Combinatorics, Probability and Computing (2018) 27, 228–244.
doi:10.1017/S0963548317000426

On Edge-Disjoint Spanning Trees in a Randomly Weighted Complete Graph

ALAN FRIEZE[1†] and TONY JOHANSSON[2‡]

[1]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
(e-mail: alan@random.math.cmu.edu)
[2]Department of Mathematics, Uppsala University, 751 06 Uppsala, Sweden
(e-mail: tony.johanssson@math.uu.se)

Received 9 November 2015; revised 28 June 2017; first published online 9 October 2017

Assume that the edges of the complete graph K_n are given independent uniform $[0,1]$ weights. We consider the expected minimum total weight μ_k of $k \geqslant 2$ edge-disjoint spanning trees. When k is large we show that $\mu_k \approx k^2$. Most of the paper is concerned with the case $k = 2$. We show that μ_2 tends to an explicitly defined constant and that $\mu_2 \approx 4.1704288\ldots$.

2010 *Mathematics subject classification*: Primary 05C80
Secondary 05C22

1. Introduction

This paper can be considered to be a contribution to the following general problem. We are given a combinatorial optimization problem where the weights of variables are random. What can be said about the random variable equal to the minimum objective value in this model? The most studied examples of this problem are those of (i) minimum spanning trees, *e.g.* Frieze [10], (ii) shortest paths, *e.g.* Janson [18], (iii) minimum cost assignment, *e.g.* Aldous [1, 2], Linusson and Wästlund [22], Nair, Prabhakar and Sharma [24] and Wästlund [31], and (iv) the travelling salesperson problem, *e.g.* Karp [20], Frieze [12] and Wästlund [32].

The minimum spanning tree problem is a special case of the problem of finding a minimum-weight basis in an element-weighted matroid. Extending the result of [10] has proved to be difficult for other matroids. We are aware of a general result due to Kordecki and Lyczkowska-Hanćkowiak [21] that expresses the expected minimum value as an integral using the Tutte polynomial. The formulae obtained, although exact, are somewhat difficult to penetrate. In this paper we consider the union of k-cycle matroids. We have a fairly simple analysis for $k \to \infty$ and a rather difficult analysis for $k = 2$.

† Research supported in part by NSF grant DMS1362785.
‡ Research supported in part by NSF grant DMS1362785.

Given a connected simple graph $G = (V,E)$ with edge lengths $\mathbf{x} = (x_e : e \in E)$ and a positive integer k, let $\mathrm{mst}_k(G,\mathbf{x})$ denote the minimum length of k edge-disjoint spanning trees of G. ($\mathrm{mst}_k(G) = \infty$ if such trees do not exist). When $\mathbf{X} = (X_e : e \in E)$ is a family of independent random variables, each uniformly distributed on the interval $[0,1]$, denote the expected value $\mathbf{E}[\mathrm{mst}_k(G,\mathbf{X})]$ by $\mathrm{mst}_k(G)$.

As previously mentioned, the case $k = 1$ has been the subject of some attention. When G is the complete graph K_n, Frieze [10] proved that

$$\lim_{n\to\infty} \mathrm{mst}_1(K_n) = \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

Generalizations and refinements of this result were subsequently given in Steele [30], Frieze and McDiarmid [14], Janson [17], Penrose [28], Beveridge, Frieze and McDiarmid [4], Frieze, Ruszinko and Thoma [15] and most recently in Cooper, Frieze, Ince, Janson and Spencer [7].

In this paper we discuss the case $k \geqslant 2$ when $G = K_n$, and define

$$\mu_k^* = \liminf_{n\to\infty} \mathrm{mst}_k(K_n) \quad \text{and} \quad \mu_k^{**} = \limsup_{n\to\infty} \mathrm{mst}_k(K_n).$$

Conjecture 1.1. $\mu_k^* = \mu_k^{**}$, *that is,* $\lim_{n\to\infty} \mathit{mst}_k(K_n)$ *exists.*

Theorem 1.2.

$$\lim_{k\to\infty} \frac{\mu_k^*}{k^2} = \lim_{k\to\infty} \frac{\mu_k^{**}}{k^2} = 1.$$

Theorem 1.3. *With* f_k *and* $c_2' \approx 3.59$ *and* $\lambda_2' \approx 2.688$ *as defined in* (2.1)*,* (2.6)*,* (5.9)*,*

$$\begin{aligned}\mu_2 &= 2c_2' - \frac{(c_2')^2}{4} \\ &\quad + \int_{\lambda=\lambda_2'}^{\infty} \left(2 - \frac{\lambda e^{\lambda}}{2f_2(\lambda)} + \frac{\lambda f_2(\lambda)}{2e^{\lambda}} - 2\frac{f_3(\lambda)}{e^{\lambda}}\right)\left(\frac{e^{\lambda}}{f_2(\lambda)} + \frac{\lambda e^{\lambda}}{f_2(\lambda)} - \frac{\lambda e^{\lambda} f_1(\lambda)}{f_2(\lambda)^2}\right) d\lambda \\ &= 4.17042881\ldots.\end{aligned}$$

There appears to be no clear connection between μ_2 and the ζ-function.

Before proceeding to the proofs of Theorems 1.2 and 1.3, we note some properties of the κ-core of a random graph.

2. The κ-core

The functions

$$f_i(\lambda) = \sum_{j=i}^{\infty} \frac{\lambda^j}{j!}, \quad i = 0,1,2,\ldots, \tag{2.1}$$

figure prominently in our calculations. For $\lambda > 0$ define

$$g_i(\lambda) = \frac{\lambda f_{2-i}(\lambda)}{f_{3-i}(\lambda)}, \quad g_i(0) = 3 - i,\ i = 0,1,2.$$

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

The κ-*core* $C_\kappa(G)$ of a graph G is the largest set of vertices that induces a graph H_κ such that the minimum degree $\delta(H_\kappa) \geqslant \kappa$. Pittel, Spencer and Wormald [29] proved that there exist constants $c_\kappa, \kappa \geqslant 3$ such that if $p = c/n$ and $c < c_\kappa$ then w.h.p. $G_{n,p}$ has no κ-core, and that if $c > c_\kappa$ then w.h.p. $G_{n,p}$ has a κ-core of linear size. We list some facts about these cores that we will need below.

Given λ, let $\text{Po}(\lambda)$ be the Poisson random variable with mean λ and let

$$\pi_r(\lambda) = \Pr\{\text{Po}(\lambda) \geqslant r\} = e^{-\lambda} f_r(\lambda).$$

Then

$$c_\kappa = \inf\left(\frac{\lambda}{\pi_{\kappa-1}(\lambda)} : \lambda > 0\right). \tag{2.2}$$

When $c > c_\kappa$, define $\lambda_\kappa(c)$ by

$$\lambda_\kappa(c) \text{ is the larger of the two roots to the equation } c = \frac{\lambda}{\pi_{\kappa-1}(\lambda)} = \frac{\lambda e^\lambda}{f_{\kappa-1}(\lambda)}. \tag{2.3}$$

Then w.v.h.p.[1] with $\lambda = \lambda_\kappa(c)$ we have that

$$C_\kappa(G_{n,p}) \text{ has } \approx \pi_\kappa(\lambda)n = \frac{f_\kappa(\lambda)}{e^\lambda}n \text{ vertices and } \approx \frac{\lambda^2}{2c}n = \frac{\lambda f_{\kappa-1}(\lambda)}{2e^\lambda}n \text{ edges.} \tag{2.4}$$

Furthermore, when κ is large,

$$c_\kappa = \kappa + (\kappa \log \kappa)^{1/2} + O(\log \kappa). \tag{2.5}$$

Łuczak [23] proved that C_κ is κ-connected w.v.h.p. when $\kappa \geqslant 3$.

