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Complexity of Ising Polynomials

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This paper deals with the partition function of the Ising model from statistical mechanics, which is used to study phase transitions in physical systems. A special case of interest is that of the Ising model with constant energies and external field. One may consider such an Ising system as a simple graph together with vertex and edge weights. When these weights are considered indeterminates, the partition function for the constant case is a trivariate polynomial $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$. This polynomial was studied with respect to its approximability by Goldberg, Jerrum and Paterson. $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ generalizes a bivariate polynomial $Z(G;\mathsf{t},\mathsf{y})$, which was studied in by Andrén and Markström.

We consider the complexity of $Z(G;\mathsf{t},\mathsf{y})$ and $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ in comparison to that of the Tutte polynomial, which is well known to be closely related to the Potts model in the absence of an external field. We show that $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is #**P**-hard to evaluate at all points in $\mathbb{Q}^3$, except those in an exceptional set of low dimension, even when restricted to simple graphs which are bipartite and planar. A counting version of the Exponential Time Hypothesis, #**ETH**, was introduced by Dell, Husfeldt and Wahlén in order to study the complexity of the Tutte polynomial. In analogy to their results, we give under #**ETH** a dichotomy theorem stating that evaluations of $Z(G;\mathsf{t},\mathsf{y})$ either take exponential time in the number of vertices of G to compute, or can be done in polynomial time. Finally, we give an algorithm for computing $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ in polynomial time on graphs of bounded clique-width, which is not known in the case of the Tutte polynomial.

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1. Introduction

An Ising system is a simple graph $G=(V,E)$ together with vertex and edge weights. Every edge $(u,v)\in E$ has an *interaction energy* and every vertex $u\in V$ has an *external magnetic field strength* associated with it. A function $\sigma : V \to \{\pm 1\}$ is a *configuration* of the system or a *spin assignment*. The partition function of an Ising system is a generating function related to the probability that the system is in a certain configuration.

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In [14], Goldberg, Jerrum and Paterson investigated the Ising polynomial in three variables $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ for the case where both the interaction energies of an edge (u, v) and the external magnetic field strength of a vertex v are constant. They consider the existence of fully polynomial randomized approximation schemes (FPRAS) for the graph parameters $Z(G; \gamma, \delta, \varepsilon)$, depending on the values of $(\gamma, \delta, \varepsilon) \in \mathbb{Q}^3$. They provide approximation schemes for some regions of $\mathbb{Q}^3$ while showing that other regions do not admit such approximation schemes. Approximation schemes for $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ were further studied in [32, 25]. Jerrum and Sinclair [17] studied the approximability and #**P**-hardness of another case of the Ising model, where weights are provided as part of the input and no external field is present. The bivariate Ising polynomial $Z(G; \mathsf{t}, \mathsf{y})$, which was studied in [1] for its combinatorial properties, is equivalent to setting $\mathsf{x} = \mathsf{z} = \mathsf{t}$ in $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$. It is shown in [1] that $Z(G; \mathsf{t}, \mathsf{y})$ encodes the matching polynomial, and is equivalent to a bivariate generalization of a graph polynomial introduced by van der Waerden [28].

The trivariate and bivariate Ising polynomials fall under the general framework of partition functions, the complexity of which has been studied extensively starting with [7] and followed by [3, 13, 27, 4]. From [27, Theorem 6.1] and implicitly from [13] we get that the complexity of evaluations of the Ising polynomials satisfies a dichotomy theorem, saying that the graph parameter $Z(G; \gamma, \delta)$ is either polynomial-time computable or #**P**-hard. However, δ must be positive here.

The q-state Potts model deals with a similar scenario to the Ising model, except that the spins are not restricted to ± 1 but instead receive one of q possible values. The complexity of the q-state Potts model has attracted considerable attention in the literature. The partition function of the Potts model in the case where no magnetic field is present is closely related to the Tutte polynomial $T(G; x, y)$. It is well known that for every $\gamma, \delta \in \mathbb{Q}$, except for points (γ, δ) in a finite union of algebraic exceptional sets of dimension at most 1, computing the graph parameter $T(G; \gamma, \delta)$ is #**P**-hard on multigraphs: see [8]. This holds even when restricted to bipartite planar graphs: see [30] and [29]. In contrast, the restriction of the Tutte polynomial to the so-called *Ising hyperbola*, which corresponds to the case of the Ising model with no external field, is tractable on planar graphs: see [9, 18, 8].

Dell, Husfeldt and Wahlén [6] introduced a counting version of the Exponential Time Hypothesis (#**ETH**), which roughly states that counting the number of satisfying assignments to a 3CNF formula requires exponential time. This hypothesis is implied by the Exponential Time Hypothesis (**ETH**) for decision problems introduced by Impagliazzo and Paturi [16]. Under #**ETH**, the authors of [6] show that the computation of the Tutte polynomial on simple graphs requires exponential time in $\frac{m_G}{\log^3 m_G}$ in general, where m_G is the number of edges of the graph. For multigraphs they show that the computation of the Tutte polynomial generally requires exponential time in m_G.

In this paper we prove that the bivariate and trivariate Ising polynomials satisfy analogues of some complexity results for the Tutte polynomial. For the bivariate Ising polynomial we show a dichotomy theorem stating that evaluations of $Z(G; \mathsf{t}, \mathsf{y})$ are either #**P**-hard or polynomial-time computable. Moreover, assuming the counting version of the Exponential Time Hypothesis, the bivariate Ising polynomial requires exponential time to compute. Let n_G be the number of vertices of G.

Theorem 1.1 (Dichotomy theorem for the bivariate Ising polynomial). *For all* $(\gamma, \delta) \in \mathbb{Q}^2$, *we have the following.*

(i) *If* $\gamma \in \{-1, 0, 1\}$ *or* $\delta = 0$, *then* $Z(G; \gamma, \delta)$ *is polynomial-time computable.*
(ii) *Otherwise:*

- $Z(G; \gamma, \delta)$ *is* #**P**-*hard on simple graphs, and*
- *unless* #**ETH** *fails, requires exponential time in* $\frac{n_G}{\log^6 n_G}$ *on simple graphs.*

We show that the evaluations of $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$, except for those in a small exceptional set $B \subseteq \mathbb{Q}^3$, are hard to compute even when restricted to simple graphs which are both bipartite and planar.

Theorem 1.2 (Hardness of the trivariate Ising polynomial). *There is a set* $B \subseteq \mathbb{Q}^3$ *such that, for every* $(\gamma, \delta, \varepsilon) \in \mathbb{Q}^3 \setminus B$, $Z(G; \gamma, \delta, \varepsilon)$ *is* #**P**-*hard on simple bipartite planar graphs. B is a finite union of algebraic sets of dimension* 2.

Although $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is hard to compute in general, its computation on restricted classes of graphs can be tractable. Computing $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is fixed-parameter tractable with respect to tree-width using the general logical framework of [19]. This implies in particular that $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is polynomial-time computable on graphs of tree-width at most k, for any fixed k, which also follows from [22]. Likewise, the Tutte polynomial is known to be polynomial-time computable on graphs of bounded tree-width: see [2, 21]. In contrast, for graphs of bounded clique-width, a width notion which generalizes tree-width, the best algorithm known for the Tutte polynomial is subexponential: see [12]. We show the following.

Theorem 1.3 (Tractability on graphs of bounded clique-width). *There exists a function* $f(k)$ *such that* $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ *is computable on graphs of clique-width at most k in running time* $O(n_G{}^{f(k)})$.

