

Counting Connected Hypergraphs via the Probabilistic Method

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In 1990 Bender, Canfield and McKay gave an asymptotic formula for the number of connected graphs on $[n] = \{1, 2, \dots, n\}$ with m edges, whenever n and the nullity $m - n + 1$ tend to infinity. Let $C_r(n, t)$ be the number of connected r -uniform hypergraphs on $[n]$ with nullity $t = (r - 1)m - n + 1$, where m is the number of edges. For $r \geq 3$, asymptotic formulae for $C_r(n, t)$ are known only for partial ranges of the parameters: in 1997 Karoński and Łuczak gave one for $t = o(\log n / \log \log n)$, and recently Behrisch, Coja-Oghlan and Kang gave one for $t = \Theta(n)$. Here we prove such a formula for any fixed $r \geq 3$ and any $t = t(n)$ satisfying $t = o(n)$ and $t \rightarrow \infty$ as $n \rightarrow \infty$, complementing the last result. This leaves open only the case $t/n \rightarrow \infty$, which we expect to be much simpler, and will consider in future work. The proof is based on probabilistic methods, and in particular on a bivariate local limit theorem for the number of vertices and edges in the largest component of a certain random hypergraph. We deduce this from the corresponding central limit theorem by smoothing techniques.

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1. Introduction

Our aim in this paper is to prove a result about r -uniform hypergraphs that can be viewed in two complementary ways, either as a probabilistic result or as an enumerative one. In this section we shall state the enumerative form; in the next section we switch to the

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probabilistic viewpoint, which we shall adopt for most of the paper, and in particular in the proofs.

If H is an r -uniform hypergraph then

$$|H| \leq c(H) + (r - 1)e(H),$$

where $|H|$ is the number of vertices of H , $e(H)$ is the number of edges, and $c(H)$ is the number of components, with equality if and only if H is a forest, *i.e.*, every component of H is a tree. Define the *nullity* $n(H)$ of H as

$$n(H) = c(H) + (r - 1)e(H) - |H|, \quad (1.1)$$

so $n(H) \geq 0$, and H is a tree if and only if $c(H) = 1$ and $n(H) = 0$. Note for later that, if H is connected, then $|H| + n(H) - 1$ must be a multiple of $r - 1$. If we replace each hyperedge of H by a tree on the same set of r vertices, then $n(H)$ is simply the nullity of the resulting (multi-) graph. Connected graphs or hypergraphs are naturally parametrized by the number of vertices and the nullity, although often the *excess* $n(H) - 1$ is considered instead.

One of the most basic questions about any class of combinatorial (or other) structures is: how many such structures are there with given ‘size’ parameters? Or, sometimes more naturally, how many ‘irreducible’ structures? For (labelled) graphs and hypergraphs, the first question is trivial, but the second, taking ‘irreducible’ to mean connected, certainly is not, and it is no surprise that it has been extensively studied. Given integers $r \geq 2$, $s \geq 1$ and $t \geq 0$, let $C_r(s, t)$ be the number of connected r -uniform hypergraphs on $[s] = \{1, 2, \dots, s\}$ having nullity t . (Thus $C_r(s, t) = 0$ if $r - 1$ does not divide $s + t - 1$.) Starting with Cayley’s formula $C_2(s, 0) = s^{s-2}$, the asymptotic evaluation of $C_2(s, t)$ was studied by Wright [28, 29, 30, 31] and others for increasingly broad ranges of $t = t(s)$, culminating in the results of Bender, Canfield and McKay [7] giving an asymptotic formula for $C_2(s, t)$ whenever $s \rightarrow \infty$, for any function $t = t(s)$.

For $r \geq 3$, much less is known. Selivanov [27] gave an exact formula for the number $C_r(s, 0)$ of trees; the remaining results we shall mention are all asymptotic, with r fixed, $s \rightarrow \infty$, and t some function of s . Karoński and Łuczak [17] gave an asymptotic formula for $C_r(s, t)$ when $t = o(\log s / \log \log s)$, so the hypergraphs counted are quite close to trees. In an extended abstract from 2006, Andriamampianina and Ravelomanana [1] outlined an extension of this to the case $t = o(s^{1/3})$. Recently, Behrisch, Coja-Oghlan and Kang [6] gave an asymptotic formula for $C_r(s, t)$ when $t = \Theta(s)$; their proof is based on probabilistic methods, which seem to work best when t is relatively large, rather than the enumerative methods most successful for small t . Independently and essentially simultaneously with the present work, Sato and Wormald [24] (see also Sato [23]) have given an asymptotic formula for $C_r(s, t)$ when $r = 3$, $t = o(s)$ and $t/(s^{1/3} \log^2 s) \rightarrow \infty$.

Our main result complements those in [6], and greatly extends those in [17, 24], covering the entire range $t \rightarrow \infty$, $t = o(s)$. The formula we obtain is rather complicated; to state it we need some definitions.

Given an integer $r \geq 2$ and a real number $0 < \rho < 1$, define

$$\Psi_r(\rho) = -\frac{r-1}{r} \frac{\log(1-\rho)}{\rho} \frac{1-(1-\rho)^r}{1-(1-\rho)^{r-1}} - 1. \quad (1.2)$$

For any $r \geq 2$ it is easy to see that $\Psi_r(\rho)$ is strictly increasing on $(0, 1)$, since each of the factors $-\log(1 - \rho)/\rho$ and $(1 - (1 - \rho)^r)/(1 - (1 - \rho)^{r-1})$ is. Since Ψ_r is continuous, considering the limits at 0 and 1 we see that Ψ_r gives a bijection from $(0, 1)$ to $(0, \infty)$.

Theorem 1.1. *Let $r \geq 2$ be fixed, and let $t = t(s)$ satisfy $t \rightarrow \infty$ and $t = o(s)$ as $s \rightarrow \infty$. Then when $s + t - 1$ is divisible by $r - 1$ the number $C_r(s, t)$ of connected r -uniform hypergraphs on $[s]$ with nullity t satisfies*

$$C_r(s, t) \sim \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{r - 1}{\sqrt{s}} \left(\frac{e(1 - (1 - \rho)^r)s^r}{m r! \rho^r} \right)^m (\rho(1 - \rho)^{(1-\rho)/\rho})^s \tag{1.3}$$

as $s \rightarrow \infty$, where $\rho > 0$ is the unique positive solution to

$$\Psi_r(\rho) = \frac{t - 1}{s}, \tag{1.4}$$

and $m = (s + t - 1)/(r - 1)$ is the number of edges of any such hypergraph. Moreover, the probability $P_r(s, t)$ that a random m -edge r -uniform hypergraph on $[s]$ is connected satisfies

$$P_r(s, t) \sim e^{r/2 + \mathbb{1}_{r=2}} \sqrt{\frac{3(r - 1)}{2}} \left(\frac{1 - (1 - \rho)^r}{\rho^r} \right)^m (\rho(1 - \rho)^{(1-\rho)/\rho})^s, \tag{1.5}$$

where $\mathbb{1}_A$ denotes the indicator function of A .

To understand this result it may help to note that $\Psi_r(x) = (r - 1)x^2/12 + O(x^3)$ as $x \rightarrow 0$, so

$$\rho \sim 2\sqrt{\frac{3}{r - 1} \frac{t}{s}}$$

when $t/s \rightarrow 0$. Also, it may be useful to note that rearranging (1.4) gives $m/s = (\Psi_r(\rho) + 1)/(r - 1)$, so we can rewrite the formulae (1.3) and (1.5) as functions of s and ρ only (or m and ρ only) if we wish.

There are many ways to write a formula such as (1.3), and checking whether two such formulae agree may require some calculation. In the appendix to [11] we present such calculations showing that Theorem 1.1 matches the results of [2, 3, 7, 17, 24] where the ranges of applicability overlap, as well as the corrected version of [6]. In particular, for the graph case (which of course is not our main focus), (1.3) is consistent with (indeed, implied by) the Bender–Canfield–McKay formula [7]. For hypergraphs, Theorem 1.1 shows that the asymptotic formula of Karoński and Łuczak [17] extends not only to $t = o(s^{1/3})$, as they suspected, but to any $t = o(s^{1/2})$ (and no further).

We shall return to the topic of estimating $C_r(s, t)$ when $t/s \rightarrow \infty$ in a future paper [12]. Although we have not yet checked all the details, this regime seems to be much easier to analyse than that considered here or by Behrisch, Coja-Oghlan and Kang. The key point is that, following the approach taken in the next section, the random hypergraph that one needs to analyse has average degree tending to infinity, which means that its behaviour is relatively simple. In particular, with high probability all small components are trees.

2. Probabilistic reformulation

In this section we shall state a probabilistic result that turns out to be equivalent to Theorem 1.1; as we shall see, the formulae in this setting are significantly simpler. In the rest of the paper we shall use probabilistic methods to prove this reformulation, deducing Theorem 1.1 in Section 12.

For $2 \leq r \leq n$ and $0 < p < 1$, let $H_{n,p}^r$ be the random r -uniform hypergraph with vertex set $[n] = \{1, 2, \dots, n\}$ in which each of the $\binom{n}{r}$ possible hyperedges is present independently with probability p . Throughout we consider $r \geq 2$ fixed, $n \rightarrow \infty$, and

$$p = p(n) = \lambda(r - 2)!n^{-r+1},$$

where $\lambda = \lambda(n) = \Theta(1)$; often, we write λ as $\lambda(n) = 1 + \varepsilon(n)$. It is well known (see Section 2.1) that the model $H_{n,p}^r$ undergoes a phase transition at $\lambda = 1$ analogous to that established by Erdős and Rényi [16] in the graph case, and indeed that the ‘window’ of this phase transition is given by $\lambda = 1 + \varepsilon$ with $\varepsilon^3 n = O(1)$; see [9]. For this reason, we call the model $H_{n,p}^r$ *subcritical* if $\lambda = 1 - \varepsilon$ with $\varepsilon = \varepsilon(n)$ satisfying $\varepsilon^3 n \rightarrow \infty$, and *supercritical* if $\lambda = 1 + \varepsilon$ with $\varepsilon^3 n \rightarrow \infty$. Here we study the supercritical phase, so throughout this paper we make the following assumption unless specified otherwise.

Assumption 2.1 (Weak Assumption). The quantities $p(n)$, $\lambda(n)$ and $\varepsilon(n) > 0$ are related by $\lambda = 1 + \varepsilon$ and $p = \lambda(r - 2)!n^{-r+1}$. Moreover, $r \geq 2$ is fixed and, as $n \rightarrow \infty$, we have $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon = O(1)$.

Much of the time we additionally suppose that $\varepsilon \rightarrow 0$, *i.e.*, we assume the following.

Assumption 2.2 (Standard Assumption). The conditions of Assumption 2.1 hold, and in addition $\varepsilon \rightarrow 0$ as $n \rightarrow \infty$.

Given a hypergraph H , let $\mathcal{L}_1(H)$ denote the component with the most vertices, chosen according to any fixed rule if there is a tie. Let $L_1(H) = |\mathcal{L}_1(H)|$, $M_1(H) = e(\mathcal{L}_1(H))$ and $N_1(H) = n(\mathcal{L}_1(H))$ be the order, size and nullity of this component. Our next result gives an asymptotic formula for the probability that the triple $(L_1(H_{n,p}^r), M_1(H_{n,p}^r), N_1(H_{n,p}^r))$ takes any specific value within the ‘typical’ range, throughout the supercritical regime. Of course, since these three parameters are dependent, the result can be stated in terms of any two of them; here we consider L_1 and N_1 . To state the result we need a few definitions.

For $\lambda > 1$ let ρ_λ be the unique positive solution to

$$1 - \rho_\lambda = e^{-\lambda\rho_\lambda}, \tag{2.1}$$

so ρ_λ is the survival probability of a Galton–Watson branching process whose offspring distribution is Poisson with mean λ , and define $\rho_{r,\lambda}$ by

$$1 - \rho_{r,\lambda} = (1 - \rho_\lambda)^{1/(r-1)}. \tag{2.2}$$

It is easy to see that $\rho_{r,\lambda}$ is the survival probability of a certain branching process naturally associated to the neighbourhood exploration process in $H_{n,p}^r$, $p = \lambda(r - 2)!n^{-r+1}$, where

each particle has a Poisson $\text{Po}(\lambda/(r - 1))$ number of groups of $r - 1$ children. From (2.1) and (2.2) it is easy to check that

$$\lambda \mapsto \rho_{r,\lambda} \text{ is a continuous function } (1, \infty) \rightarrow (0, 1). \tag{2.3}$$

Turning to the analogous parameter relevant to $N_1(H_{n,p}^r)$, set

$$\rho_{r,\lambda}^* = \frac{\lambda}{r} (1 - (1 - \rho_{r,\lambda})^r) - \rho_{r,\lambda}. \tag{2.4}$$

As noted in [10], if $\lambda = 1 + \varepsilon$ then, as $\varepsilon \rightarrow 0$ from above, we have

$$\rho_{r,\lambda} \sim \frac{2\varepsilon}{r - 1} \quad \text{and} \quad \rho_{r,\lambda}^* \sim \frac{2}{3(r - 1)^2} \varepsilon^3. \tag{2.5}$$

Theorem 2.3. *Let $r \geq 2$ be fixed, let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$, set $\lambda = \lambda(n) = 1 + \varepsilon$ and define $\rho_{r,\lambda}$ and $\rho_{r,\lambda}^*$ as above. Then, whenever $x_n = \rho_{r,\lambda} n + O(\sqrt{n/\varepsilon})$ and $y_n = \rho_{r,\lambda}^* n + O(\sqrt{\varepsilon^3 n})$ with $x_n + y_n - 1$ divisible by $r - 1$, we have*

$$\mathbb{P}(L_1(H_{n,p}^r) = x_n, N_1(H_{n,p}^r) = y_n) \sim \frac{r - 1}{\sigma_n \sigma_n^*} f\left(\frac{x_n - \rho_{r,\lambda} n}{\sigma_n}, \frac{y_n - \rho_{r,\lambda}^* n}{\sigma_n^*}\right) \tag{2.6}$$

as $n \rightarrow \infty$, where $\sigma_n = \sqrt{2n/\varepsilon}$, $\sigma_n^* = \sqrt{10/3}(r - 1)^{-1} \sqrt{\varepsilon^3 n}$, and

$$f(a, b) = \frac{1}{2\pi\sqrt{2/5}} \exp\left(-\frac{5}{4}(a^2 - 2\sqrt{3/5}ab + b^2)\right) \tag{2.7}$$

is the probability density function of a bivariate Gaussian distribution with mean 0, unit variances, and covariance $\sqrt{3/5}$.

We shall comment briefly on the uniformity of the asymptotics in (2.6) above in Remark 2.7 below. For ease of comparison with other results, note that combining (2.6) and (2.7) results in the expression

$$\frac{\sqrt{6}}{8\pi} \frac{(r - 1)^2}{\varepsilon n} \exp\left(-\frac{5}{4}(a^2 - 2\sqrt{3/5}ab + b^2)\right), \tag{2.8}$$

with a and b the arguments of f in (2.6).

The probability that the largest component of $H_{n,p}^r$ has ℓ vertices and m edges is very closely related to the number of connected hypergraphs with ℓ vertices and m edges. This relationship was used by Karoński and Łuczak [18] to prove the special case of Theorem 2.3 when $\varepsilon^3 n \rightarrow \infty$ but $\varepsilon^3 n = o(\log n / \log \log n)$. Behrisch, Coja-Oghlan and Kang [4, 5] used probabilistic methods to prove a result corresponding to Theorem 2.3 but with $\varepsilon = \Theta(1)$ (i.e., roughly speaking, the case $\lambda > 1$ constant), and then, in [6], used this to deduce their enumerative result mentioned in the previous section. We shall deduce Theorem 1.1 from Theorem 2.3 in Section 12.

At a very high level, the strategy of the proof of Theorem 2.3 is similar to that followed by Behrisch, Coja-Oghlan and Kang [5] for the case $\varepsilon = \Theta(1)$: we start from the corresponding central limit theorem (proved very recently in [10]), and apply ‘smoothing’ arguments to deduce the local limit theorem. However, the details are very different:

Behrisch, Coja-Oghlan and Kang apply this technique to a univariate result for L_1 only, and then use a different argument going via the hypergraph model analogous to $G(n, m)$ to deduce a bivariate result. This method does not appear to work when $\varepsilon \rightarrow 0$. Instead, we apply two smoothing arguments; one to handle the nullity (or excess), and then one for the number of vertices.

Bivariate local limit results do not necessarily imply the corresponding univariate local limit results, due to the possibility of a ‘bad’ event \mathcal{B} on which one of the two parameters takes a ‘typical’ value and the other does not, with $\mathbb{P}(\mathcal{B}) = o(1)$ but $\mathbb{P}(\mathcal{B})$ large compared to the relevant point probabilities. However, the method used to prove Theorem 2.3 gives the following local limit results for $L_1(H_{n,p}^r)$ and $N_1(H_{n,p}^r)$ separately.

Theorem 2.4. *Let $r \geq 2$ be fixed, and let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$. Set $\lambda = \lambda(n) = 1 + \varepsilon$ and define $\rho_{r,\lambda}$ as in (2.2). Then whenever $x_n = \rho_{r,\lambda}n + O(\sqrt{n/\varepsilon})$ we have*

$$\mathbb{P}(L_1(H_{n,p}^r) = x_n) \sim \frac{1}{2\sqrt{\pi n/\varepsilon}} \exp\left(-\frac{(x_n - \rho_{r,\lambda}n)^2}{4n/\varepsilon}\right)$$

as $n \rightarrow \infty$.

Theorem 2.5. *Let $r \geq 2$ be fixed, let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n) \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$, and set $\lambda = \lambda(n) = 1 + \varepsilon$. For any $t_n \geq 0$ we have*

$$\mathbb{P}(N_1(H_{n,p}^r) = t_n) = \frac{1}{\sigma_n^* \sqrt{2\pi}} \exp\left(-\frac{(t_n - \rho_{r,\lambda}^*n)^2}{2(\sigma_n^*)^2}\right) + o(1/\sigma_n^*),$$

where $\rho_{r,\lambda}^*$ is defined in (2.4) and $\sigma_n^* = \sqrt{10/3}(r - 1)^{-1} \sqrt{\varepsilon^3 n}$.

Our main results assume our Standard Assumption 2.2; however, all our arguments can be extended, with varying amounts of additional work (and more complicated statements), to require only our Weak Assumption 2.1. Since the results of Behrisch, Coja-Oghlan and Kang [4, 5] cover the case $\varepsilon = \Theta(1)$, we assume $\varepsilon \rightarrow 0$ much of the time for simplicity.

In the probabilistic setting, a local (central) limit theorem is not the last possible word. One could ask for moderate and/or large deviation results (indeed, Eyal Lubetzky has asked us this question). We have not pursued these questions, but for a wide range of the parameters Lemma 8.4 shows that the probability that the largest component of $H_{n,p}^r$ has s vertices and nullity t is asymptotic to the expected number of components of $H_{n,p}^r$ with these parameters. This expectation can of course be calculated using Theorem 1.1. This method should give tight results for all moderate deviations and some (but not all) large deviations.

Remark 2.6. Instead of the model $H_{n,p}^r$ one could consider the analogue $H_{n,m}^r$ of the original Erdős–Rényi size model, where we select an m -edge r -uniform hypergraph on $[n]$ uniformly at random. Relating m and p by $p = m/\binom{n}{r}$, Theorem 2.3 implies an analogous result for this model. (This is not completely obvious, but can be shown using Theorem 1.1 as an intermediate step; alternatively, one can use Lemma 8.4 and its analogue for $H_{n,m}^r$,

and directly relate the expected number of s -vertex k -edge components in the models $H_{n,p}^r$ and $H_{n,m}^r$.) Behrisch, Coja-Oghlan and Kang [5] prove such a result in the denser setting, *i.e.*, when $\lambda > 1$ is constant. Here, unlike in [5], the parameters of the local limit theorem in $H_{n,m}^r$ are exactly the same as those in $H_{n,p}^r$. Very informally this should be no surprise, since the conversion between models corresponds to changing the number of edges by a random number of order $O(\sqrt{n})$. Such a change changes the typical size of the giant component by $O(\sqrt{n})$ vertices, which (in our range) is small compared to the standard deviation $\sqrt{n/\varepsilon}$. Similarly, the change in the nullity from switching from one model to the other is $O(\varepsilon^2 \sqrt{n}) = o(\sqrt{\varepsilon^3 n})$.

2.1. Related work

We have already mentioned a number of previous enumerative results related to Theorem 1.1. In this subsection we shall outline a number of previous probabilistic results related to Theorem 2.3, but first we introduce some general terminology.

Let (A_n) be a sequence of integer-valued random variables. We say that (A_n) satisfies a *global limit theorem* with parameters μ_n and σ_n if $(A_n - \mu_n)/\sigma_n$ converges in distribution to some distribution Z on the reals whose density function $\phi(x)$ is continuous and strictly positive. We say that (A_n) satisfies the corresponding *local limit theorem* if, for any sequence (x_n) of integers with $x_n = \mu_n + O(\sigma_n)$, we have

$$\mathbb{P}(A_n = x_n) \sim \frac{\phi((x_n - \mu_n)/\sigma_n)}{\sigma_n} \tag{2.9}$$

as $n \rightarrow \infty$. In the examples considered here, Z will always be the standard normal distribution $N(0, 1)$, but this is not necessary for the general arguments. These definitions extend in a natural way to *bivariate* global and local limit theorems for sequences (A_n, B_n) . In these terms, Theorem 2.3 is a bivariate local limit theorem for the pair $(L_1(H_{n,p}^r), N_1(H_{n,p}^r))$.

Remark 2.7. Let us comment in some detail on the issue of uniformity in asymptotics such as (2.9) above, since this may perhaps cause some confusion. In general, we adopt the approach of quantifying over sequences, since this seems intuitive and avoids lengthy sequences of quantifiers. For example, writing $\eta(n, x_n)$ for the ratio of the two sides of (2.9) above, the precise interpretation of (2.9) is the following: for any sequence (x_n) with the property that $\sup_n |x_n - \mu_n|/\sigma_n < \infty$, we have $\eta(n, x_n) \rightarrow 1$ as $n \rightarrow \infty$. Thus the rate at which $\eta(n, x_n)$ tends to 1 is allowed to depend on the choice of the sequence (x_n) .

Of course, such a statement automatically gives a certain kind of uniformity: given a constant C , for each n let x_n^\pm denote the choices of x_n with $|x_n - \mu_n| \leq C\sigma_n$ that maximize/minimize the ratio $\eta(n, x_n)$. Applying (2.9) to the sequences (x_n^+) and (x_n^-) gives $\eta(n, x_n^\pm) \rightarrow 1$, so we have the uniform statement

$$\max_{x : |x - \mu_n| \leq C\sigma_n} \eta(n, x) \rightarrow 1$$

as $n \rightarrow \infty$, and the same for min.

In most of our results, we quantify over $r \geq 2$, the choice of a sequence $(p(n))$ satisfying certain assumptions, and then perhaps additional sequences such as the sequences (x_n)

and (y_n) appearing in Theorem 2.3. The results then state that with all these choices fixed, a certain sequence indexed by n is $O(1)$ or $o(1)$. As above, although the bounds are not claimed to be uniform, bounds that are uniform over suitable sets of choices follow immediately.

As usual we say that an event $E = E_n$ (formally a sequence (E_n) of events) holds *with high probability*, or w.h.p., if $\mathbb{P}(E_n) \rightarrow 1$ as $n \rightarrow \infty$. Analogous to the classical 1960 result of Erdős and Rényi [16] for the case of graphs, in 1985 Schmidt-Pruzan and Shamir [25] showed that if $r \geq 2$ is constant (which we assume throughout) and $p = p(n) = \lambda(r-2)!n^{-r+1}$, then the random hypergraph $H_{n,p}^r$ undergoes a phase transition at $\lambda = 1$: for $\lambda < 1$ constant, w.h.p. $L_1(H_{n,p}^r)$ is at most a constant times $\log n$, if $\lambda = 1$ then $L_1(H_{n,p}^r)$ is of order $n^{2/3}$, and if $\lambda > 1$ is constant then w.h.p. $L_1(H_{n,p}^r) \geq c_{r,\lambda}n$ for some constant $c_{r,\lambda} > 0$. The model studied in [25] is in fact more general, allowing edges of different sizes up to $O(\log n)$.

