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On the Number of Elements in Matroids with Small Circuits or Cocircuits

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It has been conjectured that a connected matroid with largest circuit size $c \geqslant 2$ and largest cocircuit size $c^* \geqslant 2$ has at most $\frac{1}{2}cc^*$ elements. Pou-Lin Wu has shown that this conjecture holds for graphic matroids. We prove two special cases of the conjecture, not restricted to graphic matroids, thereby providing the first nontrivial evidence that the conjecture is true for non-graphic matroids. Specifically, we prove the special case of the conjecture in which $c = 4$ or $c^* = 4$. We also prove the special case for binary matroids with $c = 5$ or $c^* = 5$.

1. Introduction

An active area of research in matroid theory is concerned with producing Ramsey theoretic results (see, for example, [1, 2, 3, 4, 5, 6, 8, 9]). The purpose of these results is to provide a deeper understanding of the structure of matroids by finding substructures that any sufficiently large matroid must contain. Circuits and cocircuits are basic substructures of matroids and hence Ramsey theoretic results that provide information on these sets are of interest.

The size of a largest circuit and cocircuit in a matroid M are denoted by c_M and ${c^*}_M$, respectively. The subscript M is often omitted. A fundamental open Ramsey question for matroids is whether the size of a connected matroid can be bounded above by a polynomial in c and c^*? It is believed that the answer to this question is 'yes' and the following specific conjecture about the size of M has been made [1].

Conjecture 1.1. *Connected matroids with largest circuit size $c \geqslant 2$ and largest cocircuit size $c^* \geqslant 2$ have at most $\frac{1}{2}cc^*$ elements.*

Bonin, McNulty and Reid [1, Theorem 1.1] showed that the conjecture holds for matroids containing a spanning circuit. Wu [9] showed that the conjecture holds for graphic matroids. For a non-graphic connected matroid M, the least upper bound in c and c^* known for the size of M is $\binom{c+c^*-2}{c-1}$, when $c, c^* \geqslant 3$ [8].

The validity of Conjecture 1.1 is difficult to establish even for specific small values of c and c^*. Specific small values for which the conjecture is known to be true are: $1 \leqslant c \leqslant 3$, $1 \leqslant c^* \leqslant 3$, or $(c, c^*) = (4, 4)$ [8]; $(c, c^*) = (4, 5)$ or $(c, c^*) = (5, 4)$ [4]; and $(c, c^*) = (5, 5)$ [1].

The two main results of the paper are given next. They provide the best evidence to date that Conjecture 1.1 holds for non-graphic matroids.

Theorem 1.2. *If M is a connected matroid with maximum circuit size $c = 4$ (or maximum cocircuit size $c^* = 4$), then $|E(M)| \leqslant \frac{1}{2}cc^*$.*

Theorem 1.3. *If M is a connected binary matroid with maximum circuit size $c = 5$ (or maximum cocircuit size $c^* = 5$), then $|E(M)| \leqslant \frac{1}{2}cc^*$.*

The proofs of these theorems are given in Section 2. The geometric structures obtained in these proofs support the plausibility of Conjecture 1.1. The increasing complexity of the cases obtained indicate that it is difficult to extend our proof technique to larger values of c and c^*. Section 1 concludes with terminology and results that are used in the proofs of the main theorems.

The terminology used here follows Oxley [7]. Let M be a matroid. Then $E(M)$ denotes the ground set of M. If $X \subseteq E(M)$, then $cl_M(X)$ and $r_M(X)$ denote the closure and rank of X in M, respectively. We denote $r_M(E(M))$ by $r(M)$. A matroid is *connected* if and only if, for each pair of distinct elements, there is a circuit containing both.

If X and Y are sets, then the *symmetric difference* of X and Y, denoted by $X \triangle Y$, is the set $(X - Y) \cup (Y - X)$. We use the following well-known fact about binary matroids.

Theorem 1.4. *A matroid M is binary if and only if the symmetric difference of any two distinct circuits is a disjoint union of circuits.* □

Suppose that $\mathscr{B} = \{C_1, C_2, \ldots, C_n\}$ is a collection of 4-element circuits of a matroid M such that there is a 2-element subset S of M that is contained in every member of $\mathscr{B}$. Then $\mathscr{B}$ is called a *book with spine* S if and only if $(C_{i+1} - S) \cap cl_M(C_1 \cup C_2 \cup \cdots \cup C_i) = \emptyset$ for all $i \in \{1, 2, \ldots, n-1\}$ (see, for example, Figure 1 with $S = \{a_0, b_0\}$ and $C_i = \{a_0, b_0, a_i, b_i\}$). The members of $\mathscr{B}$ are called the *pages* of the book. A 2-element independent set in M is called a *book with no pages*. As each page is added to a book, the rank of the book is raised by one. This implies that $r_M(\mathscr{B}) = r_M(\cup_{i=1}^{n} C_i) = n + 2$.