In particular, $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ can be computed in polynomial time on graphs of clique-width[1] at most k, for any fixed k. On the other hand, it follows from [11] that, unless **FPT** = **W[1]**, $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is not *fixed-parameter tractable with respect to clique-width, i.e.*, there is no algorithm for $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ which runs in time $O(q(n_G) \cdot f(k))$ on graphs G of clique-width at most k for every k such that q is a polynomial.

2. Preliminaries

2.1. Definitions of the Ising polynomials

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote $n_G = |V(G)|$ and $m_G = |E(G)|$. All graphs in this paper are simple and undirected unless otherwise stated.

[1] Rank-width can replace clique-width here and in Theorem 1.3, since the clique-width of a graph is bounded by a function of the rank-width of the graph.

Given $S \subseteq V(G)$, we denote by $E_G(S)$ the set of edges in the graph induced by S in G and by $E_G(\bar{S})$ the set of edges in the graph obtained from G by deleting the vertices of S and their incident edges. We may omit the subscript and write, for example, $E(S)$ when the graph G is clear from the context.

Definition 2.1 (The trivariate Ising polynomial). The trivariate Ising polynomial is

$$Z(G;\mathsf{x},\mathsf{y},\mathsf{z}) = \sum_{S\subseteq V(G)} \mathsf{x}^{|E_G(S)|}\mathsf{y}^{|S|}\mathsf{z}^{|E_G(\bar{S})|}.$$

For every G, $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is a polynomial in $\mathbb{Z}[\mathsf{x},\mathsf{y},\mathsf{z}]$ with positive coefficients.

Definition 2.2 (The bivariate Ising polynomial). The bivariate Ising polynomial is obtained from $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ by setting $\mathsf{x} = \mathsf{z} = \mathsf{t}$. In other words,

$$Z(G;\mathsf{t},\mathsf{y}) = \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)|+|E_G(\bar{S})|}\mathsf{y}^{|S|}.$$

The *cut* $[S,\bar{S}]_G$ is the set of edges with one end-point in S and the other in $\bar{S} = V(G)\setminus S$. The bivariate Ising polynomial can be rewritten as follows, using that $E_G(S)$, $E_G(\bar{S})$ and $[S,\bar{S}]_G$ form a partition of $E(G)$:

$$Z(G;\mathsf{t},\mathsf{y}) = t^{m_G} \sum_{S\subseteq V(G)} \mathsf{t}^{-|[S,\bar{S}]_G|}\mathsf{y}^{|S|}. \tag{2.1}$$

The bivariate Ising polynomial is defined in this paper in a way which is slightly different from, and yet equivalent to, the way it was defined in [1]. The definition in [1] is reminiscent of equation (2.1).

In Section 3 we use a generalization of the bivariate Ising polynomial.

Definition 2.3. For every $B, C \subseteq V(G)$ such that $B \cap C = \emptyset$, we define

$$Z(G;B,C;\mathsf{t},\mathsf{y}) = \sum_{B\subseteq S\subseteq V(G)\setminus C} \mathsf{t}^{|E_G(S)|+|E_G(\bar{S})|}\mathsf{y}^{|S|}.$$

Clearly,

$$Z(G;\emptyset,\emptyset;\mathsf{t},\mathsf{y}) = Z(G;\mathsf{t},\mathsf{y}).$$

In Section 4 we use a multivariate version of $Z(G;B,C;\mathsf{x},\mathsf{y},\mathsf{z})$. In Section 5 we use a different multivariate generalization of $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$.

We denote by $[i]$ the set $\{1,\ldots,i\}$ for every $i \in \mathbb{N}^+$.

2.2. Complexity of the Ising polynomial

Here we collect complexity results from the literature in order to discuss the complexity of computing, for every graph G, the trivariate (bivariate) polynomial $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ ($Z(G;\mathsf{t},\mathsf{y})$).

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By *computing the polynomial* we mean computing the list of coefficients of monomials $\mathsf{x}^i\mathsf{y}^j\mathsf{z}^k$ such that $i, k \in \{0, 1, \ldots, m_G\}$ and $j \in \{0, 1, \ldots, n_G\}$.

In [1] it is shown that several graph invariants are encoded in $Z(G; \mathsf{t}, \mathsf{y})$.

Proposition 2.4. *The following are polynomial-time computable in the presence of an oracle to the bivariate polynomial* $Z(G; \mathsf{t}, \mathsf{y})$. *The oracle receives a graph* G *as input and returns the matrix of coefficients of terms* $\mathsf{t}^i\mathsf{y}^j$ *in* $Z(G; \mathsf{t}, \mathsf{y})$,

- *the matching polynomial and the number of perfect matchings,*
- *the number of maximum cuts,*

and, for regular graphs,

- *the independent set polynomial and the vertex cover polynomial.*

The following propositions apply two hardness results from the literature to $Z(G; \mathsf{t}, \mathsf{y})$ using Proposition 2.4.

Proposition 2.5. $Z(G; \mathsf{t}, \mathsf{y})$ *is #**P**-hard to compute, even when restricted to simple* 3*-regular bipartite planar graphs.*

Proof. The proposition follows from a result in [31] which states that it is #**P**-hard to compute #3RBP − VC, the number of vertex covers on input graphs restricted to be 3-regular, bipartite and planar. □

For the next proposition we need the following definition, introduced in [6] following [16].

Definition 2.6 (# Exponential Time Hypothesis (#ETH)). Let s be the infimum of the set

$$\{c : \text{there exists an algorithm for \#3SAT which runs in time } O(c^{n_G})\}.$$

The # *Exponential Time Hypothesis* is the conjecture that $s > 1$.

Proposition 2.7. *There exists* $c > 1$ *such that the computation of* $Z(G; \mathsf{t}, \mathsf{y})$ *requires* $\Omega(c^{n_G})$ *time on simple graphs, unless #**ETH** fails.*

Proof. The claim follows from a result of [6] which states that there exists $c > 1$ for which computing the number of maximum cuts in simple graphs G takes at least $\Omega(c^{m_G})$ time, unless the #**ETH** fails. It is easy to see that the problem of computing the number of maximum cuts of disconnected graphs can be reduced to that of connected graphs and so no subexponential algorithm exists for connected graphs, and the proposition follows since for connected graphs $n_G = O(m_G)$. □

On the other hand, $Z(G; \mathsf{t}, \mathsf{y})$ and $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ can be computed naïvely in time which is exponential in n_G.

The three above propositions apply to $Z(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ as well.

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2.3. Clique-width

Let $[k] = \{1,\dots,k\}$. A k-graph is a tuple $(G,\bar{c})$ which consists of a simple graph G together with labels $c_v \in [k]$ for every $v \in V(G)$. The class $CW(k)$ of k-graphs of clique-width at most k is defined inductively. Singletons belong to $CW(k)$, and $CW(k)$ is closed under disjoint union $\sqcup$ and two other operations, $\rho_{i\to j}$ and $\mu_{i,j}$, to be defined next. For any $i,j \in [k]$, $\rho_{i\to j}(G,\bar{c})$ is obtained by relabelling any vertex with label i to label j. For any $i,j \in [k]$, $\mu_{i,j}(G,\bar{c})$ is obtained by adding all possible edges (u,v) such that $c_u = i$ and $c_v = j$. The clique-width of a graph G is the minimal k such that there exists a labelling $\bar{c}$ for which $(G,\bar{c})$ belongs to $CW(k)$. We denote the clique-width of G by $cw(G)$.