The case where the ‘branching factor’ λ is bounded and bounded away from 1 is essentially equivalent to that where $\lambda > 1$ is constant; we shall not distinguish them in this discussion. Still considering this case, in 2007 Coja-Oghlan, Moore and Sanwalani [14] refined the results of Schmidt-Pruzan and Shamir, finding in particular the asymptotic value $\rho_{r,\lambda}n$ of $L_1(H_{n,p}^r)$ in the supercritical case, and giving an asymptotic formula for its variance. In 2010 Behrisch, Coja-Oghlan and Kang [4] went further when they established the limiting distribution of $L_1(H_{n,p}^r)$ in the regime $\lambda > 1$ constant: they used random walk and martingale methods to establish a central limit theorem, and then a smoothing technique, combined with multi-round exposure (ideas that appear in a slightly different form in [14]), to deduce the corresponding local limit theorem. In [5] they deduced from this a *bivariate* local limit theorem for $L_1(H_{n,p}^r)$ and $M_1(H_{n,p}^r)$ (equivalent to one for $L_1(H_{n,p}^r)$ and $N_1(H_{n,p}^r)$) under the same assumption $\lambda > 1$ constant. This result is directly analogous to Theorem 2.3 except that $\varepsilon = \Theta(1)$ rather than $\varepsilon \rightarrow 0$, and, as shown in [6], leads to an enumerative result analogous to Theorem 1.1, but for hypergraphs with nullity $\Theta(s)$, where s is the number of vertices.

Turning to the case where $\lambda = \lambda(n) \rightarrow 1$, let us write λ as $1 + \varepsilon$ with $\varepsilon = \varepsilon(n)$. Building on enumerative results of theirs [17] from 1997, in 2002 Karoński and Łuczak [18] proved a bivariate local limit theorem for $L_1(H_{n,p}^r)$ and $N_1(H_{n,p}^r)$ just above the ‘critical window’ $\varepsilon = O(n^{-1/3})$ of the phase transition, in the range where $\varepsilon^3 n \rightarrow \infty$ but $\varepsilon^3 n = o(\log n / \log \log n)$. In an extended abstract from 2006, Andriamampianina and Ravelomanana [1] outlined an extension of the enumerative results of Karoński and Łuczak [17] to treat hypergraphs with much larger excess (or nullity); this implies an extension of the local limit theorem of [18] to the range where $\varepsilon^3 n \rightarrow \infty$ but $\varepsilon^4 n \rightarrow 0$. These results illustrate a general phenomenon in this field: it seems that the barely supercritical case is more accessible to enumerative methods, and the strongly supercritical case ($\lambda > 1$ constant) to probabilistic methods.

In the special case of graphs, even more detailed results have been proved. Following many earlier results (see, for example, the references in [21]), in 2006 Łuczak and Łuczak proved a local limit theorem for $L_1(H_{n,p}^2)$ throughout the entire supercritical regime, *i.e.*, when $\lambda = 1 + \varepsilon$ with $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon = O(1)$, as part of a more general result about the

random cluster model. Slightly earlier, Pittel and Wormald [21] had come very close to proving a *trivariate* local limit theorem for $L_1(H_{n,p}^2)$, $N_1(H_{n,p}^2)$ and a third parameter, the number of vertices in the ‘core’. More precisely, they proved a trivariate local limit theorem for the conditional distribution of these parameters where the conditioning is on the event that there is a unique giant component of approximately the right size, an event that holds with probability $1 - o(1)$. With hindsight it is easy to remove the conditioning using, for example, Lemma 8.4.

Returning to hypergraphs, if we ask for results covering the entire (weakly) supercritical regime $\varepsilon^3 n \rightarrow \infty$, $\varepsilon \rightarrow 0$, it is only recently that anything non-trivial has been proved about the giant component. Indeed, as far as we are aware, the first result of this type is the central limit theorem for $L_1(H_{n,p}^r)$ proved in [9], using random walk and martingale arguments. A bivariate central limit theorem for $L_1(H_{n,p}^r)$ and $N_1(H_{n,p}^r)$ was proved very recently in [10], using similar methods. Here we shall use smoothing ideas as in [14, 5], but applied in a very different way, to deduce the corresponding bivariate local limit theorem, Theorem 2.3; Theorem 1.1 will then follow easily.

The methods of Sato and Wormald [24] are extensions of those used by Pittel and Wormald [21] and so, in the range in which they apply (*i.e.*, $r = 3$, and $p = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n) \rightarrow 0$ but $\varepsilon^4 n / \log^{3/2} n \rightarrow \infty$), may potentially lead to a trivariate local limit result for L_1 , N_1 and the number of vertices in the core. As far as we are aware, whether such a result can be proved throughout the range $\varepsilon \rightarrow 0$ but $\varepsilon^3 n \rightarrow \infty$, or for $r > 3$, is currently open.

In the next section we illustrate the basic strategy of our proof of Theorem 2.3 by showing how the same idea can be applied in a much simpler setting. Then, in Section 3.1, we describe some of the complications that will arise when we implement this idea to prove Theorem 2.3. Only then, in Section 3.2, do we describe the organization of the rest of the paper. The reason for this is that almost all of the paper is devoted to the proof of Theorem 2.3, and our description of the key steps in and structure of this proof will only make sense after the discussion earlier in Section 3. Formally, next to nothing in Section 3 is required in the later sections; the exception is that we use Proposition 3.1 in the proof of Theorem 2.5.

3. Smoothing: a simple example

The following trivial, standard observation captures the intuition that ‘local smoothness’ is what is needed to pass from a global limit theorem to the corresponding local one.

Proposition 3.1. *Suppose that a sequence (A_n) of random variables satisfies a global limit theorem with parameters μ_n and σ_n , and that $\mathbb{P}(A_n = x_n) - \mathbb{P}(A_n = x'_n) = o(1/\sigma_n)$ as $n \rightarrow \infty$ whenever $x_n = \mu_n + O(\sigma_n)$ and $x_n - x'_n = o(\sigma_n)$. Then (A_n) satisfies the corresponding local limit theorem.*

Once again, we quantify over sequences: the precise assumption is that for every pair of sequences (x_n) and (x'_n) such that $(x_n - x'_n)/\sigma_n \rightarrow 0$ and $\sup_n |x_n - \mu_n|/\sigma_n < \infty$, we have $\sigma_n(\mathbb{P}(A_n = x_n) - \mathbb{P}(A_n = x'_n)) \rightarrow 0$.

Proof. Let $\phi(x)$ be the density function associated to the global limit theorem, and $\Phi(x) = \int_{y < x} \phi(y)dy$ the corresponding distribution function. Fix a sequence (x_n) with $x_n = \mu_n + O(\sigma_n)$; by our definition of a local limit theorem it suffices to show that $\mathbb{P}(A_n = x_n) \sim \phi((x_n - \mu_n)/\sigma_n)/\sigma_n$. Let $C = 2 \sup_n |x_n - \mu_n|/\sigma_n$, which is finite by assumption.

The global limit theorem implies that for any fixed $x \in [-C, C]$ we have

$$\mathbb{P}(A_n \leq \mu_n + x\sigma_n) = \Phi(x) + o(1)$$

as $n \rightarrow \infty$; since $\Phi(x)$ is continuous the same estimate holds uniformly in $x \in [-C, C]$. It follows that if $\delta_n \rightarrow 0$ slowly enough, then

$$\begin{aligned} \mathbb{P}(x_n - \delta_n\sigma_n < A_n \leq x_n + \delta_n\sigma_n) &\sim \Phi\left(\frac{x_n - \mu_n}{\sigma_n} + \delta_n\right) - \Phi\left(\frac{x_n - \mu_n}{\sigma_n} - \delta_n\right) \\ &\sim 2\delta_n\phi\left(\frac{x_n - \mu_n}{\sigma_n}\right). \end{aligned}$$

Let I_n be the set of integers x with $x_n - \delta_n\sigma_n < x \leq x_n + \delta_n\sigma_n$, and let $x_n^\pm \in I_n$ be chosen to maximize and minimize $\mathbb{P}(A_n = x)$. Since $x_n^+ = \mu_n + O(\sigma_n)$ and $x_n^+ - x_n^- = o(\sigma_n)$, by assumption $\mathbb{P}(A_n = x_n^+)$ and $\mathbb{P}(A_n = x_n^-)$ differ by $o(1/\sigma_n)$. It follows that all $2\delta_n\sigma_n + O(1)$ values of $\mathbb{P}(A_n = x)$ for $x \in I$ are within $o(1/\sigma_n)$ of each other and hence of their average, which is $(1 + o(1))\phi((x_n - \mu_n)/\sigma_n)/\sigma_n$. □

A standard technique for establishing the smoothness required by Proposition 3.1 is to find a ‘smooth part’ within the distribution of A_n . Given a sequence (σ_n) of positive real numbers, we call a sequence (\mathcal{D}_n) of sets of probability distributions on the integers σ_n -smooth if the following conditions hold whenever (Y_n) is a sequence of random variables such that the distribution of Y_n is in \mathcal{D}_n :

$$\text{if } y_n - y'_n = o(\sigma_n) \text{ then } |\mathbb{P}(Y_n = y_n) - \mathbb{P}(Y_n = y'_n)| = o(1/\sigma_n). \tag{3.1}$$

To give a simple example of a smooth sequence, suppose that $\sigma_n \rightarrow \infty$, fix a constant $c > 0$, and let \mathcal{D}_n be the family of all binomial distributions with variance at least $c\sigma_n^2$. It is easy to check that (\mathcal{D}_n) is σ_n -smooth, for example directly from the formula for the binomial distribution. Note that the number of trials in the binomial distributions need not be n , or even $\Theta(n)$.

The following trivial observation describes at a high level the general strategy that we shall use to prove Theorem 2.3; of course there will be many complications to overcome.

Lemma 3.2. *Let (σ_n) be a sequence of positive reals, and let (\mathcal{D}_n) be σ_n -smooth. Let (\mathcal{F}_n) be a sequence of σ -algebras, and suppose that we can write A_n as $X_n + Y_n$, where X_n and Y_n are integer-valued, X_n is \mathcal{F}_n -measurable, and the conditional distribution of Y_n given \mathcal{F}_n is always in \mathcal{D}_n . If (A_n) satisfies a global limit theorem with parameters μ_n and σ_n , then (A_n) satisfies the corresponding local limit theorem.*

Proof. Let (x_n) and (x'_n) be sequences of integers with $x_n - x'_n = o(\sigma_n)$. (We may also assume $x_n = \mu_n + O(\sigma_n)$, but do not need this assumption.) Writing Ω_n for the probability

space on which A_n is defined, by (3.1) we have

$$\begin{aligned} & \sup_{\Omega_n} |\mathbb{P}(A_n = x_n \mid \mathcal{F}_n) - \mathbb{P}(A_n = x'_n \mid \mathcal{F}_n)| \\ & \leq \sup_{\Omega_n} \sup_{a \in \mathbb{Z}} |\mathbb{P}(Y_n = a \mid \mathcal{F}_n) - \mathbb{P}(Y_n = a + x'_n - x_n \mid \mathcal{F}_n)| = o(1/\sigma_n). \end{aligned}$$

(As usual, to obtain this uniform bound we consider $a_n \in \mathbb{Z}$ and $\omega_n \in \Omega_n$ (almost) achieving the supremum over a and Ω_n above; then we apply (3.1) with $y_n = a_n$ and $y'_n = a_n + x'_n - x_n$, to the conditional distribution of Y_n given \mathcal{F}_n evaluated at ω_n .) It follows that $|\mathbb{P}(A_n = x_n) - \mathbb{P}(A_n = x'_n)| = o(1/\sigma_n)$, so we may apply Proposition 3.1. \square

This ‘smooth part’ technique is easiest to apply in the case of sums of independent variables; in this setting McDonald [20], for example, used it with each \mathcal{D}_n consisting of a single binomial distribution with appropriate parameters. Similar ideas in a combinatorial setting were used by Scott and Tatenos [26]. Behrisch, Coja-Oghlan and Kang [4] used it to prove the special case of Theorem 2.4 where $\varepsilon = \Theta(1)$, with the σ -algebra \mathcal{F}_n corresponding to the first part of a multi-round exposure of the edges of $H_{n,p}^r$. Their particular decomposition cannot be used to prove Theorem 2.4, since the variance of the relevant variable Y_n is too small when $\varepsilon \rightarrow 0$; we return to this later.

Remark. A variant of the method above is to replace the condition (3.1) by the stronger condition $\mathbb{P}(Y_n = y_n + 1) = \mathbb{P}(Y_n = y_n) + O(1/\sigma_n^2)$, as in Davis and McDonald [15], for example. In situations where Y_n has a simple distribution, this condition may be just as easy to verify as (3.1); applying it leads to a slightly simpler argument overall. In more complicated situations, including those where the decomposition $X_n + Y_n$ in Lemma 3.2 holds only most of the time, rather than always, it is likely to be better to consider probabilities of values $o(\sigma_n)$ apart, as above. Then the error bounds needed in the estimates of the point probabilities are looser; this is vital in our argument in Section 11, for example.

As a simple warm-up for our main result, let us outline how Lemma 3.2 may be applied to the variable $A_n = L_1(G_n)$, where $G_n = H_{n,p}^2 = G(n, p)$ is the standard Erdős–Rényi (binomial) random graph with $p = p(n) = \lambda/n$ with $\lambda > 1$ constant. Since the result here is not new, and our aim is to illustrate in a simple setting some of the ideas we shall use later, we shall assume the following fact without proof. Recall that the 2-core, or simply core, $C(G)$ of a graph G , introduced in [8], is the maximal subgraph with minimum degree at least 2.

Proposition 3.3. *Let $\lambda > 1$ be constant. There is a constant $c = c(\lambda) > 0$ such that $G_n = G(n, \lambda/n)$ has the following properties with probability $1 - o(n^{-1/2})$: the core $C(G_n)$ of G_n has a unique component \mathcal{C}_1 with at least cn vertices, and \mathcal{C}_1 is a subgraph of the largest component of G_n ; furthermore, G_n has at least cn isolated vertices.* \square

Here then is our illustration of smoothing for the Erdős–Rényi model, in the simple case of constant branching factor. In this case the central limit theorem was established

by Pittel and Wormald [21] and the local one by Łuczak and Łuczak [19]; our aim here is to show how one can deduce one from the other.

Theorem 3.4. *Let $p = p(n) = \lambda/n$ where $\lambda > 1$ is constant, set $G_n = G(n, p)$ and let $A_n = L_1(G_n)$. If (A_n) satisfies a global limit theorem with $\sigma_n = \Theta(\sqrt{n})$ then it satisfies the corresponding local limit theorem.*

Proof. Given any graph G , let G^- be the *reduced graph* obtained from G by deleting all pendant edges incident with the core $C(G)$ of G . In other words, G^- is the spanning subgraph of G obtained by deleting those edges $e = vw$ in which v has degree 1 and w is in $C(G)$. Note that G and G^- have the same core. It follows that if H is any graph that can arise as G^- for some G , then a graph G with $V(G) = V(H)$ has $G^- = H$ if and only if G is formed from H in the following way: for each isolated vertex v of H , either do nothing, or add an edge from v to some vertex w of the core $C(H)$ of H . Since the probability of a graph G in the model $G(n, p)$ is proportional to $(p/(1 - p))^{e(G)}$, it follows that for any graph H whose core $C(H)$ has m vertices, the conditional distribution of $G_n = G(n, p)$ given that $G_n^- = H$ may be described as follows.

For each isolated vertex v of H , with probability $pm/(pm + 1 - p)$ pick a uniformly random vertex w of $C(H)$ and join v to w ; otherwise do nothing. The decisions associated to different v are independent.

Let \mathcal{F}_n be the σ -algebra generated by the random variable G_n^- , let X_n be the number of vertices in the component of $H = G_n^-$ containing the largest component \mathcal{C}_1 of its core (chosen according to any fixed rule if there is a tie), and let Y_n be the number of vertices ‘rejoined’ to this component \mathcal{C}_1 when constructing G_n from G_n^- as above. Let $A'_n = X_n + Y_n$, noting that whenever \mathcal{C}_1 is a subgraph of the largest component of G_n , we have $A'_n = L_1(G_n)$. Clearly, X_n is \mathcal{F}_n -measurable. Moreover, from the independence over vertices v , the conditional distribution of Y_n given \mathcal{F}_n is the binomial distribution $\text{Bin}(i(G_n^-), \pi)$ where $i(H)$ denotes the number of isolated vertices of a graph H and $\pi = \pi(G_n^-) = p|\mathcal{C}_1|/(p|C(G_n^-)| + 1 - p)$.

Let $c > 0$ be the constant appearing in Proposition 3.3. Let E_n be the event that the core $C(G_n) = C(G_n^-)$ has a unique component with at least cn vertices, and that $i(G_n^-) \geq cn$. Note that $E_n \in \mathcal{F}_n$. Also, since $i(G_n^-) \geq i(G_n)$, by Proposition 3.3 we have $\mathbb{P}(E_n) = 1 - o(n^{-1/2})$. Whenever E_n holds we have $c \leq p|\mathcal{C}_1| \leq p|C(G_n^-)| = O(1)$ so, since $1 - p \sim 1$, the probability π is bounded away from 0 and 1. Hence, since $i(G_n^-) \geq cn$, the variance $i(G_n^-)\pi(1 - \pi)$ of the (binomial) conditional distribution of Y_n is at least an for some constant $a > 0$. Letting \mathcal{D}_n be the family of all binomial distributions with variance at least an , then whenever E_n holds, the conditional distribution of Y_n given \mathcal{F}_n is in \mathcal{D}_n . As noted above, the sequence (\mathcal{D}_n) is \sqrt{n} -smooth.

Recall that $A'_n = X_n + Y_n$ is the number of vertices in the component of G_n containing the largest component \mathcal{C}_1 of $C(G_n) = C(G_n^-)$ (chosen according to any fixed rule if there is a tie) so, by Proposition 3.3, $A'_n = L_1(G_n)$ with probability at least $1 - o(n^{-1/2})$. Since E_n holds w.h.p., the conditional distribution of A'_n given E_n satisfies the same global limit theorem as the unconditional distribution of $A_n = L_1(G_n)$; let μ_n and $\sigma_n = \Theta(\sqrt{n})$

be the parameters of this global limit theorem, and let ϕ be the associated limiting density function. Having conditioned on E_n , we now apply Lemma 3.2, which involves conditioning further on \mathcal{F}_n and using the fact that (\mathcal{D}_n) is \sqrt{n} -smooth.¹ We obtain the result that for any x_n satisfying $x_n - \mu_n = O(\sqrt{n})$ we have

$$\mathbb{P}(A'_n = x_n \mid E_n) = \frac{\phi((x_n - \mu_n)/\sigma_n)}{\sigma_n} + o(n^{-1/2}).$$

Since $\mathbb{P}(E_n) = 1 - o(n^{-1/2})$ and $\mathbb{P}(A'_n \neq A_n) = o(n^{-1/2})$ we have $\mathbb{P}(A_n = x_n) = \mathbb{P}(A'_n = x_n \mid E_n) + o(n^{-1/2})$, giving the result. □

3.1. Smoothing in the proof of Theorem 2.3

In the rest of the paper we shall use a version of the above technique to prove Theorem 2.3. Since this proof is rather long, and on reading (or writing!) it for the first time one might wonder why it is so complicated, in this section we outline some of the problems that occur when adapting the proof of Theorem 3.4. Some of these concern the transition from graphs to hypergraphs, some arise when allowing $\varepsilon \rightarrow 0$, and some concern the extension to a bivariate result. It is allowing $\varepsilon \rightarrow 0$ that turns out to cause by far the most difficulty. (Recall that $p = \lambda(n)(r - 2)!n^{-r+1}$ where $\lambda(n) = 1 + \varepsilon(n)$ is the ‘branching factor’.)

Firstly, it turns out that (in both the graph and hypergraph cases) the number of vertices of degree 1 joined directly to the core is $\Theta(\varepsilon^2 n)$. This means that the variance obtained by deleting and reattaching such vertices will be $\Theta(\varepsilon^2 n)$, which is much smaller than the variance $\Theta(n/\varepsilon)$ of $L_1 = L_1(H_{n,p}^r)$ when $\varepsilon \rightarrow 0$. For this reason we need to remove and reattach larger trees; indeed, it turns out that we need to consider trees up to size $\Theta(\varepsilon^{-2})$, which is essentially the largest size that appears. (The bulk of the variance comes from the large trees.) This complicates things, since each tree contributes a different number of vertices to the giant component.

Secondly, there are various ‘good events’ E that we need to hold for various parts of our smoothing argument. As in the simple example above, one is that the core is not too much smaller than it should be, and another is that the largest component of the core is contained in the largest component of the whole graph. Some of the bad events E^c turn out to have probability $\exp(-\Theta(\varepsilon^3 n))$ (since the core is really characterized by the kernel, which has $\Theta(\varepsilon^3 n)$ vertices). So if $\varepsilon^3 n \rightarrow \infty$ slowly, the unconditional probabilities of these events may be much larger than the probabilities such as $\mathbb{P}(L_1 = x_n) = \Theta(\sqrt{\varepsilon/n})$ that we wish to estimate. The solution is to show that $\mathbb{P}(E \mid L_1 = x_n) = 1 - o(1)$, so $\mathbb{P}(L_1 = x_n) \sim \mathbb{P}(\{L_1 = x_n\} \cap E)$. Then we can effectively condition on E (though being careful to keep independence where it is needed).

Thirdly, unlike for graphs, in the hypergraph case, even the simple operation of deleting all ‘pendant edges’ attached to the core (*i.e.*, hyperedges with one vertex in the core and

¹ To spell this out, let (Ω_n, \mathbb{P}_n) be the (finite) probability space on which G_n is defined, and let \mathbb{Q}_n be the probability measure $\mathbb{P}_n(\cdot \mid E_n)$ on Ω_n . We apply Lemma 3.2 to the sequence of probability spaces (Ω_n, \mathbb{Q}_n) , on which the random variables A'_n satisfy the required global limit theorem. Since $E_n \in \mathcal{F}_n$, then when $\omega \in E_n$ we have $\mathbb{Q}_n(\cdot \mid \mathcal{F}_n)(\omega) = \mathbb{P}_n(\cdot \mid \mathcal{F}_n)(\omega)$ (by the tower-law). So, working on (Ω_n, \mathbb{Q}_n) , when $\omega \in E_n$ the conditional distribution of Y_n given \mathcal{F}_n is in \mathcal{D}_n ; what happens when $\omega \notin E_n$ is irrelevant since $\mathbb{Q}_n(E_n^c) = 0$. Hence Lemma 3.2 gives an asymptotic formula for $\mathbb{Q}_n(A'_n = x_n) = \mathbb{P}_n(A'_n = x_n \mid E_n)$.

the other vertices in no other hyperedges) is not so simple to invert. The inverse involves selecting *disjoint* sets of $r - 1$ isolated vertices to rejoin to the core. The condition that the sets must be disjoint means that the number that do rejoin no longer has a binomial distribution. We deal with this by randomly ‘marking’ some vertices throughout the graph. Roughly speaking, we detach pendant edges attached either to the core or to marked vertices, meaning that we remember that a certain $(r - 1)$ -tuple was attached either to the core or to a marked vertex. Then all choices of *where* to reattach the tuples do turn out to be independent. Of course, we actually detach larger trees, not just pendant edges. In fact, rather than consider individual trees, we shall directly study the forests attached to the core and to a suitable set of marked vertices.

Finally, for the bivariate result we need to show that the nullity N_1 of the largest component also has a smooth distribution; for this we use the same basic smoothing technique applied in a different (and much simpler) way than for L_1 . Fortunately, since our smoothing argument for L_1 involves operations on the hypergraph that do not affect N_1 , these two separate smoothing arguments combine to give the *joint* smoothness of L_1 and N_1 needed to prove Theorem 2.3.

One might wonder whether our approach is really easier than (or indeed different from) proving a local limit theorem directly. Whether or not it is easier, the fact remains that the local limit theorem was previously only known for restricted ranges of the key parameter $\varepsilon(n)$. As to whether the approaches are genuinely different, we believe that the answer is ‘yes’. A key observation is that we study only *part* of the variation in the size of the giant component. The general method means that, writing σ_n^2 for the variance of the quantity (L_1 or N_1) we are studying, our ‘smoothing distribution’ needs variance $\Theta(\sigma_n^2)$, but it can be an arbitrarily small constant times σ_n^2 . This is vital since it means that in many of our estimates we have a constant factor elbow room. This is unlikely to be the case in any direct proof of the local limit theorem, since it would lead to a significant error in the variance of L_1 or N_1 . Here the variances of L_1 and N_1 are part of the *input* (the global limit assumption), and we really are establishing only smoothness, rather than re-evaluating the whole distribution.