2. The proofs

The following geometric lemmas are used in the proofs of our main results. The first of these follows from basic properties of circuits (see Figure 2(a)).

Lemma 2.1. *If C is a maximum size circuit of a matroid M, then there is no triangle T of M such that $|C \cap T| = 1$ and $(T - C) \cap cl_M(C) = \emptyset$.* □

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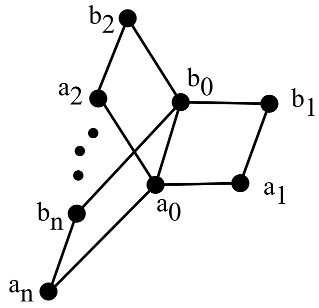

Figure 1

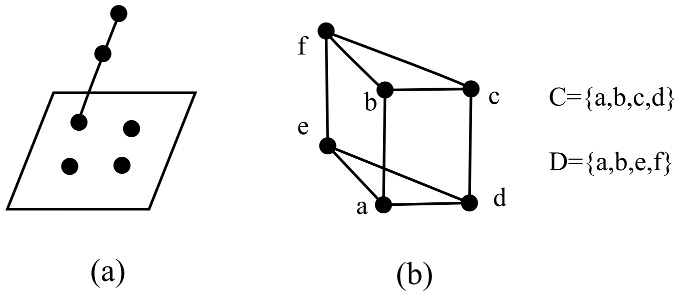

Figure 2

Lemma 2.2. *Let M be a connected matroid with $c_M = 4$. Let C be a 4-element circuit of M and $e \in E(M)\backslash cl_M(C)$. Then there is a circuit D containing e such that $\{C, D\}$ is a book. Moreover, if $\{C, D\}$ is a book, then $C \triangle D$ is a circuit of M.*

Proof. Let $C = \{a, b, c, d\}$. There is a circuit C' containing d and e because M is connected. There are at least two elements in each of C and C' not in the intersection of their closures, say a and b in C and e and f in C'. The 5-element set $\{a, b, d, e, f\}$ contains a circuit D. This circuit meets each of $C\backslash(cl_M(C) \cap cl_M(C'))$ and $C'\backslash(cl_M(C) \cap cl_M(C'))$ at least twice. Thus $D = \{a, b, e, f\}$ (see Figure 2(b)). Note that $\{C, D\}$ is a book. Moreover, the above argument may be applied to the pair C and D to obtain that $C \triangle D$ is a circuit of M. □

Suppose that $\mathscr{B}$ is a book in a connected matroid M as pictured in Figure 1. Suppose that M is binary. Then, for any fixed $i \in \{0, 1, \ldots, n\}$, $\{a_i, b_i\}$ is the spine of a book with pages $\{a_i, b_i, a_j, b_j\}$, $j \in \{0, 1, \ldots, n\}\backslash\{i\}$. Lemma 2.2 implies that this holds for non-binary matroids N with $c_N = 4$ as well. This symmetry, where the spine of the book $\mathscr{B}$ may be taken as any of the sets $\{a_i, b_i\}$, will be invoked in the proofs of our main results.

Lemma 2.3. *Let M be a connected matroid with C_1 a circuit of maximum size $c_M \in \{4, 5\}$. Suppose that C_2 and C_3 are 4-element circuits of M that meet C_1 twice such that $r_M(C_1 \cup C_2 \cup C_3) = r_M(C_1 \cup C_2) + 1 = r_M(C_1) + 2$. Then $C_1 \cap C_2$ and $C_1 \cap C_3$ either agree or are disjoint.*

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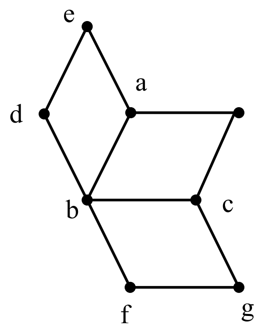

Figure 3

Proof. Suppose otherwise. Then $C_1 \cap C_2$ and $C_1 \cap C_3$ meet in exactly one element. Suppose that a, b, and c are distinct elements of C_1 such that $C_2 = \{a, b, d, e\}$ and $C_3 = \{b, c, f, g\}$ (see Figure 3).

