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Completing a $(k-1)$-Assignment

SVANTE LINUSSON[1*] and JOHAN WÄSTLUND[2†]

[1]Department of Mathematics, KTH, SE-100 44 Stockholm, Sweden
(e-mail: linusson@math.kth.se)

[2]Department of Mathematics, Linköping University, 581 83 Linköping, Sweden
(e-mail: jowas@mai.liu.se)

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We consider the distribution of the value of the optimal k-assignment in an $m \times n$ matrix, where the entries are independent exponential random variables with arbitrary rates. We give closed formulas for both the Laplace transform of this random variable and for its expected value under the condition that there is a zero-cost $(k-1)$-assignment.

1. Introduction

Let M be an $m \times n$ matrix. A *k-assignment* is a set of k matrix entries of which no two are in the same row or the same column. The *value* of a k-assignment is the sum of its entries. A k-assignment is *optimal* if its value is no larger than the value of any other k-assignment. If the entries of the matrix M are random variables then so is the value of the optimal k-assignment, here denoted by $\min_k(M)$.

The study of the optimal k-assignment has been pursued by researchers from different fields and with different random variables as entries in M. The main focus has been to estimate the size of the expected value of $\min_k(M)$. For references and more details on the history see [5] or [9].

In 1998 Giorgio Parisi [12] conjectured that if M is a $k \times k$ matrix with independent exponential random variables of rate 1, then the expected value of the optimal k-assignment is

$$\mathrm{E}(\min_k(M)) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}.$$

Two quite different proofs of this conjecture were announced simultaneously in March 2003 [9, 11]. Meanwhile David Aldous had proved that the limit as $k \to \infty$ is $\zeta(2) = \pi^2/6$ [1, 2]. The beautiful conjecture of Parisi inspired much work on exact formulas and many

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different generalizations were studied [3, 4, 5, 6, 7, 8, 14]. In [9] a formula for the expected value is given when the matrix entries are exp(1) or 0.

In this note we investigate the problem from a different extreme. We allow the rates of the exponential random variables to be arbitrary positive numbers. We include infinity as a possible rate, which corresponds to the entry being constant zero. We prove exact formulas for $\mathrm{E}(\min_k(M))$ and for the Laplace transform $L(\min_k(M))$ assuming that $\mathrm{E}(\min_{k-1}(M)) = 0$. Formulas for completing a zero-cost $(k-1)$-assignment were considered previously in [8, 6] in the case when all rates are equal to 1. The Laplace transform for some special cases when the rates are all equal to 1 have been determined in [3].

We first prove the slightly easier result on expected value and then compute the Laplace transform for the entire distribution. In theory one should be able to deduce the first result from the second but it seems easier to compute them separately.

2. Preliminaries

As in [8, 9], the concept of row and column covers of zeros will be important. The set of zeros in the matrix M will be denoted by Z and we let $k-1$ be the maximal size of an assignment consisting of positions in Z. We consider sets of rows and columns in the $m \times n$ matrix. A set α of rows and columns is said to *cover* Z if every entry in Z is either in a row or in a column in α. A cover with $k-1$ rows and columns is called a $(k-1)$-cover. For many readers it might be convenient to translate the matrix to a bipartite graph with the two vertex sets corresponding to rows and columns respectively, and the edges corresponding to matrix positions. In that setting our covers are so-called vertex covers. By the König–Egervary theorem (see for instance [10]), the maximal size of an assignment consisting of positions in Z is the same as the minimal size of a cover, in our case $k-1$.

If α is a set of rows and columns, then the *rectangle* $R(\alpha)$ is the submatrix not covered by α. If α is a $(k-1)$-cover of the zeros then the corresponding rectangle $R(\alpha)$ is called *critical*. Let $Q(k,Z)$ be the set of all critical rectangles in a matrix M with zeros in Z. We define a partial ordering on $Q(k,Z)$ as follows. For $R(\alpha), R(\beta) \in Q(k,Z)$, let $R(\alpha) \leqslant R(\beta)$ if the set of columns in α is a subset of the set of columns in β, or equivalently the set of rows in α is a superset of the set of rows in β (two other equivalent statements are that the set of rows in $R(\alpha)$ is a subset of the set of rows in $R(\beta)$, and that the set of columns in $R(\alpha)$ is a superset of the set of columns in $R(\beta)$).

A random variable X is said to be *exponential of rate* a, which we write $X \sim \exp(a)$, if $P(X > t) = e^{-at}$ for $t \geqslant 0$. In this case $\mathrm{E}(X) = 1/a$. If $X_1, \ldots, X_n$ are independent and $X_i \sim \exp(a_i)$ then

$$\min_i X_i \sim \exp\left(\sum_i a_i\right).$$

Let the *rate* $I(R)$ of a rectangle R be the sum of the rates of the individual entries.