3.2. Organization of the rest of the paper

The rest of the paper is organized as follows. In Section 4 we state two results from [10] that we shall need; one of these is the global (central) limit theorem corresponding to Theorem 2.3. Then we state two key intermediate results, Theorems 4.3 and 4.4. The first establishes smoothness of N_1 , showing (a little more than) that nearby values have almost equal probabilities. The second establishes (essentially) smoothness of the distribution of L_1 conditional on N_1 ; as we note in the next section, these results easily imply Theorem 2.3.

In Section 5 we prove Theorem 4.3, using multi-round exposure arguments reminiscent of those used by Behrisch, Coja-Oghlan and Kang [5]. In the subsequent sections we prepare the ground for the (much more complicated) proof of Theorem 4.4. First, in Section 6 we present a result of Selivanov [27] enumerating hypergraph forests subject to certain constraints, and a simple consequence concerning random forests. Then, in Section 7, we use Selivanov’s formula to show that a certain distribution associated to detaching and reattaching forests from the core and ‘marked’ vertices is \sqrt{n}/ε -smooth

as defined earlier in this section, so it can play the role of Y_n above when studying the distribution of L_1 . Next, in Section 8, we state a precise form of the supercritical/subcritical duality result for the random hypergraph $H_{n,p}^r$; in Section 9 we use this to establish some properties of the ‘small’ components of $H_{n,p}^r$ that we shall need later. In Section 10 we formally define ‘marked vertices’ and the extended core of $H_{n,p}^r$, and show that with high conditional probability it has the properties we need. After this preparation, in Section 11 we prove Theorem 2.3; in Section 12 we show that Theorem 1.1 follows. Finally, in the appendix to an earlier version [11] of this paper, we give detailed calculations comparing our formulae with those in [2, 3, 6, 7, 17, 24].

4. The key ingredients

In this section we state two results from [10] that we shall need as ‘inputs’ to our smoothing arguments. Then we state our two main intermediate results, and show how they combine to give Theorem 2.3.

4.1. Inputs

Building on methods we used in [9] to prove the central limit theorem for $L_1 = L_1(H_{n,p}^r)$, in [10] we proved the following bivariate (global) central limit theorem for the order L_1 and nullity N_1 of the largest component of $H_{n,p}^r$. Here, and throughout, $\rho_{r,\lambda}$ and $\rho_{r,\lambda}^*$ are as defined in (2.2) and (2.4).

Theorem 4.1. *Let $r \geq 2$ be fixed, and $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$, where $\varepsilon = \varepsilon(n) \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$. Let L_1 and N_1 be the order and nullity of the largest component \mathcal{L}_1 of $H_{n,p}^r$. Then*

$$\left(\frac{L_1 - \rho_{r,\lambda} n}{\sqrt{2n/\varepsilon}}, \frac{N_1 - \rho_{r,\lambda}^* n}{\sqrt{10/3(r-1)^{-1}\varepsilon^3 n}} \right) \xrightarrow{d} (Z_1, Z_2),$$

where \xrightarrow{d} denotes convergence in distribution, and (Z_1, Z_2) has a bivariate Gaussian distribution with mean 0, $\text{Var}[Z_1] = \text{Var}[Z_2] = 1$ and $\text{Cov}[Z_1, Z_2] = \sqrt{3}/5$. □

In particular, recalling (2.5), L_1 is asymptotically Gaussian with mean $\Theta(\varepsilon n)$ and variance $\Theta(n/\varepsilon)$, and N_1 is asymptotically Gaussian with mean $\Theta(\varepsilon^3 n)$ and variance $\Theta(\varepsilon^3 n)$.

In Section 5 we shall need the following large-deviation bounds on L_1 and L_2 , the order of the second largest component of $H_{n,p}^r$; this result is also proved in [10].

Theorem 4.2. *Let $r \geq 2$ be fixed, and let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = O(1)$ and $\varepsilon^3 n \rightarrow \infty$. If $\omega = \omega(n) \rightarrow \infty$ and $\omega = O(\sqrt{\varepsilon^3 n})$ then*

$$\mathbb{P}(|L_1(H_{n,p}^r) - \rho_{r,\lambda} n| \geq \omega \sqrt{n/\varepsilon}) = \exp(-\Omega(\omega^2)). \tag{4.1}$$

Moreover, if $L = L(n)$ satisfies $\varepsilon^2 L \rightarrow \infty$ and $L = O(\varepsilon n)$, then

$$\mathbb{P}(L_2(H_{n,p}^r) > L) \leq C \frac{\varepsilon n}{L} \exp(-c\varepsilon^2 L),$$

for some constants $c, C > 0$. □

Here, as usual, the constants c, C and the implicit constant in the Ω notation in (4.1) are allowed to depend on all previous choices: on r , the function $p(n)$, and the functions $\omega(n)$ and $L(n)$; see Remark 2.7.

4.2. Main steps

Theorem 2.3 is the bivariate local limit version of Theorem 4.1. To deduce it from Theorem 4.1, we must show that ‘nearby’ potential values of the pair (L_1, N_1) have essentially the same probability. (Recalling (1.1), for (s, t) to be a potential value, $r - 1$ must divide $s + t - 1$.) We proceed in two stages. In the first, we show that N_1 has a smooth distribution, which will already allow us to prove Theorem 2.5. More precisely, we shall prove the following result in Section 5. We consider the pair $(L_1 - (r - 2)N_1, N_1)$ rather than (L_1, N_1) for technical reasons that will become clear during the proof; this makes little difference, since the standard deviation of N_1 is much smaller than that of L_1 .

Theorem 4.3. *Let $r \geq 2$ be fixed, and let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n) = O(1)$ and $\varepsilon^3 n \rightarrow \infty$. For any sequences (t_n) and (t'_n) with $t_n, t'_n \geq 0$ and $t_n - t'_n = o(\sqrt{\varepsilon^3 n})$, and any $I_n \subset \mathbb{Z}$, we have*

$$\mathbb{P}(N_1 = t_n \text{ and } L_1 - (r - 2)N_1 \in I_n) - \mathbb{P}(N_1 = t'_n \text{ and } L_1 - (r - 2)N_1 \in I_n) = o((\varepsilon^3 n)^{-1/2}).$$

By Proposition 3.1, Theorems 4.1 and 4.3 imply Theorem 2.5. Indeed, Theorem 4.1 immediately implies that $N_1 = N_1(H_{n,p}^r)$ satisfies a central limit theorem with parameters $\rho_{r,\lambda}^* n$ for the mean and $\sigma_n^* = \sqrt{10/3}(r - 1)^{-1} \sqrt{\varepsilon^3 n}$ for the standard deviation. Since $\sigma_n^* = \Theta(\sqrt{\varepsilon^3 n})$, taking $I_n = \mathbb{Z}$ in Theorem 4.3 we see that if $t_n - t'_n = o(\sigma_n^*)$ then $\mathbb{P}(N_1 = t_n) - \mathbb{P}(N_1 = t'_n) = o(1/\sigma_n^*)$. Hence Theorem 2.5 follows by Proposition 3.1.

In the next result, and much of the rest of the paper, we only consider potential values of L_1 in a ‘typical’ range. To be precise, having fixed a function $p(n)$ (and thus $\varepsilon(n)$ and $\lambda(n)$) satisfying our Weak Assumption 2.1, let $\delta = \delta(n)$ satisfy

$$\delta \rightarrow 0 \quad \text{and} \quad \delta \geq (\varepsilon^3 n)^{-1/3}, \tag{4.2}$$

and let

$$R = R_n = R_{n,p} = [(1 - \delta)\rho_{r,\lambda} n, (1 + \delta)\rho_{r,\lambda} n]. \tag{4.3}$$

(To be concrete, we may just set $\delta = (\varepsilon^3 n)^{-1/3}$, but the precise value is irrelevant as long as the conditions above hold.) Recalling (2.5) and (2.3), under our Weak Assumption 2.1 we have $\rho_{r,\lambda} = \Theta(\varepsilon)$ and $\rho_{r,\lambda}$ bounded away from 1. Hence there are constants $c, C > 0$

(depending on the function $\varepsilon(n)$) such that, for n large enough,

$$R_n \subseteq [c\varepsilon n, C\varepsilon n] \quad \text{and} \quad R_n \subseteq [c\varepsilon n, (1 - c)n]. \tag{4.4}$$

By Theorem 4.2, applied with $\omega = \omega(n) = \delta \rho_{r,\lambda} n / (\sqrt{n/\varepsilon}) = \Theta(\delta \sqrt{\varepsilon^3 n})$, under our Weak Assumption 2.1 we have

$$\mathbb{P}(L_1(H_{n,p}^r) \notin R) \leq \exp(-c\delta^2 \varepsilon^3 n) \leq \exp(-c(\varepsilon^3 n)^{1/3}) = O(1/(\varepsilon^3 n)). \tag{4.5}$$

The bulk of the paper will be devoted to the proof of the following result establishing, essentially, smoothness of the conditional distribution of L_1 given N_1 .

Theorem 4.4. *Let $r \geq 2$ be fixed, let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$, and set $L_1 = L_1(H_{n,p}^r)$. Define $R = R_n$ as in (4.3). If (x_n) , (y_n) and (t_n) are sequences of integers with $x_n, y_n \in R_n$, $x_n - y_n = o(\sqrt{n/\varepsilon})$, $t_n \geq 2$, and*

$$x_n \equiv y_n \equiv 1 - t_n \quad \text{modulo } (r - 1),$$

then

$$\mathbb{P}(L_1 = x_n, N_1 = t_n) - \mathbb{P}(L_1 = y_n, N_1 = t_n) = o(1/(\varepsilon n)).$$

Theorems 4.3 and Theorem 4.4 will be proved in Sections 5–11. First, let us show how they imply Theorem 2.3. Although the argument is straightforward, since Theorem 4.4 is our main result, we shall spell out the details.

Proof of Theorem 2.3. Throughout we fix $r \geq 2$, and a function $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ such that $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$. Let

$$\sigma_n = \sqrt{2n/\varepsilon} \quad \text{and} \quad \sigma_n^* = \sqrt{10/3}(r - 1)^{-1} \sqrt{\varepsilon^3 n} = \Theta(\sqrt{\varepsilon^3 n}).$$

Indicating the dependence on n for once, let $L_{1,n} = L_1(H_{n,p}^r)$ and $N_{1,n} = N_1(H_{n,p}^r)$. It will be convenient to consider the linear combination

$$\tilde{L}_{1,n} = L_{1,n} - (r - 2)N_{1,n}.$$

Recalling the definitions (2.2) and (2.4) of $\rho_{r,\lambda}$ and $\rho_{r,\lambda}^*$, set

$$\tilde{\rho}_{r,\lambda} = \rho_{r,\lambda} - (r - 2)\rho_{r,\lambda}^*.$$

Since $\sigma_n^* = o(\sigma_n)$, Theorem 4.1 immediately implies that

$$\left(\frac{\tilde{L}_{1,n} - \tilde{\rho}_{r,\lambda} n}{\sigma_n}, \frac{N_{1,n} - \rho_{r,\lambda}^* n}{\sigma_n^*} \right) \xrightarrow{d} (Z_1, Z_2), \tag{4.6}$$

where (Z_1, Z_2) has a bivariate Gaussian distribution with mean 0, $\text{Var}[Z_1] = \text{Var}[Z_2] = 1$ and $\text{Cov}[Z_1, Z_2] = \sqrt{3/5}$; the probability density function $f(a, b)$ of this distribution is given in (2.7).

Let (x_n) and (y_n) be sequences with $x_n = \rho_{r,\lambda} n + O(\sigma_n)$ (i.e., $\sup_n |x_n - \rho_{r,\lambda} n|/\sigma_n < \infty$) and $y_n = \rho_{r,\lambda}^* n + O(\sigma_n^*)$, such that $x_n + y_n - 1$ is a multiple of $r - 1$ for all n ; our aim is

to prove (2.6) for these sequences. By a standard subsequence argument, we may assume without loss of generality that

$$\frac{x_n - \rho_{r,\lambda}n}{\sigma_n} \rightarrow x \quad \text{and} \quad \frac{y_n - \rho_{r,\lambda}^*n}{\sigma_n^*} \rightarrow y$$

for some $x, y \in \mathbb{R}$. Since the density $f(a, b)$ is continuous and strictly positive, what we must show is exactly that

$$\mathbb{P}(L_{1,n} = x_n, N_{1,n} = y_n) = \frac{(r - 1)f(x, y) + o(1)}{\sigma_n \sigma_n^*}. \tag{4.7}$$

(As usual, the $o(1)$ term represents a quantity that tends to 0 as $n \rightarrow \infty$; the rate may depend on all the choices made so far.)

It will be convenient to consider more explicit reformulations of Theorems 4.3 and 4.4. By Theorem 4.3, for every constant $\alpha > 0$ there is a constant $\beta > 0$ and an integer n_0 such that the following holds: whenever $n \geq n_0$, $t, t' \geq 0$ with $|t - t'| \leq \beta\sigma_n^*$, and $I \subset \mathbb{Z}$, then

$$\left| \mathbb{P}(N_{1,n} = t, \tilde{L}_{1,n} \in I) - \mathbb{P}(N_{1,n} = t', \tilde{L}_{1,n} \in I) \right| \leq \alpha/\sigma_n^*. \tag{4.8}$$

Indeed, if (4.8) does not hold, then picking an α for which it fails, for each k we may find an $n_k > n_{k-1}$ and I_{n_k} , t_{n_k} and t'_{n_k} such that $|t_{n_k} - t'_{n_k}| \leq \sigma_n^*/k$ and $\mathbb{P}(N_{1,n_k} = t_{n_k}, \tilde{L}_{1,n_k} \in I_{n_k})$ and $\mathbb{P}(N_{1,n_k} = t'_{n_k}, \tilde{L}_{1,n_k} \in I_{n_k})$ differ by at least α/σ_n^* . Completing the sequences t_n , t'_n and I_n appropriately gives a counterexample to Theorem 4.3.

Similarly, since $\sigma_n = \Theta(\sqrt{n/\varepsilon})$ and $\sigma_n \sigma_n^* = \Theta(\varepsilon n)$, Theorem 4.4 implies that for any constant $\eta > 0$ there are $\gamma_1 > 0$ and n_0 such that whenever $n \geq n_0$, $t \geq 2$ and $s, s' \in R_n$ with $|s - s'| \leq \gamma_1 \sigma_n$ and $s \equiv s' \equiv 1 - t$ modulo $r - 1$, then

$$\left| \mathbb{P}(L_{1,n} = s, N_{1,n} = t) - \mathbb{P}(L_{1,n} = s', N_{1,n} = t) \right| \leq \frac{\eta}{\sigma_n \sigma_n^*}. \tag{4.9}$$

Let $\eta > 0$ be constant. We shall show that if n is large enough, then

$$\left| \mathbb{P}(L_{1,n} = x_n, N_{1,n} = y_n) - \frac{(r - 1)f(x, y)}{\sigma_n \sigma_n^*} \right| \leq \frac{4r\eta}{\sigma_n \sigma_n^*}, \tag{4.10}$$

proving (4.7) and thus Theorem 2.3.

Define γ_1 as in (4.9). Since $f(\cdot, \cdot)$ is continuous at (x, y) , we may choose $\gamma_2 > 0$ such that whenever $|a - x| \leq \gamma_2$ and $|b - y| \leq \gamma_2$, we have $|f(a, b) - f(x, y)| \leq \eta$. Set $\gamma = \min\{\gamma_1, \gamma_2\}$ and let

$$I_n = [\tilde{\rho}_{r,\lambda}n + (x - \gamma/2)\sigma_n, \tilde{\rho}_{r,\lambda}n + (x + \gamma/2)\sigma_n].$$

For $n \geq 1$ and $t \geq 0$ let

$$\pi_{n,t} = \mathbb{P}(N_{1,n} = t, \tilde{L}_{1,n} \in I_n).$$

By (4.8), applied with $\alpha = \eta\gamma$, there is a constant $\beta > 0$, which we may assume to be less than γ_2 , such that for all large enough n we have

$$|\pi_{n,t} - \pi_{n,t'}| \leq \eta\gamma/\sigma_n^* \tag{4.11}$$

whenever t, t' lie in the interval

$$J_n = [\rho_{r,\lambda}^* n + (y - \beta/2)\sigma_n^*, \rho_{r,\lambda}^* n + (y + \beta/2)\sigma_n^*].$$

(Here we have used the fact that for n large J_n consists only of positive integers, which holds since $\sigma_n^* = o(\rho_{r,\lambda}^* n)$.) Let

$$a_n = \frac{1}{|J_n|} \sum_{t \in J_n} \pi_{n,t} = \frac{1}{|J_n|} \mathbb{P}((\tilde{L}_{1,n}, N_{1,n}) \in I_n \times J_n).$$

Since $\sigma_n^* \rightarrow \infty$ and β is constant, we have $|J_n| \sim \beta\sigma_n^*$. It follows from (4.6) that

$$a_n \sim \frac{1}{\beta\sigma_n^*} \int_{a=x-\gamma/2}^{x+\gamma/2} \int_{b=y-\beta/2}^{y+\beta/2} f(a, b) da db.$$

Since β and γ are at most γ_2 , for all (a, b) in the region of area $\beta\gamma$ over which we integrate we have $|f(a, b) - f(x, y)| \leq \eta$. Hence, for n large enough,

$$|a_n - f(x, y)\gamma/\sigma_n^*| \leq 2\eta\gamma/\sigma_n^*.$$

Now a_n is the average of the values $\pi_{n,t}$ over $t \in J_n$, so the bound (4.11) implies that all of these values are within $\eta\gamma/\sigma_n^*$ of a_n . For n large enough, $y_n \in J_n$, so

$$|\pi_{n,y_n} - f(x, y)\gamma/\sigma_n^*| \leq 3\eta\gamma/\sigma_n^*. \tag{4.12}$$

Since the component of $H_{n,p}^r$ with $L_{1,n}$ vertices and nullity $N_{1,n}$ is by definition connected, (1.1) gives $L_{1,n} + N_{1,n} \equiv 1$ modulo $r - 1$. Hence

$$\pi_{n,y_n} = \mathbb{P}(N_{1,n} = y_n, L_{1,n} - (r - 2)y_n \in I_n) = \sum_{s \in S_n} \mathbb{P}(L_{1,n} = s, N_{1,n} = y_n) \tag{4.13}$$

where S_n consists of all integers in $I_n + (r - 2)y_n$ congruent to $1 - y_n$ modulo $r - 1$. Hence

$$|S_n| = \frac{|I_n|}{r - 1} + O(1) = \frac{\gamma\sigma_n}{r - 1} + O(1) \sim \frac{\gamma\sigma_n}{r - 1}. \tag{4.14}$$

Recall that $x_n = \rho_{r,\lambda} n + x\sigma_n + o(\sigma_n)$ and $y_n = \rho_{r,\lambda}^* n + O(\sigma_n^*) = \rho_{r,\lambda}^* n + o(\sigma_n)$. Thus we have $x_n - (r - 2)y_n = \tilde{\rho}_{r,\lambda} n + (x + o(1))\sigma_n$, and so for sufficiently large n we have $x_n - (y - 2)y_n \in I_n$ and so $x_n \in S_n$. Further, $s \in S_n$ implies $|s - \rho_{r,\lambda} n| \leq |x_n - \rho_{r,\lambda} n| + \gamma\sigma_n = O(\sigma_n)$. Hence, for n large enough, $S_n \subseteq R_n$. It follows by (4.9) that the probabilities summed in (4.13) are all within $\eta/(\sigma_n\sigma_n^*)$ of each other and hence of their average, which by (4.12) and (4.14) is within $3r\eta/(\sigma_n^*\sigma_n)$ of $(r - 1)f(x, y)/(\sigma_n\sigma_n^*)$. Since $x_n \in S_n$ this concludes the proof of (4.10) and hence that of Theorem 2.3. □

5. Smoothing the excess: multi-round exposure

In this section we prove Theorem 4.3. The arguments in this section do not obviously simplify in the case $\varepsilon \rightarrow 0$, so throughout we work with our Weak Assumption 2.1, *i.e.*, we let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon = O(1)$ and $\varepsilon^3 n \rightarrow \infty$.

Set

$$p_1 = (1 + \varepsilon/2)(r - 2)!n^{-r+1}$$

and define p_2 by $p = p_1 + p_2 - p_1 p_2$, noting that

$$p_2 \sim (\varepsilon/2)(r - 2)!n^{-r+1} = \Theta(\varepsilon n^{-r+1}). \tag{5.1}$$

Using a now standard idea originally due to Erdős and Rényi [16], we shall view $H_{n,p}^r$ as $H_1 \cup H_2$ where H_1 and H_2 are independent, and H_i has the distribution H_{n,p_i}^r . To prove Theorem 4.3 we first ‘reveal’ (i.e., condition on) H_1 . Then we reveal many but not all edges of H_2 . We do this in such a way that the remaining edges of H_2 must be of a simple type. We then show that the conditional distribution of the number of these edges present is essentially binomial. Since each will contribute 1 to $N_1 = n(H_{n,p}^r)$, this will allow us to prove the result. The strategy is inspired by a related argument of Behrisch, Coja-Oghlan and Kang [4], itself based on ideas of Coja-Oghlan, Moore and Sanwalani [14], though the details are very different since the objective is different. (Their argument is used to ‘smooth’ L_1 rather than N_1 , and requires ε bounded away from zero.)

We start with a simple lemma showing that the distribution we shall use for smoothing is indeed smooth in the relevant sense.

Lemma 5.1. *Let $r \geq 3$ be fixed. Given integers $i, \ell > 0$ and a real number $0 < \pi < 1$, for $0 \leq a \leq i/(r - 2)$ let*

$$n_a = n_{i,\ell,a} = \frac{1}{a!} \binom{i}{r-2} \binom{i-(r-2)}{r-2} \cdots \binom{i-(a-1)(r-2)}{r-2} \binom{\ell}{2}^a, \tag{5.2}$$

and let $Y_{i,\ell,\pi}$ be the probability distribution on the non-negative integers defined by

$$\mathbb{P}(Y_{i,\ell,\pi} = a) = p_a = p_{i,\ell,\pi,a} = \pi^a n_a / \sum_{b=0}^{i/(r-2)} \pi^b n_b.$$

Let $\varepsilon = \varepsilon(n)$ satisfy $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon = O(1)$, set $\sigma_0 = \sigma_0(n) = \sqrt{\varepsilon^3 n}$, and let $i = i(n)$, $\ell = \ell(n)$ and $\pi = \pi(n)$ satisfy $i = \Theta(n)$, $\ell = \Theta(\varepsilon n)$ and $\pi = \Theta(\varepsilon n^{-r+1})$. Then, whenever (y_n) and (y'_n) satisfy $y_n - y'_n = o(\sigma_0)$, we have

$$\mathbb{P}(Y_n = y_n) - \mathbb{P}(Y_n = y'_n) = o(1/\sigma_0), \tag{5.3}$$

where $Y_n = Y_{i(n),\ell(n),\pi(n)}$.

Although the reader need not check this, Lemma 5.1 says that certain sequences (\mathcal{D}_n) of sets of probability distributions of the type $Y_{i,\ell,\pi}$ are $\sigma_0(n)$ -smooth in the sense of (3.1).

Proof. Fix sequences $\varepsilon(n)$, $i(n)$, $\ell(n)$ and $\pi(n)$ satisfying the conditions above; in what follows, much of the time we suppress the dependence on n in the notation.

Let $(x)_y$ denote the falling factorial $x(x - 1) \cdots (x - y + 1)$. Then, with n fixed, for $a + 1 \leq i/(r - 2)$ we have

$$q_a = \frac{p_{a+1}}{p_a} = \frac{1}{a+1} \frac{\pi}{(r-2)!} \binom{\ell}{2} (i - a(r-2))_{r-2}. \tag{5.4}$$

The sequence (q_a) is strictly decreasing, so (p_a) is unimodal.

For $a = a(n)$ satisfying $i - a(r - 2) = \Omega(n)$, by the assumptions on i, ℓ and π above we have

$$q_a = \Theta((a + 1)^{-1}(\varepsilon n^{-r+1})(\varepsilon n)^2 n^{r-2}) = \Theta(\varepsilon^3 n / (a + 1)).$$

For $i - a(r - 2) = o(n)$ it is easy to see that $q_a = o(1)$. Let $a_0 = a_0(n)$ be the minimal integer such that $q_{a_0} \leq 1$. Then we have $a_0 = \Theta(\varepsilon^3 n)$ and hence $i - a_0 = \Omega(n)$.