A k-expression is a term t which consists of singletons, disjoint unions $\sqcup$, relabelling $\rho_{i\to j}$ and edge creations $\mu_{i,j}$, which witnesses that the graph val(t) obtained by performing the operations on the singletons is of clique-width at most k. Every graph of tree-width at most k is of clique-width at most $2^{k+1}+1$: see [5]. While computing the clique-width of a graph is **NP**-hard, Oum and Seymour showed that given a graph of clique-width k, finding a $(2^{3k+2}-1)$-expression is fixed-parameter tractable with clique-width as parameter: see [23, 24].

3. Exponential time lower bound

In this section we prove that in general the evaluations $(\gamma,\delta) \in \mathbb{Q}^2$ of $Z(G;\mathsf{t},\mathsf{y})$ require exponential time to compute under **#ETH**. In analogy with the use of Theta graphs to deal with the complexity of the Tutte polynomial, we define Phi graphs and use them to interpolate the indeterminate t in $Z(G;\mathsf{t},\mathsf{y})$. We interpolate y by a simple construction.

3.1. Phi graphs

Our goal in this subsection is to define Phi graphs $\Phi_{\mathcal{H}}$ and compute the bivariate Ising polynomial at $\mathsf{y}=1$ on graphs $G \otimes \Phi_{\mathcal{H}}$ to be defined below. In order to define Phi graphs we must first define L_h-graphs. For every $h \in \mathbb{N}$, the graph L_h is obtained from the path P_{h+1} with h edges as follows. Let hd(h) denote one of the end-points of P_{h+1}. Let $\mathrm{tr}_1(h)$ and $\mathrm{tr}_2(h)$ be two new vertices. L_h is obtained from P_{h+1} by adding edges to make both $\mathrm{tr}_1(h)$ and $\mathrm{tr}_2(h)$ adjacent to all the vertices of P_{h+1}.

We can also construct L_h recursively from L_{h-1} by

- adding a new vertex hd(h) to L_{h-1},
- renaming $\mathrm{tr}_i(h-1)$ to $\mathrm{tr}_i(h)$ for $i=1,2$, and
- adding three edges to make hd(h) adjacent to hd$(h-1)$, $\mathrm{tr}_1(h)$ and $\mathrm{tr}_2(h)$.

Figure 1 shows L_5.

We denote by $B \sqcup C$ a partition of the set $\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\}$. We let $B(h)$ denote the subset of $\{\mathrm{tr}_1(h),\mathrm{tr}_2(h),\mathrm{hd}(h)\}$ which corresponds to B, and let $C(h)$ be defined similarly. We have that $B(h)$ and $C(h)$ form a partition of $\{\mathrm{tr}_1(h),\mathrm{tr}_2(h),\mathrm{hd}(h)\}$.

Definition 3.1. We denote $b_{B,C}(h) = Z(L_h;B(h),C(h);\mathsf{t},1)$.

The next two lemmas are devoted to computing $b_{B,C}(h)$.

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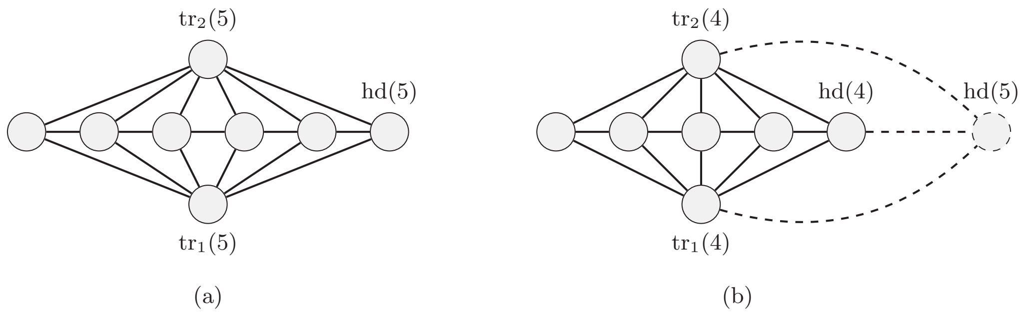

Figure 1. The graph L_5 and the construction of L_5 from L_4. L_5 is obtained from L_4 by adding the vertex hd(5) and its incident edges, and renaming $\mathrm{tr}_1(4)$ and $\mathrm{tr}_2(4)$ to $\mathrm{tr}_1(5)$ and $\mathrm{tr}_2(5)$ respectively.

Lemma 3.2.

$$b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) = b_{\{\mathrm{tr}_2,\mathrm{hd}\},\{\mathrm{tr}_1\}}(h) = b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h) = b_{\{\mathrm{tr}_2\},\{\mathrm{tr}_1,\mathrm{hd}\}}(h) = (\mathrm{t}^2+\mathrm{t})^h \cdot \mathrm{t}.$$

Proof. We have

$$b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) = b_{\{\mathrm{tr}_2,\mathrm{hd}\},\{\mathrm{tr}_1\}}(h) = b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h) = b_{\{\mathrm{tr}_2\},\{\mathrm{tr}_1,\mathrm{hd}\}}(h),$$

by symmetry. We compute $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h)$ by finding a simple linear recurrence relation which it satisfies, and solving it. We divide the sum $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h)$ into two sums,

$$\begin{aligned} b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) = {} & Z(L_h;\{\mathrm{tr}_1(h),\mathrm{hd}(h),\mathrm{hd}(h-1)\},\{\mathrm{tr}_2(h)\};\mathrm{t},1) \\ & + Z(L_h;\{\mathrm{tr}_1(h),\mathrm{hd}(h)\},\{\mathrm{tr}_2(h),\mathrm{hd}(h-1)\};\mathrm{t},1), \end{aligned}$$

depending on whether $\mathrm{hd}(h-1)$ is in the iteration variable S of the sum $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h)$ (as in Definition 2.3). These two sums can be obtained from

$$b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1) \quad \text{and} \quad b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h-1)$$

by adjusting for the addition of $\mathrm{hd}(h)$ and its incident edges.

- The case $\mathrm{hd}(h-1) \in S$. Adding $\mathrm{hd}(h)$ (to the graph and to S) puts two new edges in $E(S) \sqcup E(\bar{S})$, namely $(\mathrm{tr}_1, \mathrm{hd}(h))$ and $(\mathrm{hd}(h-1), \mathrm{hd}(h))$. Hence,

$$Z(L_h;\{\mathrm{tr}_1(h),\mathrm{hd}(h),\mathrm{hd}(h-1)\},\{\mathrm{tr}_2(h)\};\mathrm{t},1) = b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1)\cdot \mathrm{t}^2.$$

- The case $\mathrm{hd}(h-1) \notin S$. Adding $\mathrm{hd}(h)$ puts just one new edge in $E(S) \sqcup E(\bar{S})$, namely $(\mathrm{tr}_1, \mathrm{hd}(h))$. Hence,

$$Z(L_h;\{\mathrm{tr}_1(h),\mathrm{hd}(h)\},\{\mathrm{tr}_2(h),\mathrm{hd}(h-1)\};\mathrm{t},1) = b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h-1)\cdot \mathrm{t}.$$

Using that $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1) = b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h-1)$, we get

$$b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) = b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1)\cdot(\mathrm{t}^2+\mathrm{t}), \tag{3.1}$$

and the lemma follows since $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(0) = \mathrm{t}$ (note that L_0 is simply a path of length 3). □