Next, let c'_κ be the threshold for the $(\kappa+1)$-core having average degree 2κ. Here (see (2.3) and (2.4))

$$c'_\kappa = \frac{\lambda e^\lambda}{f_\kappa(\lambda)} \quad \text{where } \frac{\lambda f_k(\lambda)}{f_{k+1}(\lambda)} = 2\kappa. \tag{2.6}$$

We have $c_2 \approx 3.35$ and $c'_2 \approx 3.59$.

3. Proof of Theorem 1.2: large k

We will prove Theorem 1.2 in this section. It is relatively straightforward. Theorem 1.3 is more involved and occupies Section 4.

In this section we assume that $k = O(1)$ and is large. Let Z_k denote the sum of the $k(n-1)$ shortest edge lengths in K_n. We have that for $n \gg k$

$$\text{mst}_k(K_n) \geqslant \mathbf{E}[Z_k] = \sum_{\ell=1}^{k(n-1)} \frac{\ell}{\binom{n}{2}+1} = \frac{k(n-1)(k(n-1)+1)}{n(n-1)+2} \in [k^2(1-n^{-1}), k^2]. \tag{3.1}$$

This gives us the lower bound in Theorem 1.2.

[1] For the purposes of this paper, a sequence of events $\mathcal{E}_n$ will be said to occur *with very high probability* (w.v.h.p.) if $\Pr\{\mathcal{E}_n\} = 1 - o(n^{-1})$. Similarly, $\mathcal{E}_n$ will be said to occur *with high probability* (w.h.p.) if $\Pr\{\mathcal{E}_n\} = 1 - o(1)$.

For the upper bound, let $k_0 = k + k^{2/3}$ and consider the random graph H generated by the $k_0(n-1)$ cheapest edges of K_n. The expected total edge weight $\overline{E}_H$ of H is at most k_0^2 (see (3.1)).

Here H is distributed as $G_{n,k_0 n}$. This is sufficiently close in distribution to $G_{n,p}, p = 2k_0/n$ that we can apply the results of Section 2 without further comment. It follows from (2.5) that $c_{2k} < 2k_0$. Putting $\lambda_0 = \lambda_{2k}(2k_0)$, we see from (2.4) that w.v.h.p. H has a $2k$-core C_{2k} with $\sim n\Pr\{\mathrm{Po}(\lambda_0) \geqslant 2k\}$ vertices. It follows from (2.3) that $\lambda_0 = 2k_0\pi_{2k-1}(2k_0) \leqslant 2k_0$, and since $\pi_{2k-1}(\lambda)$ increases with λ and

$$\pi_{2k-1}(2k+k^{2/3}) = \Pr\{\mathrm{Po}(2k+k^{2/3}) \geqslant 2k-1\} \geqslant 1 - e^{-c_1 k^{1/3}}$$

for some constant $c_1 > 0$, we see that

$$\frac{2k+k^{2/3}}{\pi_{2k-1}(2k+k^{2/3})} \leqslant 2k_0,$$

and so $\lambda_0 \geqslant 2k + k^{2/3}$.

A theorem of Nash-Williams [25] states that a $2k$-edge connected graph contains k edge-disjoint spanning trees. Applying the result of Łuczak [23], we see that w.v.h.p. C_{2k} contains k edge-disjoint spanning trees $T_1, T_2, \ldots, T_k$. It remains to argue that we can cheaply augment these trees to spanning trees of K_n. Since $|C_{2k}| \sim n\Pr\{\mathrm{Po}(\lambda) \geqslant 2k\}$ w.v.h.p., we see that w.v.h.p. $D_{2k} = [n] \setminus C_{2k}$ satisfies $|D_{2k}| \leqslant 2ne^{-c_1 k^{1/3}}$.

For each $v \in D_{2k}$ we let S_v be the k shortest edges from v to C_{2k}. We can then add v as a leaf to each of the trees $T_1, T_2, \ldots, T_k$ by using one of these edges. What is the total weight of the edges Y_v, $v \in D_{2k}$? We can bound this probabilistically by using the following lemma from Frieze and Grimmett [13].

Lemma 3.1. *Suppose that $k_1 + k_2 + \cdots + k_M \leqslant a$, and $Y_1, Y_2, \ldots, Y_M$ are independent random variables with Y_i distributed as the k_ith minimum of N independent uniform [0,1] random variables. If $\mu > 1$, then*

$$Pr\left\{Y_1 + \cdots + Y_M \geqslant \frac{\mu a}{N+1}\right\} \leqslant e^{a(1+\ln\mu-\mu)}.$$

Let $\varepsilon = 2e^{-c_1 k^{1/3}}$ and $\mu = 10\ln 1/\varepsilon$, and let $M = k\varepsilon n$, $N = (1-\varepsilon)n$, $a = k^2\varepsilon n$. Let $\mathcal{B}_0$ be the event that there exists a set S of size εn such that the sum over $v \in S$ of the lengths of the k shortest edges from v to $[n] \setminus S$ exceeds $\mu a/(N+1)$. Next let $\mathcal{B}$ be the event that the sum over $v \in S$ of the length of the kth shortest edge from v to $[n] \setminus S$ exceeds $\mu a/(k(N+1))$. We have $\mathcal{B}_0 \subseteq \mathcal{B}$, and applying Lemma 3.1 we see that

$$\Pr\{\mathcal{B}\} \leqslant \binom{n}{\varepsilon n} \exp\{k\varepsilon n(1+\ln\mu-\mu)\} \leqslant \left(\frac{e}{\varepsilon} \cdot e^{-\mu k/2}\right)^{\varepsilon n} = o(n^{-1}).$$

It follows that

$$\mathrm{mst}_k(K_n) \leqslant o(1) + k_0^2 + \frac{\mu a}{N+1} \leqslant k^2 + 3k^{5/3}.$$

The $o(1)$ term is a bound $kn \times o(n^{-1})$, to account for the cases that occur with probability $o(n^{-1})$.

Combining this with (3.1), we see that

$$k^2 \leqslant \mu_k \leqslant k^2 + 3k^{5/3},$$

which proves Theorem 1.2.

4. Proof of Theorem 1.3: $k = 2$

For this case we use the fact that, for any graph $G = (V,E)$, the collection of subsets $I \subseteq E$ that can be partitioned into two edge-disjoint forests form the independent sets in a matroid, this being the matroid which is the union of two copies of the cycle matroid of G. See for example Oxley [27] or Welsh [33]. Let r_2 denote the rank function of this matroid, when $G = K_n$. If G is a subgraph of K_n, then $r_2(G)$ is the rank of its edge-set.

We will follow the proof method in [3], [4] and [17]. Let F denote the random set of edges in the minimum-weight pair of edge-disjoint spanning trees. For any $0 \leqslant p \leqslant 1$, let G_p denote the graph induced by the edges e of K_n which satisfy $X_e \leqslant p$. Note that G_p is distributed as $G_{n,p}$.

For any $0 \leqslant p \leqslant 1$, $\sum_{e \in F} 1_{(X_e > p)}$ is the number of edges of F which are not in G_p, which equals $2n - 2 - r_2(G_p)$. So,

$$\mathrm{mst}_2(K_n, \mathbf{X}) = \sum_{e \in F} X_e = \sum_{e \in F} \int_{p=0}^{1} 1_{(X_e > p)} dp = \int_{p=0}^{1} \sum_{e \in F} 1_{(X_e > p)} dp.$$

Hence, on taking expectations we obtain

$$\mathrm{mst}_2(K_n) = \int_{p=0}^{1} (2n - 2 - \mathbf{E}[r_2(G_p)]) dp. \tag{4.1}$$

It remains to estimate $\mathbf{E}[r_2(G_p)]$. The main contribution to the integral in (4.1) comes from $p = c/n$, where c is constant. Estimating $\mathbf{E}[r_2(G_p)]$ is easy enough for sufficiently small c, but it becomes more difficult for $c > c_2'$ (see (2.6)). When $p = c/n$ for $c > c_k$, we will need to be able to estimate $\mathbf{E}[r_k(C_{k+1}(G_{n,p}))]$. We give partial results for $k \geqslant 3$ and complete results for $k = 2$. We begin with a simple observation.

