\epsilon})$ for $|u| \leqslant 2r^{1+\epsilon}$ by an easy calculation, we have

$$|F(e^{-r+iu})| \ll f(|r-iu|) = f\big(r+O(r^{1+2\epsilon})\big) = f\left(\frac{\pi}{\sqrt{6n}} + O(n^{-1/2-\epsilon})\right)$$

within this interval. Hence it suffices to show that

$$\int_{|u|\leqslant 2r^{1+\epsilon}} |e^{rn-inu}P(e^{-r+iu})|\,du \ll p(n).$$

To this end, note that by (1.3),

$$\begin{aligned}
|e^{rn-inu}P(e^{-r+iu})| &= e^{rn}\sqrt{\frac{|r-iu|}{2\pi}}\exp\left(\mathrm{Re}\left(\frac{\pi^2}{6(r-iu)}-\frac{r-iu}{24}\right)\right)|P(e^{-4\pi^2/(r-iu)})| \\
&\ll e^{rn}\sqrt{r}\exp\left(\frac{\pi^2 r}{6(r^2+u^2)}\right)
\end{aligned}$$

$$= e^{rn}\sqrt{r}\exp\left(\frac{\pi^2}{6r} - \frac{\pi^2u^2}{6r(r^2+u^2)}\right)$$
$$\leqslant e^{rn}\sqrt{r}\exp\left(\frac{\pi^2}{6r} - \frac{\pi^2u^2}{30r^3}\right)$$

for $|u| \leqslant 2r^{1+\epsilon}$, so that

$$\int_{|u|\leqslant 2r^{1+\epsilon}} |e^{rn-inu}P(e^{-r+iu})|\,du \ll \sqrt{r}\exp\left(rn+\frac{\pi^2}{6r}\right)\int_{|u|\leqslant 2r^{1+\epsilon}}\exp\left(-\frac{\pi^2u^2}{30r^3}\right)du$$
$$\ll n^{-1/4}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{\pi^2u^2}{30r^3}\right)du$$
$$= n^{-1/4}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)\sqrt{\frac{30r^3}{\pi}}$$
$$\ll n^{-1}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \ll p(n),$$

which completes the proof. □

Proof of Theorem 2.2. The proof proceeds in the same way as the previous one. We split the integral (2.2) into the part for which $|u| \leqslant Ar^{1+\epsilon}$ and the remaining two intervals. The latter only contribute an error term of the form $O\big(\exp(-Bn^{1/2-\epsilon})\big)$, and the remaining part of the integral is given by

$$\frac{1}{2\pi}\int_{|u|\leqslant Ar^{1+\epsilon}} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du$$
$$= F(e^{-r})\frac{1}{2\pi}\int_{|u|\leqslant Ar^{1+\epsilon}} e^{rn-inu}P(e^{-r+iu})\,du$$
$$+\,o\left(|F(e^{-r})|\int_{|u|\leqslant Ar^{1+\epsilon}}|e^{rn-inu}P(e^{-r+iu})|\,du\right)$$
$$= F(e^{-r})p(n)(1+o(1)) + o\big(|F(e^{-r})|p(n)\big)$$

by the same estimates as before. □

Proof of Theorem 2.3. Recall the integral representation

$$[x^n]P(x)F(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} z^{-n-1}P(z)F(z)\,dz,$$

where we take $\mathcal{C}$ to be the circle of radius e^{-r} around 0. The change of variable $z = e^{-t}$ yields

$$[x^n]P(x)F(x) = \frac{1}{2\pi i}\int_{r-i\pi}^{r+i\pi} e^{nt}P(e^{-t})F(e^{-t})\,dt.$$

In view of Theorem 2.1 (and its proof), we may now restrict ourselves to the part of the integral where $|\operatorname{Im} t| \leqslant 2r^{1+\epsilon}$, and replace $F(e^{-t})$ by at^b there. We are left with an integral

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of the form

$$\frac{a}{2\pi i}\int_{r-2ir^{1+\epsilon}}^{r+2ir^{1+\epsilon}} e^{nt}P(e^{-t})t^b\,dt.$$

Now we make use of the functional equation (1.3) once again to obtain

$$\frac{a}{\sqrt{2\pi}}\cdot\frac{1}{2\pi i}\int_{r-2ir^{1+\epsilon}}^{r+2ir^{1+\epsilon}} t^{b+1/2}\exp\biggl(\biggl(n-\frac{1}{24}\biggr)t+\frac{\pi^2}{6t}\biggr)P(e^{-4\pi^2/t})\,dt.$$

Since $P(z)=1+O(z)$ as $z\to 0$, we can replace the factor $P(e^{-4\pi^2/t})$ by 1 at the expense of an error term of order

$$\exp(-4\pi^2/r+O(r^{-1+\epsilon})),$$

which is negligible. Hence we are left with an integral over an elementary function. Depending on b, we have to distinguish three cases:

- If $b<-3/2$, then we complete the integral to obtain

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} t^{b+1/2}\exp\biggl(\biggl(n-\frac{1}{24}\biggr)t+\frac{\pi^2}{6t}\biggr)\,dt,$$

which, after the change of variables $(n-1/24)t=u$, yields

$$\biggl(n-\frac{1}{24}\biggr)^{-b-3/2}\cdot\frac{1}{2\pi i}\int_{L-i\infty}^{L+i\infty} u^{b+1/2}\exp\biggl(u+\frac{\pi^2(n-1/24)}{6u}\biggr)\,du$$

for $L=(n-1/24)r$. This is a well-known integral representation for the modified Bessel function $I_{-b-3/2}$ (see [27]), so we finally get

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} t^{b+1/2}\exp\biggl(\biggl(n-\frac{1}{24}\biggr)t+\frac{\pi^2}{6t}\biggr)\,dt = \biggl(\frac{4\pi^2}{24n-1}\biggr)^{b/2+3/4} I_{-b-3/2}\biggl(\sqrt{\frac{2\pi^2}{3}\biggl(n-\frac{1}{24}\biggr)}\biggr).$$