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We are left with two distinct cases of $b_{B,C}(h)$ to compute, since by symmetry,

$$b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h) = b_{\emptyset,\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\}}(h) \quad \text{and} \quad b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h) = b_{\{\mathrm{hd}\},\{\mathrm{tr}_1,\mathrm{tr}_2\}}(h).$$

Lemma 3.3. *Let*

$$\begin{aligned}
\lambda_{1,2} &= \frac{\mathrm{t}}{2}\left(1+\mathrm{t}^2 \pm \sqrt{5-2\mathrm{t}^2+\mathrm{t}^4}\right),\\
c_1 &= \mathrm{t}^2 - c_2,\\
c_2 &= \frac{\mathrm{t}\left(-\mathrm{t}^3-2+\mathrm{t}+\mathrm{t}\sqrt{5-2\mathrm{t}^2+\mathrm{t}^4}\right)}{2\sqrt{5-2\mathrm{t}^2+\mathrm{t}^4}},\\
d_1 &= 1-d_2,\\
d_2 &= \frac{-1-2\mathrm{t}+\mathrm{t}^2+\sqrt{5-2\mathrm{t}^2+\mathrm{t}^4}}{2\sqrt{5-2\mathrm{t}^2+\mathrm{t}^4}}.
\end{aligned}$$

Here λ_1 corresponds to the + case. If $\mathrm{t} \in \mathbb{R}$ then $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$, and

$$\begin{aligned}
b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h) &= c_1\lambda_1^h + c_2\lambda_2^h,\\
b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h) &= d_1\lambda_1^h + d_2\lambda_2^h.
\end{aligned}$$

Proof. The content of the square root is always strictly positive for $\mathrm{t} \in \mathbb{R}$. Hence, $\lambda_1 \neq \lambda_2$ and $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2 \in \mathbb{R}$.

The sequences $b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h)$ and $b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h)$ satisfy a mutual linear recurrence as follows:

$$\begin{aligned}
b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h) &= b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h-1)\cdot \mathrm{t}^3 + b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h-1)\cdot \mathrm{t}^2,\\
b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h) &= b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h-1) + b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h-1)\cdot \mathrm{t}.
\end{aligned}$$

This implies that both $b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h)$ and $b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h)$ satisfy linear recurrence relations with the following initial conditions:

$$\begin{aligned}
b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(0) = t^2 \quad &\text{and} \quad b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(1) = t^5 + t^2,\\
b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(0) = 1 \quad &\text{and} \quad b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(1) = t^2 + t.
\end{aligned}$$

These recurrences can be calculated and solved using standard methods: see, *e.g.*, [10] or [15]. □

Using the previous two lemmas, we get the following result.

Lemma 3.4.

$$\begin{aligned}
&Z(L_h;\{\mathrm{tr}_1\},\{\mathrm{tr}_2\};\mathrm{t},1) = Z(L_h;\{\mathrm{tr}_2\},\{\mathrm{tr}_1\};\mathrm{t},1) = 2\mathrm{t}(\mathrm{t}^2+\mathrm{t})^h,\\
&Z(L_h;\{\mathrm{tr}_1,\mathrm{tr}_2\},\emptyset;\mathrm{t},1) = Z(L_h;\emptyset,\{\mathrm{tr}_1,\mathrm{tr}_2\};\mathrm{t},1) = (c_1+d_1)\lambda_1^h + (c_2+d_2)\lambda_2^h,
\end{aligned}$$

where $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2$ are as in Lemma 3.3.

Proof. The lemma follows from Lemmas 3.2 and 3.3. □

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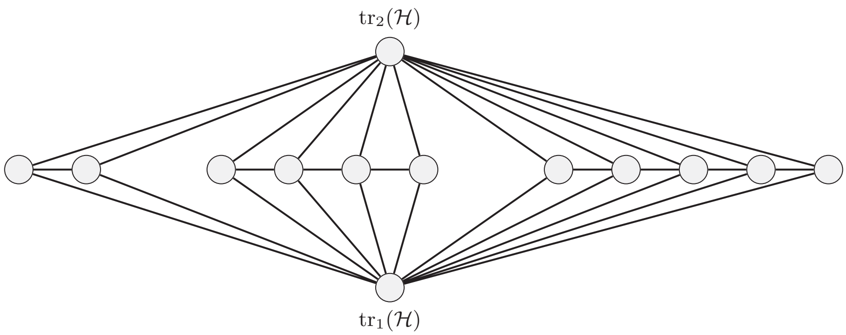

Figure 2. An example of a Phi graph: the graph $\Phi_{\mathcal{H}}$ for $\mathcal{H} = \{1, 3, 4\}$.

Definition 3.5 (Phi graphs). Let $\mathcal{H}$ be a finite set of positive integers. We denote by $\Phi_{\mathcal{H}}$ the graph obtained from the disjoint union of the graphs $L_h : h \in \mathcal{H}$ as follows. For each $i = 1, 2$, the vertices $\mathrm{tr}_i(h)$, $h \in \mathcal{H}$, are identified as one vertex denoted $\mathrm{tr}_i(\mathcal{H})$.

The number of vertices in $\Phi_{\mathcal{H}}$ is $2 + \sum_{h\in\mathcal{H}}(h + 1)$. Figure 2 shows $\Phi_{\{1,3,4\}}$.

Lemma 3.6. *Let $\mathcal{H}$ be a finite set of positive integers. Then*

$$Z(\Phi_{\mathcal{H}}; \{\mathrm{tr}_1(\mathcal{H})\}, \{\mathrm{tr}_2(\mathcal{H})\}; \mathrm{t}, 1) = (2\mathrm{t})^{|\mathcal{H}|} \prod_{h\in\mathcal{H}} (\mathrm{t}^2 + \mathrm{t})^h,$$

and

$$\begin{aligned} Z(\Phi_{\mathcal{H}}; \{\mathrm{tr}_1(\mathcal{H}), \mathrm{tr}_2(\mathcal{H})\}, \emptyset; \mathrm{t}, 1) &= Z(\Phi_{\mathcal{H}}; \emptyset, \{\mathrm{tr}_1(\mathcal{H}), \mathrm{tr}_2(\mathcal{H})\}; \mathrm{t}, 1) \\ &= \prod_{h\in\mathcal{H}} \left((c_1 + d_1)\lambda_1^h + (c_2 + d_2)\lambda_2^h\right). \end{aligned}$$

Proof. The proof follows from Lemma 3.4 using that all edges are contained in some L_h. □

We can now define the graphs $G \otimes \mathcal{H}$.

Definition 3.7 ($G \otimes \mathcal{H}$). Let $\mathcal{H}$ be a finite set of positive integers. Let G be a graph. For every edge $e = (u_1, u_2) \in E(G)$, let $\Phi_{\mathcal{H},e}$ be a new copy of $\Phi_{\mathcal{H}}$, where we denote $\mathrm{tr}_1(\mathcal{H})$ and $\mathrm{tr}_2(\mathcal{H})$ for $\Phi_{\mathcal{H},e}$ by $\mathrm{tr}_1(\mathcal{H}, e)$ and $\mathrm{tr}_2(\mathcal{H}, e)$. Let $G \otimes \Phi_{\mathcal{H}} = G \otimes \mathcal{H}$ be the graph obtained from the disjoint union of the graphs

$$\Phi_{\mathcal{H},e} : e \in E(G)$$

by identifying $\mathrm{tr}_i(\mathcal{H}, e)$ with u_i, $i = 1, 2$, for every edge $e = (u_1, u_2) \in E(G)$.[2]

[2] It does not matter how we identify u_1 and u_2 with $\mathrm{tr}_1(\mathcal{H}, e)$ and $\mathrm{tr}_2(\mathcal{H}, e)$, since the two possible alignments will give rise to isomorphic graphs.