Lemma 4.1. *Let $k \geqslant 2$. Let $C_{k+1} = C_{k+1}(G)$ denote the graph induced by the $(k+1)$-core of graph G (it may be an empty subgraph). Let $E_k(G)$ denote the set of edges that are* not *contained in C_{k+1}. Then*

$$r_k(G) = |E_k(G)| + r_k(C_{k+1}). \tag{4.2}$$

Proof. We use induction on $|V(G)|$. It is trivial if $|V(G)| = 1$, so assume that $|V(G)| > 1$. If $\delta(G) \geqslant k+1$ then $G = C_{k+1}$ and there is nothing to prove. Otherwise, G contains a vertex v of degree $d_G(v) \leqslant k$. Now $G - v$ has the same $(k+1)$-core as G. If $F_1, \ldots, F_k$ are edge-disjoint forests such that $r_k(G) = |F_1| + \cdots + |F_k|$, then by removing v we see, inductively, that

$$|E_k(G - v)| + r_k(C_{k+1}) = r_k(G - v) \geqslant |F_1| + \cdots + |F_k| - d_G(v) = r_k(G) - d_G(v).$$

On the other hand $G - v$ contains k forests $F_1', \ldots, F_k'$ such that

$$r_k(G - v) = |F_1'| + \cdots + |F_k'| = |E_k(G - v)| + r_k(C_{k+1}).$$

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

We can then add v as a vertex of degree one to $d_G(v)$ of the forests $F_1', \ldots, F_k'$, implying that

$$r_k(G) \geqslant d_G(v) + |E_k(G-v)| + r_k(C_{k+1}).$$

Thus,

$$r_k(G) = d_G(v) + |E_k(G-v)| + r_k(C_{k+1}) = |E_k(G)| + r_k(C_{k+1}). \qquad \square$$

Lemma 4.2. *Let $k \geqslant 2$. If $c_k < c < c_k'$, then w.h.p.*

$$|E(G_{n,c/n})| - o(n) \leqslant r_k(G_{n,c/n}) \leqslant |E(G_{n,c/n})|. \tag{4.3}$$

Proof. We will show that when $c < c_k'$ we can find k disjoint forests $F_1, F_2, \ldots, F_k$ contained in C_{k+1} such that

$$|E(C_{k+1})| - \sum_{i=1}^{k} |E(F_i)| = o(n). \tag{4.4}$$

This implies that $r_k(C_{k+1}) \geqslant |E(C_{k+1})| - o(n)$, and because $r_k(C_{k+1}) \leqslant |E(C_{k+1})|$, the lemma follows from this and Lemma 4.1.

Gao, Pérez-Giménez and Sato [16] show that when $c < c_k'$, no subgraph of $G_{n,p}$ has average degree more than $2k$, w.h.p. Fix $\varepsilon > 0$. Cain, Sanders and Wormald [6] proved that if the average degree of the $(k+1)$-core is at most $2k - \varepsilon$, then w.h.p. the edges of $G_{n,p}$ can be oriented so that no vertex has indegree more than k. It is clear from (2.4) that the edge density of the $(k+1)$-core increases smoothly w.h.p., so we can apply the result of [6] for some value of ε.

It then follows that the edges of $G_{n,p}$ can be partitioned into k sets $\Phi_1, \Phi_2, \ldots, \Phi_k$ where each subgraph $H_i = ([n], \Phi_i)$ can be oriented so that each vertex has indegree at most one. We call such a graph a *partial functional digraph*, or PFD. Each component of a PFD is either a tree or contains exactly one cycle. We obtain $F_1, F_2, \ldots, F_k$ by removing one edge from each such cycle. We must show that w.h.p. we remove $o(n)$ vertices in total. Observe that if Z denotes the number of edges of $G_{n,p}$ that are on cycles of length at most $\omega_0 = \frac{1}{3}\log_c n$, then

$$\mathbf{E}[Z] \leqslant \sum_{\ell=3}^{\omega_0} \ell! \binom{n}{\ell} \ell p^\ell \leqslant \omega_0 c^{\omega_0} \leqslant n^{1/2}.$$

The Markov inequality implies that $Z \leqslant n^{2/3}$ w.h.p. The number of edges removed from the larger cycles to create $F_1, F_2, \ldots, F_k$ can be bounded by $kn/\omega_0 = o(n)$, and this proves (4.4) and the lemma. $\square$

Lemma 4.3. *If $c > c_2'$, then w.h.p. the 3-core of $G_{n,c/n}$ contains two edge-disjoint forests of total size $2|V(C_3)| - o(n)$. In particular, $r_2(C_3(G_{n,c/n})) = 2|V(C_3)| - o(n)$.*

The proof of Lemma 4.3 is postponed to Section 6. We can now prove Theorem 1.3.

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

5. Proof of Theorem 1.3

As noted in (4.1),

$$\mathrm{mst}_2(K_n) = \int_{p=0}^{1} (2n-2-\mathbf{E}[r_2(G_p)])dp. \tag{5.1}$$

A crude calculation shows that if c is large then

$$p \geqslant \frac{c}{n} \text{ implies that } \Pr\{r_2(G_p) < 2n-2-nAc^6e^{-c}\} = o(1), \tag{5.2}$$

for some absolute constant $A > 0$.

Indeed, we know that if $p = c/n$ and c is sufficiently large, then G_p contains a pair of edge-disjoint cycles, each of length at least $n(1-c^6e^{-c})$ with probability $1-\varepsilon_1$, where $\varepsilon_1 = O(n^{-\alpha})$, for some absolute constant $\alpha > 0$: see Frieze [11]. If $p_1 = c_1/n$ and $p_2 = Kp_1$, then

$$\Pr\{r_2(G_{p_2}) < 2n-2-nc^6e^{-c}\} \leqslant \varepsilon_1^{p_2/p_1} = O(n^{-K\alpha}),$$

since G_{p_2} can be generated by adding edges to p_2/p_1 independent copies of G_{p_1}. This confirms (5.2).

So, for large c,

$$\mathrm{mst}_2(K_n) = \int_{p=0}^{c/n} (2n-2-\mathbf{E}[r_2(G_p)])dp + \varepsilon_c, \tag{5.3}$$

where

$$0 \leqslant \varepsilon_c \leqslant An\int_{p=c/n}^{1} (np)^6 e^{-np}dp = A\int_{x=c}^{n} x^6e^{-x}dx \leqslant A\int_{x=c}^{\infty} x^6e^{-x}dx < c^7e^{-c},$$

after changing variables to $x = pn$. Doing this once more, we have

$$\mathrm{mst}_2(K_n) = \int_{x=0}^{c} (2-2n^{-1}-n^{-1}\mathbf{E}[r_2(G_{x/n})])dx + \varepsilon_c. \tag{5.4}$$

By Lemmas 4.1 and 4.2, for $x < c_2'$ we have

$$n^{-1}\mathbf{E}[r_2(G_{x/n})] = n^{-1}\mathbf{E}[|E(G_{x/n})|] - \xi(x,n) = x/2 - \xi(x,n),$$

where $\lim_{n\to\infty} \xi(x,n) = 0$. Now $n^{-1}\mathbf{E}[r_2(G_{x/n})], n = 1,2,\ldots,$ is a sequence of bounded monotone increasing continuous functions of x. This sequence converges pointwise to a continuous function f, and so it converges uniformly to f. Thus we can bound $\max_{0\leqslant x\leqslant c_2'} \xi(x,n) \leqslant \eta(n)$, where $\lim_{n\to\infty} \eta(n) = 0$. Clearly $f(x) = x/2$, and so

$$\int_{x=0}^{c_2'} n^{-1}\mathbf{E}[r_2(G_{x/n})]dx = \int_{x=0}^{c_2'} \frac{x}{2}dx + o(1).$$