It remains to show that the parts of the integral that were added only contribute to the error term. Consider the integral

$$\mathcal{I}=\int_{r+2ir^{1+\epsilon}}^{r+i\infty} t^{b+1/2}\exp\biggl(\biggl(n-\frac{1}{24}\biggr)t+\frac{\pi^2}{6t}\biggr)\,dt,$$

which we estimate as follows:

$$\begin{aligned}|\mathcal{I}| &\leqslant \int_{r+2ir^{1+\epsilon}}^{r+i\infty}\biggl|t^{b+1/2}\exp\biggl(\biggl(n-\frac{1}{24}\biggr)t+\frac{\pi^2}{6t}\biggr)\biggr|\,dt\\ &= \int_{r+2ir^{1+\epsilon}}^{r+i\infty} |t|^{b+1/2}\exp\biggl(\biggl(n-\frac{1}{24}\biggr)r+\mathrm{Re}\biggl(\frac{\pi^2}{6t}\biggr)\biggr)\,dt\\ &\leqslant \exp\biggl(\biggl(n-\frac{1}{24}\biggr)r\biggr)\int_{2r^{1+\epsilon}}^{\infty} u^{b+1/2}\exp\biggl(\mathrm{Re}\biggl(\frac{\pi^2}{6(r+iu)}\biggr)\biggr)\,du\\ &= \exp\biggl(\biggl(n-\frac{1}{24}\biggr)r\biggr)\int_{2r^{1+\epsilon}}^{\infty} u^{b+1/2}\exp\biggl(\frac{\pi^2 r}{6(r^2+u^2)}\biggr)\,du\end{aligned}$$

$$\begin{aligned}
&= \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}\right)\int_{2r^{1+\epsilon}}^{\infty} u^{b+1/2}\exp\left(-\frac{\pi^2u^2}{6r(r^2+u^2)}\right)du\\
&\leqslant \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}-\frac{2\pi^2r^{2+2\epsilon}}{3r(r^2+4r^{2+2\epsilon})}\right)\int_{2r^{1+\epsilon}}^{\infty} u^{b+1/2}\,du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}-\frac{2\pi^2}{3}r^{2\epsilon-1}+O\left(r^{4\epsilon-1}+\log(1/r)\right)\right)\\
&\ll p(n)\exp(-n^{1/2-\epsilon}),
\end{aligned}$$

completing the proof in this case.

- For $b=-3/2$, we have to be slightly more careful in the estimation of the integral, but otherwise the proof remains the same. We have to consider

$$\mathcal{I}=\int_{r+2ir^{1+\epsilon}}^{r+i\infty}\frac{1}{t}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt,$$

which we split into two integrals $\mathcal{I}_1$ and $\mathcal{I}_2$, corresponding to $\operatorname{Im} t\leqslant 1$ and $\operatorname{Im} t\geqslant 1$ respectively. For $\mathcal{I}_1$, the same argument as above yields

$$\begin{aligned}
|\mathcal{I}_1| &\leqslant \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}-\frac{2\pi^2r^{2+2\epsilon}}{6r(r^2+4r^{2+2\epsilon})}\right)\int_{2r^{1+\epsilon}}^{1}\frac{1}{u}\,du\\
&\ll p(n)\exp(-n^{1/2-\epsilon}),
\end{aligned}$$

while $\mathcal{I}_2$ can be estimated as follows:

$$\begin{aligned}
\mathcal{I}_2 &= \int_1^{\infty}(r+iu)^{-1}\exp\left(\left(n-\frac{1}{24}\right)(r+iu)+\frac{\pi^2}{6(r+iu)}\right)du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\int_1^{\infty}(r+iu)^{-1}\exp\left(\left(n-\frac{1}{24}\right)iu+\frac{\pi^2}{6(r+iu)}\right)du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\left(\int_1^{\infty}(iu)^{-1}\exp\left(\left(n-\frac{1}{24}\right)iu\right)du+O\left(\int_1^{\infty}u^{-2}\,du\right)\right)\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\cdot O(1)\\
&\ll np(n)\exp\left(-\sqrt{\frac{\pi^2n}{6}}\right).
\end{aligned}$$

- For $b>-3/2$, the same method would lead to non-convergent integrals, so we have to use another change of variables first. In the integral

$$\frac{1}{2\pi i}\int_{r-iR}^{r+iR}t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt,$$

where $R=2r^{1+\epsilon}$, set $\pi^2/6t=w$ to get

$$\left(\frac{\pi^2}{6}\right)^{b+3/2}\cdot\frac{1}{2\pi i}\int_{s-iS}^{s+iS}w^{-b-5/2}\exp\left(w+\frac{\pi^2(n-1/24)}{6w}\right)dw,$$

where

$$s=\frac{\pi^2r}{6(r^2+R^2)}\quad\text{and}\quad S=\frac{\pi^2R}{6(r^2+R^2)}.$$

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Now we complete the integral as in the first two cases (note that the exponent $-b-5/2$ is less than -1, so the same steps can be applied) to obtain

$$\left(\frac{4\pi^2}{24n-1}\right)^{b/2+3/4} I_{b+3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)$$

plus an error term of the desired order. Note in particular that we obtain the first term of Rademacher's series (1.2) in the special case $b=0$. Dividing by $p(n)$, we finally end up with the stated formula.

The second part of the theorem can be reduced to the first part quite easily. We now have to consider the integral

$$\frac{1}{2\pi i}\int_{r-2ir^{1+\epsilon}}^{r+2ir^{1+\epsilon}} t^{b+1/2}\log\frac{1}{t}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt.$$

For our purposes, it is convenient to take $r=2\pi/\sqrt{24n-1}$ here rather than $r=\pi/\sqrt{6n}$, which is only a minor modification and does not alter the rest of the argument. Now write

$$\log\frac{1}{t}=\log\frac{1}{r}-\log\left(1-\left(1-\frac{t}{r}\right)\right)=\log\frac{1}{r}+\sum_{k=1}^{L}\frac{1}{k}\left(1-\frac{t}{r}\right)^k+O(r^{\epsilon(L+1)})$$
$$=\log\frac{1}{r}+\sum_{k=1}^{L}\frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{t^j}{r^j}+O(r^{\epsilon(L+1)}).$$

Noting that the error term can be made arbitrarily small by expanding sufficiently far, we can now apply the first part of the theorem to each of the resulting summands, which completes the proof. □

4. Examples

To demonstrate the power of our results, we now look at different examples of interesting partition statistics and apply our general scheme. The order differs slightly from the order in the Introduction (where the simplest statistics from a combinatorial point of view came first), since we start with the analytically simplest examples. Let us first consider some instances in which the asymptotic expansion of $F(e^{-t})$ can be obtained directly.