Lemma 3.8. *Let $\mathcal{H}$ be a finite set of positive integers. Let $f_{\mathrm{t},\mathcal{H}}$ and $g_{p,\mathcal{H}}$ be the following functions:*

$$f_{\mathrm{t},\mathcal{H}}(e_1,e_2,r_1,r_2) = \prod_{h\in\mathcal{H}}(e_1 r_1^h + e_2 r_2^h),$$
$$f_{p,\mathcal{H}}(\mathrm{t}) = \left((2\mathrm{t})^{|\mathcal{H}|}\prod_{h\in\mathcal{H}}(\mathrm{t}^2+\mathrm{t})^h\right)^{m_G}.$$

Then

$$Z(G\otimes\mathcal{H};\mathrm{t},1) = f_{p,\mathcal{H}}(\mathrm{t})\cdot Z\left(G; f_{\mathrm{t},\mathcal{H}}\left(\frac{c_1+d_1}{2t},\frac{c_2+d_2}{2t},\frac{\lambda_1}{\mathrm{t}^2+\mathrm{t}},\frac{\lambda_2}{\mathrm{t}^2+\mathrm{t}}\right),1\right).$$

Proof. Let $\tilde{G} = G\otimes\mathcal{H}$. By definition,

$$Z(\tilde{G};\mathrm{t},1) = \sum_{S\subseteq V(\tilde{G})}\mathrm{t}^{|E_{\tilde{G}}(S)\sqcup E_{\tilde{G}}(\bar{S})|}.$$

We can rewrite this sum as

$$Z(\tilde{G};\mathrm{t},1) = \sum_{S\subseteq V(G)}\left(\prod_{e\in[S,\bar{S}]_G} Z(\Phi_{\mathcal{H},e};\{\mathrm{tr}_1(\mathcal{H},e)\},\{\mathrm{tr}_2(\mathcal{H},e)\};\mathrm{t},1)\right)$$
$$\cdot\left(\prod_{e\in E_G(S)\sqcup E_G(\bar{S})} Z(\Phi_{\mathcal{H},e};\{\mathrm{tr}_1(\mathcal{H},e),\mathrm{tr}_2(\mathcal{H},e)\},\emptyset;\mathrm{t},1)\right),$$

since edges only occur within some $\Phi_{\mathcal{H},e}$. Using Lemma 3.6, the sum in the last equation can be written as

$$\sum_{S\subseteq V(G)}\left((2\mathrm{t})^{|\mathcal{H}|}\prod_{h\in\mathcal{H}}(\mathrm{t}^2+\mathrm{t})^h\right)^{|[S,\bar{S}]_G|}\cdot\left(\prod_{h\in\mathcal{H}}((c_1+d_1)\lambda_1^h+(c_2+d_2)\lambda_2^h)\right)^{|E_G(S)\sqcup E_G(\bar{S})|}.$$

Since $|[S,\bar{S}]_G = m_G - |E_G(S)\sqcup E_G(\bar{S})|$, we can rewrite the last equation as

$$\left((2\mathrm{t})^{|\mathcal{H}|}\prod_{h\in\mathcal{H}}(\mathrm{t}^2+\mathrm{t})^h\right)^{m_G}\cdot\sum_{S\subseteq V(G)}\left(\frac{\prod_{h\in\mathcal{H}}((c_1+d_1)\lambda_1^h+(c_2+d_2)\lambda_2^h)}{(2\mathrm{t})^{|\mathcal{H}|}\prod_{h\in\mathcal{H}}(\mathrm{t}^2+\mathrm{t})^h}\right)^{|E_G(S)\sqcup E_G(\bar{S})|}.$$

The last sum can be rewritten as

$$\sum_{S\subseteq V(G)}\left[\prod_{h\in\mathcal{H}}\left(\frac{c_1+d_1}{2t}\left(\frac{\lambda_1}{\mathrm{t}^2+\mathrm{t}}\right)^h+\frac{c_2+d_2}{2t}\left(\frac{\lambda_2}{\mathrm{t}^2+\mathrm{t}}\right)^h\right)\right]^{|E_G(S)\sqcup E_G(\bar{S})|},$$

and the lemma follows. □

The construction described above will be useful for dealing with the evaluation of $Z(G;\mathrm{t},\mathrm{y})$ with $\mathrm{y}=-1$ due to the following lemma. For a graph G, let $G_{(1)}$ be the graph obtained from G by adding, for each $v\in V(G)$, a new vertex v' and an edge (v,v'). So v' is adjacent to v only. $G_{(1)}$ is a graph with $2n_G$ vertices.

Lemma 3.9. $Z(G;\mathrm{t},1) = (\mathrm{t}-1)^{-n_G}Z(G_{(1)};\mathrm{t},-1)$.

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Proof. By definition we have

$$Z(G_{(1)};\mathsf{t},-1) = \sum_{S\subseteq V(G_{(1)})} \mathsf{t}^{|E_{G_{(1)}}(S)\sqcup E_{G_{(1)}}(\bar{S})|}(-1)^{|S|}$$
$$= \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|}(\mathsf{t}-1)^{|S|}(\mathsf{t}-1)^{n_G-|S|},$$

where the last equality is by considering the contribution of v' for each $v \in V(G)$: if $v \in S$ then v' contributes either $-\mathsf{t}$ or 1; if $v \notin S$ then v' contributes either t or -1. The last expression in the equation above equals

$$(\mathsf{t}-1)^{n_G} \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|} = (\mathsf{t}-1)^{n_G} \cdot Z(G;\mathsf{t},1). \qquad \square$$

3.2. The Ising polynomials of certain trees

We denote by S_n the star with n leaves. Let $\mathrm{cent}(S_n)$ be the central vertex of the star S_n. A construction based on stars will be used to interpolate the y indeterminate from $Z(G;\gamma,\delta)$. First, notice the following.

Proposition 3.10. *For every $n \in \mathbb{N}^+$,*

$$Z(S_n;\{\mathrm{cent}(S_n)\},\emptyset;\mathsf{t},\mathsf{y}) = \mathsf{y}\cdot(\mathsf{y}\mathsf{t}+1)^n,$$
$$Z(S_n;\emptyset,\{\mathrm{cent}(S_n)\};\mathsf{t},\mathsf{y}) = (\mathsf{y}+\mathsf{t})^n.$$

Proof. By definition,

$$Z(S_n;\{\mathrm{cent}\, S_n\},\emptyset;\mathsf{t},\mathsf{y}) = \sum_{S:\{\mathrm{cent}(S_n)\}\subseteq S\subseteq V(S_n)} \mathsf{t}^{|E_{S_n}(S)\sqcup E_{S_n}(\bar{S})|}\mathsf{y}^{|S|},$$
$$Z(S_n;\emptyset,\{\mathrm{cent}\, S_n\};\mathsf{t},\mathsf{y}) = \sum_{S\subseteq V(S_n)\setminus\{\mathrm{cent}(S_n)\}} \mathsf{t}^{|E_{S_n}(S)\sqcup E_{S_n}(\bar{S})|}\mathsf{y}^{|S|}.$$