Let $C = \{a, d, e, c, f, g\}$. Whether c_M is 4 or 5, it follows that $r(C) = 5$. However, every 5-element subset of C spans $cl_M(C)$, proving that all proper subsets of C are independent. This gives the contradiction that the 6-element set C is a circuit. □

Proof of Theorem 1.2. We may assume that $c_M = 4$ by duality. Hence $r(M) \geqslant 3$. If $r(M) = 3$, then M has a spanning circuit. Thus the result holds by [1, Theorem 1.1]. Hence we may assume that $r(M) \geqslant 4$.

We first argue that the theorem will follow if M contains two disjoint hyperplanes. Let X and Y be disjoint hyperplanes of M and $\mathscr{H} = \{X, Y\}$. Suppose that N denotes the number of pairs (H, e) such that $H \in \mathscr{H}$ and $e \in H$. Each member of $\mathscr{H}$ is a hyperplane containing at least $|E(M)| - c^*$ elements. Thus $N \geqslant 2 \cdot (|E(M)| - c^*)$. Each member of $E(M)$ is in at most one hyperplane of $\mathscr{H}$. Thus $N \leqslant |E(M)|$. It follows from combining these two inequalities that $|E(M)| \leqslant 2c^* = \frac{1}{2}cc^*$. The remainder of the proof is devoted to producing two disjoint hyperplanes in M.

Lemma 2.4. *There is a book $\mathscr{B}$ of M with $r(\mathscr{B}) = r(M)$.*

Proof. It follows from Lemma 2.2 that M contains a book with at least two pages. Suppose that $\mathscr{B}$ is a book of M of largest rank. Assume that $r(\mathscr{B}) < r(M)$. Then there exists $e \in E(M) \backslash cl_M(\mathscr{B})$. Let C and D be distinct pages of $\mathscr{B}$. By Lemma 2.2, there is a 4-element circuit P of M containing e such that $\{C, P\}$ is a book of M. By Lemma 2.3, the spines $C \cap D$ and $C \cap P$ of $\{C, D\}$ and $\{C, P\}$, respectively, are either identical or are disjoint. If the former occurs, then $\mathscr{B} \cup \{P\}$ is a book of M. Suppose the latter occurs. By Lemma 2.2, $C \triangle P$ is a 4-element circuit of M. Hence $\mathscr{B} \cup \{C \triangle P\}$ is a book of M. In either case we obtain the contradiction that there exists a book of M with larger rank than $\mathscr{B}$. Thus $r(\mathscr{B}) = r(M)$. □

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Let $\mathscr{B} = \{C_1, C_2, \ldots, C_{r(M)-2}\}$ be a book of M of rank $r(M)$. Suppose that $\{a_0, b_0\}$ is the spine of $\mathscr{B}$ and $C_i - \{a_0, b_0\} = \{a_i, b_i\}$ for $i \in \{1, 2, \ldots, r(M) - 2\}$ (see Figure 1). Let $A = \{a_0, a_1, \ldots, a_{r(M)-2}\}$ and $B = \{b_0, b_1, \ldots, b_{r(M)-2}\}$.

Lemma 2.5. *Let $C \subseteq A \cup B$. Then C is a circuit of M if and only if $C = \{a_i, a_j, b_i, b_j\}$ for distinct $i, j \in \{0, 1, \ldots, r(M) - 2\}$.*

Proof. Any set $C = \{a_i, a_j, b_i, b_j\}$ for distinct $i, j \in \{0, 1, \ldots, r(M) - 2\}$ is a circuit by the construction of the book and Lemma 2.2. Suppose T is a subset of $A \cup B$ that contains no set of the form $C = \{a_i, a_j, b_i, b_j\}$ for i and j distinct. We complete the proof by showing that T is independent. This is accomplished by inducting on the number of pages in the book. The result is easily established if the book has two pages. Suppose the number of pages exceeds two and the result holds for books with fewer pages. There is an $i \in \{0, 1, \ldots, r(M) - 2\}$ such that T meets $\{a_i, b_i\}$ at most once. If T does not meet $\{a_i, b_i\}$, then the result follows by induction. If T meets $\{a_i, b_i\}$ in a single element e, then $T - e$ is independent by the induction hypothesis. But e is not in the closure of $T - e$ by the construction of the book. Thus T is independent. □

Lemma 2.6. $cl_M(A) \cap cl_M(B) = \emptyset$.