4.1. Number of distinct parts

Of all the examples presented in the Introduction, this one is probably the simplest. We obtain the following result.

Proposition 4.1. *The average number of distinct parts* (*which equals the average number of ones*) *in a random partition of n is*

$$\frac{\sqrt{6n}}{\pi}+\frac{6-\pi^2}{2\pi^2}+\frac{2\pi^4-3\pi^2+216}{24\pi^3\sqrt{6n}}+\frac{54-\pi^4}{12\pi^4 n}+O(n^{-3/2}).$$

Proof. As explained in the Introduction, the function F is in this case

$$F(e^{-t}) = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + O(t^3).$$

By Theorem 2.3, this directly translates to the following formula for the average number of distinct parts in a random partition of n:

$$\frac{\sqrt{24n-1}}{2\pi}\frac{I_{1/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} - \frac{1}{2} + \frac{1}{12}\frac{2\pi}{\sqrt{24n-1}}\frac{I_{5/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + O(n^{-3/2}),$$

Making use of the asymptotic expansions for Bessel functions mentioned in Remark 3, this can be further simplified to

$$\frac{\sqrt{24n-1}}{2\pi}\frac{m}{m-1} - \frac{1}{2} + \frac{\pi}{6\sqrt{24n-1}}\frac{m}{m-1}\left(1-\frac{3}{m}+\frac{3}{m^2}\right) + O(n^{-3/2}),$$

with

$$m = \sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}.$$

Finally, one can turn this into an asymptotic expansion in powers of n (which is best done by means of computer algebra) to obtain the final result. □

Remark 4. Of course, it is possible to determine arbitrarily many terms of the asymptotic expansion in this and later examples.

4.2. Number of parts of a given size

This is only a slight generalization of the previous example. The expansion of $F(e^{-t})$ around $t = 0$ is given by

$$F(e^{-t}) = \frac{e^{-dt}}{1-e^{-dt}} = \frac{1}{e^{dt}-1} = \frac{1}{dt} - \frac{1}{2} + \frac{dt}{12} + O(t^3),$$

to the effect that we get the following result.

Proposition 4.2. *The average number of parts of size d in a random partition of n is*

$$\frac{\sqrt{6n}}{\pi d} + \frac{6-\pi^2 d}{2\pi^2 d} + \frac{2\pi^4 d^2 - 3\pi^2 + 216}{24\pi^3 d\sqrt{6n}} + \frac{54-\pi^4 d^2}{12\pi^4 dn} + O(n^{-3/2}).$$

4.3. Number of parts with given multiplicity

This is our last example in which the asymptotic expansion of $F(e^{-t})$ can be determined in such a direct way, since $F(x)$ is a simple rational function as shown in the Introduction. As was also mentioned there, the main term of the mean has already been determined in [4, 15]. For the sake of the example, we also include an asymptotic formula for the variance, which shows the typical cancellation in the main term.

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Proposition 4.3. *The average number of parts of multiplicity d in a random partition of n is*

$$\frac{\sqrt{6n}}{\pi d(d+1)} + \frac{3}{\pi^2 d(d+1)} + \frac{2\pi^4 d(d+1) - 3\pi^2 + 216}{24\pi^3 d(d+1)\sqrt{6n}} + \frac{54 + \pi^4 d(d+1)}{12\pi^4 d(d+1)n} + O(n^{-3/2}),$$

and the variance of this parameter is given by

$$\frac{(4\pi^2 d^3 + 5\pi^2 d^2 + \pi^2 d - 12d - 6)\sqrt{6n}}{2\pi^3 d^2 (d+1)^2 (2d+1)} + \frac{3(4\pi^2 d^3 + 5\pi^2 d^2 + \pi^2 d - 24d - 12)}{2\pi^4 d^2 (d+1)^2 (2d+1)} + O(n^{-1/2}).$$

Proof. This is again based on Theorem 2.3, making use of the expansion

$$F(e^{-t}) = \frac{(1-e^{-t})e^{-dt}}{(1-e^{-dt})(1-e^{-(d+1)t})} = \frac{1}{e^{dt}-1} - \frac{1}{e^{(d+1)t}-1} = \frac{1}{d(d+1)t} - \frac{t}{2} + O(t^3).$$

The variance is treated by the same technique. □

4.4. Largest part, largest repeated part

Quite frequently, the Mellin transform comes to our aid in the computation of asymptotic expansions. As our first example, let us study the sum

$$\sum_{j=1}^{\infty} \frac{e^{-jt}}{1-e^{-jt}} = \sum_{j=1}^{\infty} \frac{1}{e^{jt}-1},$$

which is needed in the analysis of the statistics 'largest part' (equivalently also the length) and 'largest repeated part'; see the discussion of the generating functions in the Introduction. The Mellin transform of this function is easily found to be $\zeta(s)^2\Gamma(s)$, which has poles at 1, 0 and all negative odd integers. By means of a standard technique involving the Mellin inversion formula (see [6] for details), the behaviour at these poles can be translated to an asymptotic expansion around $t = 0$:

$$\sum_{j=1}^{\infty} \frac{1}{e^{jt}-1} = \frac{\log(1/t) + \gamma}{t} + \frac{1}{4} - \frac{t}{144} + O(t^3).$$

Now application of Theorem 2.3 immediately yields an asymptotic expansion for the mean of the largest part that is repeated at least d times.

Proposition 4.4. *The largest part in a random partition of n that is repeated at least d times is on average equal to*

$$\frac{\sqrt{6n}}{2\pi d}\left(\log n + 2\gamma - \log\left(\frac{\pi^2 d^2}{6}\right)\right) + \frac{3}{2\pi^2 d}\log n + \frac{1}{4} + \frac{3(1 + 2\gamma - \log(\pi^2 d^2/6))}{2\pi^2 d} + O\left(\frac{\log n}{n}\right).$$

The result of Husimi [12] and Kessler and Livingston [14] that was mentioned in the Introduction is included as a special case ($d = 1$). For $d = 2$, the formula agrees with the asymptotic expansion found in [10].

4.5. Moments of a partition

This example shows an interesting phenomenon: we obtain an asymptotic expansion of $F(e^{-t})$ with a particularly strong error term, which in turn leads to strong asymptotic estimates for the mean of the kth moment in the case that k is odd. Specifically, we have the following result; note that the error term is much stronger than in all previous examples.