Consider a leaf v of S_n. For $Z(S_n;\{\mathrm{cent}(S_n)\},\emptyset;\mathsf{t},\mathsf{y})$, a leaf v has two options: either $v \in S$, in which case it contributes the weight of its incident edge, so its contribution is $\mathsf{y}\mathsf{t}$; or $v \notin S$, in which case it contributes 1. For $Z(S_n;\emptyset,\{\mathrm{cent}(S_n)\};\mathsf{t},\mathsf{y})$, v has two options: either $v \in S$, in which case it does not contribute the weight of its edge, so its contribution is y; or $v \notin S$, in which case its edge contributes t. $\square$

Definition 3.11 (The graph $S_{\mathcal{H}}$). Let $\mathcal{H}$ be a set of positive integers. The graph $S_{\mathcal{H}}$ is obtained from the disjoint union of $S_n : n \in \mathcal{H}$ and a new vertex $\mathrm{cent}(\mathcal{H})$ by adding edges between $\mathrm{cent}(\mathcal{H})$ and the centres $\mathrm{cent}(S_n)$ of all the stars $S_n : n \in \mathcal{H}$.

See Figure 3(a) for an example.

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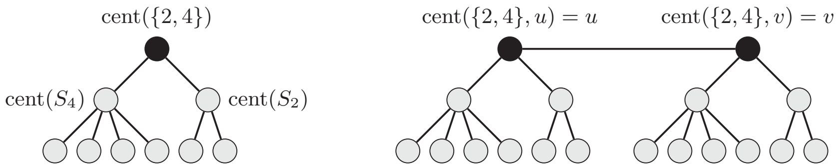

(a) The graph $S_{\{2,4\}}$ (b) The graph $S_{\{2,4\}}(K_2)$

Figure 3. Examples of $S_{\mathcal{H}}$ and $S_{\mathcal{H}}(G)$. In (b), the black vertices u, v are the vertices of K_2. They are also denoted cent$(\mathcal{H}, u)$ and cent$(\mathcal{H}, v)$ respectively.

Proposition 3.12. *Let $\mathcal{H}$ be a set of positive integers. Then,*

$$Z(S_{\mathcal{H}}; \{\text{cent}(\mathcal{H})\}, \emptyset; \mathsf{t}, \mathsf{y}) = \mathsf{y} \cdot \prod_{h \in \mathcal{H}} \big(\mathsf{yt} \cdot (\mathsf{yt}+1)^h + (\mathsf{y}+\mathsf{t})^h\big),$$

$$Z(S_{\mathcal{H}}; \emptyset, \{\text{cent}(\mathcal{H})\}; \mathsf{t}, \mathsf{y}) = \prod_{h \in \mathcal{H}} \big(\mathsf{y} \cdot (\mathsf{yt}+1)^h + \mathsf{t} \cdot (\mathsf{y}+\mathsf{t})^h\big).$$

Proof. We have

$$Z(S_{\mathcal{H}}; \{\text{cent}(\mathcal{H})\}, \emptyset; \mathsf{t}, \mathsf{y}) = \mathsf{y} \cdot \prod_{h \in \mathcal{H}} \big(\mathsf{t} \cdot Z(S_h; \{\text{cent}(S_h)\}, \emptyset; \mathsf{t}, \mathsf{y}) + Z(S_h; \emptyset, \{\text{cent}(S_h)\}; \mathsf{t}, \mathsf{y})\big),$$

$$Z(S_{\mathcal{H}}; \emptyset, \{\text{cent}(\mathcal{H})\}; \mathsf{t}, \mathsf{y}) = \prod_{h \in \mathcal{H}} \big(Z(S_h; \{\text{cent}(S_h)\}, \emptyset; \mathsf{t}, \mathsf{y}) + \mathsf{t} \cdot Z(S_h; \emptyset, \{\text{cent}(S_h)\}; \mathsf{t}, \mathsf{y})\big),$$

and by Proposition 3.10 the claim follows. □

Definition 3.13 (The graph $S_{\mathcal{H}}(G)$). Let $\mathcal{H}$ be a set of positive integers and let G be a graph. For every vertex v of G, let $S_{\mathcal{H},v}(G)$ be a new copy of $S_{\mathcal{H}}$. We denote the centre of each such copy of $S_{\mathcal{H}}$ by cent$(\mathcal{H}, v)$. Let $S_{\mathcal{H}}(G)$ be the graph obtained from the disjoint union of the graphs in the set

$$\{G\} \cup \{S_{\mathcal{H},v} : v \in V(G)\}$$

by identifying the pairs of vertices v and cent$(\mathcal{H}, v)$.

In other words, $S_{\mathcal{H}}(G)$ is the *rooted product* of G and $(S_{\mathcal{H}}, \text{cent}(\mathcal{H}))$. See Figure 3(b) for an example.

Proposition 3.14. *Let $\mathcal{H}$ be a set of positive integers. Let*

$$g_{p,\mathcal{H}}(\mathsf{t}, \mathsf{y}) = \left(\prod_{h \in \mathcal{H}} \big(\mathsf{y} \cdot (\mathsf{yt}+1)^h + \mathsf{t} \cdot (\mathsf{y}+\mathsf{t})^h\big)\right)^{|V(G)|},$$

$$g_{\mathsf{y},\mathcal{H}}(\mathsf{t}, \mathsf{y}) = \mathsf{y} \prod_{h \in \mathcal{H}} \frac{\mathsf{yt} \cdot (\mathsf{yt}+1)^h + (\mathsf{y}+\mathsf{t})^h}{\mathsf{y} \cdot (\mathsf{yt}+1)^h + \mathsf{t} \cdot (\mathsf{y}+\mathsf{t})^h}.$$

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Then

$$Z(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) = g_{p,\mathcal{H}}(\mathsf{t},\mathsf{y}) \cdot Z(G;\mathsf{t},g_{\mathsf{y},\mathcal{H}}(\mathsf{t},\mathsf{y})).$$

Proof. By definition

$$Z(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) = \sum_{S\subseteq V(S_{\mathcal{H}}(G))} \mathsf{t}^{|E_{S_{\mathcal{H}}(G)}(S)\sqcup E_{S_{\mathcal{H}}(G)}(\bar{S})|}\mathsf{y}^{|S|}.$$

We would like to rewrite this sum as a sum over $S \subseteq V(G)$. By the structure of $S_{\mathcal{H}}(G)$,

$$Z(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) = \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|} \left(\prod_{v\in S} Z(S_{\mathcal{H},v};\{\mathrm{cent}(\mathcal{H},v)\},\emptyset;\mathsf{t},\mathsf{y}) + \prod_{v\in\bar{S}} Z(S_{\mathcal{H},v};\emptyset,\{\mathrm{cent}(\mathcal{H},v)\};\mathsf{t},\mathsf{y})\right).$$

By Proposition 3.12,

$$Z(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) = \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|} \left(\left(\mathsf{y}\cdot\prod_{h\in\mathcal{H}}(\mathsf{yt}\cdot(\mathsf{yt}+1)^h + (\mathsf{y}+\mathsf{t})^h)\right)^{|S|}\left(\prod_{h\in\mathcal{H}}(\mathsf{y}\cdot(\mathsf{yt}+1)^h + \mathsf{t}\cdot(\mathsf{y}+\mathsf{t})^h)\right)^{|V(G)\setminus|S|}\right),$$

and the claim follows. □

The following propositions will be useful.