Proof. Suppose e is in the closure of both A and B. Let C_A and C_B be subsets of A and B, respectively, such that $C_A \cup e$ and $C_B \cup e$ are circuits of M. By circuit elimination and Lemma 2.5, there are distinct i and j such that $\{a_i, a_j, b_i, b_j\} \subseteq C_A \cup C_B$. In particular, both C_A and C_B have at least two elements. Lemma 2.1 implies that C_A and C_B do not have cardinality two. Thus $c_M \leqslant 4$ implies that both of these sets contain exactly three elements. Let $k, l \in \{0, 1, \ldots, r(M) - 2\}$ be such that $C_A = \{a_i, a_j, a_k\}$ and $C_B = \{b_i, b_j, b_l\}$, where $k = l$ is possible. Then the circuits $\{a_i, a_j, a_k, e\}$ and $\{a_i, a_j, b_i, b_j\}$ form the pages of a book so that Lemma 2.2 implies that $\{a_k, b_i, b_j, e\}$ is a circuit of M. By circuit elimination, there is a circuit contained in $(\{a_k, b_i, b_j, e\} \cup \{b_i, b_j, b_l, e\}) - \{e\} = \{a_k, b_i, b_j, b_l\}$. But this contradicts that B is independent and $a_k \notin cl_M(B)$. □

We have shown that M contains disjoint hyperplanes $cl_M(A)$ and $cl_M(B)$. This completes the proof of Theorem 1.2 by the remarks made near the beginning of the proof. □

Proof of Theorem 1.3. We may assume that $c_M = 5$ by duality. Let Z be a 5-element circuit of M. If Z is a spanning circuit of M, then the result holds by [1, Theorem 1.1]. Assume that Z is not a spanning circuit of M. Hence $r(M) \geqslant 5$.

This proof is similar in structure to the proof of Theorem 1.2. Here we show that there is a set $\mathscr{H}$ of five hyperplanes of M such that each element of M is in at most three of the hyperplanes. As before, N denotes the number of pairs (H, e) such that $H \in \mathscr{H}$ and $e \in H$. Each hyperplane of $\mathscr{H}$ contains at least $|E(M)| - c^*$ elements. Thus $N \geqslant |\mathscr{H}| \cdot (|E(M)| - c^*) = 5 \cdot (|E(M)| - c^*)$. Each element of $E(M)$ is in at most three hyperplanes of $\mathscr{H}$. Thus $N \leqslant 3|E(M)|$. Hence $3|E(M)| \geqslant 5 \cdot (|E(M)| - c^*)$ implies that $|E(M)| \leqslant \frac{5}{2}c^* = \frac{1}{2}cc^*$. The remainder of the proof is devoted to constructing $\mathscr{H}$.

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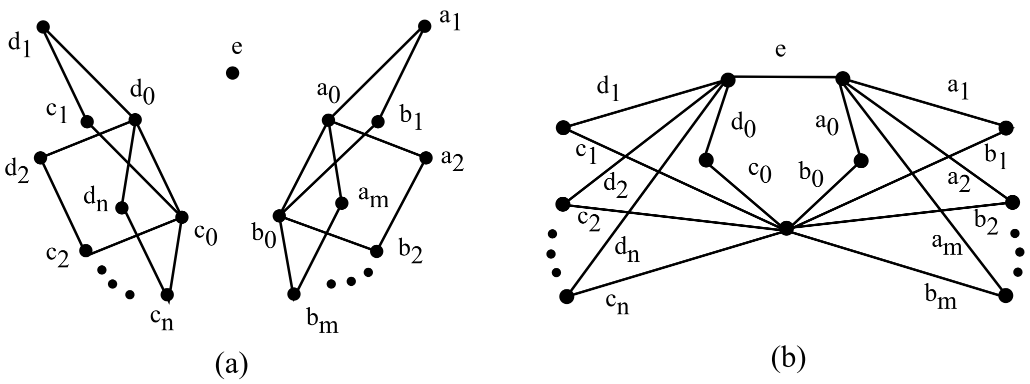

Figure 4

We next present several lemmas that specify geometric substructures of M.