Proposition 4.5. *For any odd integer $k > 1$, the average kth moment of a random partition of n is given by*

$$k!\zeta(k+1)\left(\frac{\sqrt{24n-1}}{2\pi}\right)^{k+1} \cdot \frac{I_{k-1/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} - \frac{\zeta(-k)}{2} + O\left(\exp(-n^{1/2-\epsilon})\right)$$

for any $\epsilon > 0$.

Proof. First of all, we find easily that the Mellin transform of

$$F(e^{-t}) = \sum_{j=1}^{\infty} \frac{j^k}{e^{jt}-1}$$

is given by $\zeta(s)\zeta(s-k)\Gamma(s)$. For even k, this yields poles at $k+1$, 1 and all negative odd integers, so that

$$F(e^{-t}) = \frac{k!\zeta(k+1)}{t^{k+1}} + \frac{\zeta(1-k)}{t} + \frac{\zeta(-k-1)t}{12} - \frac{\zeta(-k-3)t^3}{720} + \cdots.$$

If k is odd, however, the zeros of $\zeta(s)\zeta(s-k)$ cancel with the poles of $\Gamma(s)$ at the negative integers, so that $s = k+1$ and $s = 0$ are the only poles (if $k > 1$), and the Mellin transform yields an error term whose order can be any power of t. In fact, one can employ a known technique (see for instance [23]) to obtain a functional equation for $f(t) = F(e^{-t})$ as follows. Shifting the path of integration in the Mellin inversion formula to the left, we obtain

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty} \zeta(s)\zeta(s-k)\Gamma(s)t^{-s}\,ds \\ &= \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \zeta(s)\zeta(s-k)\Gamma(s)t^{-s}\,ds. \end{aligned}$$

Now we replace s by $k+1-s$ to obtain

$$f(t) = \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty} \zeta(k+1-s)\zeta(1-s)\Gamma(k+1-s)t^{s-k-1}\,ds.$$

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The functional equations of the ζ-function and the Γ-function now yield

$$\begin{aligned}
&\zeta(k+1-s)\zeta(1-s)\\
&\quad= 2^{1+k-s}\pi^{k-s}\cos\left(\frac{\pi(s-k)}{2}\right)\Gamma(s-k)\zeta(s-k)\cdot 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)\\
&\quad= 2^{2+k-2s}\pi^{k-2s}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right)\Gamma(s-k)\zeta(s)\zeta(s-k)\Gamma(s)\\
&\quad= 2^{1+k-2s}\pi^{k-2s}\sin\left(\frac{\pi k}{2}\right)\sin(\pi s)\Gamma(s-k)\zeta(s)\zeta(s-k)\Gamma(s)\\
&\quad= 2^{1+k-2s}\pi^{k-2s}\sin\left(\frac{\pi k}{2}\right)\sin(\pi s)\cdot\frac{\pi}{\sin(\pi(s-k))\Gamma(k+1-s)}\cdot\zeta(s)\zeta(s-k)\Gamma(s)\\
&\quad= (2\pi)^{k+1-2s}\sin\left(\frac{\pi k}{2}\right)\sin(\pi s)\cdot\frac{1}{-\sin(\pi s)\Gamma(k+1-s)}\cdot\zeta(s)\zeta(s-k)\Gamma(s)\\
&\quad= (-1)^{(k+1)/2}(2\pi)^{k+1-2s}\cdot\frac{1}{\Gamma(k+1-s)}\cdot\zeta(s)\zeta(s-k)\Gamma(s),
\end{aligned}$$

so that finally

$$\begin{aligned}
f(t) &= \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \left(\frac{2\pi i}{t}\right)^{k+1}\cdot\frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty}\zeta(s)\zeta(s-k)\Gamma(s)\left(\frac{4\pi^2}{t}\right)^{-s}ds\\
&= \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \left(\frac{2\pi i}{t}\right)^{k+1} f\left(\frac{4\pi^2}{t}\right).
\end{aligned}$$

Since

$$f\left(\frac{4\pi^2}{t}\right) \ll \exp\left(-\frac{4\pi^2}{t}\right),$$

Theorem 2.3 now shows that the average kth moment (k odd, $k>1$) of a random partition of n is given by

$$k!\zeta(k+1)\left(\frac{\sqrt{24n-1}}{2\pi}\right)^{k+1}\cdot\frac{I_{k-1/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} - \frac{\zeta(-k)}{2} + O\left(\exp(-n^{1/2-\epsilon})\right)$$

for any $\epsilon>0$, completing our proof. □

4.6. Longest run

Our final example is also the most complicated one. As in the previous example, we encounter the situation that the Mellin transform of $F(e^{-t})$ has only finitely many poles. The technique to obtain a very precise asymptotic expansion is similar to the previous example, but the details are somewhat more intricate. The final result is stated in the following proposition; a weak version (first-order asymptotics only) was given in [25].

Proposition 4.6. *The average length of the longest run in a random partition of n is*

$$\frac{(2\sqrt{3}\pi-9)(24n-1)}{3(\pi\sqrt{24n-1}-6)} - \frac{1}{2} + O\left(\exp(-Cn^{1/4})\right), \tag{4.1}$$

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and the variance is given by

$$\frac{(24n-1)^{3/2}((135\pi-12\sqrt{3}\pi^2-7\pi^3)\sqrt{24n-1}+(18\pi^2+144\sqrt{3}\pi-972))}{3\pi(\pi\sqrt{24n-1}-6)^2} + \frac{1}{12}+O\big(\exp(-Cn^{1/4})\big) \tag{4.2}$$

for some $C>0$.

It is actually possible to determine the asymptotic behaviour of all moments and thus to determine the limiting distribution (different approaches were used in [17] and [25]).