By Lemma 4.3, for $x > c_2'$ we have

$$\mathbf{E}[r_2(C_3(G_{x/n}))] = \mathbf{E}[2|V(C_3)|] - o(n).$$

So, by Lemma 4.1,

$$\mathbf{E}[r_2(G_{x/n})] = \mathbf{E}[|E(G_{x/n})| - |E(C_3)| + 2|V(C_3)|] - o(n),$$

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

and

$$\mu_2 = \int_{x=0}^{c_2'} \left(2-\frac{x}{2}\right)dx + \int_{x=c_2'}^{c} \left(2-\frac{1}{n}\left(\frac{xn}{2} - \mathbf{E}[|E(C_3(G_{x/n}))|] + \mathbf{E}[2|V(C_3(G_{x/n}))|]\right)\right)dx + \varepsilon_c + o(1). \tag{5.5}$$

From (2.4), for $p = x/n$ we have

$$\frac{1}{n}\mathbf{E}[|V(C_3)|] = \frac{f_3(\lambda)}{e^\lambda} + o(1),$$
$$\frac{1}{n}\mathbf{E}[|E(C_3)|] = \frac{\lambda f_2(\lambda)}{2e^\lambda} + o(1),$$

where λ is the largest solution to $\lambda e^\lambda / f_2(\lambda) = x$. Thus,

$$\mu_2 = \lim_{n\to\infty} \mathrm{mst}_2(K_n) = \int_{x=0}^{c_2'} \left(2-\frac{x}{2}\right)dx + \int_{x=c_2'}^{c} \left(2-\frac{x}{2}+\frac{\lambda f_2(\lambda)}{2e^\lambda} - 2\frac{f_3(\lambda)}{e^\lambda}\right)dx + \varepsilon_c. \tag{5.6}$$

To calculate this, note that

$$\frac{dx}{d\lambda} = \frac{e^\lambda}{f_2(\lambda)} + \frac{\lambda e^\lambda}{f_2(\lambda)} - \frac{\lambda e^\lambda f_1(\lambda)}{f_2(\lambda)^2} \tag{5.7}$$

so

$$\int_{x=c_2'}^{c} \left(2-\frac{x}{2}+\frac{\lambda f_2(\lambda)}{2e^\lambda} - 2\frac{f_3(\lambda)}{e^\lambda}\right)dx \tag{5.8}$$
$$= \int_{\lambda(c_2')}^{\lambda(c)} \left(2-\frac{\lambda e^\lambda}{2f_2(\lambda)}+\frac{\lambda f_2(\lambda)}{2e^\lambda} - 2\frac{f_3(\lambda)}{e^\lambda}\right)\left(\frac{e^\lambda}{f_2(\lambda)} + \frac{\lambda e^\lambda}{f_2(\lambda)} - \frac{\lambda e^\lambda f_1(\lambda)}{f_2(\lambda)^2}\right)d\lambda + \varepsilon_c,$$

where $\lambda(x)$ is the unique solution to $\lambda e^\lambda / f_2(\lambda) = x$.

Note that

$$\lambda(c_2') \approx 2.688 \quad \text{and} \quad \lambda(c) > \frac{c}{2} \quad \text{for large } c. \tag{5.9}$$

Now for large λ we can bound

$$\left(2-\frac{\lambda e^\lambda}{2f_2(\lambda)}+\frac{\lambda f_2(\lambda)}{2e^\lambda} - 2\frac{f_3(\lambda)}{e^\lambda}\right)\left(\frac{e^\lambda}{f_2(\lambda)} + \frac{\lambda e^\lambda}{f_2(\lambda)} - \frac{\lambda e^\lambda f_1(\lambda)}{f_2(\lambda)^2}\right)$$

from above by $\lambda^3 e^{-\lambda}$. So the range in the integral in (5.8) can be extended to ∞ at the cost of adding an amount δ_c where $0 \leqslant \delta_c \leqslant c^4 e^{-c}$. Using the fact that we can make ε_c, δ_c arbitrarily close to zero by making c arbitrarily large, we obtain the expression for μ_2 claimed in Theorem 1.3.

Attempts to transform the integral in the theorem into an explicit integral with explicit bounds have been unsuccessful. Numerical calculations give

$$\mu_2 \approx 4.1704288\ldots. \tag{5.10}$$

The Inverse Symbolic Calculator[2] has yielded no symbolic representation of this number. An apparent connection to the ζ-function lies in its representation as

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_{\lambda=0}^{\infty} \frac{\lambda^{x-1}}{e^{\lambda}-1} d\lambda, \tag{5.11}$$

which is somewhat similar to terms of the form

$$\int_{\lambda=\lambda_2'}^{\infty} \frac{\text{poly}(\lambda)}{e^{\lambda}-1-\lambda} d\lambda \tag{5.12}$$

appearing in μ_2, but no real connection has been found.

6. Proof of Lemma 4.3

6.1. More on the 3-core

Suppose now that $c > c_2'$ and that the 3-core C_3 of $G_{n,p}$ has $N = \Omega(n)$ vertices and M edges. It will be distributed as a random graph uniformly chosen from the set of graphs with vertex set $[N]$ and M edges and minimum degree at least three. This is an easy well-known observation, and follows from the fact that each such graph H can be extended in the same number of ways to a graph G with vertex set $[n]$ and m edges, and such that H is the 3-core of G. For convenience we will now assume that $V(C_3) = [N]$.

The degree sequence $d(v), v \in [N]$ can be generated as follows. We independently choose for each $v \in V(C_3)$ a truncated Poisson random variable with parameter λ satisfying $g_0(\lambda) = 2M/N$, conditioned on $d(v) \geqslant 3$. So for $v \in [N]$,

$$\Pr\{d(v) = k\} = \frac{\lambda^k}{k! f_3(\lambda)}, \quad k = 3,4,5,\ldots, \quad \lambda = g_0^{-1}\left(\frac{2M}{N}\right). \tag{6.1}$$

Properties of the functions f_i, g_i are derived in Appendix B. In particular, the g_i are strictly increasing by Lemma **??** (Appendix C), so g_0^{-1} is well-defined.

These independent variables are further conditioned so that the event

$$\mathcal{D} = \left\{ \sum_{v \in [N]} d(v) = 2M \right\} \tag{6.2}$$

occurs. Now λ has been chosen so that $\mathbf{E}[d(v)] = 2M/N$, and then the local central limit theorem implies that $\Pr\{\mathcal{D}\} = \Omega(1/N^{1/2})$; see for example Durrett [8]. It follows that

$$\Pr\{\mathcal{E} \mid \mathcal{D}\} \leqslant O(n^{1/2}) \Pr\{\mathcal{E}\}, \tag{6.3}$$

for any event $\mathcal{E}$ that depends on the degree sequence of C_3.

In what follows we use the configuration model of Bollobás [5] to analyse C_3 after we have fixed its degree sequence. Thus, for each vertex v we define a set W_v of *points* such that $|W_v| = d(v)$, and write $W = \bigcup_v W_v$. A random configuration F is generated by selecting a random partition of W into M pairs. A pair $\{x,y\} \in F$ with $x \in W_u, y \in W_v$ yields an edge $\{u,v\}$ of the associated (multi-)graph Γ_F.

[2] https://isc.carma.newcastle.edu.au/

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

The key properties of F that we need are as follows: (i) conditional on F having no loops or multiple edges, it is equally likely to be any simple graph with the given degree sequence; (ii) for the degree sequences of interest, the probability that Γ_F is simple will be bounded away from zero. This is because the degree sequence in (6.3) has exponential tails. Thus we only need to show that Γ_F has certain properties w.h.p.