Lemma 2.7. *If $f \in E(M)\backslash cl_M(Z)$, then there is a 4-element circuit of M containing f that meets Z twice.*

Proof. It follows from M being connected that there is a circuit D of M containing f that meets Z. Let D be such a circuit with $|D - Z|$ minimum. Then, as $f \notin cl_M(Z)$, $|D - Z| \geqslant 2$.

Suppose $|D - Z| \geqslant 3$. Then $|D \cap Z| \leqslant 2$ implies that $|D \triangle Z| \geqslant 6$. By Theorem 1.4, $D \triangle Z$ is the disjoint union of at least two circuits. Each circuit contained in $D \triangle Z$ meets both $D - Z$ and $Z - D$ as both of these sets are independent. Thus there is a circuit $X \subseteq (D \triangle Z)$ that contains f such that $|X - Z| < |D - Z|$. This contradicts the choice of D. Thus $|D - Z| = 2$.

It follows from Lemma 2.1 that $|D \cap Z| \neq 1$. If $|D \cap Z| = 2$, then D has the required properties. If $|D \cap Z| = 3$, then Theorem 1.4 implies that $D \triangle Z$ is a 4-element circuit of M having the required properties. □

The main lemma used in the proof of Theorem 1.3 is given next. The geometric structure obtained in this lemma is depicted in Figure 4(a). This set is spanning in M. The 5-element set $Z = \{a_0, b_0, c_0, d_0, e\}$ is the circuit mentioned near the beginning of the proof of Theorem 1.3. These five elements are drawn as coplanar in Figure 4(a) in order to simplify the drawing even though they are not coplanar in M. When the elements of Figure 4(a) are mentioned as members of circuits and hyperplanes of M in the remainder of the proof of Theorem 1.3, the reader may find it helpful to view these elements as edges in the graph of Figure 4(b) even though the matroid M may not be graphic. For example, the edge set consisting of those edges labelled b_i, $0 \leqslant i \leqslant m$ together with edge e forms a bond in the graph of Figure 4(b). This illustrates that the complementary edges in that graph, those labelled by an a, c, or d with a subscript, can be shown to span a hyperplane in the matroid M.

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Lemma 2.8. *There are disjoint books $\mathscr{B}_1$ and $\mathscr{B}_2$ of M with spines being subsets of Z and $r_M(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z) = r(M)$ satisfying the following independence condition. If P is a page of $\mathscr{B}_1 \cup \mathscr{B}_2$, then the two elements of $P \backslash Z$ are not in $cl_M(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z) \backslash (P \backslash Z)$.*

Proof. By Lemma 2.7, there are books $\mathscr{B}_1$ and $\mathscr{B}_2$ satisfying all of the conditions of the present lemma except perhaps the condition $r(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z) = r(M)$. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be books satisfying these conditions with the rank of $\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z$ maximum.

Suppose that $r(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z) < r(M)$. Hence there exists $f \in E(M) \backslash cl_M(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z)$. By Lemma 2.7, there is a 4-element circuit P of M that meets Z twice. It follows from Lemma 2.3 (possibly by relabelling the elements of Z if one of the two books has no pages), that $P \cap Z$ is the spine of $\mathscr{B}_1$ or $\mathscr{B}_2$. Suppose the former by symmetry. Then $\mathscr{B}_1 \cup P$ and $\mathscr{B}_2$ are a pair of books satisfying the independence condition stated in the lemma and the rank of these sets together with Z exceeds the rank of $\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z$. This contradicts the choice of $\mathscr{B}_1$ and $\mathscr{B}_2$. Hence $r_M(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z) = r(M)$. □

It follows from the above lemma that we may assume that the elements of Z are labelled by a_0, b_0, c_0, d_0, and e so that there are books $\mathscr{B}_1$ and $\mathscr{B}_2$ with pages $\{a_0, b_0, a_i, b_i\}$, $1 \leqslant i \leqslant m$, and $\{c_0, d_0, c_j, d_j\}$, $1 \leqslant j \leqslant n$, respectively, with $r(\mathscr{B}_1 \cup \mathscr{B}_2 \cup Z) = r(M)$ and the pages of the books satisfy the independence condition given in the statement of the lemma (see Figure 4). Let $A = \{a_0, a_1, \ldots, a_m\}$, $B = \{b_0, b_1, \ldots, b_m\}$, $C = \{c_0, c_1, \ldots, c_n\}$, and $D = \{d_0, d_1, \ldots, d_n\}$. Let $m = 0$ or $n = 0$ as appropriate if $\mathscr{B}_1$ or $\mathscr{B}_2$ have no pages. At least one of these books will contain pages by Lemma 2.7.