Proposition 4.7. *Let* R_n *denote the longest run of a random partition of n. The random variable* $n^{-1/2}R_n$ *converges weakly to a limiting distribution with distribution function*

$$\Psi(x)=\prod_{j=1}^{\infty}(1-e^{-\pi jx/\sqrt{6}}).$$

Proof of Proposition 4.6. Recall the generating function for the mth moment, given by

$$P(x)\cdot\sum_{k=1}^{\infty}(k^m-(k-1)^m)\big(1-P(x^k)^{-1}\big).$$

We are therefore interested in the behaviour of functions of the form

$$\sum_{k=1}^{\infty}k^r\big(1-P(e^{-kt})^{-1}\big)$$

as $t\to 0$. In the following, it will be convenient to replace t by $2\pi t$, so we study the function

$$G_r(t)=\sum_{k=1}^{\infty}k^rH(kt),$$

where

$$H(t)=1-P(e^{-2\pi t})^{-1}=1-\prod_{j=1}^{\infty}\big(1-e^{-2\pi jt}\big).$$

Let us first consider the Mellin transform $H^*(s)$ of H. By (1.3), H satisfies the functional equation

$$1-H(t)=\frac{1}{\sqrt{t}}\exp\left(\frac{\pi}{12}\left(t-\frac{1}{t}\right)\right)\left(1-H\left(\frac{1}{t}\right)\right).$$

Therefore, for $\operatorname{Re}s>0$,

$$\begin{aligned}H^*(s)&=\int_0^{\infty}t^{s-1}H(t)\,dt=\int_0^1 t^{s-1}H(t)\,dt+\int_1^{\infty}t^{s-1}H(t)\,dt\\&=\int_1^{\infty}t^{-s-1}H\left(\frac{1}{t}\right)dt+\int_1^{\infty}t^{s-1}H(t)\,dt\end{aligned}$$

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$$
\begin{aligned}
&= \int_1^\infty t^{-s-1}\left[1-\sqrt{t}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(1-H(t)\right)\right]dt + \int_1^\infty t^{s-1}H(t)\,dt \\
&= \frac{1}{s} - \int_1^\infty t^{-s-1/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)dt \\
&\quad + \int_1^\infty \left[t^{-s-1/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)+t^{s-1}\right]H(t)\,dt.
\end{aligned}
$$

Since $H(t)$ decreases exponentially as $t\to\infty$, the integrals converge for any $s\in\mathbb{C}$, and so this gives us a meromorphic continuation of $H^*(s)$ to $\mathbb{C}$ (with a simple pole at $s=0$). We may thus replace s by $1-s$ to obtain

$$
\begin{aligned}
H^*(1-s) &= \frac{1}{1-s} - \int_1^\infty t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)dt \\
&\quad + \int_1^\infty \left[t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)+t^{-s}\right]H(t)\,dt \\
&= -\int_1^\infty t^{-s}\,dt - \int_1^\infty t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)dt \\
&\quad + \int_1^\infty \left[t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)+t^{-s}\right]H(t)\,dt \\
&= \int_1^\infty t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(H(t)-1\right)dt + \int_1^\infty t^{-s}\left(H(t)-1\right)dt \\
&= \int_1^\infty t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(H(t)-1\right)dt + \int_0^1 t^{s-2}\left(H\left(\frac{1}{t}\right)-1\right)dt \\
&= \int_1^\infty t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(H(t)-1\right)dt \\
&\quad + \int_0^1 t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(H(t)-1\right)dt \\
&= \int_0^\infty t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(H(t)-1\right)dt
\end{aligned}
$$

for $\operatorname{Re} s > 1$, which is exactly the Mellin transform $K^*(s)$ of

$$
K(t) = \frac{1}{\sqrt{t}}\exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)\left(H(t)-1\right) = \frac{1}{t}\left(H\left(\frac{1}{t}\right)-1\right) = -\frac{1}{t}\prod_{j=1}^{\infty}(1-e^{-2j\pi/t}).
$$

Returning to $G_r(t)$, it is clear that the Mellin transform of this function is

$$
G_r^*(s) = H^*(s)\zeta(s-r),
$$

and thus

$$
G_r(t) = \frac{1}{2\pi i}\int_{r+2-i\infty}^{r+2+i\infty} H^*(s)\zeta(s-r)t^{-s}\,ds.
$$

We shift the path of integration to the left, picking up residues at $s=r+1$ and $s=0$.

- At $s = r + 1$, the residue is

$$\frac{H^*(r+1)}{t^{r+1}}.$$

Since $H(t)$ can also be written as

$$H(t) = \sum_{m=1}^{\infty}(-1)^{m-1}(e^{-m(3m-1)\pi t} + e^{-m(3m+1)\pi t})$$

by virtue of Euler's pentagonal theorem, we also have

$$H^*(s) = \pi^{-s}\Gamma(s)\sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{1}{(m(3m-1))^s} + \frac{1}{(m(3m+1))^s}\right)$$

and thus

$$H^*(r+1) = \frac{r!}{\pi^{r+1}}\sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{1}{(m(3m-1))^{r+1}} + \frac{1}{(m(3m+1))^{r+1}}\right).$$

In particular,

$$H^*(1) = \frac{2\sqrt{3}\pi - 9}{3\pi} \quad \text{and} \quad H^*(2) = \frac{54 - 8\sqrt{3}\pi - \pi^2}{2\pi^2}$$

after a few manipulations.

- The residue of $H^*(s)$ at $s = 0$ is 1 by the above calculations, and therefore the residue of $H^*(s)\zeta(s-r)t^{-s}$ at $s = 0$ is $\zeta(-r)$.

Hence we have

$$\begin{aligned}G_r(t) &= \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} H^*(s)\zeta(s-r)t^{-s}\,ds \\ &= \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H^*(1-s)\zeta(1-s-r)t^{s-1}\,ds \\ &= \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) \\ &\quad + \frac{1}{(2\pi)^r t}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} 2\cos\left(\frac{\pi(s+r)}{2}\right)\Gamma(s+r)\zeta(s+r)K^*(s)\left(\frac{2\pi}{t}\right)^{-s} ds\end{aligned}$$

by the functional equation of the zeta function and the equation for $H^*(1-s)$ that was deduced above. This can be written as

$$G_r(t) = \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{(2\pi)^r t}\sum_{k=1}^{\infty} k^{-r}\Phi_r\left(\frac{2\pi k}{t}\right),$$

where Φ_r is the function whose Mellin transform is

$$\Phi_r^*(s) = 2\cos\left(\frac{\pi(s+r)}{2}\right)\Gamma(s+r)K^*(s).$$

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Noting that $\Gamma(s+r)$ is the Mellin transform of $t^r e^{-t}$, while $K^*(s)$ is the Mellin transform of