Writing $\sigma_0 = \sigma_0(n) = \sqrt{\varepsilon^3 n}$, it follows from (5.4) that for $a = a(n) = a_0 + O(\sigma_0)$ we have

$$q_a = q_{a_0}(1 + O(\sigma_0/a_0)) = q_{a_0}(1 + O(\sigma_0^{-1})) = 1 + O(\sigma_0^{-1}). \tag{5.5}$$

Since $q_a = p_{a+1}/p_a$, this has the following consequence: for any sequences $a_1 = a_1(n)$ and $a_2 = a_2(n)$ such that $a_i = a_0 + O(\sigma_0)$, $a_1 - a_2 = o(\sigma_0)$ and $a_1 < a_2$, we have²

$$p_{a_2}/p_{a_1} = \prod_{a_1 \leq a < a_2} q_a = (1 + O(\sigma_0^{-1}))^{o(\sigma_0)} = 1 + o(1). \tag{5.6}$$

From the unimodality of (q_a) and the definition of a_0 we have $\max_a p_a = p_{a_0}$. It is easy to see that $p_{a_0} = O(1/\sigma_0)$: otherwise, we could use (5.6) to deduce that $\sum_a p_a > 1$, a contradiction. Hence, $\max_a p_a = p_{a_0} = O(1/\sigma_0)$. Thus, from (5.6), for $a_i = a_0 + O(\sigma_0)$ we have

$$a_1 - a_2 = o(\sigma_0) \implies p_{a_2} - p_{a_1} = o(1/\sigma_0). \tag{5.7}$$

For $a > a_0$, by unimodality we have

$$1 = \sum_b p_b \geq \sum_{a_0 < b \leq a} p_b \geq (a - a_0)p_a,$$

so if $(a - a_0)/\sigma_0 \rightarrow \infty$ then $p_a = o(1/\sigma_0)$. Similarly, if $(a_0 - a)/\sigma_0 \rightarrow \infty$ then $p_a = o(1/\sigma_0)$. It follows that (5.7) holds for any sequences $a_1(n), a_2(n)$ with $a_1 - a_2 = o(1/\sigma_0)$, which is exactly (5.3). □

Proof of Theorem 4.3. Define p_1, p_2, H_1 and H_2 as at the start of the section, and set

$$\sigma_0 = \sqrt{\varepsilon^3 n}.$$

(Recall that, up to a constant factor, σ_0^2 is the variance of $N_1(H_{n,p}^r)$.) We shall first apply Theorem 4.2 to H_1 , noting that $(\varepsilon/2)^3 n \rightarrow \infty$. Let \mathcal{C}_1 be the component of H_1 with the most vertices (chosen according to any rule if there is a tie). Since $\rho_{r,1+\varepsilon/2} = \Theta(\varepsilon)$, by Theorem 4.2 there are constants $0 < c < C$ such that the event

$$\mathcal{E}_1 = \{c\varepsilon n \leq |\mathcal{C}_1| \leq C\varepsilon n\}$$

satisfies

$$\mathbb{P}(\mathcal{E}_1^c) = \exp(-\Omega(\varepsilon^3 n)) = o(1/\sigma_0).$$

² To deduce (5.6) we need (5.5) to hold uniformly in a with $a_1(n) \leq a < a_2(n)$. To see that it does, choose the ‘worst-case’ $a = a(n)$ in this range for each n and apply (5.5) to the resulting sequence.

By the last part of Theorem 4.2,

$$\mathbb{P}(L_2(H_{n,p}^r) \geq c\epsilon n) \leq \exp(-\Omega(\epsilon^3 n)) = o(1/\sigma_0).$$

Let \mathcal{E}_2 be the event that \mathcal{C}_1 is contained in the largest component \mathcal{L}_1 of $H_{n,p}^r = H_1 \cup H_2$. Since $H_1 \subset H_{n,p}^r$, we have

$$\mathbb{P}(\mathcal{E}_2^c) \leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(L_2(H_{n,p}^r) \geq c\epsilon n) = o(1/\sigma_0).$$

Let $i(H)$ denote the number of isolated vertices in a hypergraph H . It is easy to see that

$$\mathbb{E}[i(H_{n,p}^r)] = \Theta(n).$$

Let c' be a constant such that $\mathbb{E}[i(H_{n,p}^r)] \geq 2c'n$ for large enough n , and let \mathcal{E}_3 be the event

$$\mathcal{E}_3 = \{i(H_{n,p}^r) \geq c'n\}. \tag{5.8}$$

Then standard concentration arguments (e.g., a simple application of the Hoeffding–Azuma inequality) show that

$$\mathbb{P}(\mathcal{E}_3^c) = \exp(-\Omega(n)) = o(1/\sigma_0).$$

Reveal all edges of H_1 , which of course determines \mathcal{C}_1 . We shall reveal some partial information about H_2 in a two-step process.

First, test r -sets (i.e., potential edges) for their presence in H_2 according to the following algorithm: if there is any untested r -set e which does not consist of two vertices in \mathcal{C}_1 and $r - 2$ vertices that are isolated in the current hypergraph H , then pick some such r -set e and test whether it is present in H_2 . Otherwise, stop. By the ‘current hypergraph’ we mean the hypergraph formed by the edges revealed so far, so $H_1 \subset H \subset H_1 \cup H_2 = H_{n,p}^r$.

Let H be the hypergraph revealed at the end of the algorithm, let \mathcal{I} be the set of isolated vertices of H , and let U be the set of untested r -sets when the algorithm stops. Then U has a very simple form: it consists precisely of all $\binom{|\mathcal{C}_1|}{2} \binom{|\mathcal{I}|}{r-2}$ r -sets with two vertices in \mathcal{C}_1 and $r - 2$ in \mathcal{I} . To see this, note first that if there were any untested r -set not of this form, the algorithm would not have stopped. Conversely, since any isolated vertices in the final hypergraph H were isolated throughout the running of the algorithm, and \mathcal{C}_1 (a component of H_1 , *not* of the current graph) does not change as the algorithm runs, any r -set of this form cannot have been tested.

At this point, each untested edge is present independently with conditional probability p_2 .

In the second step, we reveal the set F of edges e in U present in H_2 with the property that some vertex of $e \cap \mathcal{I}$ is incident with one or more other edges of H_2 . Let \mathcal{I}' be the set of vertices in \mathcal{I} not incident with edges in F .

Let \mathcal{F} denote the the σ -algebra generated by all the information revealed so far, and let F' be the set of edges of H_2 not yet revealed. Then F' consists of edges with two vertices in \mathcal{C}_1 and $r - 2$ in \mathcal{I}' , with the corresponding subsets of \mathcal{I}' disjoint. Furthermore, given \mathcal{F} (which determines \mathcal{C}_1 and \mathcal{I}'), any set F' of edges satisfying this description is possible. Let $Y_n = |F'|$; this will be our smoothing random variable. Recalling the definition (5.2) of $n_{i',a}$, there are exactly $n_{|\mathcal{I}'|,|\mathcal{C}_1|,a}$ possible sets F' with a edges. Let $\pi = p_2/(1 - p_2)$. Since

the probability of a hypergraph in the model H_{n,p_2}^r is proportional to π raised to the power of the number of edges, we see that (for $r \geq 3$) the conditional distribution of $Y_n = |F'|$ given \mathcal{F} is exactly the distribution $Y_{|\mathcal{I}'|,|\mathcal{C}_1|,\pi}$ defined in Lemma 5.1.

Let \mathcal{E} be the event

$$\mathcal{E} = \mathcal{E}_1 \cap \{|\mathcal{I}'| \geq c'n\},$$

where c' is as in the definition (5.8) of \mathcal{E}_3 . Note that \mathcal{E} is \mathcal{F} -measurable. Since every isolated vertex of $H_{n,p}^r$ is in \mathcal{I}' , we have

$$\mathbb{P}(\mathcal{E}^c) \leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_3^c) = o(1/\sigma_0). \tag{5.9}$$

When \mathcal{E} holds, then $|\mathcal{C}_1| = \Theta(\epsilon n)$ and $|\mathcal{I}'| = \Theta(n)$; from (5.1) we always have $\pi = p_2/(1 - p_2) = \Theta(\epsilon n^{-r+1})$. Let (ω_n) be a sequence of elements of the probability space(s) on which $H_{n,p}^r$ is defined, with $\omega_n \in \mathcal{E} = \mathcal{E}_n$. By Lemma 5.1,³ for any such sequence (ω_n) and for any sequences y_n, y'_n with $y_n - y'_n = o(\sigma_0)$ we have

$$\mathbb{P}(Y_n = y_n \mid \mathcal{F})(\omega_n) - \mathbb{P}(Y_n = y'_n \mid \mathcal{F})(\omega_n) = o(1/\sigma_0). \tag{5.10}$$

Fix sequences $t_n, t'_n \geq 0$ with $t_n - t'_n = o(\sigma_0)$ and a sequence (I_n) of subsets of \mathbb{Z} . Our aim is to show that

$$\mathbb{P}(N_1 = t_n, L_1 - (r - 2)N_1 \in I_n) - \mathbb{P}(N_1 = t'_n, L_1 - (r - 2)N_1 \in I_n) = o(1/\sigma_0). \tag{5.11}$$

Let \mathcal{C} be the component of $H \supset H_1$ containing \mathcal{C}_1 , and \mathcal{C}' the component of $H_{n,p}^r$ containing \mathcal{C} (and hence \mathcal{C}_1). Let

$$X_n = n(\mathcal{C}) \quad \text{and} \quad Z_n = |\mathcal{C}| - (r - 2)n(\mathcal{C}) = |\mathcal{C}| - (r - 2)X_n.$$

Then X_n and Z_n are \mathcal{F} -measurable, so from (5.10), for any $\omega_n \in \mathcal{E}$ we have

$$\mathbb{P}(X_n + Y_n = t_n, Z_n \in I_n \mid \mathcal{F})(\omega_n) - \mathbb{P}(X_n + Y_n = t'_n, Z_n \in I_n \mid \mathcal{F})(\omega_n) = o(1/\sigma_0).$$

As usual, this bound holds uniformly in $\omega_n \in \mathcal{E} = \mathcal{E}_n$, since we are free to choose ω_n to maximize the difference. Taking the expectation, and recalling that \mathcal{E} is \mathcal{F} -measurable and $\mathbb{P}(\mathcal{E}^c) = o(1/\sigma_0)$, it follows that

$$\mathbb{P}(X_n + Y_n = t_n, Z_n \in I_n) - \mathbb{P}(X_n + Y_n = t'_n, Z_n \in I_n) = o(1/\sigma_0). \tag{5.12}$$

Now each edge in F' meets \mathcal{C} in two vertices, and has no vertices outside \mathcal{C} in common with any other edge of F' . Thus

$$n(\mathcal{C}') = X_n + Y_n \quad \text{and} \quad |\mathcal{C}'| = |\mathcal{C}| + (r - 2)Y_n,$$

so

$$|\mathcal{C}'| - (r - 2)n(\mathcal{C}') = |\mathcal{C}| - (r - 2)X_n = Z_n.$$

³ For $r = 2$ (which is not our main focus) we cannot apply Lemma 5.1. However, in this case F' is simply the set of edges of H_2 with both ends in \mathcal{C}_1 . This has a binomial distribution with parameters $\Theta(\epsilon^2 n^2)$ and $\Theta(\epsilon n^{-1})$; the family of such distributions is σ_0 -smooth, so (5.10) holds in this case also.

When \mathcal{E}_2 holds, then $C' = \mathcal{L}_1$. Hence, whenever \mathcal{E}_2 holds, we have

$$N_1 = X_n + Y_n \quad \text{and} \quad L_1 - (r - 2)N_1 = Z_n. \tag{5.13}$$

Recalling that $\mathbb{P}(\mathcal{E}_2) = 1 - o(1/\sigma_0)$, our aim (5.11) follows from (5.12) and (5.13), completing the proof of Theorem 4.3. □

6. Trees and forests

For $m \geq 2$, an m -cycle in a hypergraph H consists of distinct vertices v_1, \dots, v_m and distinct edges e_1, \dots, e_m such that each e_i contains both v_i and v_{i+1} , with v_{m+1} defined to be v_1 . Thus a 2-cycle consists of two edges sharing at least two vertices. Note that an m -cycle corresponds to a cycle of length $2m$ in the bipartite vertex–edge incidence graph $G_{\text{inc}}(H)$ associated to H .

A hypergraph H is a *tree* if it is connected and contains no cycles, or, equivalently, if H can be built up by starting with a single vertex, and adding new edges one by one so that each meets the current hypergraph in exactly one vertex. Note that H is a tree if and only if $G_{\text{inc}}(H)$ is a tree.

By an r -tree we simply mean an r -uniform hypergraph that is a tree. An r -forest is a vertex-disjoint union of r -trees. For $A \subset V$, an A -rooted r -forest on V is an r -forest with vertex set V such that each component contains exactly one vertex from A ; in particular, there are $|A|$ components. Note that A -rooted r -forests on V exist if and only if $|V| = |A| + (r - 1)k$ for some integer $k \geq 0$ (the number of edges). For $r = 2$, the formula an^{n-a-1} for the number of $[a]$ -rooted 2-forests on $[n]$ was observed by Cayley [13] and proved by Rényi [22]. We shall make repeated use of the following generalization to hypergraphs, due to Selivanov [27].

Lemma 6.1. *Let $r \geq 2$, $a \geq 1$ and $k \geq 0$ be integers, and set $n = a + (r - 1)k$. The number $F_{a,k} = F_{a,k}^{(r)}$ of $[a]$ -rooted r -forests on $[n] = \{1, 2, \dots, n\}$ satisfies*

$$F_{a,k} = an^{k-1} \{k : r - 1\}, \tag{6.1}$$

where

$$\{k : t\} = \frac{(kt)!}{k! t!^k}$$

is the number of partitions of a set of size kt into k parts of size t . □

For completeness we give a proof in the appendix to [11], since the original source is perhaps a little obscure. (We only became aware of it from Karoński and Łuczak [17]).

One consequence of Lemma 6.1 is the following surprisingly simple bound on the expected number of vertices at a given distance from the root set in a random $[a]$ -rooted r -forest. Recall that $(x)_y$ denotes the falling factorial $x(x - 1) \cdots (x - y + 1)$.

Lemma 6.2. *Let $r \geq 2$, $a \geq 1$ and $k, \ell \geq 0$ be integers, and set $n = a + (r - 1)k$. Choosing an $[a]$ -rooted r -forest on $[n]$ uniformly at random, the expected number of vertices at graph*

distance exactly ℓ from $[a]$ is equal to

$$(a + (r - 1)\ell) \frac{(r - 1)^\ell (k)_\ell}{n^\ell}$$

and is (hence) at most $a + (r - 1)\ell$.

Proof. Let N be the number of ordered pairs (F, v) where F is an $[a]$ -rooted r -forest on $[n]$ and $v \in [n]$ is at graph distance ℓ from $[a]$ in F . Since there is a unique path from v to $[a]$ in F , we can instead view N as the number of tuples $(F, v_0, e_1, \dots, v_{\ell-1}, e_\ell, v_\ell)$ where F is an $[a]$ -rooted r -forest on $[n]$, $v_0 \in [a]$, and $v_0 e_1 \dots e_\ell v_\ell$ is a path in F . (The bijection from such tuples to pairs (F, v) maps v_ℓ to v .)

With F not yet determined, there are a choices for v_0 , then $\binom{(r-1)k}{r-1}$ choices for the remaining vertices that with v_0 make up e_1 . Then there are $r - 1$ choices for v_1 , then $\binom{(r-1)(k-1)}{r-1}$ choices for the rest of e_2 , and so on, giving

$$\begin{aligned} N_1 &= a(r - 1)^\ell \binom{(r - 1)k}{r - 1} \dots \binom{(r - 1)(k - \ell + 1)}{r - 1} \\ &= a(r - 1)^\ell \frac{((r - 1)k)!}{((r - 1)(k - \ell))!(r - 1)^\ell} \end{aligned}$$

choices for $(v_0, e_1, \dots, e_\ell, v_\ell)$. Now we must choose an $[a]$ -rooted r -forest F on $[n]$ containing the edges e_1, \dots, e_ℓ ; this is the same as choosing an $[S]$ -rooted r -forest F' on $[n]$ where $S = [a] \cup e_1 \cup \dots \cup e_\ell$ is a set of $a + (r - 1)\ell$ vertices. By Lemma 6.1 we thus have

$$\begin{aligned} N &= (a + (r - 1)\ell)n^{k-\ell-1} \frac{((r - 1)(k - \ell))!}{(k - \ell)!(r - 1)^{k-\ell}} N_1 \\ &= (a + (r - 1)\ell)n^{k-\ell-1} a(r - 1)^\ell \frac{((r - 1)k)!}{(k - \ell)!(r - 1)^{k-\ell}}. \end{aligned}$$

The expectation we wish to calculate is precisely N divided by the number of $[a]$ -rooted r -forests on $[n]$. By Lemma 6.1 the expectation is thus

$$\begin{aligned} (a + (r - 1)\ell)n^{-\ell}(r - 1)^\ell \frac{k!}{(k - \ell)!} &= (a + (r - 1)\ell) \frac{(r - 1)^\ell (k)_\ell}{n^\ell} \\ &\leq (a + (r - 1)\ell)((r - 1)k/n)^\ell \\ &\leq a + (r - 1)\ell, \end{aligned}$$

as required. □

Note that, surprisingly, k does not appear in the final upper bound in the lemma above.

7. The smoothing distribution

Given positive integers m and a , let $A_1 \subset A \subset V$ with $|A_1| = a$, $|A| = 2a$ and $|V| = 2a + (r - 1)m$, and let F be an A -rooted r -forest on V chosen uniformly at random. Let $Y_{m,a}$ be the total number of edges of F in components rooted in A_1 . Note that F has m edges, so $0 \leq Y_{m,a} \leq m$.

Lemma 7.1. *Let $m = m(n)$ and $a = a(n)$ satisfy $m = o(a^2)$ and $m = \Omega(a)$, let $Y_n = Y_{m,a}$, and set $\sigma_n = m^{3/2}a^{-1}$. Then for any integers x_n, y_n with $x_n - y_n = o(\sigma_n)$ we have*

$$\mathbb{P}(Y_n = x_n) - \mathbb{P}(Y_n = y_n) = o(1/\sigma_n),$$

and $\mathbb{P}(Y_n = x_n) = O(1/\sigma_n)$.

In the terminology of Section 3, the sequence of distributions $Y_{m(n),a(n)}$ is σ_n -smooth.

Proof. As usual, we suppress the dependence on n in the notation, for example writing σ for σ_n .

Note first that our assumptions imply that $a = O(m) = o(a^2)$, so certainly $a \rightarrow \infty$ and thus $m \rightarrow \infty$. Note for later that $\sigma/m = m^{1/2}a^{-1} = \sqrt{m/a^2}$, so

$$\sigma = o(m).$$

Let $p_k = p_{n,k} = \mathbb{P}(Y_n = k)$. Considering first the choices for the vertices outside A appearing in the subforest rooted at A_1 , we see that

$$p_k = \binom{(r-1)m}{(r-1)k} \frac{F_{a,k}F_{a,m-k}}{F_{2a,m}},$$

where $F_{a,k}$ denotes the number of X -rooted r -forests on Y when $X \subset Y$ with $|X| = a$ and $|Y| = a + (r-1)k$. From now on, let us write t for $r-1$, since this will appear so often in the following calculations. By Lemma 6.1, writing ℓ for $m-k$, for $0 \leq k \leq m$ we have

$$\begin{aligned} p_k &= \binom{tm}{tk} \frac{a(a+tk)^{k-1}(tk)!k!^{-1}t!^{-k}a(a+t\ell)^{\ell-1}(t\ell)! \ell!^{-1}t!^{-\ell}}{2a(2a+tm)^{m-1}(tm)!m!^{-1}t!^{-m}} \\ &= \frac{a}{2} \binom{m}{k} \frac{(a+tk)^{k-1}(a+t\ell)^{\ell-1}}{(2a+tm)^{m-1}}. \end{aligned} \tag{7.1}$$

We shall prove the following three statements concerning functions k, k_1 and k_2 of n bounded between 0 and $m(n)$, where $\sigma = \sigma(n) = m^{3/2}a^{-1}$:

$$\text{If } k_1 = k_2 + o(\sigma) \text{ and } k_1, k_2 = m/2 + O(\sigma) \text{ then } p_{k_1} \sim p_{k_2}. \tag{7.2}$$

$$\text{If } k = m/2 + O(\sigma) \text{ then } p_k = O(1/\sigma). \tag{7.3}$$

$$\text{If } |k - m/2|/\sigma \rightarrow \infty \text{ then } p_k = o(1/\sigma). \tag{7.4}$$

(As usual, we quantify over sequences here: e.g., the formal statement of (7.3) is that for any sequence $k(n)$ such that $\limsup_n |k(n) - m(n)/2|/\sigma(n) < \infty$, we have $\limsup_n p_{n,k(n)}\sigma(n) < \infty$.)

Suppose for the moment that (7.2)–(7.4) hold, and consider sequences $k_1 = k_1(n)$ and $k_2 = k_2(n)$ with $k_1 - k_2 = o(\sigma)$. The lemma asserts that then

$$p_{k_1} - p_{k_2} = o(1/\sigma) \quad \text{and} \quad p_{k_1} = O(1/\sigma). \tag{7.5}$$

In the special case where $k_1 = m/2 + O(\sigma)$, the relations (7.2) and (7.3) give (7.5). In the special case where $|k_1 - m/2|/\sigma \rightarrow \infty$, then also $|k_2 - m/2|/\sigma \rightarrow \infty$, so by (7.4) both p_{k_1} and p_{k_2} are $o(1/\sigma)$, and (7.5) follows. The general case now follows by a standard subsequence argument: a counterexample would have a subsequence falling into one of these two special cases.

Our aim is now to prove (7.2)–(7.4). Let us first deal with the extreme values, *i.e.*, cases where k is very close to 0 or to m . We shall show that when $k \leq c_0 a$ for some constant c_0 , then $p_{k+1} \geq p_k$, so if we can show that $p_k = o(1/\sigma)$ for $k = \lceil c_0 a \rceil$, then the same bound for $k < \lceil c_0 a \rceil$ follows. Here c_0 may depend on the sequences $m(n)$ and $a(n)$, but not on $k(n)$.

From (7.1) we see that for $0 \leq k < m$ we have

$$\begin{aligned} \frac{p_{k+1}}{p_k} &= \frac{\ell}{k+1} \frac{(a+t(k+1))^k}{(a+tk)^{k-1}} \frac{(a+t(\ell-1))^{\ell-2}}{(a+t\ell)^{\ell-1}} \\ &= \frac{\ell}{k+1} \frac{a+tk}{a+t(\ell-1)} \left(1 + \frac{t}{a+tk}\right)^k \left(1 + \frac{t}{a+t(\ell-1)}\right)^{-(\ell-1)} \\ &= \frac{a+tk}{k+1} \frac{\ell}{a+t(\ell-1)} \Theta(1), \end{aligned}$$

since $(1+x)^i = \exp(O(ix)) = \Theta(1)$ when $x \geq 0$ and $|ix| \leq 1$. For $k \leq m/2$, say, we have $\ell = m - k = \Theta(m)$ and $a + t(\ell - 1) = \Theta(a + m) = \Theta(m)$, so $p_{k+1}/p_k = \Theta((a + tk)/(k + 1))$. It follows that there exists a constant c_0 such that for $k \leq c_0 a$ we have $p_{k+1}/p_k \geq 1$, so

$$\max_{k \leq c_0 a} p_k \leq p_{\lceil c_0 a \rceil}. \tag{7.6}$$

Since $m = \Omega(a)$, we may choose c_0 small enough that $\lceil c_0 a \rceil \leq m/4$, say. In proving (7.4), we may assume by symmetry that $k \leq m/2$. Since $\sigma = o(m)$, we have $|\lceil c_0 a \rceil - m/2|/\sigma \geq m/(4\sigma) \rightarrow \infty$, so in the light of (7.6), to prove (7.4) it suffices to show that

$$\text{If } (m/2 - k)/\sigma \rightarrow \infty \text{ and } k \geq c_0 a \text{ then } p_k = o(1/\sigma). \tag{7.7}$$

From this point our aim is to prove (7.2), (7.3) and (7.7). Since all three statements only involve $k = k(n)$ such that $k, \ell = \Omega(a)$, from now on we impose this condition. In this case, from (7.1) and Stirling’s formula we have

$$p_k \sim \frac{a}{2\sqrt{2\pi}} \frac{m^m}{k^k \ell^\ell} \sqrt{\frac{m}{k\ell}} \frac{tm + 2a}{(tk + a)(t\ell + a)} \frac{(tk + a)^k (t\ell + a)^\ell}{(tm + 2a)^m}.$$

Roughly speaking, we shall write this expression as a polynomial factor times an exponential factor. Then we expand the function inside the exponential around $k = m/2$ to see that p_k is small when k is far from $m/2$, and does not change too rapidly when k is close to $m/2$. The complication is that the polynomial factor ‘blows up’ as k/m approaches 0 or 1, and it is only the condition $m = o(a^2)$ that ensures that this ‘blow up’ is beaten by the exponential factor.