6.2. Setting up the main calculation

Suppose now that $p = c/n$ where $c > c_2'$. We will show that w.h.p., for any fixed $\varepsilon > 0$,

$$i(S) = |\{e \in E(C_3) : e \cap S \neq \emptyset\}| \geqslant (2-\varepsilon)|S| \quad \text{for all } S \subseteq [N]. \tag{6.4}$$

Proving this is the main computational task of the paper. In principle, it is just an application of the first moment method. We compute the expected number of S that violate (6.4) and show that this expectation tends to zero. On the other hand, a moment's glance at the expression $f(\mathbf{w})$ below will show that this is unlikely to be easy, and it takes more than half of the paper to verify (6.4).

It follows from (6.4) that

$$E(C_3) \text{ can be oriented so that at least } (1-\varepsilon)N \text{ vertices have indegree at least two.} \tag{6.5}$$

To see this, consider the following network flow problem. We have a source s and a sink t plus a vertex for each $v \in [N]$ and a vertex for each edge $e \in E(C_3)$. The directed edges are: (i) $(s,v), v \in [N]$ of capacity two; (ii) (u,e), where $u \in e$ of infinite capacity; (iii) $(e,t), e \in E(C_3)$ of capacity one. An $s-t$ flow decomposes into paths s,u,e,t corresponding to orienting the edge e into u. A flow thus corresponds to an orientation of $E(C_3)$. The condition (6.4) implies that the minimum cut in the network has capacity at least $(2-\varepsilon)N$. This implies that there is a flow of value at least $(2-\varepsilon)N$ and then the orientation claimed in (6.5) exists.

Thus w.h.p. C_3 contains two edge-disjoint PFDs, each containing $(1-\varepsilon)N$ edges. Arguing as in the proof of Lemma 4.2, we see that we can w.h.p. remove $o(N)$ edges from the cycles of these PFDs and obtain forests. Thus w.h.p. C_3 contains two edge-disjoint forests of total size at least $2(1-\varepsilon)N - o(N)$. This implies that

$$\mathbf{E}[r_2(C_3(G_{n,c/n}))] \geqslant 2(1-\varepsilon)N - o(N),$$

and since $N = \Omega(n)$, we can have

$$\mathbf{E}[r_2(C_3(G_{n,c/n}))] = 2(1-\varepsilon)N - o(n).$$

Because ε is arbitrary, this implies $r_2(C_3(G_{n,c/n})) = 2N - o(n)$ whenever $c > c_2'$.

6.3. Proof of (6.4): small S

It will be fairly easy to show that (6.5) holds w.h.p. for all $|S| \leqslant s_\varepsilon$ where

$$s_\varepsilon = \left(\frac{1+\varepsilon}{e^{2+\varepsilon}c}\right)^{1/\varepsilon} n.$$

We claim that w.h.p.

$$|S| \leqslant s_\varepsilon \text{ implies } e(S) < (1+\varepsilon)|S| \text{ in } G_{n,p}. \tag{6.6}$$

Here $e(S) = |\{e \in E(G_{n,p}) : e \subseteq S\}|$.

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

Indeed,

$$\begin{aligned}\Pr\{\exists S \text{ violating (6.6)}\} &\leqslant \sum_{s=4}^{s_\varepsilon} \binom{n}{s} \binom{\binom{s}{2}}{(1+\varepsilon)s} p^{(1+\varepsilon)s} \\ &\leqslant \sum_{s=4}^{s_\varepsilon} \left(\frac{ne}{s}\right)^s \left(\frac{sec}{2(1+\varepsilon)n}\right)^{(1+\varepsilon)s} \\ &= \sum_{s=4}^{s_\varepsilon} \left(\left(\frac{s}{n}\right)^\varepsilon \frac{e^{2+\varepsilon}c}{2(1+\varepsilon)}\right)^s = o(1).\end{aligned}$$

For sets A, B of vertices and $v \in A$ we will let $d_B(v)$ denote the number of neighbours of v in B. We then let $d_B(A) = \sum_{v\in A} d_B(v)$. We will drop the subscript B when $B = [N]$.

Suppose then that (6.6) holds and that $|S| \leqslant s_\varepsilon$ and $i(S) \leqslant (2-\varepsilon)|S|$. Then if $\bar{S} = [N] \setminus S$, we have

$$e(S) + d_{\bar{S}}(S) \leqslant (2-\varepsilon)|S| \quad \text{and} \quad d(S) = 2e(S) + d_{\bar{S}}(S) \geqslant 3|S|,$$

which implies that $e(S) \geqslant (1+\varepsilon)|S|$, contradiction.

6.4. Proof of (6.4): large S

Suppose now that C_3 contains an S such that $i(S) < (2-\varepsilon)|S|$. Let such sets be *bad*. Let S be a minimal bad set, and write $T = [N] \setminus S$. For any $v \in S$, we have $i(S \setminus v) \geqslant (2-\varepsilon)|S \setminus v|$ while $i(S) < (2-\varepsilon)|S|$. This implies $d_T(v) = i(S) - i(S \setminus v) < 2$.

We will start with a minimal bad set and then carefully add more vertices. Consider a set S such that $i(S) < 2|S|$ and $d_T(v) \leqslant 2$ for all $v \in S$. If there is a $w \in T$ such that $d_T(w) \leqslant 2$, let $S' = S \cup \{w\}$. We have $i(S') \leqslant i(S) + 2 < 2|S'|$. This means we may add vertices to S in this fashion to acquire a partition $[N] = S \cup T$ where $d_T(v) \leqslant 2$ for all $v \in S$ and $d_T(v) \geqslant 3$ for all $v \in T$. We further partition $S = S_0 \cup S_1 \cup S_2$ so that $d_T(v) = i$ if and only if $v \in S_i$. Denote the size of any set by its lower-case equivalent, that is, $|S_i| = s_i$ and $|T| = t$.

We now start to use the configuration model. Partition each point set into $W_v = W_v^S \cup W_v^T$, where a point is in W_v^S if and only if it is matched to a point in $\cup_{u\in S} W_u$. The sizes of W_v^S, W_v^T uniquely determine $\mathbf{w} = (s_0, s_1, s_2, D_0, D_1, D_2, D_3, t, M)$. Here $D_i = d_S(S_i), i = 0,1,2$ and $D_3 = d_T(T)$.

6.4.1. Estimating the probability of w.

By construction, $D_i \geqslant (3-i)s_i$ for $i = 0,1,2$ and $D_3 \geqslant 3t$. Define degree sequences $(d_i^1, \ldots, d_i^{s_i})$ for $S_i, i = 0,1,2$ and $(d_3^1, \ldots, d_3^t)$ for T. Furthermore, let $\widehat{d_1^j} = d_1^j - 1$, $\widehat{d_2^j} = d_2^j - 2$ and $\widehat{d_3^j} \geqslant 0$ be the S-degrees of vertices in S_1, S_2, T, respectively.