$$K(t) = -\frac{1}{t}\prod_{j=1}^{\infty}(1-e^{-2j\pi/t}),$$

we find that $\Gamma(s+r)K^*(s)$ is the transform of the convolution

$$\begin{aligned}
L_r(t) &= -\int_0^\infty x^r\, e^{-x}\frac{x}{t}\prod_{k=1}^{\infty}(1-e^{-2k\pi x/t})\,\frac{dx}{x}\\
&= -\frac{1}{t}\int_0^\infty x^r\, e^{-x}\sum_{m\in\mathbb{Z}}(-1)^m\, e^{-m(3m-1)\pi x/t}\,dx\\
&= -\frac{r!}{t}\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{(1+m(3m-1)\pi/t)^{r+1}} = -r!t^r\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{(t+m(3m-1)\pi)^{r+1}},
\end{aligned}$$

making use of Euler's pentagonal theorem again. This can be simplified further:

$$\begin{aligned}
L_r(t) &= -(-t)^r\cdot\frac{d^r}{dt^r}\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{t+m(3m-1)\pi} = -\frac{(-t)^r}{3\pi}\cdot\frac{d^r}{dt^r}\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{m^2-\frac{m}{3}+\frac{t}{3\pi}}\\
&= -\frac{(-t)^r}{3\pi}\cdot\frac{d^r}{dt^r}\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{\left(m-\frac{1}{6}+\sqrt{\frac{1}{36}-\frac{t}{3\pi}}\right)\left(m-\frac{1}{6}-\sqrt{\frac{1}{36}-\frac{t}{3\pi}}\right)}\\
&= (-t)^r\cdot\frac{d^r}{dt^r}\frac{1}{\pi\sqrt{1-\frac{12t}{\pi}}}\sum_{m\in\mathbb{Z}}(-1)^m\left(\frac{1}{m-\frac{1}{6}+\sqrt{\frac{1}{36}-\frac{t}{3\pi}}}-\frac{1}{m-\frac{1}{6}-\sqrt{\frac{1}{36}-\frac{t}{3\pi}}}\right)\\
&= (-t)^r\cdot\frac{d^r}{dt^r}\frac{1}{\pi\sqrt{1-\frac{12t}{\pi}}}\left(\pi\csc\left(\frac{\pi}{6}+\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)-\pi\csc\left(\frac{\pi}{6}-\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)\right)\\
&= (-t)^r\cdot\frac{d^r}{dt^r}\frac{1}{\sqrt{1-\frac{12t}{\pi}}}\left(\csc\left(\frac{\pi}{6}+\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)-\csc\left(\frac{\pi}{6}-\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)\right)
\end{aligned}$$

by virtue of the partial fraction decomposition of the cosecant. This function is meromorphic on the entire complex plane, in spite of the square root. Finally, since

$$2\cos\left(\frac{\pi(s+r)}{2}\right) = i^{-s-r}+(-i)^{-s-r},$$

we find that $\Phi_r^*(s)$ is the Mellin transform of

$$\Phi_r(t) = i^{-r}L_r(it)+(-i)^{-r}L_r(-it).$$

To justify this formally, note that the function L_r can be estimated by $L_r(z)\ll\exp(-C\sqrt{|z|})$ for some constant C inside the cone $\{z : |\arg z|\leqslant 3\pi/4\}$. The Mellin transform of $L_r(it)$ is given by

$$\int_0^\infty t^{s-1}L_r(it)\,dt = i^{-s}\int_0^{i\infty}t^{s-1}L_r(t)\,dt = i^{-s}\lim_{R\to\infty}\int_0^{iR}t^{s-1}L_r(t)\,dt.$$

Now replace the line segment from 0 to iR by a line segment from 0 to R and a quarter-circle with parametrization $t = R\,e^{i\theta}$. Then we get

$$\int_0^\infty t^{s-1} L_r(it)\,dt = i^{-s} \lim_{R\to\infty} \left(\int_0^R t^{s-1} L_r(t)\,dt + i^{1-s} R^s \int_0^{\pi/2} e^{i\theta s} L_r(R\,e^{i\theta})\,d\theta \right).$$

The integral along the quarter-circle tends to zero as $R \to \infty$ by the estimate given above. Hence we end up with $i^{-s} L_r^*(s)$, as claimed. The Mellin transform of $L_r(-it)$ is treated analogously.

It is not difficult to show now that

$$\Phi_r(t) = O\left(t^{(r-1)/2} \exp\left(-\sqrt{\frac{\pi t}{6}} \right) \right)$$

as $t \to \infty$, so that we finally obtain

$$G_r(t) = \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + O\left(t^{-(r+1)/2} \exp\left(-\sqrt{\frac{\pi^2}{3t}} \right) \right)$$

as $t \to 0$. Now we can return to the study of the moments of the longest run in integer partitions. As mentioned at the beginning, the generating function for the mth moment is given by

$$\begin{aligned} P(x) \cdot \sum_{k=1}^{\infty} (k^m - (k-1)^m) \left(1 - \frac{1}{P(x^k)} \right) \\ = P(x) \cdot \sum_{r=0}^{m-1} (-1)^{m-r-1} \binom{m}{r} \sum_{k \geqslant 1} k^r \left(1 - \frac{1}{P(x^k)} \right), \end{aligned}$$

which is of the form $P(x)F(x)$ with

$$\begin{aligned} F(e^{-t}) &= \sum_{r=0}^{m-1} (-1)^{m-r-1} \binom{m}{r} G_r\left(\frac{t}{2\pi} \right) \\ &= \sum_{r=0}^{m-1} (-1)^{m-r-1} \binom{m}{r} \left(\frac{(2\pi)^{r+1} H^*(r+1)}{t^{r+1}} + \zeta(-r) \right) \\ &\quad + O\left(t^{-m/2} \exp\left(-\sqrt{\frac{2\pi^3}{3t}} \right) \right). \end{aligned}$$

Now we are able to apply Theorem 2.3 and obtain a very strong error term again, as in the previous example. In particular, we find that the mean is given by

$$\frac{(2\sqrt{3}\pi - 9)(24n-1)}{3(\pi\sqrt{24n-1} - 6)} - \frac{1}{2} + O\big(\exp(-Cn^{1/4})\big),$$

as stated. The same applies to the variance. □

Proof of Proposition 4.7. Going on to higher moments, we also obtain an alternative approach to the limiting distribution of the longest run that was determined in [17]. Considering only the most significant term, we see that the mth moment of our random

Table 1. Numerical values of p_k.