The following lemma is well known: see, *e.g.*, [10].

Lemma 2.1. *Suppose that the set Z of zeros of M contains a $(k-1)$-assignment, but no k-assignment. Then $Q(k,Z)$ is a lattice. In particular, it has unique maximal and minimal elements.*

The least upper bound of two rectangles is obtained by taking the union of row-sets and intersection of column-sets. Conversely the greatest lower bound is obtained by taking the intersection of row-sets and union of column-sets.

The following is Theorem 2.9 of [8].

Lemma 2.2. *Suppose that a row r belongs to every $(k-1)$-cover of the zeros Z in the matrix M. Then every optimal k-assignment contains a zero from row r. This means that we can remove row r from M and obtain a matrix M^r with the property that* $\min_k(M) = \min_{k-1}(M^r)$.

3. The incidence algebra

The *incidence algebra* over a poset P is the algebra of functions defined on the intervals of P, that is, on pairs $x, y \in P$ such that $x \leqslant y$, and taking values in a given field, in our case the real numbers (see [13]). The elements of the incidence algebra are multiplied by convolution. One way to define this is to let a function f be represented by a matrix F with rows and columns indexed by the elements of P. The entry in position (x, y) of F is $f(x, y)$ if $x \leqslant y$ and 0 otherwise. If the rows and columns are arranged according to a linear extension of P, then every matrix in the incidence algebra is upper triangular. Multiplication of functions corresponds to matrix multiplication, and it is straightforward to check that the incidence algebra is closed under multiplication. The identity element of the incidence algebra is the function that assigns the value 1 to single-element intervals, that is, $f(x,x) = 1$, and we obtain the inverse of f by inverting the matrix F. Hence, in order for f to be invertible in the incidence algebra, the diagonal elements of F have to be nonzero, *i.e.*, $f(x,x) \neq 0$ for every $x \in P$. It follows from Lemma 3.1 below that the inverse of F actually belongs to the incidence algebra whenever F is invertible, that is, $F^{-1}(x, y) = 0$ unless $x \leqslant y$.

In principle the following lemma is well known, but we include a proof since we have not found it stated in precisely this form elsewhere.

Lemma 3.1. *Let f be a function in the incidence algebra over a poset P and let $x \leqslant y \in P$ be arbitrary elements. The inverse of f is determined by*

$$f^{-1}(x,y) = \sum_{x=z_0<z_1<\cdots<z_s=y} (-1)^s \frac{f(z_0,z_1)f(z_1,z_2)\cdots f(z_{s-1},z_s)}{f(z_0,z_0)f(z_1,z_1)\cdots f(z_s,z_s)},$$

where the sum is taken over all chains in the interval $[x, y]$ beginning with x and ending with y.

Proof. Let F be the upper triangular matrix corresponding to f as described in Section 2. Let D be the diagonal matrix with the values of $f(z,z)$ on the diagonal and zeros elsewhere. Let N be the nilpotent matrix that has zeros on the diagonal and agrees with F at all other positions. We then have

$$F = D + N.$$

The equation

$$F^{-1} = D^{-1} - D^{-1}ND^{-1} + D^{-1}ND^{-1}ND^{-1} - \cdots \tag{3.1}$$

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can be verified by multiplying the right-hand side by $D+N$. The matrix $D^{-1}N$ has zeros on and below the diagonal, and is therefore nilpotent. Hence the right-hand side of (3.1) only has a finite number of terms. For each chain $x = z_0 < z_1 < \cdots < z_s = y$ we get a contribution to the entry $F^{-1}(x, y)$ from

$$D^{-1}(z_0, z_0)N(z_0, z_1)D^{-1}(z_1, z_1)N(z_1, z_2)\cdots D^{-1}(z_s, z_s).$$

Here $N(z_i, z_{i+1}) = F(z_i, z_{i+1})$, and $D^{-1}(z_i, z_i) = 1/F(z_i, z_i)$. Since inverting f in the incidence algebra is the same thing as inverting the matrix F, the lemma follows. □

4. The expected value

We define a function f in the incidence algebra over $Q(k, Z)$ by

$$f(R(\alpha), R(\beta)) = I(R(\alpha) \cap R(\beta)).$$

Since $I(R(\alpha)) > 0$ for all critical rectangles, f is invertible in the incidence algebra.