Dealing with S_0. Ignoring for the moment that we must condition on the event $\mathcal{D}$ (see (6.2)), the probability that S_0 has degree sequence $(d_0^1, \ldots, d_0^{s_0})$, $d_0^i \geqslant 3$ for all i, is given by

$$\prod_{i=1}^{s_0} \frac{\lambda^{d_0^i}}{d_0^i! f_3(\lambda)}, \tag{6.7}$$

where λ is the solution to

$$g_0(\lambda) = \frac{2M}{N}.$$

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

Hence, letting $[x^D]f(x)$ denote the coefficient of x^D in the power series $f(x)$, the probability $\pi_0(S_0, D_0)$ that $d(S_0) = D_0$ is bounded by

$$\begin{aligned}
\pi_0(S_0,D_0) &\leqslant \sum_{\substack{d_0^1+\cdots+d_0^{s_0}=D_0 \\ d_0^i \geqslant 3}} \prod_{i=1}^{s_0} \frac{\lambda^{d_0^i}}{d_0^i!\,f_3(\lambda)} \\
&= \frac{\lambda^{D_0}}{f_3(\lambda)^{s_0}} \sum_{\substack{d_0^1+\cdots+d_0^{s_0}=D_0 \\ d_0^i \geqslant 3}} \prod_{i=1}^{s_0} \frac{1}{d_0^i!} \\
&= \frac{\lambda^{D_0}}{f_3(\lambda)^{s_0}} [x^{D_0}] \left(\sum_{d_0 \geqslant 3} \frac{x^{d_0}}{d_0!} \right)^{s_0} \\
&= \frac{\lambda^{D_0}}{f_3(\lambda)^{s_0}} [x^{D_0}] f_3(x)^{s_0} \\
&\leqslant \frac{\lambda^{D_0}}{f_3(\lambda)^{s_0}} \frac{f_3(\lambda_0)^{s_0}}{\lambda_0^{D_0}}, \qquad (6.8)
\end{aligned}$$

for all λ_0. Here we use the fact that for any function f and any $y > 0$, $[x^{D_0}]f(x) \leqslant f(y)/y^{D_0}$. To minimize (6.8) we choose λ_0 to be the unique solution to

$$g_0(\lambda_0) = \frac{D_0}{s_0}. \qquad (6.9)$$

If $D_0 = 3s_0$ then $\lambda_0 = 0$. In this case, since

$$f_3(\lambda_0) = \frac{\lambda_0^3(1+O(\lambda_0))}{6},$$

we have

$$\pi_0(S_0,D_0) \leqslant \left(\frac{\lambda^3}{6 f_3(\lambda)} \right)^{s_0}, \quad \text{when } D_0 = 3s_0. \qquad (6.10)$$

Dealing with S_1. For each $v \in S_1$, we have $W_v = W_v^S \cup W_v^T$ where $|W_v^T| = 1$. Hence, the probability $\pi_1(S_1, D_1)$ that $d(S_1) = D_1 + s_1$ is bounded by

$$\begin{aligned}
\pi_1(S_1,D_1) &\leqslant \sum_{\substack{\widehat{d}_1^1+\cdots+\widehat{d}_1^{s_1}=D_1 \\ \widehat{d}_1^i \geqslant 2}} \prod_{i=1}^{s_1} \binom{\widehat{d}_1^i+1}{1} \frac{\lambda^{\widehat{d}_1^i+1}}{(\widehat{d}_1^i+1)!\,f_3(\lambda)} \\
&= \frac{\lambda^{D_1+s_1}}{f_3(\lambda)^{s_1}} \sum_{\substack{\widehat{d}_1^1+\cdots+\widehat{d}_1^{s_1}=D_1 \\ \widehat{d}_1^i \geqslant 2}} \prod_{i=1}^{s_1} \frac{1}{\widehat{d}_1^i!} \\
&= \frac{\lambda^{D_1+s_1}}{f_3(\lambda)^{s_1}} [x^{D_1}] f_2(x)^{s_1} \\
&\leqslant \frac{\lambda^{D_1+s_1}}{f_3(\lambda)^{s_1}} \frac{f_2(\lambda_1)^{s_1}}{\lambda_1^{D_1}}. \qquad (6.11)
\end{aligned}$$

We choose λ_1 to satisfy the equation

$$g_1(\lambda_1) = \frac{D_1}{s_1}. \tag{6.12}$$

Similarly to what happens in (6.10), we have $\lambda_1 = 0$ when $D_1 = 2s_1$ and

$$f_2(\lambda_1) = \frac{\lambda_1^2(1+O(\lambda_1))}{2},$$

so

$$\pi_1(S_1, D_1) \leqslant \left(\frac{\lambda^3}{2f_3(\lambda)}\right)^{s_1}, \quad \text{when } D_1 = 2s_1. \tag{6.13}$$

Dealing with S_2. For $v \in S_2$, we choose two points from W_v to be in W_v^T, so the probability $\pi_2(S_2, D_2)$ that $d(S_2) = D_2 + 2s_2$ is bounded by

$$\pi_2(S_2, D_2) \leqslant \sum_{\substack{\widehat{d}_2^1+\cdots+\widehat{d}_2^{s_2}=D_2 \\ \widehat{d}_2^i \geqslant 1}} \prod_{i=1}^{s_2} \binom{\widehat{d}_2^i+2}{2} \frac{\lambda^{\widehat{d}_2^i+2}}{(\widehat{d}_2^i+2)!f_3(\lambda)} \leqslant \frac{\lambda^{D_2+2s_2}}{f_3(\lambda)^{s_2}} \frac{f_1(\lambda_2)^{s_2}}{\lambda_2^{D_2}} 2^{-s_2}, \tag{6.14}$$

where we choose λ_2 to satisfy the equation

$$g_2(\lambda_2) = \frac{D_2}{s_2}. \tag{6.15}$$

Similarly to what happens in (6.10), we have $\lambda_2 = 0$ when $D_2 = s_2$ and $f_1(\lambda_2) = \lambda_2(1+O(\lambda_2))$, so

$$\pi_2(S_2, D_2) \leqslant \left(\frac{\lambda^3}{2f_3(\lambda)}\right)^{s_2}, \quad \text{when } D_2 = s_2. \tag{6.16}$$

Dealing with T. Finally, the degree of vertex i in T can be written as $d_3^i = \widehat{d}_3^i + \overline{d}_3^i$, where $\widehat{d}_3^i \geqslant 0$ is the S-degree and $\overline{d}_3^i \geqslant 3$ is the T-degree. Here, with $t = |T|$, we have

$$\sum_{i=1}^{t} \widehat{d}_3^i = d_S(T) = s_1 + 2s_2$$

by the definition of S_0, S_1, S_2. So the probability $\pi_3(T, D_3)$ that $d_T(T) = D_3$, given s_1, s_2 can be bounded by

$$\begin{aligned}
\pi_3(T, D_3) &\leqslant \sum_{\substack{\widehat{d}_3^1+\cdots+\widehat{d}_3^t=s_1+2s_2 \\ \widehat{d}_3^i \geqslant 0}} \sum_{\substack{\overline{d}_3^1+\cdots+\overline{d}_3^t=D_3 \\ \overline{d}_3^i \geqslant 3}} \prod_{i=1}^{t} \binom{\widehat{d}_3^i+\overline{d}_3^i}{\widehat{d}_3^i} \frac{\lambda^{\widehat{d}_3^i+\overline{d}_3^i}}{(\widehat{d}_3^i+\overline{d}_3^i)!f_3(\lambda)} \\
&= \frac{\lambda^{D_3+s_1+2s_2}}{f_3(\lambda)^t} \sum_{\substack{\widehat{d}_3^1+\cdots+\widehat{d}_3^t=s_1+2s_2 \\ \widehat{d}_3^i \geqslant 0}} \sum_{\substack{\overline{d}_3^1+\cdots+\overline{d}_3^t=D_3 \\ \overline{d}_3^i \geqslant 3}} \prod_{i=1}^{t} \frac{1}{\widehat{d}_3^i!\overline{d}_3^i!}
\end{aligned}$$

$$= \frac{\lambda^{D_3+s_1+2s_2}}{f_3(\lambda)^t}([x^{D_3}]f_3(x)^t)([x^{s_1+2s_2}]e^x)$$
$$\leqslant \frac{\lambda^{D_3+s_1+2s_2}}{f_3(\lambda)^t}\frac{f_3(\lambda_3)^t}{\lambda_3^{D_3}}\frac{t^{s_1+2s_2}}{(s_1+2s_2)!}, \tag{6.17}$$

where we choose λ_3 to satisfy the equation

$$g_0(\lambda_3) = \frac{D_3}{t}. \tag{6.18}$$

Similarly to what happens in (6.10), we have $\lambda_3 = 0$ when $D_3 = 3t$ and

$$f_3(\lambda_3) = \frac{\lambda_3^3(1+O(\lambda_1))}{6},$$

so

$$\pi_3(T,D_3) \leqslant \frac{\lambda^{D_3+s_1+2s_2}}{(6f_3(\lambda))^t}\frac{t^{s_1+2s_2}}{(s_1+2s_2)!}, \quad \text{when } D_3 = 3t.$$