Setting

$$x = k/m \quad \text{and} \quad \beta = a/(tm),$$

and noting that by assumption $\beta = O(1)$, we have

$$\begin{aligned}
 p_k &\sim \frac{a}{2\sqrt{2\pi}} \sqrt{\frac{1}{x(1-x)m} \frac{1}{tm} \frac{1+2\beta}{(x+\beta)(1-x+\beta)}} \\
 &\quad \left(x^{-x}(1-x)^{-(1-x)} \frac{(x+\beta)^x(1-x+\beta)^{1-x}}{1+2\beta} \right)^m \\
 &= \frac{a(1+2\beta)}{2\sqrt{2\pi}tm^{3/2}} f(x) \exp(-mg(x)) \\
 &= \frac{c}{\sigma} f(x) \exp(-mg(x)),
 \end{aligned} \tag{7.8}$$

where $c = (1+2\beta)/(2t\sqrt{2\pi}) = \Theta(1)$ is independent of k ,

$$f(x) = x^{-1/2}(1-x)^{-1/2}(x+\beta)^{-1}(1-x+\beta)^{-1}, \tag{7.9}$$

and

$$g(x) = x \log x + (1-x) \log(1-x) - x \log(x+\beta) - (1-x) \log(1-x+\beta) + \log(1+2\beta).$$

It is easy to see that $g(1/2) = 0$. Moreover,

$$g'(x) = \log x - \log(1-x) - \log(x+\beta) + \log(1-x+\beta) + \frac{\beta}{x+\beta} - \frac{\beta}{1-x-\beta}$$

is also zero at $x = 1/2$, and (after a little calculation) we see that

$$g''(x) = \beta^2 \left(\frac{1}{x(x+\beta)^2} + \frac{1}{(1-x)(1-x+\beta)^2} \right) > 0. \tag{7.10}$$

Since $\beta = O(1)$, for $1/8 \leq x \leq 7/8$, say, the bracket in (7.10) is uniformly $\Theta(1)$, so we have $g''(x) = \Theta(\beta^2)$. Integrating twice, we see that for $x \in [1/8, 7/8]$ we have

$$|g'(x)| = \Theta(\beta^2|x-1/2|) \quad \text{and} \quad g(x) = \Theta(\beta^2(x-1/2)^2). \tag{7.11}$$

Recalling that $\beta = a/(tm)$ and $\sigma = m^{3/2}/a$, note that

$$\beta^2 \sigma^2 / m^2 = \frac{a^2}{t^2 m^2} \frac{m^3}{a^2} \frac{1}{m^2} = \frac{1}{t^2 m} = \Theta(m^{-1}). \tag{7.12}$$

Let k_1 and k_2 satisfy $k_i = m/2 + O(\sigma)$ and $k_1 - k_2 = o(\sigma)$, and set $x_i = k_i/m$. Then $x_i = 1/2 + O(\sigma/m)$, and $x_1 - x_2 = o(\sigma/m)$. By the Mean Value Theorem, there is some $\xi = 1/2 + O(\sigma/m)$ for which

$$|g(x_1) - g(x_2)| = |g'(\xi)||x_1 - x_2| = O(\beta^2|\xi - 1/2||x_1 - x_2|) = o(\beta^2 \sigma^2 / m^2) = o(1/m),$$

from (7.11) and (7.12). From (7.9), since $x_1, x_2 \sim 1/2$ we have $f(x_1) \sim f(1/2) \sim f(x_2)$, and it follows from (7.8) that $p_{k_1} \sim p_{k_2}$, proving (7.2). For (7.3), simply note that $g(x) \geq 0$ always, while if $k = m/2 + O(\sigma)$ then $x = k/m$ satisfies $x = 1/2 + O(\sigma/m) = 1/2 + o(1)$, so x is bounded away from 0 and 1 and (7.9) gives $f(x) = O(1)$. Hence (7.8) gives $p_k = O(1/\sigma)$, proving (7.3).

Finally, we turn to the proof of (7.7), considering k ‘far’ from $m/2$, but not too close to 0 or to m . First, note that since $\beta = O(a/m)$ and, by assumption, $m = o(a^2)$, we have

$$\beta^2 m \rightarrow \infty.$$

Let $c_0a \leq k \leq m/2$ with $(m/2 - k)/\sigma \rightarrow \infty$ and set $x = k/m$, so that $x < 1/2$ and $(1/2 - x)/(\sigma/m) \rightarrow \infty$. If $x \geq 1/8$ then $f(x) = \Theta(1)$ while from (7.11) we have $g(x) = \Omega(\beta^2)$ and hence $mg(x) \rightarrow \infty$. Thus (7.8) gives $p_k = o(1/\sigma)$, as required.

Suppose instead that $x < 1/8$; note that $x = k/m \geq c_0a/m = c_1\beta$, where $c_1 = c_0t$ is a positive constant. For $y \geq c_1\beta$ we have $\beta = O(y)$ and hence $y^{-1}(y + \beta)^{-2} = \Omega(y^{-3})$, so in this range (7.10) gives $g''(y) \geq c\beta^2y^{-3}$ for some constant $c > 0$. It follows easily that there is a constant c' such that for $c_1\beta \leq x \leq 1/8$ we have $g(x) \geq c'\beta^2x^{-1}$. (Indeed, for $c_1\beta \leq y \leq 1/4$ we have $-g'(y) = \int_y^{1/2} g''(z)dz \geq \int_y^{2y} g''(z)dz = \Omega(\beta^2y^{-2})$, and then $g(x) = \int_x^{1/2} (-g'(y))dy \geq \int_x^{2x} (-g'(y))dy = \Omega(\beta^2x^{-1})$.) Hence, for $c_1\beta \leq x \leq 1/8$ we have

$$f(x) \exp(-mg(x)) = O(x^{-3/2}) \exp(-mg(x)) = O(x^{-3/2} \exp(-c'\beta^2mx^{-1})).$$

Since $\beta^2m \rightarrow \infty$, it follows that $f(x) \exp(-mg(x)) \rightarrow 0$ uniformly in this range, which with (7.8) gives $p_k = o(1/\sigma)$, completing the proof of (7.7) and hence of the lemma. \square

With a small amount of further work, the argument above extends to show that (under the given assumptions) $Y_{m,a}$ satisfies a Gaussian local limit theorem. We shall not need this, so we omit the details.

8. Discrete duality

Recall that $H_{n,p}^r$ denotes the random r -uniform hypergraph on $[n]$ in which each of the $\binom{n}{r}$ possible edges is present independently with probability p . As in the Introduction, we write $p = p(n)$ as $\lambda(n)(r - 2)!n^{-r+1}$, so $\lambda = 1$ corresponds to the critical point of the phase transition. More generally, for any r, n and p we call

$$\lambda = pn^{r-1}/(r - 2)! \tag{8.1}$$

the *branching factor* of $H_{n,p}^r$. For $\lambda > 1$ recall that $\rho_{r,\lambda}$, defined in (2.1), is the survival probability of a certain branching process associated to $H_{n,p}^r$. In particular, when $r = 2$ this process is just a Galton–Watson process with a Poisson offspring distribution with mean λ ; we write $\rho_\lambda = \rho_{2,\lambda}$ for its survival probability.

Given any $\lambda > 1$, define $\lambda_* < 1$, the parameter *dual* to λ , by

$$\lambda_* e^{-\lambda_*} = \lambda e^{-\lambda}. \tag{8.2}$$

It is easy to check that $\lambda_* = \lambda(1 - \rho_\lambda)$, where $\rho_\lambda = \rho_{2,\lambda}$, and that for $\lambda > 1$ with $\lambda = O(1)$ we have

$$\lambda_* = 1 - \Theta(\lambda - 1) \quad \text{and} \quad \lambda_* = \Theta(1). \tag{8.3}$$

In other words, for any $A > 0$ there exist $c, C > 0$ such that $\lambda \in (1, A]$ implies $(1 - \lambda_*)/(\lambda - 1) \in [c, C]$ and $\lambda_* \in [c, 1)$ (recall that $\lambda_* < 1$ by definition). The second, crude bound in (8.3) is only relevant when λ is large.

In the regime we are interested in, we have $\lambda = 1 + \varepsilon$ with $\varepsilon = \varepsilon(n)$ bounded and $\varepsilon^3 n \rightarrow \infty$, so by the results of [18, 9], $H_{n,p}^r$ is supercritical. Defining $\delta = \delta(n) \geq (\varepsilon^3 n)^{-1/3}$ and $R = R_n = [(1 - \delta)\rho_{r,\lambda}n, (1 + \delta)\rho_{r,\lambda}n]$ as in (4.3), by (4.5) we have

$$\mathbb{P}(L_1 \in R) = 1 - O(1/(\varepsilon^3 n)) = 1 - o(1). \tag{8.4}$$

We shall only consider possible values of L_1 lying in R . We start with a simple calculation, showing that if $s \in R$ then $H_{n-s,p}^r$ is subcritical (but not too strongly so).

Lemma 8.1. *Under our Weak Assumption 2.1, for any $s = s(n) \in R$, the branching factor $\lambda' = \lambda(H_{n-s,p}^r)$ of the random hypergraph $H_{n-s,p}^r$ satisfies $\lambda' = 1 - \Theta(\varepsilon)$ and $\lambda' = \Theta(1)$.*

Proof. Let $\mu_n = \rho_{r,\lambda} n$. Ignoring the fact that μ_n need not be an integer, if we define the branching factor $\lambda(H_{n-\mu_n,p}^r)$ by (8.1), with $n - \mu_n$ in place of n , then

$$\lambda(H_{n-\mu_n,p}^r) = (1 - \mu_n/n)^{r-1} \lambda = (1 - \rho_{r,\lambda})^{r-1} \lambda = (1 - \rho_\lambda) \lambda = \lambda_*,$$

which is $1 - \Theta(\varepsilon)$ by (8.3). For $s \in R$ we have $(n - s)/(n - \mu_n) = 1 + O(\delta\varepsilon) = 1 + o(\varepsilon)$, so, since r is constant, $\lambda' = \lambda(H_{n-s,p}^r) = (1 + o(\varepsilon))^{r-1} \lambda(H_{n-\mu_n,p}^r) = 1 - \Theta(\varepsilon)$ also. To see that $\lambda' = \Theta(1)$ (i.e., is bounded away from zero), recall from (4.4) that $s \in R$ implies $s \leq (1 - c)n$ for some constant $c > 0$. Then $\lambda' = (1 - s/n)^{r-1} \lambda \geq c^{r-1} \lambda \geq c^{r-1}$. □

Note that here we do not really need δ to tend to zero: it would suffice to assume that δ is at most some small constant depending on the upper bound on ε .

A simple consequence of the fact that $H_{n-s,p}^r$ is subcritical is that it is unlikely to contain a component with s or more vertices. We state a convenient form of this result rather than the strongest version possible.

Lemma 8.2. *Under our Weak Assumption 2.1, for any $s = s(n) \in R$, w.h.p. $L_1(H_{n-s,p}^r) < n^{2/3} < s$.*

Proof. From either Karoński and Łuczak [18, Theorem 6] or [10, Theorem 2] (which gives a better probability bound but a worse constant c_r), there is a constant $c_r > 0$ such that if $H_{m,p}^r$ has branching factor $1 - \eta$ where $\eta^3 m \rightarrow \infty$, then w.h.p.

$$L_1(H_{m,p}^r) \leq c_r \eta^{-2} \log(\eta^3 m) = o(m^{2/3}).$$

For $s \in R$, by Lemma 8.1 the branching factor of $H_{n-s,p}^r$ is $1 - \eta$ with $\eta = \Theta(\varepsilon)$. Since $m = n - s = \Theta(n)$ and $\varepsilon^3 n \rightarrow \infty$, we have $\eta^3 m \rightarrow \infty$, so w.h.p. $L_1(H_{m,p}^r) < m^{2/3} < n^{2/3}$. The result follows since $s = \Theta(\varepsilon n)$, so $s/n^{2/3} \rightarrow \infty$ and in particular $s > n^{2/3}$ if n is large enough. □

Let \mathcal{L}_1 be the component of $H_{n,p}^r$ with the most vertices, if there is a unique such component. In the case of ties we order (the vertex sets of) possible components arbitrarily (e.g., by the lowest numbered vertex present), and use this order to break the tie. Of course $|\mathcal{L}_1| = L_1$. The following explicit version of the discrete duality principle says that we may treat the graph outside \mathcal{L}_1 as a subcritical instance of the same hypergraph model. We write \mathcal{H}_s for the set of all labelled r -uniform hypergraphs with exactly s vertices. We always assume implicitly that any conditional probability is defined: i.e., if the event being conditioned on has probability 0, there is nothing to prove.

Lemma 8.3. *Suppose that our Weak Assumption 2.1 holds, and define $R = R_n$ as in (4.3). Let \mathcal{Q} be any isomorphism-invariant property of hypergraphs, and f any isomorphism-invariant function from hypergraphs to the non-negative reals. Then, for any $s = s(n) \in R$ and any $\mathcal{P} = \mathcal{P}(n) \subset \mathcal{H}_s$, we have*

$$\mathbb{P}(H_{n,p}^r \setminus \mathcal{L}_1 \text{ has } \mathcal{Q} \mid \mathcal{L}_1 \in \mathcal{P}) \leq (1 + o(1))\mathbb{P}(H_{n-s,p}^r \text{ has } \mathcal{Q})$$

and

$$\mathbb{E}(f(H_{n,p}^r \setminus \mathcal{L}_1) \mid \mathcal{L}_1 \in \mathcal{P}) \leq (1 + o(1))\mathbb{E}(f(H_{n-s,p}^r)),$$

as $n \rightarrow \infty$, with the error terms uniform over all $s \in R$ and $\mathcal{P} \subset \mathcal{H}_s$.

The most natural case here is $\mathcal{P} = \mathcal{H}_s$, in which case we are simply conditioning on the event $L_1 = s$. Often we shall take \mathcal{P} to be the set of hypergraphs with s vertices and nullity t ; then we are conditioning on the event $\{L_1 = s, N_1 = t\}$.

Proof. Although we have emphasized the uniformity of the error terms for clarity, this uniformity is automatic, considering the worst-case choice of $s = s(n)$ and $\mathcal{P} = \mathcal{P}(n)$.

Without loss of generality \mathcal{P} consists of a single hypergraph H_s with vertex set $S \subset [n]$ with $|S| = s$. From the definitions of $H_{n,p}^r$ and of \mathcal{L}_1 , the conditional distribution of $H_{n,p}^r \setminus \mathcal{L}_1$ given that $\mathcal{L}_1 = H_s$ is that of the random hypergraph $H' = H_{n-s,p}^r$ on the vertex set $[n] \setminus S$ conditioned on the event \mathcal{E} that

- (i) H' contains no component with more than s vertices, and
- (ii) H' has no s -vertex component that beats H_s in the tie-break order used in defining \mathcal{L}_1 .

By Lemma 8.2, $\mathbb{P}(\mathcal{E}) = 1 - o(1)$. Hence,

$$\begin{aligned} \mathbb{P}(H_{n,p}^r \setminus \mathcal{L}_1 \text{ has } \mathcal{Q} \mid \mathcal{L}_1 = H_s) &= \mathbb{P}(H' \text{ has } \mathcal{Q} \mid \mathcal{E}) \\ &\leq \frac{\mathbb{P}(H' \text{ has } \mathcal{Q})}{\mathbb{P}(\mathcal{E})} = (1 + o(1))\mathbb{P}(H' \text{ has } \mathcal{Q}), \end{aligned}$$

proving the first statement. For the second, argue similarly, or express $\mathbb{E}(f(H))$ as $\int_0^\infty \mathbb{P}(f(H) \geq t)dt$ and apply the first statement. □

A variant of the argument above gives the following result, which may be seen as an extension of an observation of Karoński and Łuczak [18, p. 133]. By a *property* of hypergraphs we simply mean a set of hypergraphs; we do not assume that this set is closed under isomorphism. As usual, let \mathcal{L}_1 be a component of $H_{n,p}^r$ with the maximal number of vertices, chosen according to any fixed rule if there is a tie.

Lemma 8.4. *Let \mathcal{Q}_s be any property of s -vertex hypergraphs, and let N_s be the expected number of components of $H_{n,p}^r$ having property \mathcal{Q}_s . Let \mathcal{U}_{big} be the event that $H_{n,p}^r$ has at most one component with more than $n^{2/3}$ vertices, and set $\mathcal{A}_s = \{N_s > 0\} \cap \mathcal{U}_{\text{big}}$ and $\mathcal{B}_s = \{N_s > 0\} \cap \mathcal{U}_{\text{big}}^c$. Under our Weak Assumption 2.1 we have*

$$\mathbb{P}(\mathcal{L}_1 \in \mathcal{Q}_s) \sim \mathbb{P}(\mathcal{A}_s) \sim \mathbb{E}[N_s] \tag{8.5}$$

and

$$\mathbb{P}(\mathcal{B}_s) = o(\mathbb{P}(\mathcal{L}_1 \in \mathcal{Q}_s)), \tag{8.6}$$

uniformly over all $s \in R$ and all properties \mathcal{Q}_s , where R is defined in (4.3).

Note that \mathcal{U}_{big} holds w.h.p. by (for example) the second statement of Theorem 4.2.

Proof. Clearly

$$\mathbb{E}[N_s] \geq \mathbb{P}(N_s > 0) \geq \mathbb{P}(\mathcal{L}_1 \in \mathcal{Q}_s) \geq \mathbb{P}(\mathcal{A}_s). \tag{8.7}$$

Let $N^+ \geq N_s$ denote the number of components of $H_{n,p}^r$ with more than $n^{2/3}$ vertices. If \mathcal{A}_s holds, then $N_s = 1$. If \mathcal{A}_s does not hold and $N_s > 0$, then $N^+ \geq 2$. Hence

$$N_s \leq \mathbb{1}_{\mathcal{A}_s} + N_s \mathbb{1}_{N^+ \geq 2}$$

and, taking expectations,

$$\mathbb{E}[N_s] \leq \mathbb{P}(\mathcal{A}_s) + \mathbb{E}[N_s \mathbb{1}_{N^+ \geq 2}]. \tag{8.8}$$

For $S \subset [n]$ with $|S| = s$, let \mathcal{Q}_S be the event that S is the vertex set of a component of $H_{n,p}^r$ having property \mathcal{Q}_s . Then

$$\begin{aligned} \mathbb{E}[N_s \mathbb{1}_{N^+ \geq 2}] &= \mathbb{E} \sum_{S : |S|=s} \mathbb{1}_{\mathcal{Q}_S} \mathbb{1}_{N^+ \geq 2} = \sum_S \mathbb{P}(\mathcal{Q}_S \cap \{N^+ \geq 2\}) \\ &= \sum_S \mathbb{P}(\mathcal{Q}_S) \mathbb{P}(N^+ \geq 2 \mid \mathcal{Q}_S) = \sum_S \mathbb{P}(\mathcal{Q}_S) \mathbb{P}(L_1(H_{n-s,p}^r) > n^{2/3}) \\ &= \mathbb{E}[N_s] \mathbb{P}(L_1(H_{n-s,p}^r) > n^{2/3}) = o(\mathbb{E}[N_s]), \end{aligned}$$

by Lemma 8.2.

From (8.8) we now obtain $\mathbb{P}(\mathcal{A}_s) \geq (1 - o(1))\mathbb{E}[N_s]$, which combined with (8.7) completes the proof of (8.5). The final statement (8.6) follows since

$$\mathbb{P}(\mathcal{B}_s) = \mathbb{P}(N_s > 0) - \mathbb{P}(\mathcal{A}_s) \leq \mathbb{E}[N_s] - \mathbb{P}(\mathcal{A}_s). \quad \square$$

9. Trees, paths and cycles outside the giant component

Throughout this section we assume our Weak Assumption 2.1. In other words we fix an integer $r \geq 2$ and a function $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n) = O(1)$ and $\varepsilon^3 n \rightarrow \infty$. We write λ for $1 + \varepsilon$, which is the branching factor of $H_{n,p}^r$ as defined in (8.1).

Our next lemma concerns trees outside the giant component. As in Section 8 we consider the hypergraph $H' = H_{m,p}^r$ where $m = n - s$ with $s \in R$, where $R = R_n$ is defined as in (4.3).

Lemma 9.1. *Let T_k denote the number of tree components of $H' = H_{n-s,p}^r$ with k edges, and $T_{k,\ell}^{(2)}$ the number of ordered pairs (T, T') of distinct tree components of H' with $e(T) = k$ and $e(T') = \ell$. Then*

$$\mu_k = \mathbb{E}[T_k] = \Theta(n(k + 1)^{-5/2}) \tag{9.1}$$

and

$$\mu_{k,\ell} = \mathbb{E}[T_{k,\ell}^{(2)}] = \mu_k \mu_\ell (1 + O(\varepsilon(k + \ell)^2 m^{-1})) \sim \mu_k \mu_\ell, \tag{9.2}$$

uniformly in $0 \leq k, \ell \leq 10/\varepsilon^2$ and $s \in R$.

Proof. It suffices to fix sequences $k = k(n)$, $\ell = \ell(n)$ and $s = s(n)$ satisfying $0 \leq k, \ell \leq 10/\varepsilon^2$ and $s \in R_n$, and prove (9.1) and (9.2) for these sequences, where in principle the implicit constants above and in the proof that follows may depend on the choice of the sequences. The claimed uniform bounds follow by considering appropriate worst-case sequences.

Suppressing the dependence on n as usual, fix sequences k, ℓ and s as above, and let $m = n - s$. Note that $m = \Theta(n)$; see (4.4). We shall apply Lemma 6.1 with $a = 1$; recall the notation $\{k : t\} = (kt)! / (k! t^k)$ used there.

Considering first the number of choices for the $k(r - 1) + 1$ vertices, then the number of trees T on the given vertex set, and finally the probability that the edges of T are present but no other edges incident with T are, we have

$$\mu_k = \binom{m}{k(r-1)+1} (k(r-1)+1)^{k-1} \{k : r-1\} p^k (1-p)^{t_{m,k}-k}, \tag{9.3}$$

where

$$t_{m,k} = \binom{m}{r} - \binom{m - k(r-1) - 1}{r}$$

is the number of potential hyperedges on an m -vertex set meeting a given $(k(r - 1) + 1)$ -vertex set at least once. Postponing the evaluation of μ_k for the moment, if we write a similar formula for $\mu_{k,\ell}$, then most terms agree with the corresponding terms in $\mu_k \mu_\ell$. Indeed, writing a for $k(r - 1) + 1$ and b for $\ell(r - 1) + 1$, it is easy to see that

$$\frac{\mu_{k,\ell}}{\mu_k \mu_\ell} = \binom{m-a}{b} \binom{m}{b}^{-1} (1-p)^{-t_{m,k,\ell}}, \tag{9.4}$$

where $t_{m,k,\ell}$ is the number of potential hyperedges meeting both a given set of a vertices and a given disjoint set of b vertices. Note that

$$t_{m,k,\ell} = ab \binom{m}{r-2} + O((a+b)^3 m^{r-3}) = ab \frac{m^{r-2}}{(r-2)!} + O((a+b)^3 m^{r-3}).$$

Writing

$$\lambda' = pm^{r-1} / (r-2)!$$

for the branching factor of $H' = H_{m,p}^r$ (see (8.1)), since $p = O(n^{-r+1}) = O(m^{-r+1})$ it follows that

$$pt_{m,k,\ell} = \lambda' ab / m + O((a+b)^3 m^{-2}).$$

Since, crudely, $p = O(1/m)$ and $ab = O((a+b)^3)$, from this it certainly follows that $p^2 t_{m,k,\ell} = O((a+b)^3 m^{-2})$, so

$$\log((1-p)^{-t_{m,k,\ell}}) = pt_{m,k,\ell} + O(p^2 t_{m,k,\ell}) = \lambda' ab / m + O((a+b)^3 m^{-2}). \tag{9.5}$$

By Lemma 8.1 we have

$$\lambda' = 1 - \Theta(\varepsilon) \quad \text{and} \quad \lambda' = \Theta(1). \tag{9.6}$$

Using the formula $\binom{m-a}{b} / \binom{m}{b} = \exp(-ab/m + O((a+b)^3/m^2))$, valid for $a, b \leq m/3$, say, from (9.4)–(9.6) we see that

$$\log\left(\frac{\mu_{k,\ell}}{\mu_k \mu_\ell}\right) = \frac{(\lambda' - 1)ab}{m} + O((a+b)^3 m^{-2}) = O(\varepsilon(a+b)^2 m^{-1}) = o(1),$$

since $a + b = O(\varepsilon^{-2}) = o(\varepsilon m)$. This proves (9.2).