Theorem 4.1. *Let M be an $m \times n$ matrix whose entries are either zero or independent exponential random variables of arbitrary positive rates. Let Z be the set of positions of the zeros. Suppose that Z contains a $(k-1)$-assignment, but no k-assignment. Then*

$$\mathrm{E}(\min_k(M)) = \sum_{R(\alpha)\leqslant R(\beta)} f^{-1}(R(\alpha), R(\beta)),$$

where the sum is over all intervals in $Q(k, Z)$ and f^{-1} is the inverse of f in the incidence algebra. Equivalently we can write

$$\mathrm{E}(\min_k(M)) = \sum_{R_0<R_1<\cdots<R_s} (-1)^s \frac{I(R_0 \cap R_1)I(R_1 \cap R_2)\cdots I(R_{s-1} \cap R_s)}{I(R_0)I(R_1)\cdots I(R_s)}, \tag{4.1}$$

where the sum is taken over all nonempty chains in $Q(k, Z)$, and in particular over all $s \geqslant 0$.

Remark 1. The second formula runs over all chains in $Q(k, Z)$, a set which in the worst case has size of order $k!$. The first formula is computationally an improvement for large k. It involves taking the inverse of a matrix indexed by the elements of $Q(k, Z)$ whose size is exponential in k in the worst case.

Proof. The equivalence of the two formulas follows from Lemma 3.1. The proof is by induction over k and we prove (4.1). The theorem is certainly true for $k = 1$. Without loss of generality we may assume that the entries $(1, 1), (2, 2), \ldots, (k-1, k-1)$ of M are zero. We may also assume that the maximal rectangle $R(\gamma)$ in $Q(k, Z)$ corresponds to the cover γ consisting of columns $1, \ldots, k-1$. If this is not the case, then by Lemma 2.1 there is a row i that belongs to every $(k-1)$-cover. By Lemma 2.2 this implies that $\mathrm{E}(\min_k(M)) = \mathrm{E}(\min_{k-1}(M^i))$, where M^i is obtained from M by deleting row i. Since $Q(k, Z) = Q(k-1, Z^i)$, where Z^i are the zeros of M^i, the result is clear by induction.

We now use the same recursion procedure as in [3] and [8] corresponding to the $(k-1)$-cover γ consisting of the first $k-1$ columns. This is based on the so-called Hungarian

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algorithm for computing the optimal assignment [10]. Let X be the minimum of all the matrix entries in $R(\gamma)$. By Lemma 2.2 of [8], exactly one entry in $R(\gamma)$ will belong to an optimal k-assignment. Moreover, $\mathrm{E}(X) = 1/I(R(\gamma))$.

We subtract X from all entries in $R(\gamma)$, and a new zero will occur at the position of the minimum. All other entries are unchanged in distribution by the memorylessness of the exponential distribution. Let $M_{i,j}$ be the matrix obtained when position (i, j) in M has been replaced with a zero.

Let K_i be the intersection of $R(\gamma)$ and row i. If $i > k-1$, then there is a zero-cost k-assignment in $M_{i,j}$. If $1 \leqslant i \leqslant k-1$, then row i has to be in every $(k-1)$-cover of $M_{i,j}$ and as above we can remove row i and this case is done by induction. Again let M^i denote the matrix with row i removed from M. Also let γ^i be the maximal $(k-1)$-cover of M which contains row i. This means that γ^i consists of row i together with the rows and columns of the maximal $(k-2)$-cover of M^i. The poset $Q(k-1, Z^i)$ is equal to the induced subposet of $Q(k, Z)$ of elements $\leqslant R(\gamma^i)$.

The probability that the zero occurs in K_i is

$$\frac{I(K_i)}{I(R(\gamma))}$$

and we get

$$\mathrm{E}(\min_k(M)) = \frac{1}{I(R(\gamma))} + \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))}\mathrm{E}(\min_{k-1}(M^i)), \tag{4.2}$$

which by induction becomes

$$\frac{1}{I(R(\gamma))} + \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \sum_{R_0 < \cdots < R_s \leqslant R(\gamma^i)} (-1)^s \frac{I(R_0 \cap R_1) \cdots I(R_{s-1} \cap R_s)}{I(R_0) \cdots I(R_s)}.$$

By changing the order of summation we obtain

$$\frac{1}{I(R(\gamma))} + \sum_{R_0 < \cdots < R_s < R(\gamma)} (-1)^s \frac{I(R_0 \cap R_1) \cdots I(R_{s-1} \cap R_s)}{I(R_0) \cdots I(R_s)} \sum_{i \notin \text{row-set}(R_s)} \frac{I(K_i)}{I(R(\gamma))}. \tag{4.3}$$