k	1	2	3	4	5
p_k	0.51609432	0.21321189	0.10730957	0.05975045	0.03548875
k	6	7	8	9	10
p_k	0.02207159	0.01421668	0.00941619	0.00638121	0.00440862

variable R_n is

$$\mathbb{E}(R_n^m) = mH^*(m)(24n)^{m/2} + O(n^{(m-1)/2}).$$

Therefore, the mth moment of the renormalized random variable $n^{-1/2}R_n$ tends to

$$mH^*(m)24^{m/2}.$$

It is not difficult to show that these are exactly the moments associated with the distribution function

$$\Psi(x) = \prod_{j=1}^{\infty}(1 - e^{-\pi jx/\sqrt{6}}).$$

The moments grow sufficiently slowly for the distribution to be characterized by its moments. Hence convergence of moments implies weak convergence of $n^{-1/2}R_n$ to this limiting distribution (see for example [8, p. 778, Theorem C.2]). □

Let us finally consider the problem of determining the probability that k is one of the parts of largest multiplicity. This problem leads to a discrete limiting distribution.

Proposition 4.8. *The probability that no part in a random partition of n has greater or equal multiplicity than k and the probability that no part has strictly greater multiplicity than k both tend to*

$$p_k = \int_0^1 \frac{kx^{k-1}}{1-x^k}\prod_{j=1}^{\infty}(1-x^j)\,dx.$$

In particular, the probability that there are two or more longest runs tends to 0. *Numerical values of p_k are given in Table 1. Asymptotically, we have*

$$p_k = \pi\sqrt{2k}e^{-\pi\sqrt{2k/3}}\left(1 + O\left(\frac{1}{k}\right)\right).$$

Remark 5. The probabilities p_k add up to 1. We have the rather curious identity

$$1 = \sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty}\int_0^1 \frac{kx^{k-1}}{1-x^k}\prod_{j=1}^{\infty}(1-x^j)\,dx = \int_0^1\left(\sum_{k=1}^{\infty}\sigma(k)x^{k-1}\right)\prod_{j=1}^{\infty}(1-x^j)\,dx,$$

where $\sigma(k)$ denotes the sum of all divisors of k. This follows directly from the fact that

$$-\frac{d}{dx}\prod_{j=1}^{\infty}(1-x^j)=\sum_{k=1}^{\infty}\frac{kx^{k-1}}{1-x^k}\prod_{j=1}^{\infty}(1-x^j),$$

as can be seen by logarithmic differentiation.

Proof. Recall the generating function for all partitions that have the first property, which is given by

$$P(x)\cdot\sum_{\ell=1}^{\infty}\frac{x^{k\ell}(1-x^k)}{(1-x^{k(\ell+1)})P(x^{\ell+1})}.$$

We have to study the behaviour of the second factor as $x\to 1$. Substitute $x=e^{-t}$ and simplify to obtain

$$\begin{aligned}F_k(e^{-t})&=\sum_{\ell=1}^{\infty}\frac{e^{-k\ell t}(1-e^{-kt})}{(1-e^{-k(\ell+1)t})P(e^{-(\ell+1)t})}\\&=\sum_{\ell=1}^{\infty}\frac{e^{kt}-1}{(e^{k(\ell+1)t}-1)P(e^{-(\ell+1)t})}\\&=(kt+O(t^2))\sum_{\ell=2}^{\infty}M_k(\ell t),\end{aligned}$$

where

$$M_k(t)=\frac{1}{(e^{kt}-1)P(e^{-t})}.$$

The Mellin transform of $M_k(t)$,

$$M_k^*(s)=\int_0^{\infty}\frac{t^{s-1}}{(e^{kt}-1)P(e^{-t})}\,dt,$$

exists for all complex s, since $P(e^{-t})$ tends to ∞ faster than any power of t as $t\to 0$ (by (1.3)) and to 1 as $t\to\infty$. Therefore, the Mellin transform of the sum

$$\sum_{\ell=2}^{\infty}M_k(\ell t)$$

is $(\zeta(s)-1)M_k^*(s)$, which only has a simple pole at $s=1$ with residue 1. Applying the inverse Mellin transform, we thus find that

$$\sum_{\ell=2}^{\infty}M_k(\ell t)\sim\frac{M_k^*(1)}{t}$$

as $t\to 0$, so that finally

$$F_k(e^{-t})\sim kM_k^*(1)=\int_0^{\infty}\frac{k}{(e^{kt}-1)P(e^{-t})}\,dt=\int_0^{1}\frac{kx^{k-1}}{1-x^k}\prod_{j=1}^{\infty}(1-x^j)\,dx.$$

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Hence, by Theorem 2.2, the probability that no part in a random partition of n has larger multiplicity than k tends to $p_k = kM_k^*(1)$. The same is true for the probability that all parts have strictly smaller multiplicity than k (by a similar argument).

It is natural to ask how the sequence p_k would behave for $k \to \infty$. This can be done by adopting ideas that are frequently used in the theory of partitions. We write

$$\begin{aligned} p_k &= \int_0^\infty \frac{k}{(e^{kt}-1)P(e^{-t})}\,dt \\ &= \int_0^\infty \frac{k\,e^{-kt}}{P(e^{-t})}\,dt + \int_0^\infty \frac{k\,e^{-2kt}}{P(e^{-t})}\,dt + \int_0^\infty \frac{k}{e^{2kt}(e^{kt}-1)P(e^{-t})}\,dt. \end{aligned} \tag{4.3}$$

The third integral can be estimated using the functional equation (1.3):

$$\begin{aligned} \int_0^\infty \frac{k}{e^{2kt}(e^{kt}-1)P(e^{-t})}\,dt &= \int_0^\infty \frac{k}{e^{2kt}(e^{kt}-1)P(e^{-4\pi^2/t})}\sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt \\ &\leqslant \int_0^\infty \sqrt{\frac{2\pi}{t^3}}\exp\left(-\frac{\pi^2}{6t}-\left(2k-\frac{1}{24}\right)t\right)dt \\ &= 2\sqrt{3}\exp\left(-\frac{\pi}{6}\sqrt{48k-1}\right) = O\left(\exp\left(-2\pi\sqrt{\frac{k}{3}}\right)\right). \end{aligned}$$