6.4.2. Putting the bounds together. For a fixed $\mathbf{w} = (s_0,s_1,s_2,D_0,D_1,D_2,D_3,t,M)$, there are $\binom{t+s}{s_0,s_1,s_2,t}$ choices for S_0,S_1,S_2,T. Having chosen these sets, we partition the $W_v, v \in S\cup T$ into $W_v^S \cup W_v^T$. Note that our expressions (6.8), (6.11), (6.14), (6.17) account for these choices. Given the partitions of the W_v, there are $(D_0+D_1+D_2)!!D_3!!(s_1+2s_2)!$ configurations, where $(2s)!! = (2s-1)\times(2s-3)\times\cdots\times 3\times 1$ is the number of ways of partitioning a set of size $2s$ into s pairs. Here $(D_0+D_1+D_2)!!$ is the number of ways of pairing up $\bigcup_{v\in S} W_v^S$, $D_3!!$ is the number of ways of pairing up $\bigcup_{v\in T} W_v^T$, and $(s_1+2s_2)!$ is the number of ways of pairing points associated with S to points associated with T. Each configuration has probability $1/(2M)!!$. So, the total probability of all configurations whose vertex partition and degrees are described by $\mathbf{w}$ can be bounded by

$$\binom{t+s}{s_0,s_1,s_2,t}\frac{\lambda^{D_0}}{f_3(\lambda)^{s_0}}\frac{f_3(\lambda_0)^{s_0}}{\lambda_0^{D_0}}\frac{\lambda^{D_1+s_1}}{f_3(\lambda)^{s_1}}\frac{f_2(\lambda_1)^{s_1}}{\lambda_1^{D_1}}\frac{\lambda^{D_2+2s_2}}{f_3(\lambda)^{s_2}}\frac{f_1(\lambda_2)^{s_2}}{\lambda_2^{D_2}}2^{-s_2}$$
$$\times\frac{\lambda^{D_3+s_1+2s_2}}{f_3(\lambda)^t}\frac{f_3(\lambda_3)^t}{\lambda_3^{D_3}}\frac{t^{s_1+2s_2}}{(s_1+2s_2)!}\frac{(D_0+D_1+D_2)!!D_3!!(s_1+2s_2)!}{(2M)!!}$$
$$= \binom{t+s}{s_0,s_1,s_2,t}\frac{\lambda^{2M}}{f_3(\lambda)^N}\frac{f_3(\lambda_0)^{s_0}}{\lambda_0^{D_0}}\frac{f_2(\lambda_1)^{s_1}}{\lambda_1^{D_1}}\frac{f_1(\lambda_2)^{s_2}}{\lambda_2^{D_2}}2^{-s_2}\frac{f_3(\lambda_3)^t}{\lambda_3^{D_3}}\frac{t^{s_1+2s_2}}{(s_1+2s_2)!}$$
$$\times\frac{(D_0+D_1+D_2)!!D_3!!(s_1+2s_2)!}{(2M)!!}$$

Write $D_i = \Delta_i s$, $|S_i| = \sigma_i s$, $t = \tau s$, $M = \mu s$ and $N = \nu s$. We have $k!! \sim \sqrt{2}(k/e)^{k/2}$ as $k\to\infty$ by Stirling's formula, so the expression above, modulo an $e^{o(s)}$ factor, can be written as

$$f(\mathbf{w})^s = \left(\frac{(\tau+1)^{\tau+1}}{\sigma_0^{\sigma_0}\sigma_1^{\sigma_1}(1-\sigma_0-\sigma_1)^{1-\sigma_0-\sigma_1}\tau^\tau}\frac{\lambda^{2\mu}}{f_3(\lambda)^\nu}\frac{f_3(\lambda_0)^{\sigma_0}}{\lambda_0^{\Delta_0}}\frac{f_2(\lambda_1)^{\sigma_1}}{\lambda_1^{\Delta_1}}\frac{f_1(\lambda_2)^{\sigma_2}}{\lambda_2^{\Delta_2}}\frac{f_3(\lambda_3)^\tau}{\lambda_3^{\Delta_3}}\frac{(\tau e)^{\sigma_1+2\sigma_2}}{2^{\sigma_2}}\right.$$
$$\left.\frac{(\Delta_0+\Delta_1+\Delta_2)^{(\Delta_0+\Delta_1+\Delta_2)/2}\Delta_3^{\Delta_3/2}}{(2\mu)^\mu}\right)^s. \tag{6.19}$$

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

We note that

$$\sigma_2 = 1 - \sigma_0 - \sigma_1, \tag{6.20}$$

$$\begin{aligned}\Delta_3 &= 2\mu - \Delta_0 - \Delta_1 - \Delta_2 - 2\sigma_1 - 4\sigma_2 \\ &= 2\mu - 4 - \Delta_0 - \Delta_1 - \Delta_2 + 4\sigma_0 + 2\sigma_1,\end{aligned} \tag{6.21}$$

$$\nu = 1 + \tau. \tag{6.22}$$

Hence σ_2, Δ_3, ν may be eliminated, and we can consider $\mathbf{w}$ to be $(\sigma_0, \sigma_1, \Delta_0, \Delta_1, \Delta_2, \tau, \mu)$. When convenient, Δ_3 may be used to denote $2\mu - 4 - \Delta_0 - \Delta_1 - \Delta_2 + 4\sigma_0 + 2\sigma_1$. Define the constraint set F to be all $\mathbf{w}$ satisfying

$$\begin{aligned}&\Delta_0 \geqslant 3\sigma_0, \quad \Delta_1 \geqslant 2\sigma_1, \quad \Delta_2 \geqslant 1 - \sigma_0 - \sigma_1, \quad \Delta_3 \geqslant 3\tau, \\ &\frac{\Delta_0 + \Delta_1 + \Delta_2}{2} + \sigma_1 + 2(1 - \sigma_0 - \sigma_1) < 2 - \varepsilon \quad \text{since } i(S) < (2-\varepsilon)|S| \ \text{(see (6.4))}, \\ &\sigma_0, \sigma_1 \geqslant 0, \quad \sigma_0 + \sigma_1 \leqslant 1, \\ &0 \leqslant \tau \leqslant (1-\varepsilon)/\varepsilon \quad \text{since } |S| \geqslant \varepsilon N, \\ &\mu \geqslant (2+\varepsilon)(1+\tau) \quad \text{since } M \geqslant (2+\varepsilon)N, \\ &\sigma_0 < 1, \quad \text{otherwise } C_3 \text{ is not connected.}\end{aligned}$$

Here ε is a sufficiently small positive constant such that we can (i) exclude the case of small S, (ii) satisfy condition (6.4), and (iii) have $M \geqslant (2+\varepsilon)N$ since $c > c_2'$.

For a given s, there are $O(\mathrm{poly}(s))$ choices of $\mathbf{w} \in F$, and the probability that the randomly chosen configuration corresponds to a $\mathbf{w} \in F$ can be bounded by

$$\sum_{s \geqslant \varepsilon N} \sum_{\mathbf{w}} O(\mathrm{poly}(s)) f(\mathbf{w})^s \leqslant \sum_s \big(e^{o(1)} \max_F f(\mathbf{w})\big)^s \leqslant N \big(e^{o(1)} \max_F f(\mathbf{w})\big)^{\varepsilon N}. \tag{6.23}$$

As $N \to \infty$, it remains to show that $f(\mathbf{w}) \leqslant 1 - \delta$ for all $\mathbf{w} \in F$, for some $\delta = \delta(\varepsilon) > 0$. At this point we remind the reader that we have so far ignored conditioning on the event $\mathcal{D}$ defined in (6.2). Inequality (6.3) implies that it is sufficient to inflate the right-hand side of (6.23) by $O(n^{1/2})$ to obtain our result.