Lemma 2.9. *Let $X \subseteq A \cup B \cup C \cup D \cup e$. Then X is a circuit of M if and only if X is a page of $\mathscr{B}_1$ or $\mathscr{B}_2$, X is the symmetric difference of two pages of one of these books, or $X = \{a_i, b_i, c_j, d_j, e\}$ for $i \in \{0, 1, \ldots, m\}$ and $j \in \{0, 1, \ldots, n\}$.*

Proof. The pages of $\mathscr{B}_1$ and $\mathscr{B}_2$ are circuits by definition. The symmetric difference of two pages of one of these books is a circuit by Lemma 2.2. Suppose that $i \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$, and $X = \{a_i, b_i, c_j, d_j, e\}$. Suppose that both i and j are 0. Then $X = Z$, so X is a circuit of M. Suppose that exactly one of i and j, say i, is 0. It follows from Theorem 1.4 and $X = \{c_0, d_0, c_j, d_j\} \triangle Z$ that X is a circuit of M. Suppose that neither i nor j is 0. Then Theorem 1.4 and $X = \{a_0, b_0, c_j, d_j, e\} \triangle \{a_0, b_0, a_i, b_i\}$ imply that X is a circuit of M. Hence Figure 4 is symmetric in the sense that any set $\{a_i, b_i\}$ may be swapped for $\{a_0, b_0\}$ as the spine of $\mathscr{B}_1$. Likewise, any set $\{c_j, d_j\}$ may be taken to be the spine of $\mathscr{B}_2$.

Suppose that S is a subset of $A \cup B \cup C \cup D \cup e$ that contains no set of the form listed in the statement of the lemma. We complete the proof by showing that S is independent; we do this by induction on the total number of pages in the two books. If there are no pages between these two books, then the result is straightforward. Suppose $m + n \geqslant 1$ and the result holds for sets as in Figure 4(a) containing fewer than $m + n$ pages in the two books. Since $m + n \geqslant 1$ and S does not contain a page of either book, it follows that either some intersection $S \cap \{a_i, b_i\}$ or $S \cap \{c_j, d_j\}$ is empty or one of these intersections is a singleton. If such an intersection is empty, then S is independent by the induction

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hypothesis. If some intersection $S \cap \{a_i, b_i\}$ or $S \cap \{c_j, d_j\}$ is a singleton f, then $S - f$ is independent by the induction hypothesis, and so S is independent by our construction. □

Let H_1, H_2, H_3, H_4, and H_5 be the five hyperplanes of M determined by the subsets $A \cup C \cup D$, $A \cup B \cup D$, $B \cup C \cup e$, $A \cup C \cup e$, and $B \cup D \cup e$, respectively. These five hyperplanes are pairwise distinct, as otherwise a spanning set for $A \cup B \cup C \cup D$ with fewer than $r(M) - 4$ elements could be constructed. Let $\mathscr{H} = \{H_1, H_2, H_3, H_4, H_5\}$.

Lemma 2.10. *Each element f of M is in at most three hyperplanes of $\mathscr{H}$.*

Proof. Suppose that f is in at least four hyperplanes in $\mathscr{H}$. Each element of $A \cup B \cup C \cup D$ is in exactly 3 hyperplanes in $\mathscr{H}$. This fact follows as otherwise, again, a spanning set for $A \cup B \cup C \cup D$ with fewer than $r(M) - 4$ elements could be constructed. Thus, f is neither in $A \cup B \cup C \cup D$ nor is it in parallel with a member of this set.

Suppose that $f \in cl_M(A)$. Then there is a circuit X containing f with at most five elements such that $X \subseteq A \cup f$ and $|A \cap X| \geqslant 2$. Assume $|A \cap X| = 2$, say $A \cap X = \{a_i, a_j\}$ for distinct i and j. Then we obtain a contradiction to Lemma 2.1 by considering the circuits $\{a_i, a_j, f\}$ and $\{a_i, b_i, c_i, d_i, e\}$. Thus $|A \cap X| \geqslant 3$.