Here row-set(R) denotes the set of rows of M that intersect the rectangle R. Now

$$\sum_{i \notin \text{row-set}(R_s)} \frac{I(K_i)}{I(R(\gamma))} = \frac{I(R(\gamma) \backslash R_s)}{I(R(\gamma))} = 1 - \frac{I(R_s \cap R(\gamma))}{I(R(\gamma))}.$$

Hence the chains ending with R_s contribute the two terms

$$(-1)^s \frac{I(R_0 \cap R_1) \cdots I(R_{s-1} \cap R_s)}{I(R_0) \cdots I(R_s)} + (-1)^{s+1} \frac{I(R_0 \cap R_1) \cdots I(R_{s-1} \cap R_s) \cdot I(R_s \cap R(\gamma))}{I(R_0) \cdots I(R_s) \cdot R(\gamma)}. \tag{4.4}$$

The second of these terms corresponds to a chain of length $s+1$ with $R_{s+1} = R(\gamma)$. Hence each chain will occur exactly once, including $\frac{1}{I(R(\gamma))}$, which means that (4.3) equals the right-hand side of (4.1). □

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5. The Laplace transform

We can use the same proof technique to get a stronger result determining the Laplace transform of $\min_k(M)$. The Laplace transform of a random variable X is defined by $L(X,t) = \mathrm{E}(e^{-tX})$. The Laplace transform has the following properties.

L1 $L(0,t) = 1$.

L2 If X and Y are independent, then

$$L(X+Y,t) = L(X,t)L(Y,t).$$

L3 Let $X_1,\ldots,X_s$ be random variables and let I be a random variable independent of them that takes the values $1,\ldots,s$, and that takes the value i with probability p_i. Then

$$L(X_I,t) = \sum_{i=1}^{s} p_i L(X_i,t).$$

L4 If $X \sim \exp(a)$, then

$$L(X,t) = \frac{a}{a+t}.$$

For a critical rectangle R we use the notation

$$\phi(R,t) = L(\min_1(R),t) = \frac{I(R)}{I(R)+t}. \tag{5.1}$$

As in Theorem 4.1 we give two statements of the same formula using Lemma 3.1. Remark 1 also applies here. Let M be a matrix with a set Z of zeros and independent exponential random variables of positive rate in the other entries. M is said to be *generic* if, for every pair $R_1 \neq R_2$ of distinct critical rectangles, we have $I(R_1) \neq I(R_2)$.

Theorem 5.1. *Let M be an $m \times n$ matrix whose entries are either zero or independent exponential random variables with arbitrary positive rates. Let Z be the set of positions of the zeros. Suppose that Z contains a $(k-1)$-assignment but no k-assignment, and that M is generic. Then the Laplace transform of* $\min_k(M)$ *can be written*

$$L(\min_k(M),t) = \sum_{R \in Q(k,Z)} c_R(M)\phi(R,t). \tag{5.2}$$

The coefficient $c_R(M)$ can be factored as

$$c_R(M) = a_R(M) \cdot b_R(M),$$

where

$$a_R(M) = \sum_{R_0 < R_1 < \cdots < R_s = R} (-1)^s \prod_{j=0}^{s-1} \frac{I(R_j \cap R_{j+1}) - I(R)}{I(R_j) - I(R)}$$

and

$$b_R(M) = \sum_{R = R_s < R_{s+1} < \cdots < R_u} (-1)^{u-s} \prod_{j=s+1}^{u} \frac{I(R_j \cap R_{j-1}) - I(R)}{I(R_j) - I(R)},$$

where the sums are taken over all chains containing R in $Q(k,Z)$.

The formula for $c_R(M)$ *can also be written*

$$c_R(M) = \left(\sum_{R(\alpha) \leqslant R} g_R^{-1}(R(\alpha), R) \right) \cdot \left(\sum_{R \leqslant R(\beta)} g_R^{-1}(R, R(\beta)) \right),$$

where the sums are over elements in $Q(k,Z)$ *and where* g_R^{-1} *is the inverse in the incidence algebra of*

$$g_R(R_i, R_j) = \begin{cases} 1, & \text{if } R_i = R_j = R, \\ 0, & \text{if } R_i \nleq R_j, \\ I(R_i \cap R_j) - I(R), & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction over k. We use the same notation as in the proof of Theorem 4.1 and compute the Laplace transform using the same recursive step (4.2). This gives (using properties L2 and L3)

$$L(\min_k(M), t) = \phi(R(\gamma), t) \left(p + \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \cdot L(\min_{k-1}(M^i), t) \right), \tag{5.3}$$

where p ($p = p \cdot L(0, t)$ by property L1) is the probability that the minimum entry in $R(\gamma)$ is located so that a zero-cost k-assignment occurs. By induction, the right-hand side of (5.3) takes the form

$$\phi(R(\gamma), t) \left(p + \sum_{i,R} \frac{I(K_i)}{I(R(\gamma))} c_R(M^i) \phi(R, t) \right), \tag{5.4}$$

where the sum is over both i and critical rectangles $R < R(\gamma)$. Notice that since row i has been deleted from M^i, $\phi(R,t)$ will not occur in the expansion of $L(\min_{k-1}(M^i), t)$ if i belongs to the set of rows in R. Therefore we may use the convention that $c_R(M^i) = 0$ whenever row i intersects R.