Let us temporarily adopt the convention of writing $f \approx g$ for $f = \Theta(g)$. Returning to μ_k , for $k = 0$ we have $\mu_k = m(1-p)^{t_{m,0}}$. Since $m \approx n$, $t_{m,0} = \binom{m-1}{r-1} \approx n^{r-1}$ and $p \approx n^{-r+1}$, it follows that $\mu_0 \approx n$, as required. From now on suppose that $1 \leq k \leq 10/\varepsilon^2$. Since $p^2 t_{m,k} = O(p^2 k m^{r-1}) = O(pk)$ and $pk = o(1)$, from (9.3) we have

$$\begin{aligned} \mu_k &\sim m \binom{m-1}{k(r-1)} (k(r-1)+1)^{k-2} \frac{(k(r-1))!}{k!(r-1)!^k} p^k \exp(-pt_{m,k}) \\ &\approx m(m-1)_{k(r-1)} \frac{k^{k-2}(r-1)^{k-2}}{k!(r-1)!^k} p^k \exp(-pt_{m,k}) \\ &\approx mk^{-2}(m-1)_{k(r-1)} \frac{k^k}{k!(r-2)!^k} p^k \exp(-pt_{m,k}), \end{aligned}$$

where, as before, $(x)_y$ denotes the falling factorial $x(x-1)\cdots(x-y+1)$. For $y \leq x/2$,

$$(x-1)_y = x^y \exp\left(-\frac{y^2}{2x} + O(y/x) + O(y^3/x^2)\right).$$

Since $m \approx n$, $\varepsilon^3 n \rightarrow \infty$ and $k \leq 10/\varepsilon^2$, both k/m and k^3/m^2 are $o(1)$. Hence

$$\begin{aligned} \mu_k &\approx mk^{-2} \left(\frac{m^{r-1}p}{(r-2)!}\right)^k \frac{k^k}{k!} \exp\left(-pt_{m,k} - \frac{(r-1)^2 k^2}{2m}\right) \\ &\approx mk^{-5/2} (\lambda')^k \exp\left(k - pt_{m,k} - \frac{(r-1)^2 k^2}{2m}\right), \end{aligned} \tag{9.7}$$

since $k^k/k! \approx e^k/\sqrt{k}$.

Now

$$\begin{aligned} t_{m,k} &= \frac{m^r - (m-k(r-1))^r}{r!} + O(m^{r-1}) \\ &= \frac{rk(r-1)m^{r-1} - \binom{r}{2}k^2(r-1)^2m^{r-2}}{r!} + O(m^{r-1} + k^3m^{r-3}) \\ &= \frac{km^{r-1}}{(r-2)!} - \frac{k^2(r-1)^2m^{r-2}}{2(r-2)!} + O(m^{r-1}). \end{aligned}$$

Since $p = \lambda'(r-2)!/m^{r-1}$, it follows that

$$pt_{m,k} = \lambda'k - \lambda' \frac{(r-1)^2 k^2}{2m} + O(1).$$

Thus, recalling that $1 - \lambda' = O(\varepsilon)$, that $k = O(\varepsilon^{-2})$, and that $\varepsilon^3 m \rightarrow \infty$, the term inside the exponential in (9.7) is

$$k(1 - \lambda') - (1 - \lambda') \frac{(r - 1)^2 k^2}{2m} + O(1) = k(1 - \lambda') + O(1).$$

Hence, from (9.7),

$$\mu_k \approx mk^{-5/2}(\lambda' e^{1-\lambda'})^k.$$

From the second bound in (9.6), λ' is bounded away from 0. Since $(1 - x)e^x = \exp(O(x^2))$ when $0 < x < 1$ is bounded away from 1, it follows that $(\lambda' e^{1-\lambda'})^k = \exp(O((1 - \lambda')^2 k)) = \exp(O(\varepsilon^2 k)) \approx 1$, completing the proof of (9.1). □

Corollary 9.2. *Suppose that our Weak Assumption 2.1 holds, and define $R = R_n$ as in (4.3). There is a constant $c > 0$ such that, for any $s = s(n) \in R$ and $t = t(n) \geq 0$,*

$$\mathbb{P}(T_{\lceil \varepsilon^{-2} \rceil, 2\lceil \varepsilon^{-2} \rceil}(H_{n,p}^r \setminus \mathcal{L}_1) \leq c\varepsilon^3 n \mid L_1 = s, N_1 = t) = o(1),$$

where $T_{k,k'}(H)$ denotes the number components of a hypergraph H that are trees with between k and k' edges (inclusive).

Proof. We must be a little careful with the uniformity in this proof: the choice of c is not allowed to depend on $s = s(n)$ and $t = t(n)$.

Let $H' = H_{n-s,p}^r$ as before and, ignoring rounding to integers, let $T = T_{n,s} = T_{\varepsilon^{-2}, 2\varepsilon^{-2}}(H')$. Defining μ_k and $\mu_{k,\ell}$ as in Lemma 9.1, by that lemma we have

$$\mathbb{E}[T] = \sum_{k=\varepsilon^{-2}}^{2\varepsilon^{-2}} \mu_k = \Theta(\varepsilon^{-2} n (\varepsilon^{-2})^{-5/2}) = \Theta(\varepsilon^3 n), \tag{9.8}$$

and

$$\mathbb{E}[T(T - 1)] = \sum_{k=\varepsilon^{-2}}^{2\varepsilon^{-2}} \sum_{\ell=\varepsilon^{-2}}^{2\varepsilon^{-2}} \mu_{k,\ell} \sim \sum_{k,\ell} \mu_k \mu_\ell = \mathbb{E}[T]^2$$

uniformly in the choice of $s = s(n) \in R_n$. Let $a > 0$ be the implicit constant in the lower bound in (9.8), which does not depend on s . Since $\mathbb{E}[T] \geq a\varepsilon^3 n \rightarrow \infty$, we have $\mathbb{E}[T^2] = \mathbb{E}[T(T - 1)] + \mathbb{E}[T] \sim \mathbb{E}[T]^2$. Hence, by Chebyshev's inequality, $\mathbb{P}(T \geq a\varepsilon^3 n/2) = 1 - o(1)$ as $n \rightarrow \infty$, uniformly in $s = s(n)$.

The result follows by Lemma 8.3, applied with \mathcal{P} the set of all s -vertex hypergraphs with nullity t . □

We shall need some further, simpler results about the part of $H_{n,p}^r$ lying outside the giant component. The first concerns (essentially) the sum of the squares of the component sizes; it is perhaps in the literature, but since it is immediate, we give a proof for completeness. Given a hypergraph H , let $N_{\text{con}}(H)$ denote the number of (ordered) pairs (v, w) of (not necessarily distinct) vertices of H with the property that v and w are connected by a path, i.e., are in the same component.

Lemma 9.3. *Suppose that our Weak Assumption 2.1 holds, and define $R = R_n$ as in (4.3). Let $s = s(n) \in R$ and $t = t(n) \geq 0$. Then $\mathbb{E}[N_{\text{con}}(H_{n-s,p}^r)] = O(n/\varepsilon)$ and*

$$\mathbb{E}[N_{\text{con}}(H_{n,p}^r \setminus \mathcal{L}_1) \mid L_1 = s, N_1 = t] = O(n/\varepsilon).$$

Proof. By Lemma 8.3 it suffices to prove the first statement. Let $H' = H_{n-s,p}^r$ and, for $\ell \geq 0$, let J_ℓ be the number of ordered pairs of vertices $v, w \in H'$ joined by a path in H' of length ℓ , so $N_{\text{con}}(H') \leq \sum_\ell J_\ell$. Set $m = n - s$. Writing a v - w path of length ℓ as $v_0 e_1 v_1 e_2 \cdots e_\ell v_\ell$, where the v_i are distinct vertices and the e_i distinct hyperedges with $v_0 = v$, $v_\ell = w$ and e_i containing v_{i-1} and v_i , there are at most $m^{\ell+1}$ choices for the v_i , then at most $\binom{m}{r-2}$ ways of extending each pair $v_{i-1}v_i$ to a hyperedge; to obtain a path these edges must be distinct, so the probability that all are present is p^ℓ . Hence,

$$\mathbb{E}[J_\ell] \leq m^{\ell+1} \frac{m^{(r-2)\ell}}{(r-2)!^\ell} p^\ell = m \left(\frac{m^{r-1} p}{(r-2)!} \right)^\ell = m \lambda(H')^\ell,$$

where $\lambda(H')$ is the ‘branching factor’ of $H' = H_{n-s,p}^r$, defined by (8.1). By Lemma 8.1 we have $\lambda(H') = 1 - \Theta(\varepsilon)$, so summing over ℓ we see that

$$\mathbb{E}[N_{\text{con}}(H')] \leq m(1 - \lambda(H'))^{-1} = O(n/\varepsilon),$$

as claimed. □

By similar arguments, one can show that the expected number of vertices on cycles is $O(\varepsilon^{-1})$, and that the expected number of vertices in components containing cycles is $O(\varepsilon^{-2})$. We do not need these bounds here.

We finish this section by considering *complex* components, *i.e.*, ones with nullity at least 2. Karoński and Łuczak [18] prove a version of the following lemma for the ‘size model’ $H_{n,m}^r$. We give a (more detailed) proof for $H_{n,p}^r$ for completeness.

Lemma 9.4. *Suppose that our Weak Assumption 2.1 holds, and define $R = R_n$ as in (4.3). The expected number of complex components of $H' = H_{n-s,p}^r$ is $O(1/(\varepsilon^3 n)) = o(1)$, for any $s = s(n) \in R$.*

Proof. Writing $G_{\text{inc}}(H)$ for the bipartite vertex–edge incidence graph of a hypergraph H , it is easy to check that $n(H) = n(G_{\text{inc}}(H))$. A minimal connected graph with nullity at least 2 clearly has nullity exactly 2 (otherwise delete an edge in a cycle), and is easily seen to be either a θ -graph, consisting of two distinct vertices joined by three internally vertex-disjoint paths, or a dumbbell, *i.e.*, two edge-disjoint cycles connected by a path of length at least 0. (The cycles are vertex-disjoint unless the connecting path has length 0.) Up to isomorphism, there are $O(\ell^2)$ such graphs with ℓ edges: having chosen whether the graph is of the θ or dumbbell type, it is specified by choosing the lengths of three paths/cycles, constrained to sum to ℓ .

Let \mathcal{G}_ℓ denote the set of isomorphism classes of ℓ -edge bipartite graphs of the form above, where we distinguish the vertex class A corresponding to hypergraph vertices from the class B corresponding to hypergraph edges; thus $|\mathcal{G}_\ell| = O(\ell^2)$. If H is a connected

hypergraph with $n(H) \geq 2$, then $n(G_{\text{inc}}(H)) \geq 2$, so $G_{\text{inc}}(H)$ contains some $G \in \bigcup_{\ell} \mathcal{G}_{\ell}$ as a subgraph. If G has vertex partition $A \cup B$, with $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_t\}$, then in particular H has a subgraph H_0 consisting of t hyperedges with $G_{\text{inc}}(H_0)$ containing G as a subgraph. Fixing G for the moment, let us estimate the expected number of such subgraphs H_0 present in $H' = H_{n-s,p}^r$.

Writing $m = n - s$, there are $m(m - 1) \dots (m - k + 1) \leq m^k$ choices for the vertices of H' corresponding to a_1, \dots, a_k . Let d_i be the degree of b_i in G . For each $1 \leq i \leq t$ we must choose $r - d_i$ further vertices (other than those already specified by the neighbours of b_i in G) to complete the hyperedge corresponding to b_i . For each i there are at most $m^{r-d_i} / (r - d_i)!$ ways of doing this. Since all but at most two vertices of G have degree 2, and $\sum_i d_i = e(G) = \ell$, this gives in total

$$O\left(\frac{m^{t-\ell}}{(r-2)^t}\right)$$

choices. Finally, the probability that the resulting subgraph H_0 is present in H' is exactly p^t . Hence, the expected number of such subgraphs H_0 corresponding to a particular G is bounded by a constant times

$$\frac{m^{k+t-\ell}}{(r-2)^t} p^t = m^{k+t-\ell} \left(\frac{m^{r-1}p}{(r-2)!}\right)^t = m^{-1} \lambda(H')^t,$$

where in the last step we used the fact that G has nullity 2, so $k + t - \ell = |G| - e(G) = -1$, and the definition of the branching factor $\lambda(H')$.

Since G has either two vertices of degree 3 or one of degree 4, and all other vertices have degree 2, we have $2t \leq \ell = \sum_i d_i \leq 2t + 2$. Hence $t \geq \ell/2 - 1$. Thus, summing over the $O(\ell^2)$ choices of $G \in \mathcal{G}_{\ell}$ and then over ℓ we see that the expectation μ of number of complex components of H' satisfies

$$\mu = O\left(\sum_{\ell \geq 2} m^{-1} \ell^2 \lambda(H')^{\ell/2-1}\right) = O\left(\sum_{\ell \geq 2} m^{-1} \ell^2 \lambda(H')^{\ell/2}\right),$$

using the bound $\lambda(H') = \Theta(1)$ from Lemma 8.1 in the last step. Now $\lambda(H') = 1 - \Theta(\varepsilon)$ by Lemma 8.1; hence $\lambda(H')^{1/2} = 1 - \Theta(\varepsilon)$. Since

$$\sum_{\ell \geq 2} \ell^2 x^{\ell} \leq 2 \sum_{\ell \geq 0} (\ell + 1)(\ell + 2)x^{\ell} / 2 = 2(1 - x)^{-3}$$

for $0 \leq x < 1$, it follows that $\mu = O(m^{-1} \varepsilon^{-3}) = O(1/(\varepsilon^3 n))$, as claimed. □

Of course, instead of considering vertex–edge incidence graphs, we could directly count the expected number of minimal complex hypergraphs present in $H_{n-s,p}^r$. However, there are significantly more classes of minimal complex hypergraphs than minimal complex graphs, because the special (degree more than 2) vertices of the corresponding bipartite incidence graph may correspond to vertices or edges of the hypergraph.

Lemma 9.5. *Suppose that our Weak Assumption 2.1 holds. Let \mathcal{U}_{cx} be the event that \mathcal{L}_1 is the unique complex component of $H_{n,p}^r$. Then for any $s = s(n) \in R$ and $t = t(n) \geq 2$ we have*

$$\mathbb{P}(\mathcal{U}_{\text{cx}} \mid L_1 = s, N_1 = t) = O(1/(\varepsilon^3 n)).$$

Furthermore, the probability that $H_{n,p}^r \setminus \mathcal{L}_1$ has a complex component is $O(1/(\varepsilon^3 n))$.

Proof. Let \mathcal{E} be the event that $H_{n,p}^r \setminus \mathcal{L}_1$ has at least one complex component. By Lemmas 8.3 and 9.4, for $s \in R$ and $t' \geq 0$ we have

$$\mathbb{P}(\mathcal{E} \mid L_1 = s, N_1 = t') = O(1/(\varepsilon^3 n)). \tag{9.9}$$

Since $N_1 = t \geq 2$ implies that \mathcal{L}_1 is complex, the first statement follows.

Since (9.9) holds for all t' , for any $s \in R$ we have

$$\mathbb{P}(\mathcal{E} \mid L_1 = s) = O(1/(\varepsilon^3 n)).$$

Recalling from (4.5) that $\mathbb{P}(L_1 \notin R) = O(1/(\varepsilon^3 n))$, it follows that $\mathbb{P}(\mathcal{E}) = O(1/(\varepsilon^3 n))$. □

10. Extended cores in hypergraphs

The strategy of our proof of Theorem 2.3 is as follows. We shall randomly mark a small (order ε^2) fraction of the vertices of $H = H_{n,p}^r$, and define the *extended core* $C^+(H)$ by repeatedly deleting edges in which at least $r - 1$ vertices are unmarked and are contained in no other edges. We shall show that, conditional on the event $\{L_1 = s, N_1 = t\}$, where s and t are in the typical range, certain events are likely to hold. In particular, it is likely that the largest component \mathcal{C}_1^+ of $C^+(H)$ is a subgraph of the largest component of H , that the number a_1 of vertices in \mathcal{C}_1^+ is $\Theta(\varepsilon^2 n)$, and that the number a_0 of isolated vertices in $C^+(H) \setminus \mathcal{C}_1^+$ is also $\Theta(\varepsilon^2 n)$. We condition on $C^+(H)$, and pick $a = \min\{a_0, a_1\}$ vertices of \mathcal{C}_1^+ and a isolated vertices of $C^+(H) \setminus \mathcal{C}_1^+$. We also condition on the set V of vertices joined by paths in H to the selected vertices, which we show satisfies $|V| = \Theta(\varepsilon n)$ with high probability. Then we show that the conditional distribution of the number of vertices in V that are joined by paths to \mathcal{C}_1^+ has a smooth distribution; it is this number that will play the role that Y_n plays in the proof of Theorem 3.4.

Turning to the details, by the *core* $C(H)$ of a hypergraph H we mean the (possibly empty) hypergraph formed from H by repeatedly deleting isolated vertices and hyperedges e in which at most one vertex is in a hyperedge other than e . Equivalently, $C(H)$ is the maximal sub-hypergraph of H without isolated vertices in which every edge contains at least two vertices in other hyperedges. Note that this is only one of several possible generalizations of the concept of the core of a graph [8]; another natural possibility is to take the maximal sub-hypergraph with minimum degree at least 2. A hypergraph H consists of its core, tree components, and the ‘mantle’, made up of trees each of which meets the core in a single vertex. It is a part of the mantle that we shall use in our smoothing argument.

Note that the core of H and that of its bipartite vertex–edge incidence graph $G_{\text{inc}}(H)$ correspond in a natural way, except that in the latter, any vertices corresponding to vertices of $C(H)$ that are in a single edge of $C(H)$ are deleted.

As discussed in Section 3, we would like to ‘detach and reattach’ the trees attached not only to the core, but also to an additional set of vertices of comparable size. To achieve this, we define an ‘extended core’, essentially by artificially placing a suitable number of extra vertices into the core; we shall call these vertices ‘marked’ vertices.

Let (H, V^*) be a *marked hypergraph*: a hypergraph $H = (V, E)$ together with a subset V^* of V . The vertices in V^* will be called *marked vertices*. The *extended core* $C^+(H, V^*)$ is the marked sub-hypergraph obtained by repeatedly deleting unmarked isolated vertices, and hyperedges in which all or all but one vertices are unmarked and have degree 1. Equivalently, $C^+(H, V^*)$ is the maximal sub-hypergraph in which every edge contains at least two vertices that are either marked or in at least one other edge, and all isolated vertices are marked. Note that the deletion operation defining the extended core preserves connectivity, so the extended core of a connected hypergraph H is either connected or, if H is a tree with no marked vertices (an ‘unmarked tree’), empty. Of course, $C^+(H, V^*)$ is the union of the extended cores of the components of H .

Proposition 10.1. *Any marked hypergraph (H, V^*) is the union of its extended core $C^+ = C^+(H, V^*)$, a set $\{T_v\}_{v \in V(C^+)}$ of trees, each with with $v \in T_v$, and a possibly empty set $\{U_i\}$ of trees, with the vertex sets $V(T_v) \setminus v$ and $V(U_i)$ disjoint from each other and from $V(C^+)$.*

In other words, noting that by definition all vertices outside $C^+(H, V^*)$ are unmarked, we may reconstruct (H, V^*) from its extended core by adding disjoint trees to each vertex v of the extended core, unmarked expect possibly at v , and possibly some further disjoint unmarked trees. Later we shall refer to the set $M^+ = \bigcup_{v \in C^+(H, V^*)} (V(T_v) \setminus v)$ as the (vertex set of) the *extended mantle* of (H, V^*) .

Proof. Simply reverse the edge-deletion algorithm defining the extended core. □

In this section and the next it will be convenient (though not essential) to assume that $\varepsilon \rightarrow 0$, *i.e.*, our Standard Assumption 2.2, as in Theorem 4.4 whose proof we are preparing for. We also consider a constant $0 < \eta < 1/100$ whose role will be explained at the start of the next section. Any implicit constants or functions may depend on the choice of the functions $\varepsilon = \varepsilon(n)$ and the constant $\eta > 0$. As we shall see in Section 11, this will cause no problems when we come to apply the results. Thus, in this section, we may regard $\varepsilon = \varepsilon(n)$ and $\eta > 0$ as given, satisfying the following condition which we state for ease of reference.

Assumption 10.2. The integer $r \geq 2$ and real number $0 < \eta < 1/100$ are fixed. The functions $p(n)$, $\lambda(n)$ and $\varepsilon(n)$ are related by $\lambda = 1 + \varepsilon$ and $p = \lambda(r - 2)!n^{-r+1}$. Moreover, as $n \rightarrow \infty$, we have $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon \rightarrow 0$.

With $\eta > 0$ and $\varepsilon(n)$ given as above, set

$$\alpha = \frac{\eta}{100r}. \tag{10.1}$$

We shall mark the vertices of our random hypergraph $H = H_{n,p}^r$ independently with probability

$$p_{\text{mark}} = \alpha\varepsilon^2 = \alpha\varepsilon(n)^2.$$

We shall treat $H_{n,p}^r$ as a marked hypergraph without explicitly indicating the set V^* of marked vertices in the notation. Let $C^+(\mathcal{L}_1)$ be the extended core of the marked hypergraph \mathcal{L}_1 , where, as usual, \mathcal{L}_1 is the largest component of $H_{n,p}^r$. Thus $C^+(\mathcal{L}_1)$ is a component of $C^+(H_{n,p}^r)$, except in the unlikely event that \mathcal{L}_1 is an unmarked tree, in which case $C^+(\mathcal{L}_1) = \emptyset$. Recall that $L_1 = |\mathcal{L}_1|$. The next few lemmas gather properties of $C^+(H_{n,p}^r)$ and its ‘mantle’ that we shall need. A key point is that these results hold conditional on the giant component \mathcal{L}_1 having a specific order s and nullity t , provided s is in the typical range R defined in (4.3). For this reason they do not obviously follow from ‘global’ results saying that w.h.p. the (extended) core has some property. Another key point is that we can afford to give up constant factors in the estimates of the size of the extended core and of its mantle. Throughout the rest of this section, p, λ, ε and η satisfy Assumption 10.2, and we define R as in (4.3). All new constants introduced below may depend on the choice of the function $\varepsilon = \varepsilon(n)$ and of η .

Lemma 10.3. *Let $r \geq 2, \eta > 0$ and $\varepsilon = \varepsilon(n)$ satisfying Assumption 10.2 be given. Then there is a constant $c_1 > 0$ such that, for n large enough, for any $s = s(n) \in R$ and $t = t(n) \geq 1$ we have*

$$\mathbb{P}(|C^+(\mathcal{L}_1)| > c_1\varepsilon^2n \mid L_1 = s, N_1 = t) \geq 1 - \eta. \tag{10.2}$$

Proof. We shall condition not only on the event $\{L_1 = s, N_1 = t\}$, but also on the vertex set of \mathcal{L}_1 and on the entire hypergraph structure of its core $C(\mathcal{L}_1)$. The extended core contains the core; if the core is not already large enough, we shall show that with conditional probability at least $1 - \eta$, the interaction of the marked vertices with the core generates an extended core of at least the required size.