Suppose $|A \cap X| = 3$, say $A \cap X = \{a_i, a_j, a_k\}$ for distinct i, j, and k. By Theorem 1.4, $\{a_i, a_j, a_k, f\} \triangle \{a_i, b_i, c_i, d_i, e\} = \{a_j, a_k, b_i, c_i, d_i, e, f\}$ is a disjoint union of circuits $Y_1, Y_2, \ldots, Y_p$. The maximum circuit size of M being five implies that $p \geqslant 2$. We may assume that $e \in Y_1$. The set $\{a_j, a_k, b_i, c_i, d_i\}$ is independent by Lemma 2.9. It follows from e not being in the closure of $\{a_j, a_k, b_i, c_i, d_i\}$ that $f \in Y_1$. We obtain the contradiction that the circuit Y_2 is a subset of the independent set $\{a_j, a_k, b_i, c_i, d_i\}$. Likewise, we obtain a contradiction if $|A \cap X| > 3$. Thus $f \notin cl_M(A)$. By a similar argument, $f \notin cl_M(B) \cup cl_M(C) \cup cl_M(D)$.

Suppose that $f \in H_1 \cap H_2$. The rank $r(M) - 2$ set $A \cup D$ spans $H_1 \cap H_2$. Thus there is a circuit X containing f with at most five elements such that $X \subseteq A \cup D \cup f$. Then $f \notin cl_M(A)$ and $f \notin cl_M(D)$ implies that X meets both A and D. Thus $|X| \geqslant 3$. Suppose $|X| = 3$, say $A \cap X = \{a_i\}$ and $D \cap X = \{d_j\}$. Suppose that $i = j$. By Lemma 2.9 and the fact that m and n are not both 0, we may choose $k \neq i$. It follows that $\{a_i, b_i, c_k, d_k, e\}$ is a circuit of M. But then $|\{a_i, d_j, f\} \triangle \{a_i, b_i, c_k, d_k, e\}| > 5$ and we obtain a contradiction by arguing as in the previous paragraph. Thus $i \neq j$. Hence we obtain a contradiction as above by considering the symmetric differences of the circuits $\{a_i, d_j, f\}$ and $\{a_i, b_i, c_i, d_i, e\}$. Thus $|X| > 3$ and proceeding as in the $|X| = 3$ case, we obtain a contradiction. Thus $f \notin H_1 \cap H_2$. But $A \cup C$ spans $H_1 \cap H_4$ and $B \cup D$ spans $H_2 \cap H_5$. Hence a similar argument to that above may be employed to show that $f \notin H_1 \cap H_4$ and $f \notin H_2 \cap H_5$. But f not being in any of $H_1 \cap H_2$, $H_1 \cap H_4$, and $H_2 \cap H_5$ contradicts that f is in at least four members of $\mathscr{H}$. □

It follows from Lemma 2.10 that $\mathscr{H}$ is a set of five hyperplanes such that each element of M is in at most three of the hyperplanes. This completes the proof of Theorem 1.3 by the remarks made at the beginning of the proof. □

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References

[1] Bonin, J., McNulty, J. and Reid, T. J. The matroid Ramsey number $n(6,6)$. *Combinatorics, Probability and Computing* **8** 229–235.

[2] Ding, G., Oporowski, B., Oxley, J. and Vertigan, D. (1996) Unavoidable minors of large 3-connected binary matroids. *J. Combin. Theory Ser. B* **66** 334–360.

[3] Ding, G., Oporowski, B., Oxley, J. and Vertigan, D. Unavoidable minors of large 3-connected matroids. *J. Combin. Theory Ser. B*, to appear.

[4] Hurst, F. and Reid, T. J. (1995) Some small circuit–cocircuit Ramsey numbers for matroids. *Combinatorics, Probability and Computing* **4** 67–80.

[5] Hurst, F. and Reid, T. J. (1997) Ramsey numbers for cocircuits in matroids. *Ars Combinatoria* **45** 181–192.

[6] Oporowski, B., Oxley, J. and Thomas, R. (1993) Typical subgraphs of 3- and 4-connected graphs. *J. Combin. Theory Ser. B* **57** 239–257.

[7] Oxley, J. (1992) *Matroid Theory*, Oxford University Press, New York.

[8] Reid, T. J. (1997) Ramsey numbers for matroids. *Europ. J. Combin.* **18** 589–595.

[9] Wu, P.-L. (1997) An upper bound on the number of edges of a 2-connected graph. *Combinatorics, Probability and Computing* **6** 107–113.