Proposition 3.15. *Let $g_{\mathsf{y},\mathcal{H}}(\mathsf{t},\mathsf{y})$ be as in Proposition 3.14. Let $h_{\mathsf{y},\mathcal{H}}$ be the function given by*

$$h_{\mathsf{y},\mathcal{H}}(e_1,e_2,r) = \prod_{h\in\mathcal{H}}\left(1+\frac{1}{e_1+e_2\cdot r^h}\right).$$

Let $\gamma,\delta \notin \{-1,0,1\}$ such that $\gamma \neq -\delta$. There exist constants h_1,u_1,u_2,w (which depend on γ and δ) such that for every two finite sets of positive even numbers $\mathcal{H}_1$ and $\mathcal{H}_2$ which satisfy

- $|\mathcal{H}_1| = |\mathcal{H}_2|$, *and* $\mathcal{H}_1,\mathcal{H}_2 \subseteq \mathbb{N}^+ \setminus \{1,\dots,h_1\}$,

we have

(i) $g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta), g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta), h_{\mathsf{y},\mathcal{H}_1}(u_1,u_2,w), h_{\mathsf{y},\mathcal{H}_2}(u_1,u_2,w) \in \mathbb{R}\setminus\{0\}$, *and*
(ii) $g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta) = g_{\mathsf{y},\mathcal{H}_2}(\gamma,\delta)$ *if and only if* $h_{\mathsf{y},\mathcal{H}_1}(u_1,u_2,w) = h_{\mathsf{y},\mathcal{H}_2}(u_1,u_2,w)$

Furthermore, u_1 and u_2 are non-zero and $w \notin \{-1,0,1\}$.

Proof. It cannot hold that $|\delta\gamma+1| = |\delta+\gamma|$. Furthermore we know that $\gamma,\delta \neq 0$. Hence, there exists h_1 such that for every even $h > h_1$, the sequences $\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h$ and $\delta\gamma(\delta\gamma+1)^h + (\delta+\gamma)^h$ are strictly ascending or descending, and in particular, are non-zero. Therefore we have $g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta), g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta) \in \mathbb{R}\setminus\{0\}$.

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We have for $i = 1,2$

$$\begin{aligned} g_{\mathsf{y},\mathcal{H}_i}(\gamma,\delta) &= \delta \prod_{h\in\mathcal{H}_i} \frac{\delta\gamma\cdot(\delta\gamma+1)^h + (\delta+\gamma)^h}{\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h} \\ &= \frac{\delta}{\gamma^{|\mathcal{H}_i|}} \prod_{h\in\mathcal{H}_i} \frac{\delta\gamma^2\cdot(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h}{\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h} \\ &= \frac{\delta}{\gamma^{|\mathcal{H}_i|}} \prod_{h\in\mathcal{H}_i} \left(1 + \frac{\delta(\gamma^2-1)\cdot(\delta\gamma+1)^h}{\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h}\right) \\ &= \frac{\delta}{\gamma^{|\mathcal{H}_i|}} \prod_{h\in\mathcal{H}_i} \left(1 + \frac{1}{\frac{1}{\gamma^2-1} + \frac{\gamma}{\delta(\gamma^2-1)}\cdot\left(\frac{\delta+\gamma}{\delta\gamma+1}\right)^h}\right). \end{aligned}$$

Let $u_1 = \frac{1}{\gamma^2-1}$, $u_2 = \frac{\gamma}{\delta(\gamma^2-1)}$ and $w = \frac{\delta+\gamma}{\delta\gamma+1}$. We have $u_1, u_2, w \in \mathbb{R}\setminus\{0\}$ and $w \notin \{-1,1\}$. Hence, we can take h_1 to be sufficiently large that $u_1 + u_2 + w^h$ non-zero. Since $u_1 + u_2 + w^h$ is strictly ascending or descending for even h, we have $h_{\mathsf{y},\mathcal{H}_1}(u_1,u_2,w), h_{\mathsf{y},\mathcal{H}_2}(u_1,u_2,w) \in \mathbb{R}\setminus\{0\}$ for large enough values of h. □

Proposition 3.16. *Let $\gamma,\delta \notin \{-1,0,1\}$ and $\gamma \neq -\delta$. Let $\mathcal{H}$ be a set of positive even integers. Let $g_{p,\mathcal{H}}(\mathsf{t},\mathsf{y}))$ be from Proposition 3.14. Then there exists h_2 such that if $\mathcal{H} \subseteq \mathbb{N}^+ \setminus \{1,\ldots,h_2\}$ then $g_{p,\mathcal{H}}(\gamma,\delta)) \neq 0$.*

Proof. Recall that

$$g_{p,\mathcal{H}}(\gamma,\delta)) = \left(\prod_{h\in\mathcal{H}} \left(\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h\right)\right)^{|V(G)|}.$$

We have that $\delta+\gamma$ is non-zero. If $\delta\gamma + 1 = 0$ then the claim holds even for $h_2 = 0$. Otherwise, using that $|\delta\gamma+1| \neq |\delta+\gamma|$, at least one of $(\delta\gamma+1)^h, (\delta+\gamma)^h$ becomes strictly larger in absolute value than the other for large enough h. □

3.3. Proof of Theorem 1.1

The following lemma is a variation of Lemma 4 in [6]. For any $\mathcal{H}$, let $\sigma(\mathcal{H}) = \sum_{h\in\mathcal{H}} h$.

Lemma 3.17. *Let $\gamma \notin \{-1,0,1\}$, $\delta \neq 0$, $e_1, e_2 \neq 0$ and $r_1, r_2 \notin \{-1,0,1\}$ such that $|r_1| \neq |r_2|$. For every positive integer q' there exist $\hat{q} = \Omega(q')$ sets of positive even integers $\mathcal{H}_0,\ldots,\mathcal{H}_{\hat{q}}$ such that*

(i) *$\sigma(\mathcal{H}_i) = O(\log^3 q')$ for all i,*
(ii) *$\sigma(\mathcal{H}_i) = \sigma(\mathcal{H}_j)$ for all $i \neq j$,*
(iii) *$f_{\mathsf{t},\mathcal{H}_i}(e_1,e_2,r_1,r_2) \neq f_{\mathsf{t},\mathcal{H}_j}(e_1,e_2,r_1,r_2)$ for $i \neq j$,*

where $f_{\mathsf{t},\mathcal{H}}(e_1,e_2,r_1,r_2)$ is from Proposition 3.8. If, in addition, $\delta \notin \{-1,1\}$ and $\gamma \neq -\delta$, we have

(iv) *$g_{\mathsf{y},\mathcal{H}_i}(\gamma,\delta) \neq g_{\mathsf{y},\mathcal{H}_j}(\gamma,\delta)$ for $i \neq j$.*
(v) *$g_{p,\mathcal{H}_i}(\gamma,\delta) \neq 0$,*

where $g_{\mathsf{y},\mathcal{H}}(e_1,e_2,r_1)$ is from Proposition 3.14.

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The sets $\mathcal{H}_i$ can be computed in polynomial time in q'.

Proof. Let $q = q' \log^3 q'$. First we define sets $\mathcal{H}'_0, \ldots, \mathcal{H}'_q$. We will use these sets to define the desired sets $\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}$.