So, let

$$\begin{aligned}&f(\Delta_0, \Delta_1, \Delta_2, \sigma_0, \sigma_1, \tau, \mu) \\ &\quad = \frac{(\tau+1)^{\tau+1}}{\sigma_0^{\sigma_0} \sigma_1^{\sigma_1} (1-\sigma_0-\sigma_1)^{1-\sigma_0-\sigma_1} \tau^\tau} \frac{\lambda^{2\mu}}{f_3(\lambda)^{\tau+1}} \frac{f_3(\lambda_0)^{\sigma_0}}{\lambda_0^{\Delta_0}} \frac{f_2(\lambda_1)^{\sigma_1}}{\lambda_1^{\Delta_1}} \frac{f_1(\lambda_2)^{1-\sigma_0-\sigma_1}}{\lambda_2^{\Delta_2}} \frac{f_3(\lambda_3)^{\tau}}{\lambda_3^{\Delta_3}} \\ &\quad\quad \times \frac{(e\tau)^{2-2\sigma_0-\sigma_1}}{2^{1-\sigma_0-\sigma_1}} \frac{(\Delta_0+\Delta_1+\Delta_2)^{(\Delta_0+\Delta_1+\Delta_2)/2} \Delta_3^{\Delta_3/2}}{(2\mu)^\mu}.\end{aligned}$$

We complete the proof of Theorem 1.3 by showing that

$$f(\mathbf{w}) \leqslant \exp\left\{-\frac{\varepsilon^2}{3}\right\} \quad \text{for all } \mathbf{w} \in F. \tag{6.24}$$

The proof of (6.24) is a very long and careful calculation. It can be found in the Arxiv version: arXiv:1505.03429.

https://doi.org/10.1017/S0963548317000426 Published online by Cambridge University Press

7. Final remarks

There are a number of loose ends to be taken care of. Is Conjecture 1.1 true? Is there a simpler expression for μ_2 of Theorem 1.3? Is it possible to get an exact expression for μ_3? On another tack, what are the expected running times of algorithms for computing these edge-disjoint trees? They are polynomial-time solvable problems, in the worst case, but maybe their average complexity is significantly better than worst case.

Acknowledgement

We thank the referee for his/her comments.

References

[1] Aldous, D. (1992) Asymptotics in the random assignment problem. *Probab. Theory Rel. Fields* **93** 507–534.
[2] Aldous, D. (2001) The $\zeta(2)$ limit in the random assignment problem. *Random Struct. Alg.* **4** 381–418.
[3] Avram, F. and Bertsimas, D. (1992) The minimum spanning tree constant in geometrical probability and under the independent model: A unified approach. *Ann. Appl. Probab.* **2** 113–130.
[4] Beveridge, A., Frieze, A. M. and McDiarmid, C. J. H. (1998) Minimum length spanning trees in regular graphs. *Combinatorica* **18** 311–333.
[5] Bollobás, B. (1980) A probabilistic proof of an asymptotic formula for the number of labelled graphs. *Europ. J. Combin.* **1** 311–316.
[6] Cain, J. A., Sanders, P. and Wormald, N. (2007) The random graph threshold for k-orientability and a fast algorithm for optimal multiple-choice allocation. In *SODA 2007: Proc. 18th Annual ACM–SIAM Symposium on Discrete Algorithms*, SIAM, pp. 469–476.
[7] Cooper, C., Frieze, A. M., Ince, N., Janson, S. and Spencer, J. (2016) On the length of a random minimum spanning tree. *Combin. Probab. Comput.* **25**, 89–107.
[8] Durrett, R. (1991) *Probability: Theory and Examples*, Wadsworth & Brooks/Cole.
[9] Fenner, T. I. and Frieze, A. M. (1982) On the connectivity of random m-orientable graphs and digraphs. *Combinatorica* **2** 347–359.
[10] Frieze, A. M. (1985) On the value of a random minimum spanning tree problem. *Discrete Appl. Math.* **10** 47–56.
[11] Frieze, A. M. (1986) On large matchings and cycles in sparse random graphs. *Discrete Math.* **59** 243–256.
[12] Frieze, A. M. (2004) On random symmetric travelling salesman problems. *Math. Oper. Res.* **29** 878–890.
[13] Frieze, A. M. and Grimmett, G. R. (1985) The shortest path problem for graphs with random arc-lengths. *Discrete Appl. Math.* **10** 57–77.
[14] Frieze, A. M. and McDiarmid, C. J. H. (1989) On random minimum length spanning trees. *Combinatorica* **9** 363–374.
[15] Frieze, A. M., Ruszinko, M. and Thoma, L. (2000) A note on random minimum length spanning trees. *Electron. J. Combin.* **7** R41.
[16] Gao, P., Pérez-Giménez, X. and Sato, C. M. (2014) Arboricity and spanning-tree packing in random graphs with an application to load balancing. Extended abstract published in *SODA 2014*, pp. 317–326. arXiv:1303.3881
[17] Janson, S. (1995) The minimal spanning tree in a complete graph and a functional limit theorem for trees in a random graph. *Random Struct. Alg.* **7** 337–355.
[18] Janson, S. (1999) One, two and three times $\log n/n$ for paths in a complete graph with random weights. *Combin. Probab. Comput.* **8** 347–361.

[19] Janson, S. and Łuczak, M. J. (2007) A simple solution to the k-core problem. *Random Struct. Alg.* **30** 50–62.
[20] Karp, R. M. (1979) A patching algorithm for the non-symmetric traveling salesman problem. *SIAM J. Comput.* **8** 561–573.
[21] Kordecki, W. and Lyczkowska-Hanćkowiak, A. (2013) Exact expectation and variance of minimal basis of random matroids. *Discussiones Mathematicae Graph Theory* **33** 277–288.
[22] Linusson, S. and Wästlund, J. (2004) A proof of Parisi's conjecture on the random assignment problem. *Probab. Theory Rel. Fields* **128** 419–440.
[23] Łuczak, T. (1991) Size and connectivity of the k-core of a random graph. *Discrete Math.* **91** 61–68.
[24] Nair, C., Prabhakar, B. and Sharma, M. (2005) Proofs of the Parisi and Coppersmith–Sorkin random assignment conjectures. *Random Struct. Alg.* **27** 413–444.
[25] Nash-Williams, C. St J. A. (1961) Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.* **36** 445–450.
[26] Nash-Williams, C. St J. A. (1964) Decomposition of finite graphs into forests. *J. London Math. Soc.* **39** 12.
[27] Oxley, J. (1992) *Matroid Theory*, Oxford University Press.
[28] Penrose, M. (1998) Random minimum spanning tree and percolation on the n-cube. *Random Struct. Alg.* **12** 63–82.
[29] Pittel, B., Spencer, J. and Wormald, N. (1996) Sudden emergence of a giant k-core in a random graph. *J. Combin. Theory Ser. B* **67** 111–151.
[30] Steele, J. M. (1987) On Frieze's $\zeta(3)$ limit for lengths of minimal spanning trees. *Discrete Appl. Math.* **18** 99–103.
[31] Wästlund, J. (2009) An easy proof of the $\zeta(2)$ limit in the random assignment problem. *Electron. Comm. Probab.* **14** 261–269.
[32] Wästlund, J. (2010) The mean field traveling salesman and related problems. *Acta Math.* **204** 91–150.
[33] Welsh, D. J. A. (1976) *Matroid Theory*, Academic Press.