Turning to the details, by (4.4) there is a constant $c_0 > 0$ that depends only on the function $\varepsilon(n)$, such that

$$s \in R_n \quad \text{implies} \quad s \geq c_0\varepsilon n. \tag{10.3}$$

We shall prove (10.2) with

$$c_1 = \frac{\alpha c_0}{4} = \frac{\eta c_0}{400r}. \tag{10.4}$$

First, by Chebyshev’s inequality, if X has a binomial distribution with mean $\mu \geq 8/\eta$ (and so variance less than μ) then $\mathbb{P}(X \geq \mu/2) \geq 1 - \eta/2$. Hence, from (10.3) and the assumption $\varepsilon^3 n \rightarrow \infty$, there is an n_0 such that for all $n \geq n_0$

$$s \in R_n \quad \text{implies} \quad \mathbb{P}(\text{Bin}(s, \alpha\varepsilon^2) \geq \alpha\varepsilon^2 s/2) \geq 1 - \eta/2. \tag{10.5}$$

By assumption $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$. Hence, increasing n_0 if necessary, for all $n \geq n_0$ we have

$$\varepsilon \leq 1/100 \quad \text{and} \quad \varepsilon^3 n \geq 8r/c_1. \tag{10.6}$$

From now on, let $n \geq n_0$, $s \in R_n$ and $t \geq 1$ be given. We condition on the event \mathcal{E} that $L_1 = s$, $N_1 = t$, the vertex set of \mathcal{L}_1 is some specific set V_1 of s vertices, and the usual (non-extended) core $C(\mathcal{L}_1)$ is some particular hypergraph with vertex set $V_2 \subset V_1$. We write $a = |V_2|$. Our aim is to show that

$$\mathbb{P}(|C^+(\mathcal{L}_1)| \leq c_1 \varepsilon^2 n \mid \mathcal{E}) \leq \eta. \tag{10.7}$$

Since \mathcal{L}_1 and $C(\mathcal{L}_1)$ have the same nullity, we may assume that $C(\mathcal{L}_1)$ has nullity t ; in fact, we only need the trivial consequence that $C(\mathcal{L}_1)$ is not empty.⁴ Since $C^+(\mathcal{L}_1) \supset C(\mathcal{L}_1)$, if $a > c_1 \varepsilon^2 n$ then the conditional probability in (10.7) is 0. Thus we may assume that

$$1 \leq a \leq c_1 \varepsilon^2 n. \tag{10.8}$$

Relabelling, let us take the vertex set of \mathcal{L}_1 to be $[s]$ and that of its core to be $[a] \subset [s]$. From the definition of the core, \mathcal{L}_1 is the union of its core and an $[a]$ -rooted r -forest F on $[s]$. Since this forest F does not affect the core, after conditioning on \mathcal{E} as above, F is uniformly random on all such forests. Recall that we mark vertices independently with probability $\alpha \varepsilon^2$, where α is given in (10.1). Since \mathcal{L}_1 and its core $C(\mathcal{L}_1)$ are defined without reference to the set V^* of marked vertices, each vertex of $[s]$ is marked independently of the others and of the random forest F .

Set $\ell = \lceil \varepsilon^{-1} \rceil$. Call a marked vertex $v \in \mathcal{L}_1$ *bad* if either

- (i) it is at distance at most ℓ from $[a] = V(C(\mathcal{L}_1))$ or
- (ii) it is joined to another marked vertex by a path in $F = \mathcal{L}_1 - C(\mathcal{L}_1)$ of length at most 2ℓ .

If v is not bad, we call it *good*.

Every marked vertex in \mathcal{L}_1 is on a path to the core $C(\mathcal{L}_1)$. The union of these paths is a subgraph F^* of the forest F , and $C^+(\mathcal{L}_1) = C(\mathcal{L}_1) \cup F^*$, with each component of F^* meeting $C(\mathcal{L}_1)$ in a single vertex. For each good marked vertex v , consider the first ℓ edges of the path to the core starting at v : these shortened paths are necessarily disjoint, so $|C^+(\mathcal{L}_1)|$ is at least ℓ times (in fact, at least $(r - 1)\ell$ times) the number of good marked vertices. As the number of marked vertices in \mathcal{L}_1 has the binomial distribution $\text{Bin}(s, \alpha \varepsilon^2)$, by (10.5) the probability that there are at least $\alpha \varepsilon^2 s / 2$ marked vertices in \mathcal{L}_1 is at least $1 - \eta / 2$. We claim that, conditional on \mathcal{E} , the expected number of bad marked vertices is at most $\eta \alpha \varepsilon^2 s / 8$. Assuming this then, by Markov's inequality, with probability at least $1 - \eta / 2$ there are at most $\alpha \varepsilon^2 s / 4$ bad marked vertices, and hence with probability at least

⁴ In proving Theorem 2.4, we do not condition on the nullity $n(\mathcal{L}_1)$. This means we cannot *a priori* assume that $C(\mathcal{L}_1)$ is non-empty. However, it is immediate from the formulae given by Karoński and Łuczak [17, Theorem 9] for the number of connected hypergraphs on s vertices with a given small excess that $\mathbb{P}(n(\mathcal{L}_1) = 0 \mid L_1 = s) = o(\mathbb{P}(n(\mathcal{L}_1) = r - 1 \mid L_1 = s))$, so $\mathbb{P}(n(\mathcal{L}_1) = 0 \mid L_1 = s) = o(1)$. Hence we can indeed assume that $a \geq 1$.

$1 - \eta$ there are at least $\alpha\epsilon^2s/2 - \alpha\epsilon^2s/4 = \alpha\epsilon^2s/4$ good marked vertices. But then, recalling (10.3) and (10.4),

$$|C^+(\mathcal{L}_1)| \geq \ell\alpha\epsilon^2s/4 \geq \alpha\epsilon s/4 \geq \alpha c_0\epsilon^2n/4 = c_1\epsilon^2n.$$

To prove the claim, let v be a vertex in $[s] = V(\mathcal{L}_1)$ chosen uniformly at random. We must show that the probability that v is a bad marked vertex is at most $\eta\alpha\epsilon^2/8$. So first condition on the event that v is marked; it remains to show that the conditional probability that (i) or (ii) holds is at most $\eta/8$.

For (i), this conditional probability is exactly $1/s$ times the expectation μ of the number of vertices in $[s]$ within distance ℓ of $[a]$. From Lemma 6.2 and (10.8),

$$\mu \leq \sum_{0 \leq j \leq \ell} (a + (r - 1)j) \leq 2a\ell + 2r\ell^2 \leq 2c_1\epsilon^2n\ell + 2r\ell^2.$$

Since $n \geq n_0$, from (10.6) we have $\ell = \lceil \epsilon^{-1} \rceil \leq 2\epsilon^{-1}$, say, and $8r/(\epsilon^3n) \leq c_1$. Thus

$$\mu \leq 4c_1\epsilon n + 8r\epsilon^{-2} \leq 5c_1\epsilon n = \frac{\eta c_0 \epsilon n}{80r} \leq \frac{\eta s}{80r},$$

recalling (10.3). Hence the conditional probability μ/s that (i) holds is at most $\eta/(80r) < \eta/16$.

For (ii), the components of the forest F give a partition of the vertex set $[s]$ of \mathcal{L}_1 into a parts (some of which may be singletons). Let us condition on the vertex v and on this partition. The component T of F containing v is then a uniformly random r -tree on its vertex set X . Viewing v as the root, we can regard this r -tree as a $\{v\}$ -rooted r -forest, and then by Lemma 6.2 the expected number of vertices $w \neq v$ joined to v by paths in F (and hence in T) of length at most 2ℓ is at most

$$\sum_{1 \leq j \leq 2\ell} (1 + (r - 1)j) \leq 4r\ell^2 = 4r\lceil 1/\epsilon \rceil^2.$$

Hence the probability that one or more such vertices are marked is at most $4r\lceil 1/\epsilon \rceil^2\alpha\epsilon^2$. From (10.6) and (10.1) this probability is at most $5r\alpha \leq \eta/16$. Thus the conditional probability that (i) or (ii) holds is at most $\eta/8$, completing the proof of the claim and hence of the lemma. □

We have shown that with high (conditional) probability, the extended core $C^+(\mathcal{L}_1)$ of the largest component is not too small. Roughly speaking, our next aim is to show that with high probability the rest of the extended core, *i.e.*, $C^+(H_{n,p}^r) \setminus C^+(\mathcal{L}_1) = C^+(H_{n,p}^r \setminus \mathcal{L}_1)$ is neither too small nor too big. While this is not too hard, it turns out that we can avoid some work by considering instead the set

$$\mathcal{I} = \{\text{isolated vertices in the hypergraph } C^+(H_{n,p}^r \setminus \mathcal{L}_1)\}. \tag{10.9}$$

By definition, an isolated vertex in $C^+(H_{n,p}^r \setminus \mathcal{L}_1)$ is marked (otherwise it would be deleted in defining the extended core). By Proposition 10.1, each $v \in \mathcal{I}$ corresponds to a tree component of $H_{n,p}^r \setminus \mathcal{L}_1$ containing exactly one marked vertex, namely v .

Lemma 10.4. *Let $r \geq 2, \eta > 0$ and $\varepsilon = \varepsilon(n)$ satisfying Assumption 10.2 be given. Then there is a constant $c_2 > 0$ such that, for any $s = s(n) \in \mathbb{R}$ and $t = t(n) \geq 0$,*

$$\mathbb{P}(c_2\varepsilon^2n \leq |\mathcal{I}| \leq 2\alpha\varepsilon^2n \mid L_1 = s, N_1 = t) = 1 - o(1).$$

Proof. The upper bound on $|\mathcal{I}|$ is trivial. Indeed, any vertex of \mathcal{I} is marked, by the definition of the extended core. Given that $L_1 = s$, and any further information about \mathcal{L}_1 , the number of marked vertices in $H_{n,p}^r \setminus \mathcal{L}_1$ has the binomial distribution $\text{Bin}(n - s, \alpha\varepsilon^2)$, with mean at most $\alpha\varepsilon^2n \rightarrow \infty$, so with high probability this number is at most $2\alpha\varepsilon^2n$.

Turning to the lower bound, by Lemma 8.3 it suffices to show that w.h.p. $H' = H_{n-s,p}^r$ (with vertices marked independently with probability $\alpha\varepsilon^2$) has at least $c_2\varepsilon^2n$ isolated vertices in its extended core. An elementary first and second moment calculation (or the case $k = 0$ of Lemma 9.1) shows that w.h.p. H' has $\Theta(n)$ isolated vertices. Since each is marked independently with probability $\alpha\varepsilon^2$ and, if marked, is an isolated vertex of $C^+(H')$, the result follows from concentration of the binomial distribution. \square

Let H be a hypergraph with extended core $C^+(H)$. We define the *mantle* $M^+(H)$ to be the set of vertices of H not in $C^+(H)$ but connected to it by paths. Thus $C^+(H) \cup M^+(H)$ includes all vertices of H except those in tree components with no marked vertices. By Proposition 10.1, each $w \in M^+(H)$ is connected by a path in the mantle to a unique vertex $v \in C^+(H)$; for $A \subset V(C^+(H))$ we write $M^+(A)$ for the set of $w \in M^+(H)$ whose corresponding core vertex v is in A .

Lemma 10.5. *Let $r \geq 2, \eta > 0$ and $\varepsilon = \varepsilon(n)$ satisfying Assumption 10.2 be given. Then there is a constant $c_3 > 0$ such that, for any $s = s(n) \in \mathbb{R}$ and $t = t(n) \geq 0$, we have*

$$\mathbb{P}(|M^+(\mathcal{I})| \geq c_3\varepsilon n \mid L_1 = s, N_1 = t) = 1 - o(1).$$

Proof. Condition on the event that $L_1 = s$ and $N_1 = t$. From Corollary 9.2, with conditional probability $1 - o(1)$ the hypergraph $H_{n,p}^r \setminus \mathcal{L}_1$ contains at least $c\varepsilon^3n$ tree components each having between $\lceil \varepsilon^{-2} \rceil$ and $2\lceil \varepsilon^{-2} \rceil$ edges, and so $\Theta(\varepsilon^{-2})$ vertices. Having revealed the graph $H_{n,p}^r$, for each such tree, the probability that it contains exactly one marked vertex is at least some constant $c' > 0$. So the conditional distribution of the number X of such trees containing exactly one marked vertex stochastically dominates a Binomial distribution with mean $cc'\varepsilon^3n$. Since $\varepsilon^3n \rightarrow \infty$, it follows that w.h.p. $X \geq cc'\varepsilon^3n/2$. Since each tree counted by X contains at least $1 + (r - 1)\lceil \varepsilon^{-2} \rceil$ vertices, and so contributes at least $(r - 1)\lceil \varepsilon^{-2} \rceil \geq \varepsilon^{-2}$ vertices to $M^+(\mathcal{I})$, the result follows. \square

Lemma 10.6. *Let $r \geq 2, \eta > 0$ and $\varepsilon = \varepsilon(n)$ satisfying Assumption 10.2 be given. Then there is a constant $c_4 > 0$ such that, for n large enough, for every $s \in \mathbb{R}$ and $t \geq 0$ we have*

$$\mathbb{P}(|M^+(\mathcal{I})| \leq c_4\varepsilon n \mid L_1 = s, N_1 = t) \geq 1 - \eta.$$

Proof. Given a marked hypergraph H , let $X(H)$ denote the number of vertices v of H with the property that v is joined to some marked vertex of H by a path in H . Note that every vertex of $M^+(\mathcal{I})$ has this property in $H^- = H_{n,p}^r \setminus \mathcal{L}_1$, so $|M^+(\mathcal{I})| \leq X(H^-)$. Hence, by Markov's inequality, it suffices to show that $\mathbb{E}[X(H^-) \mid L_1 = s, N_1 = t] = O(\varepsilon n)$.⁵ Now $X(H^-)$ is at most the number of ordered pairs (v, w) of vertices of H^- with v marked and v, w joined by a path, so

$$\mathbb{E}[X(H^-) \mid L_1 = s, N_1 = t] \leq \alpha \varepsilon^2 \mathbb{E}[N_{\text{con}}(H^-) \mid L_1 = s, N_1 = t]$$

which, by Lemma 9.3, is $O(\varepsilon^2 n / \varepsilon) = O(\varepsilon n)$. □

11. The core smoothing argument

In this section we prove Theorem 4.4; this is all that remains to complete the proof of Theorem 2.3. The strategy that we follow is outlined at the start of Section 10. Recall that we always relate $p = p(n)$ and $\varepsilon = \varepsilon(n)$ by

$$\lambda(n) = 1 + \varepsilon(n) \quad \text{and} \quad p(n) = \lambda(n)(r - 2)!n^{-r+1}.$$

Define $R = R_n$ as in (4.3); in this section we shall consider sequences (x_n) , (y_n) and (t_n) of integers such that

$$t_n \geq 2, \quad x_n, y_n \in R_n, \quad x_n - y_n = o(\sqrt{n/\varepsilon}), \quad \text{and} \quad x_n \equiv y_n \equiv 1 - t_n \pmod{r - 1}. \quad (11.1)$$

(The congruence condition arises since otherwise there are no r -uniform hypergraphs with nullity t_n and x_n or y_n vertices.) The following lemma captures (a particular form of) what is needed to prove Theorem 4.4. Here $\alpha \pm \beta$ denotes a quantity in the range $[\alpha - \beta, \alpha + \beta]$.

Lemma 11.1. *Suppose that $p(n)$ satisfies our Standard Assumption 2.2, that $0 < \eta < 1/100$ is constant, and that the sequences (x_n) , (y_n) and (t_n) satisfy (11.1). Then*

$$\mathbb{P}(L_1 = y_n, N_1 = t_n) = O(1/(\varepsilon n)), \quad (11.2)$$

and, for n large enough,

$$\mathbb{P}(L_1 = x_n, N_1 = t_n) = (1 \pm 30\eta)(\mathbb{P}(L_1 = y_n, N_1 = t_n) \pm \eta/(\varepsilon n)). \quad (11.3)$$

As usual, the implicit constant in (11.2) may depend on all choices so far, *i.e.*, on the sequences $(p(n))$, (x_n) , (y_n) and (t_n) and constants r and η , just of course not on n . (See Remark 2.7.) The same applies to the implicit constant n_0 in ‘for n large enough’.

Before proving Lemma 11.1, which will take most of the section, we show that it implies Theorem 4.4.

Proof of Theorem 4.4, assuming Lemma 11.1. Theorem 4.4 asserts that, given $r \geq 2$, a sequence $(p(n))$ (and hence $\varepsilon(n)$) satisfying Assumption 2.2, and sequences (x_n) , (y_n) and

⁵ We need this bound to hold uniformly over $s \in R_n$ and $t \geq 0$; for this we just consider the worst-case $s(n)$ and $t(n)$.

(t_n) satisfying (11.1), we have

$$\mathbb{P}(L_1 = x_n, N_1 = t_n) - \mathbb{P}(L_1 = y_n, N_1 = t_n) = o(1/(\epsilon n)). \tag{11.4}$$

In proving this we may of course fix $r \geq 2$, $(p(n))$, (x_n) , (y_n) and (t_n) as above, and $0 < \delta \leq 1$, say. Then we must show that for all large enough n (depending on all choices so far), we have

$$\left| \mathbb{P}(L_1 = x_n, N_1 = t_n) - \mathbb{P}(L_1 = y_n, N_1 = t_n) \right| \leq \delta/(\epsilon n). \tag{11.5}$$

By the first part of Lemma 11.1, applied with $\eta = 1/200$, say, there is a constant C (which may depend on all choices so far) such that $\mathbb{P}(L_1 = y_n, N_1 = t_n) \leq C/(\epsilon n)$. We may assume $C > 1$. Let $\eta = \delta/(60C) \leq \delta/4$. By the second part of Lemma 11.1, if n is large enough then

$$\mathbb{P}(L_1 = x_n, N_1 = t_n) = (1 \pm 30\eta)(\mathbb{P}(L_1 = y_n, N_1 = t_n) \pm \delta/(4\epsilon n)).$$

Since $(1 + 30\eta) \leq 2$, this gives

$$\mathbb{P}(L_1 = x_n, N_1 = t_n) = \mathbb{P}(L_1 = y_n, N_1 = t_n) \pm (30\eta C/(\epsilon n) + \delta/(2\epsilon n)),$$

which implies (11.5) since $30\eta C = \delta/2$. □

It remains to prove Lemma 11.1. In doing so we may of course fix $r \geq 2$, sequences $(p(n))$, (x_n) , (y_n) , (t_n) , and a real number $0 < \eta < 1/100$ such that our Standard Assumption 2.2 holds, as does (11.1). Any new constants introduced may depend on these choices. Note that Assumption 10.2 of Section 10 holds.

Define the largest component \mathcal{L}_1 of $H_{n,p}^r$ as before, and the extended core $C^+(H_{n,p}^r)$ and the set \mathcal{I} as in Section 10 (see (10.9)). Define R as in (4.3). By (4.4), there are constants $c_0 > 0$ and c_5 such that, for n large,

$$R = [(1 - \delta)\rho_{r,\lambda}n, (1 + \delta)\rho_{r,\lambda}n] \subset [c_0\epsilon n, c_5\epsilon n].$$

Set

$$c = \min\{c_0, c_1, c_2, c_3\} \quad \text{and} \quad C = \max\{c_4, c_5\},$$

where the constants c_i , $1 \leq i \leq 4$, are as in Lemmas 10.3–10.6.

Let \mathcal{A} be the event that the following conditions hold:

- (i) $|C^+(\mathcal{L}_1)| \geq c\epsilon^2n$,
- (ii) $c\epsilon^2n \leq |\mathcal{I}| \leq 2\alpha\epsilon^2n$,
- (iii) $|M^+(\mathcal{I})| \geq c\epsilon n$,
- (iv) $|M^+(\mathcal{I})| \leq C\epsilon n$ and
- (v) $c\epsilon n \leq |C^+(\mathcal{L}_1)| + |M^+(C^+(\mathcal{L}_1))| \leq C\epsilon n$.

Claim 11.2. For n sufficiently large, for any $s \in R$ and any $t \geq 2$ we have

$$\mathbb{P}(\mathcal{A} \mid L_1 = s, N_1 = t) \geq 1 - 3\eta. \tag{11.6}$$

Proof. Lemmas 10.3, 10.4, 10.5 and 10.6 imply that properties (i)–(iv) hold with conditional probability at least $1 - (2\eta + o(1)) \geq 1 - 3\eta$ for n large. Whenever (i) holds then in particular $C^+(\mathcal{L}_1)$ is not empty. But then by, Proposition 10.1, $|C^+(\mathcal{L}_1)| + |M^+(C^+(\mathcal{L}_1))| = |\mathcal{L}_1| = L_1 = s \in R$, so (v) holds. \square

As before, let \mathcal{U}_{cx} be the event

$$\mathcal{U}_{cx} = \{\mathcal{L}_1 \text{ is the unique complex component of } H_{n,p}^r\},$$

so \mathcal{U}_{cx} holds w.h.p. by Lemma 9.5. Let \mathcal{C}_1^+ be the component of $C^+(H_{n,p}^r)$ with the highest nullity/excess, chosen according to any fixed rule if there is a tie, and let \mathcal{I}' be the set of isolated vertices of $C^+(H_{n,p}^r) \setminus \mathcal{C}_1^+$. Note that if \mathcal{U}_{cx} holds, then $C^+(H_{n,p}^r)$ has a unique complex component, and we have $\mathcal{C}_1^+ = C^+(\mathcal{L}_1)$ and so $\mathcal{I}' = \mathcal{I}$. We shall define an event \mathcal{B} that is closely related to \mathcal{A} , but defined using \mathcal{C}_1^+ and \mathcal{I}' in place of \mathcal{C}_1 and \mathcal{I} . The point is that we would like to condition on the extended core (and some further information), and then use the remaining randomness concerning which parts of the mantle are joined to the largest component as our smoothing distribution. But until this remaining randomness has been revealed, we do not know which component is largest, so we cannot easily condition on \mathcal{A} .

Let $a_1 = |\mathcal{C}_1^+|$, $a_0 = |\mathcal{I}'|$, and $a = \min\{a_1, a_0\}$. Given the entire extended core, pick sets $A_1 \subset V(\mathcal{C}_1^+)$ and $A_0 \subset \mathcal{I}'$ with $|A_1| = |A_0| = a$, for example by choosing in each case the first a eligible vertices in a fixed order. (This is mostly a convenience; with a little more work we could work directly with \mathcal{C}_1^+ and \mathcal{I}' .) Let \mathcal{B} be the event that the following hold:

- (I) $c\varepsilon^2 n \leq a \leq 2\alpha\varepsilon^2 n$ and
- (II) $c\varepsilon n/2 \leq |M^+(A_1 \cup A_0)| \leq 2C\varepsilon n$.

Claim 11.3. *If n is large enough, then whenever $\mathcal{A} \cap \mathcal{U}_{cx}$ holds, so does \mathcal{B} .*

Proof. Suppose that $\mathcal{A} \cap \mathcal{U}_{cx}$ holds. Then, since \mathcal{U}_{cx} holds, $\mathcal{C}_1^+ = C^+(\mathcal{L}_1)$. Since $|C^+(\mathcal{L}_1)| \geq c\varepsilon^2 n$ by condition (i) of \mathcal{A} , we have $a_1 \geq c\varepsilon^2 n$. Also, $a_0 = |\mathcal{I}'| = |\mathcal{I}|$ is between $c\varepsilon^2 n$ and $2\alpha\varepsilon^2 n$ by (ii). Since $a = \min\{a_1, a_0\}$, this gives (I). Consider next the upper bound in (II). Since \mathcal{U}_{cx} holds, $A_1 \cup A_0 \subset \mathcal{C}_1^+ \cup \mathcal{I}' = C^+(\mathcal{L}_1) \cup \mathcal{I}$, so $M^+(A_1 \cup A_0) \subset M^+(C^+(\mathcal{L}_1)) \cup M^+(\mathcal{I})$, and (iv) and (v) imply $|M^+(A_1 \cup A_0)| \leq 2C\varepsilon n$. For the lower bound we have two cases: if $a_0 \leq a_1$ then $A_0 = \mathcal{I}' = \mathcal{I}$ so $|M^+(A_1 \cup A_0)| \geq |M^+(A_0)| = |M^+(\mathcal{I})| \geq c\varepsilon n$ by (iii). If $a_1 \leq a_0$ then $A_1 = \mathcal{C}_1^+ = C^+(\mathcal{L}_1)$, so from (v) we have

$$|M^+(A_1 \cup A_0)| \geq |M^+(C^+(\mathcal{L}_1))| \geq c\varepsilon n - a_1 = c\varepsilon n - a \geq c\varepsilon n - 2\alpha\varepsilon^2 n,$$

by (I). Since $\alpha \leq 1$ and $\varepsilon \rightarrow 0$, if n is large enough then it follows that $|M^+(A_1 \cup A_0)| \geq c\varepsilon n/2$, so (II) holds. \square

At this point the reader may forget the definition of \mathcal{A} ; we work with \mathcal{B} from now on.

Claim 11.4. For n sufficiently large, for any $s \in R$ and any $t \geq 2$ we have

$$\mathbb{P}(\mathcal{B} \cap \mathcal{U}_{\text{cx}} \mid L_1 = s, N_1 = t) \geq 1 - 4\eta, \tag{11.7}$$

$$\mathbb{P}(\mathcal{B}) \geq 1 - 5\eta \tag{11.8}$$

and

$$\mathbb{P}(\mathcal{B}) \geq 1/2. \tag{11.9}$$

Proof. For any $s \in R$ and $t \geq 2$, by Lemma 9.5,

$$\mathbb{P}(\mathcal{U}_{\text{cx}} \mid L_1 = s, N_1 = t) = 1 - o(1). \tag{11.10}$$

Since $\mathcal{A} \cap \mathcal{U}_{\text{cx}}$ implies \mathcal{B} , it follows from this and (11.6) that, if n is large enough, then (11.7) holds. In turn, we deduce that

$$\mathbb{P}(\mathcal{B}) \geq \mathbb{P}(\mathcal{B} \cap \mathcal{U}_{\text{cx}}) \geq (1 - 4\eta)\mathbb{P}(L_1 \in R, N_1 \geq 2).$$

Since $L_1 \in R$ w.h.p. (from (8.4)) and (by Theorem 4.1, say) $N_1 \geq 2$ w.h.p., it follows that $\mathbb{P}(\mathcal{B}) \geq 1 - 4\eta - o(1)$. Hence (11.8) holds for n large enough. Of course (11.9) (stated only for convenient reference later) follows, since $\eta \leq 1/100$. □

We now have the pieces in place to complete the proof of Lemma 11.1 and hence of Theorem 4.4.