The remaining integrals are again treated by using the functional equation (1.3):

$$\begin{aligned} \int_0^\infty \frac{k\,e^{-kt}}{P(e^{-t})}\,dt &= \int_0^\infty \frac{k\,e^{-kt}}{P(e^{-4\pi^2/t})}\sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt \\ &= \int_0^\infty k\,e^{-kt}\sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt \\ &\quad + \int_0^\infty k\,e^{-kt}\sqrt{\frac{2\pi}{t}}\left(\frac{1}{P(e^{-4\pi^2/t})}-1\right)\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt. \end{aligned}$$

The first of these two integrals evaluates to

$$\frac{4\pi\sqrt{3}k\exp\left(-\frac{\pi}{6}\sqrt{24k-1}\right)}{\sqrt{24k-1}},$$

whereas the second integral is estimated as

$$O\left(\sqrt{k}\exp\left(-5\pi\sqrt{\frac{2k}{3}}\right)\right)$$

using

$$1 - 1/P(x) \leqslant \frac{3}{2}x.$$

The middle integral in (4.3) can be treated analogously, and it gives a contribution of

$$O\left(\sqrt{k}\exp\left(-2\pi\sqrt{\frac{k}{3}}\right)\right).$$

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Thus we have found

$$p_k = \frac{4\pi\sqrt{3k}\exp\left(-\frac{\pi}{6}\sqrt{24k-1}\right)}{\sqrt{24k-1}} + O\left(\sqrt{k}\exp\left(-2\pi\sqrt{\frac{k}{3}}\right)\right)$$
$$= \pi\sqrt{2k}e^{-\pi\sqrt{2k/3}}\left(1 + O\left(\frac{1}{k}\right)\right).$$

It is clear that the above computations can be extended to yield an asymptotic expansion of p_k. □

5. Conclusion

As the examples in the previous section show, our results are widely applicable to a variety of different partition statistics, and there are certainly many more examples. Once an asymptotic expansion of $F(e^{-t})$ has been found, everything else is essentially automatic and can be done by a computer algebra system. It is remarkable that one obtains very precise asymptotic formulae with error terms that decrease superpolynomially in certain cases (as in (4.1) and (4.2)). In some instances, as we have seen, limiting distributions can be determined as well. Let us finally mention that our method should also apply to partitions into distinct parts and other families of partitions.

References

[1] Andrews, G. E. (1998) *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press.

[2] Apostol, T. M. (1990) *Modular Functions and Dirichlet Series in Number Theory*, second edition, Vol. 41 of *Graduate Texts in Mathematics*, Springer.

[3] Brennan, C., Knopfmacher, A. and Wagner, S. (2008) The distribution of ascents of size *d* or more in partitions of *n*. *Combin. Probab. Comput.* **17** 495–509.

[4] Corteel, S., Pittel, B., Savage, C. and Wilf, H. (1999) On the multiplicity of parts in a random partition. *Random Struct. Alg.* **14** 185–197.

[5] Erdős, P. and Lehner, J. (1941) The distribution of the number of summands in the partitions of a positive integer. *Duke Math. J.* **8** 335–345.

[6] Flajolet, P., Gourdon, X. and Dumas, P. (1995) Mellin transforms and asymptotics: Harmonic sums. *Theoret. Comput. Sci.* **144** 3–58.

[7] Flajolet, P. and Odlyzko, A. (1990) Singularity analysis of generating functions. *SIAM J. Discrete Math.* **3** 216–240.

[8] Flajolet, P. and Sedgewick, R. (2009) *Analytic Combinatorics*, Cambridge University Press.

[9] Goh, W. M. Y. and Schmutz, E. (1995) The number of distinct part sizes in a random integer partition. *J. Combin. Theory Ser. A* **69** 149–158.

[10] Grabner, P. J. and Knopfmacher, A. (2006) Analysis of some new partition statistics. *Ramanujan J.* **12** 439–454.

[11] Han, G.-N. (2008) An explicit expansion formula for the powers of the Euler product in terms of partition hook lengths. `arXiv:0804.1849v3 [math.CO]`

[12] Husimi, K. (1938) Partitio numerorum as occurring in a problem of nuclear physics. In *Proc. Physico-Mathematical Society of Japan* **20** 912–925.

[13] Hwang, H.-K. and Yeh, Y.-N. (1997) Measures of distinctness for random partitions and compositions of an integer. *Adv. Appl. Math.* **19** 378–414.

[14] Kessler, I. and Livingston, M. (1976) The expected number of parts in a partition of *n*. *Monatsh. Math.* **81** 203–212.

[15] Knopfmacher, A. and Munagi, A. O. (2009) Successions in integer partitions. *Ramanujan J.* **18** 239–255.

[16] Knopfmacher, A. and Warlimont, R. (2006) Gaps in integer partitions. *Util. Math.* **71** 257–267.

[17] Mutafchiev, L. R. (2005) On the maximal multiplicity of parts in a random integer partition. *Ramanujan J.* **9** 305–316.

[18] Rademacher, H. (1973) *Topics in Analytic Number Theory* (E. Grosswald, J. Lehner and M. Newman, eds), Vol. 169 of *Die Grundlehren der Mathematischen Wissenschaften*, Springer.

[19] Ralaivaosaona, D. (2012) On the distribution of multiplicities in integer partitions. *Ann. Combin.* **16** 871–889.

[20] Richmond, L. B. (1974/75) The moments of partitions I. *Acta Arith.* **26** 411–425.

[21] Stanley, R. P. (1997) *Enumerative Combinatorics 1*, Vol. 49 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press. Corrected reprint of the 1986 original.

[22] Szekeres, G. (1990) Asymptotic distribution of partitions by number and size of parts. In *Number Theory I: Budapest 1987*, Vol. 51 of *Colloq. Math. Soc. János Bolyai*, North-Holland, pp. 527–538.

[23] Szpankowski, W. (2001) *Average Case Analysis of Algorithms on Sequences*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience.

[24] Wagner, S. (2006) A class of trees and its Wiener index. *Acta Appl. Math.* **91** 119–132.

[25] Wagner, S. (2009) On the distribution of the longest run in number partitions. *Ramanujan J.* **20** 189–206.

[26] Wagner, S. (2011) Limit distributions of smallest gap and largest repeated part in integer partitions. *Ramanujan J.* **25** 229–246.

[27] Watson, G. N. (1995) *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library. Cambridge University Press. Reprint of the second (1944) edition.