Because of (5.1), the assumption that M is generic and the identity

$$\frac{a}{a+t} \cdot \frac{b}{b+t} = \frac{ab/(b-a)}{a+t} + \frac{ab/(a-b)}{b+t}$$

(for $a \neq b$), a term $\phi(R(\gamma), t) \cdot \phi(R, t)$ can be decomposed as

$$\frac{I(R)}{I(R) - I(R(\gamma))} \phi(R(\gamma), t) + \frac{I(R(\gamma))}{I(R(\gamma)) - I(R)} \phi(R, t).$$

Therefore a term $\phi(R, t)$ in the expansion of $L(\min_{k-1}(M^i), t)$ yields a coefficient of

$$\frac{I(R(\gamma))}{I(R(\gamma)) - I(R)}$$

for $\phi(R, t)$. This proves that $L(\min_k(M))$ can be written

$$\sum_{R \in Q(k,Z)} c_R(M) \phi(R, t),$$

for some $c_R(M)$.

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First we treat the case $R \neq R(\gamma)$. We determine the coefficient $c_R(M)$ by extracting the terms involving $\phi(R,t)$. This gives

$$c_R(M) = \frac{I(R(\gamma))}{I(R(\gamma)) - I(R)} \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \cdot c_R(M^i). \tag{5.5}$$

Obviously $a_R(M) = a_R(M^i)$ if R does not intersect row i.

Since the factor $I(R(\gamma))$ in (5.5) cancels, we can inductively write

$$c_R(M) = \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma)) - I(R)} a_R(M) \cdot \sum_{R=R_s<R_{s+1}<\cdots<R_u} (-1)^{u-s} \prod_{j=s+1}^{u} \frac{I(R_j \cap R_{j-1}) - I(R)}{I(R_j) - I(R)},$$

where the sum is taken over all chains where R_u does not intersect row i. By changing the order of summation, we get

$$c_R(M) = a_R(M) \sum_{R=R_s<R_{s+1}<\cdots<R_u} (-1)^{u-s} \cdot \prod_{j=s+1}^{u} \frac{I(R_j \cap R_{j-1}) - I(R)}{I(R_j) - I(R)} \cdot \sum_{i \notin \text{row-set}(R_u)} \frac{I(K_i)}{I(R(\gamma)) - I(R)}.$$

We note that

$$\sum_{i \notin \text{row-set}(R_u)} \frac{I(K_i)}{I(R(\gamma)) - I(R)} = \frac{I(R(\gamma)) - I(R(\gamma) \cap R_u)}{I(R(\gamma)) - I(R)} = 1 - \frac{I(R(\gamma) \cap R_u) - I(R)}{I(R(\gamma)) - I(R)},$$

and proceed as in the proof of Theorem 4.1.

Finally, if $R = R(\gamma)$ we have $b_R(M) = 1$. We can then proceed in exactly the same way but instead use a recursive step (5.3) based on the minimal cover (*i.e.*, the cover with the minimum number of columns, corresponding to the smallest element of $Q(k,Z)$) instead of γ, to prove the formula for $a_R(M)$. This proves the first formula for $c_R(M)$.

The function g_R is invertible if $I(R) \neq I(R(\alpha))$ for all $R(\alpha) \in Q(k,Z)$ such that $R < R(\alpha)$ or $R(\alpha) < R$, which is the case since M is assumed to be generic. The equivalence of the two formulas for the coefficients then follows from Lemma 3.1. □

Remark 2. The main theorem in [9], where the exponential random variables all have rate 1 or infinity, has a reformulation in [8] in terms of the Möbius function of a certain poset called P. The Möbius function is the inverse of the function ζ that takes the value 1 on every interval of P. The poset P is different and much larger than $Q(k,Z)$ in Theorem 4.1. More precisely, the elements of $Q(k,Z)$ are the atoms of P. All our efforts to join the two theorems to one for arbitrary rates have so far been fruitless. It would be very interesting indeed if a unification could be found.

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