Proof of Lemma 11.1. We start by revealing the following partial information about our random marked hypergraph $H = H'_{n,p}$. First reveal $C^+(H)$, and in particular which vertices are marked. Define C_1^+ , A_1 and A_0 as above, noting that these depend only on $C^+(H)$. Reveal $M^+(A_1 \cup A_0)$, the set of non-core vertices joined by paths to $A_1 \cup A_0$. Also (although this is not necessary), reveal all hyperedges outside $C^+(H) \cup M^+(A_1 \cup A_0)$. We write $\mathcal{F} = \mathcal{F}_n$ for the σ -algebra generated by the information revealed so far. Note that the event \mathcal{B} defined above is \mathcal{F} -measurable.

What have we not yet revealed? Let F be the subgraph of H induced by $V = A_1 \cup A_0 \cup M^+(A_1 \cup A_0)$ with any edges inside $A_1 \cup A_0$ removed (these removed edges are in $C^+(H)$). By Proposition 10.1 and the definition of $M^+(A_1 \cup A_0)$, the hypergraph F is an $(A_1 \cup A_0)$ -rooted r -forest on V . Moreover, replacing one such forest by another does not affect $C^+(H)$, or indeed any information revealed earlier. Thus, conditional on \mathcal{F} , the distribution of F is uniform over all $(A_1 \cup A_0)$ -rooted r -forests on V ; this uniform choice of the forest F is the only remaining randomness.

When \mathcal{B} holds, $|A_1| = |A_0| = a = \Theta(\varepsilon^2 n)$, while $m = e(F) = |M^+(A_1 \cup A_0)|/(r - 1) = \Theta(\varepsilon n)$. Since $\varepsilon = O(1)$ we have $m = \Omega(a)$. Also, $a^2/m = \Theta(\varepsilon^3 n) \rightarrow \infty$, so $m = o(a^2)$. Hence the conditions of Lemma 7.1 are satisfied. Let $Y_n = |M^+(A_1)|$ be the number of vertices in $V \setminus (A_1 \cup A_0)$ joined to A_1 (rather than to A_0). Since $m^{3/2}a^{-1} = \Theta(\sqrt{n/\varepsilon})$, Lemma 7.1 tells us that when \mathcal{B} holds and $x'_n - y'_n = o(\sqrt{n/\varepsilon})$, then

$$\mathbb{P}(Y_n = x'_n \mid \mathcal{F}) - \mathbb{P}(Y_n = y'_n \mid \mathcal{F}) = o(\sqrt{\varepsilon/n}) \tag{11.11}$$

and

$$\mathbb{P}(Y_n = y'_n \mid \mathcal{F}) = O(\sqrt{\varepsilon/n}). \tag{11.12}$$

Let \mathcal{C}^* be the component of $H = H_{n,p}^r$ containing \mathcal{C}_1^+ , and let $L^* = |\mathcal{C}^*|$ and N^* denote the order and nullity of \mathcal{C}^* . Since \mathcal{C}^* consists of \mathcal{C}_1^+ with a forest attached, N^* is also the nullity of \mathcal{C}_1^+ and so is an \mathcal{F} -measurable random variable. Let \mathcal{E} denote the event that $H_{n,p}^r \setminus \mathcal{L}_1$ has a complex component, so $\{N^* = t_n\} \subset \{N_1 = t_n\} \cup \mathcal{E}$. Theorem 2.5 implies that $\mathbb{P}(N_1 = t_n) = O((\varepsilon^3 n)^{-1/2})$. By the last part of Lemma 9.5, we have $\mathbb{P}(\mathcal{E}) = O(1/(\varepsilon^3 n))$, so

$$\mathbb{P}(N^* = t_n) \leq \mathbb{P}(N_1 = t_n) + \mathbb{P}(\mathcal{E}) = O((\varepsilon^3 n)^{-1/2}).$$

It follows from this and (11.9) that

$$\mathbb{P}(N^* = t_n \mid \mathcal{B}) \leq 2\mathbb{P}(N^* = t_n) = O((\varepsilon^3 n)^{-1/2}). \tag{11.13}$$

Given \mathcal{F} , the only uncertainly (*i.e.*, not-yet-revealed information) affecting L^* is which vertices of $M^+(A_1 \cup A_0)$ join to A_1 rather than to A_0 . Thus we may write L^* as $X_n + Y_n$ where X_n is \mathcal{F} -measurable and Y_n is defined as above. Hence, when \mathcal{B} holds,

$$\begin{aligned} \mathbb{P}(L^* = x_n \mid \mathcal{F}) - \mathbb{P}(L^* = y_n \mid \mathcal{F}) \\ = \mathbb{P}(Y_n = x_n - X_n \mid \mathcal{F}) - \mathbb{P}(Y_n = y_n - X_n \mid \mathcal{F}) = o(\sqrt{\varepsilon/n}), \end{aligned} \tag{11.14}$$

by (11.11) with $x'_n = x_n - X_n$ and $y'_n = y_n - X_n$. Taking the expectation⁶ over the \mathcal{F} -measurable event $\mathcal{B} \cap \{N^* = t_n\}$, it follows that

$$\begin{aligned} \mathbb{P}(L^* = x_n, N^* = t_n \mid \mathcal{B}) - \mathbb{P}(L^* = y_n, N^* = t_n \mid \mathcal{B}) \\ = o(\sqrt{\varepsilon/n} \mathbb{P}(N^* = t_n \mid \mathcal{B})) = o(1/(\varepsilon n)), \end{aligned} \tag{11.15}$$

where the last step is from (11.13). Similarly, from (11.12) and (11.13) we see that

$$\mathbb{P}(L^* = y_n, N^* = t_n \mid \mathcal{B}) = O(\sqrt{\varepsilon/n} \mathbb{P}(N^* = t_n \mid \mathcal{B})) = O(1/(\varepsilon n)). \tag{11.16}$$

It remains to remove the conditioning, and to replace L^* by L_1 .

Recall that when \mathcal{U}_{cx} holds, then $\mathcal{C}_1^+ = C^+(\mathcal{L}_1)$, so $\mathcal{C}^* = \mathcal{L}_1$, and hence $L_1 = L^*$ and $N_1 = N^*$. Let $s = s(n) \in \mathbb{R}$, and let $t = t(n) \geq 2$. If $(L^*, N^*) = (s, t)$ but $(L_1, N_1) \neq (s, t)$, then there is a component with s vertices and nullity t which is not the unique largest component. By Lemma 8.4 (in particular from (8.6)), we thus have

$$\mathbb{P}((L^*, N^*) = (s, t), (L_1, N_1) \neq (s, t)) = o(\mathbb{P}((L_1, N_1) = (s, t))).$$

Using (11.7) for the first inequality, and recalling that

$$\{L_1 = s, N_1 = t\} \cap \mathcal{B} \cap \mathcal{U}_{cx} = \{L^* = s, N^* = t\} \cap \mathcal{B} \cap \mathcal{U}_{cx},$$

⁶ Again, this requires a uniform bound, but we have that by considering the worst-case $\omega_n \in \mathcal{B}$ in (11.11) and (11.14).

we have

$$\begin{aligned}
 (1 - 4\eta)\mathbb{P}(L_1 = s, N_1 = t) &\leq \mathbb{P}(\{L_1 = s, N_1 = t\} \cap \mathcal{B} \cap \mathcal{U}_{\text{cx}}) \\
 &= \mathbb{P}(\{L^* = s, N^* = t\} \cap \mathcal{B} \cap \mathcal{U}_{\text{cx}}) \\
 &\leq \mathbb{P}(\{L^* = s, N^* = t\} \cap \mathcal{B}) \\
 &\leq \mathbb{P}(L^* = s, N^* = t) \\
 &\leq \mathbb{P}(L_1 = s, N_1 = t) \\
 &\quad + \mathbb{P}(L^* = s, N^* = t, (L_1, N_1) \neq (s, t)) \\
 &= (1 + o(1))\mathbb{P}(L_1 = s, N_1 = t).
 \end{aligned}$$

Hence, for n large,

$$\mathbb{P}(\{L^* = s, N^* = t\} \cap \mathcal{B}) = (1 \pm 4\eta)\mathbb{P}(L_1 = s, N_1 = t). \tag{11.17}$$

Relations (11.17) and (11.8) imply that

$$\mathbb{P}(L^* = s, N^* = t \mid \mathcal{B}) = \frac{\mathbb{P}(\{L^* = s, N^* = t\} \cap \mathcal{B})}{\mathbb{P}(\mathcal{B})} = (1 \pm 10\eta)\mathbb{P}(L_1 = s, N_1 = t), \tag{11.18}$$

since $0 < \eta < 1/50$. Applying (11.18) (backwards) with $s = x_n$ and $t = t_n$, then (11.15), then (11.18) with $s = y_n$ and $t = t_n$, we deduce that

$$\mathbb{P}(L_1 = x_n, N_1 = t_n) = (1 \pm 10\eta)^{-1}((1 \pm 10\eta)\mathbb{P}(L_1 = y_n, N_1 = t_n) + o(1/(\varepsilon n))).$$

Since $0 < \eta < 1/30$ this implies (11.3) for n large enough. Similarly, from (11.18) and (11.16) we deduce (11.2), completing the proof of Theorem 4.4. □

Finally, let us comment briefly on the proof of Theorem 2.4. The arguments in this section and the previous one can be modified to prove Theorem 2.4, by omitting all conditioning on N_1 , and replacing the quantity $1/(\varepsilon n)$ where it appears as the order of a point probability (for example in (11.15) and (11.16)) by $\sqrt{\varepsilon/n}$, which is (within a constant factor) the probability that L_1 takes a given typical value. At almost all points nothing needs to be added to the argument. Two exceptions are in the proof of Lemma 10.3, and that in place of (11.10) we need $\mathbb{P}(\mathcal{U}_{\text{cx}} \mid L_1 = s) = 1 - o(1)$. See the footnote to the proof of Lemma 10.3 for an argument covering both of these.

12. Proof of Theorem 1.1

In this section we shall deduce Theorem 1.1 from Theorem 2.3. The only additional result needed for this is Lemma 8.4; however, the formulae are rather messy and we will devote some space to calculations aimed at simplifying them.

Proof of Theorem 1.1. Let $r \geq 2$ be fixed, and suppose that $t = t(s) \rightarrow \infty$ as $s \rightarrow \infty$; our aim is to give an asymptotic formula for the number $C_r(s, t)$ of connected r -uniform hypergraphs on $[s]$ having nullity t . From (1.1) the number m of edges of any such hypergraph satisfies

$$m = \frac{s + t - 1}{r - 1}.$$

In particular, we must have $s + t$ congruent to 1 modulo $r - 1$ for $C_r(s, t)$ to be non-zero. We assume this from now on. We also assume that $t = o(s)$ and $t \rightarrow \infty$. More precisely, we fix a function $t = t(s)$ with these properties; we shall define a number of other quantities in terms of s and t . Except where otherwise specified, all limits and asymptotic notation then refer to $s \rightarrow \infty$.

The function $\Psi_r(x)$ defined in (1.2) is continuous on $(0, 1)$ and tends to 0 as $x \rightarrow 0$ and to infinity as $x \rightarrow 1$. Also, as mentioned in the Introduction, $\Psi_r(x)$ is strictly increasing on $(0, 1)$; hence, for s large enough that $t \geq 2$, the equation (1.4) has a unique positive solution $\rho = \rho(s)$. Expanding about $x = 0$ we see that

$$\Psi_r(x) = \frac{r-1}{12}x^2 + O(x^3),$$

uniformly in $0 < x \leq 1/2$, say. Thus

$$\rho \sim 2\sqrt{\frac{3}{r-1} \frac{t}{s}} \tag{12.1}$$

as $s \rightarrow \infty$.

Define

$$\rho_2 = \rho_2(s) = 1 - (1 - \rho)^{r-1} \tag{12.2}$$

and

$$\lambda = \lambda(s) = \frac{-\log(1 - \rho_2)}{\rho_2} = \frac{-(r-1)\log(1 - \rho)}{1 - (1 - \rho)^{r-1}}. \tag{12.3}$$

Note that $\lambda > 1$; comparing (12.2) and (12.3) with (2.1) and (2.2) we see that in the notation of the rest of the paper,

$$\rho_2 = \rho_\lambda = \rho_{2,\lambda} \quad \text{and} \quad \rho = \rho_{r,\lambda}.$$

As $s \rightarrow \infty$, from (12.1) we have $\rho = \rho(s) \rightarrow 0$. Thus, from (12.2), $\rho_2 \sim (r - 1)\rho$. Hence

$$\lambda = 1 + \rho_2/2 + O(\rho_2^2)$$

and

$$\varepsilon = \lambda - 1 \sim \frac{\rho_2}{2} \sim \frac{r-1}{2}\rho \sim \sqrt{3(r-1)} \frac{t}{s} \rightarrow 0. \tag{12.4}$$

Set

$$n = n(s) = \lfloor s/\rho \rfloor \quad \text{and} \quad p = p(s) = \lambda \frac{(r-2)!}{n^{r-1}}.$$

Since $\rho \rightarrow 0$ as $s \rightarrow \infty$, certainly $n \rightarrow \infty$ and $n \sim s/\rho$. Hence, from (12.4),

$$\varepsilon n \sim \frac{r-1}{2} s.$$

From (12.4) we also have $\varepsilon \rightarrow 0$. In addition,

$$\varepsilon^3 n = \varepsilon^2(\varepsilon n) = \Theta((t/s)s) = \Theta(t) \rightarrow \infty.$$

Hence our Standard Assumption 2.2 is satisfied, *i.e.*, we have the conditions needed to apply Theorem 2.3. (Of course, here we consider a sequence $(n(s), \varepsilon(s))_{s \geq 1}$ of values rather

than a sequence $(n, \varepsilon(n))_{n \geq 1}$. This causes no problems since we can pass to subsequences on which $n(s)$ is strictly increasing.)

We have chosen the parameters n and p so that the ‘typical’ order and nullity of the largest component of $H_{n,p}^r$ will be very close to s and t , respectively. More precisely, for the ‘typical’ number $\rho_{r,\lambda} n$ of vertices we have

$$\rho_{r,\lambda} n = \rho n = \rho \lfloor s/\rho \rfloor = s + O(\rho) = s + O(\varepsilon).$$

For the nullity, recalling (2.4) and (12.3) we see that the formula (1.2) defining Ψ_r may be written as

$$\Psi_r(\rho_{r,\lambda}) = \rho_{r,\lambda}^* / \rho_{r,\lambda}. \tag{12.5}$$

Indeed, this is how we arrived at this formula. Since $\rho_{r,\lambda} = \rho$ it follows using (1.4) that

$$\rho_{r,\lambda}^* n = \Psi_r(\rho) \rho n = \frac{t-1}{s} \rho n = t - 1 + O(\varepsilon^3) = t + O(1).$$

The standard deviations σ_n and σ_n^* appearing in Theorem 2.3 tend to infinity, so certainly we have $s = \rho_{r,\lambda} n + o(\sigma_n)$ and $t = \rho_{r,\lambda}^* n + o(\sigma_n^*)$. Hence, by Theorem 2.3, and in particular the formula (2.8) (with $a = b = 0$),

$$\mathbb{P}(L_1(H_{n,p}^r) = s, N_1(H_{n,p}^r) = t) \sim \frac{\sqrt{6} (r-1)^2}{8\pi \varepsilon n} \sim \frac{\sqrt{6} r - 1}{4\pi s}. \tag{12.6}$$

On the other hand, applying Lemma 8.4 with \mathcal{Q}_s the set of all r -uniform hypergraphs with s vertices and nullity t , writing $N_{s,t}$ for the number of components of $H_{n,p}^r$ with the property \mathcal{Q}_s , we have

$$\mathbb{P}(L_1(H_{n,p}^r) = s, N_1(H_{n,p}^r) = t) \sim \mathbb{E}[N_{s,t}]. \tag{12.7}$$

By linearity of expectation,

$$\mathbb{E}[N_{s,t}] = \binom{n}{s} C_r(s, t) p^m (1-p)^{\binom{n}{r} - \binom{n-s}{r} - m}. \tag{12.8}$$

Combining (12.6)–(12.8) we see that

$$C_r(s, t) \sim \frac{\sqrt{6} r - 1}{4\pi s} \binom{n}{s}^{-1} p^{-m} (1-p)^{-\left(\binom{n}{r} - \binom{n-s}{r} - m\right)}. \tag{12.9}$$

In the rest of this section we simplify this formula, in particular by showing that we can replace $n = \lfloor s/\rho \rfloor$ by s/ρ , for example.

Working in terms of n and ε (the more familiar parameters from the bulk of the paper), we have

$$s = \Theta(\varepsilon n), \quad t = \Theta(\varepsilon^3 n), \quad m = \Theta(\varepsilon n), \quad p = O(n^{-r+1}) = O(n^{-1}).$$

It follows immediately that $pm = o(1)$, so $(1-p)^m \sim 1$. Also,

$$r! \binom{n}{r} = n(n-1) \cdots (n-r+1) = n^r - \binom{r}{2} n^{r-1} + O(n^{r-2})$$

and

$$r! \binom{n-s}{r} = (n-s)^r - \binom{r}{2} (n-s)^{r-1} + O(n^{r-2}) = (n-s)^r - \binom{r}{2} n^{r-1} + O(sn^{r-2}).$$

Subtracting, we see that

$$\binom{n}{r} - \binom{n-s}{r} = \frac{n^r - (n-s)^r}{r!} + O(sn^{r-2}) = \frac{n^r - (n-s)^r}{r!} + o(n^{r-1}) = o(n^r).$$

Since $\log((1-p)^k) = -pk + O(p^2k)$, it follows easily that

$$\begin{aligned} a &= -\log((1-p)^{\binom{n}{r} - \binom{n-s}{r}}) = p \frac{n^r - (n-s)^r}{r!} + o(1) \\ &= \frac{\lambda s}{r(r-1)} (1 - (1-s/n)^r) = \frac{\lambda s}{r(r-1)} f(s/n), \end{aligned}$$

where $f(x) = x^{-1}(1 - (1-x)^r)$. Since $f'(x) = O(1)$ for $x = O(1)$ and $s/n - \rho = O(\varepsilon/n)$ we have $f(s/n) - f(\rho) = O(\varepsilon/n)$, so $sf(s/n) - sf(\rho) = O(\varepsilon^2) = o(1)$. Hence

$$a = \frac{\lambda s}{r(r-1)} f(\rho) + o(1).$$

From (1.2), (12.3) and (1.4) it follows that

$$a = s \frac{\Psi_r(\rho) + 1}{r-1} + o(1) = s \frac{(t-1)/s + 1}{r-1} + o(1) = \frac{s+t-1}{r-1} + o(1) = m + o(1).$$

From (12.9) we now obtain the formula

$$C_r(s, t) \sim \frac{\sqrt{6}}{4\pi} \frac{r-1}{s} e^m p^{-m} \binom{n}{s}^{-1}.$$

By Stirling's formula,

$$\binom{n}{s}^{-1} \sim \sqrt{2\pi} \sqrt{\frac{s(n-s)}{n}} \frac{s^s (n-s)^{n-s}}{n^n} \sim \sqrt{2\pi s} \left(\frac{s}{n}\right)^s \left(1 - \frac{s}{n}\right)^{n-s}.$$

Since $s = \rho n + O(\varepsilon)$, we have $s/n = \rho(1 + O(1/n))$. Also, $1 - s/n = (1 - \rho)(1 + O(\varepsilon/n))$, and it follows that

$$\binom{n}{s}^{-1} \sim \sqrt{2\pi s} \rho^s (1 - \rho)^{s(1-\rho)/\rho}.$$

Using again that $s/n = \rho(1 + O(1/n))$, and that $m = O(\varepsilon n) = o(n)$, we have

$$p^{-m} = \frac{n^{(r-1)m}}{\lambda^m (r-2)!^m} \sim \frac{s^{(r-1)m}}{\lambda^m (r-2)!^m \rho^{(r-1)m}}.$$

Next, we shall eliminate λ from this expression. From (1.4), (12.5) and (2.4) we have

$$\frac{(r-1)m}{s} = \frac{s+t-1}{s} = 1 + \Psi_r(\rho) = 1 + \frac{\rho_{r,\lambda}^*}{\rho} = \frac{\lambda}{r\rho} (1 - (1-\rho)^r).$$

Hence

$$\lambda^m = ((r-1)m/s)^m r^m \rho^m (1 - (1-\rho)^r)^{-m}.$$

Putting the pieces together we obtain the asymptotic formula

$$\begin{aligned}
 C_r(s, t) &\sim \frac{\sqrt{6}}{4\pi} \frac{r-1}{s} e^m \frac{s^{(r-1)m}}{\lambda^m (r-2)!^m \rho^{(r-1)m}} \sqrt{2\pi s} \rho^s (1-\rho)^{s(1-\rho)/\rho} \\
 &= \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{r-1}{\sqrt{s}} e^m \frac{s^{(r-1)m}}{\lambda^m (r-2)!^m \rho^{(r-1)m}} \rho^s (1-\rho)^{s(1-\rho)/\rho} \\
 &= \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{r-1}{\sqrt{s}} e^m \frac{(1-(1-\rho)^r)^m s^{(r-1)m}}{((r-1)m/s)^m r^m \rho^m (r-2)!^m \rho^{(r-1)m}} \rho^s (1-\rho)^{s(1-\rho)/\rho} \\
 &= \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{r-1}{\sqrt{s}} e^m \frac{(1-(1-\rho)^r)^m s^{rm}}{m^m r!^m \rho^{rm}} \rho^s (1-\rho)^{s(1-\rho)/\rho}, \tag{12.10}
 \end{aligned}$$

proving the main formula (1.3) of Theorem 1.1.

Turning to (1.5), let

$$N = \binom{s}{r} = \frac{s(s-1)\cdots(s-r+1)}{r!} = \frac{s^r}{r!} e^{-\binom{s}{2}/s + O(s^{-2})}.$$

Since $m \sim s/(r-1)$, it follows that

$$N^m \sim \frac{s^{rm}}{r!^m} e^{-\binom{s}{2}m/s} \sim \frac{s^{rm}}{r!^m} e^{-r/2}.$$

Since $N = \Theta(s^r)$, for $r \geq 3$ we have $m^2 = o(N)$, and it follows that

$$\binom{N}{m} = \frac{N(N-1)\cdots(N-m+1)}{m!} \sim \frac{N^m}{m!}.$$

On the other hand, if $r = 2$ then $m \sim s$ and $N \sim s^2/2$, so

$$\binom{N}{m} = \frac{N^m}{m!} e^{-\binom{m}{2}/N + o(1)} \sim e^{-1} \frac{N^m}{m!}.$$

We may write the last two formulae together as $\binom{N}{m} \sim e^{-\mathbb{1}_{r=2}} N^m/m!$, where $\mathbb{1}_A$ denotes the indicator function of A . Hence, using Stirling's formula, and recalling that $m = (s+t-1)/(r-1) \sim s/(r-1)$,

$$\binom{N}{m} \sim \frac{e^{-r/2-\mathbb{1}_{r=2}}}{\sqrt{2\pi m}} \frac{e^m s^{rm}}{m^m r!^m} \sim \frac{e^{-r/2-\mathbb{1}_{r=2}}}{\sqrt{2\pi s/(r-1)}} \frac{e^m s^{rm}}{m^m r!^m}.$$

From this and (12.10) we obtain the expression

$$\begin{aligned}
 P_r(s, t) &\sim e^{r/2+\mathbb{1}_{r=2}} \sqrt{2\pi s/(r-1)} \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{r-1}{\sqrt{s}} \frac{(1-(1-\rho)^r)^m}{\rho^{rm}} \rho^s (1-\rho)^{s(1-\rho)/\rho} \\
 &= e^{r/2+\mathbb{1}_{r=2}} \sqrt{\frac{3(r-1)}{2}} \left(\frac{1-(1-\rho)^r}{\rho^r} \right)^m (\rho(1-\rho)^{(1-\rho)/\rho})^s,
 \end{aligned}$$

completing the proof. □

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