For $i = 0, \ldots, q$, let $i[0], \ldots, i[\ell] \in \{0,1\}$ be the binary expansion of i, where $\ell = \lfloor \log q \rfloor$.[3] Let Δ denote a positive even integer to be chosen later. Let $\tau \in \{1,2\}$ be such that $|r_\tau| = \max\{|r_1|,|r_2|\}$. Then $|r_{3-\tau}| = \min\{|r_1|,|r_2|\}$. Let m_0 be an even integer such that $m_0 > h_1$ from Proposition 3.15 and $m_0 > h_2$ from Proposition 3.16. We choose $\mathcal{H}'_i$ as follows:

$$\mathcal{H}'_i = \{m_0 + \Delta \lceil \log q \rceil \cdot (2j + i[j]) : 0 \leqslant j \leqslant \ell\}.$$

The sets $\mathcal{H}'_i$ satisfy the following:

(a) they are distinct,
(b) they have equal cardinality $\ell+1$,
(c) they contain only positive even integers between m_0 and $m_0 + \Delta(\log q + 1)(2 \log q + 1)$, and
(d) for i, j and any $a \in \mathcal{H}'_i$ and $b \in \mathcal{H}'_j$, either $a = b$ or $|a - b| \geqslant \Delta \log q$.

It is easy to see that $\sigma(\mathcal{H}'_i) = \Omega(\log q)$, $i = 0, \ldots, q$. On the other hand, since all the numbers in each of the $\mathcal{H}'_i$ are bounded by $O(\log^2 q)$ and the size of each $\mathcal{H}'_i$ is $O(\log q)$, we get that $\sigma(\mathcal{H}'_i) = O(\log^3 q)$ for each i. From this we get that at least

$$\hat{q} = \Omega\left(\frac{q' \log^3 q' + 1}{\log^3 q'}\right) = \Omega(q')$$

of the sets $\mathcal{H}'_0, \ldots, \mathcal{H}'_q$ have the same sum value $\sigma(\mathcal{H}'_i)$. Let $\{\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}\}$ be a subset of $\{\mathcal{H}'_0, \ldots, \mathcal{H}'_q\}$ such that all the sets in $\{\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}\}$ have the same sum value $\sigma(\mathcal{H}_i)$. We have (i), (ii) and (v) for $\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}$.

We now turn to (iii) and (iv). The proofs of (iii) and (iv) are similar but not identical. Let $0 \leqslant i \neq j \leqslant \hat{q}$, $\mathcal{H}_{i \setminus j} = \mathcal{H}_i \setminus \mathcal{H}_j$ and $\mathcal{H}_{j \setminus i} = \mathcal{H}_j \setminus \mathcal{H}_i$. Notice that $\mathcal{H}_{i \setminus j} \cap \mathcal{H}_{j \setminus i} = \emptyset$. Let $\sigma = \sigma(\mathcal{H}_{i \setminus j}) = \sigma(\mathcal{H}_{j \setminus i})$ and let $d = |\mathcal{H}_{i \setminus j}| = |\mathcal{H}_{j \setminus i}|$.

(iii) We write $f_{\mathrm{t},\mathcal{H}_i}$ for short instead of $f_{\mathrm{t},\mathcal{H}_i}(e_1, e_2, r_1, r_2)$ in this proof. When other parameters are used instead of e_1, e_2, r_1, r_2, we write them explicitly. Since $f_{\mathrm{t},\mathcal{H}_i} = f_{\mathrm{t},\mathcal{H}_{i \setminus j}} \cdot f_{\mathrm{t},\mathcal{H}_i \cap \mathcal{H}_j}$, $f_{\mathrm{t},\mathcal{H}_j} = f_{\mathrm{t},\mathcal{H}_{j \setminus i}} \cdot f_{\mathrm{t},\mathcal{H}_i \cap \mathcal{H}_j}$ and $f_{\mathrm{t},\mathcal{H}_i \cap \mathcal{H}_j} \neq 0$, it is enough to show that $f_{\mathrm{t},\mathcal{H}_{i \setminus j}} - f_{\mathrm{t},\mathcal{H}_{j \setminus i}} \neq 0$.

Since $\sigma(\mathcal{H}_{i \setminus j}) = \sigma(\mathcal{H}_{j \setminus i})$ we have

$$f_{\mathrm{t},\mathcal{H}_{i \setminus j}} = f_{\mathrm{t},\mathcal{H}_{j \setminus i}} \quad \text{if and only if} \quad f_{\mathrm{t},\mathcal{H}_{i \setminus j}}(e_1, e_2, r_2^{-1}, r_1^{-1}) = f_{\mathrm{t},\mathcal{H}_{j \setminus i}}(e_1, e_2, r_2^{-1}, r_1^{-1}).$$

Hence we can assume from now on that $|r_\tau| > 1$ (otherwise we look at r_1^{-1} and r_2^{-1} instead).

[3] In fact we will also need that ℓ is larger than a constant depending on e_1, but this is true for large enough values of q.

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For every $\mathcal{H}$, $f_{t,\mathcal{H}}$ can be rewritten as follows:

$$f_{t,\mathcal{H}} = \prod_{h \in \mathcal{H}} (e_\tau r_\tau^h + e_{3-\tau} r_{3-\tau}^h) = e_{3-\tau}^{\ell+1} \sum_{X \subseteq \mathcal{H}} s_{\mathcal{H}}(X),$$

where

$$s_{\mathcal{H}}(X) = \left(\frac{e_\tau}{e_{3-\tau}}\right)^{|X|} r_\tau^{\sigma(X)} r_{3-\tau}^{\sigma(\mathcal{H}\setminus X)}.$$

We think of $h \in X$ (respectively $h \in \mathcal{H} \setminus X$) as corresponding to $e_\tau r_\tau^h$ (respectively $e_{3-\tau} r_{3-\tau}^h$).

It suffices to show that

$$\sum_{X_1 \subseteq \mathcal{H}_{i\setminus j}} s_{\mathcal{H}_{i\setminus j}}(X) - \sum_{X_2 \subseteq \mathcal{H}_{j\setminus i}} s_{\mathcal{H}_{j\setminus i}}(X) \neq 0. \tag{3.2}$$

It holds that

$$s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j}) = s_{\mathcal{H}_{j\setminus i}}(\mathcal{H}_{j\setminus i}) = \left(\frac{e_\tau}{e_{3-\tau}}\right)^{\ell+1} r_\tau^\sigma.$$

Hence, $s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j})$ and $s_{\mathcal{H}_{j\setminus i}}(\mathcal{H}_{j\setminus i})$ cancel out in (3.2). Similarly, $s_{\mathcal{H}_{i\setminus j}}(\emptyset) = s_{\mathcal{H}_{j\setminus i}}(\emptyset) = r_{3-\tau}^\sigma$ cancel out. Let m_1 be the minimal element in $\mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i}$. Without loss of generality, assume $m_1 \in \mathcal{H}_{i\setminus j}$. We have

$$s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} \setminus \{m_1\}) = \left(\frac{e_\tau}{e_{3-\tau}}\right)^{\ell} r_\tau^{\sigma - m_1} r_{3-\tau}^{m_1}.$$

Here $s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} \setminus \{m_1\})$ has the largest exponent of r_τ out of all the monomials in both of the sums in inequality (3.2), and any other exponent of r_τ is smaller by at least $\Delta \log q$. We will demonstrate that (3.2) holds by showing the following:

$$|s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} \setminus \{m_1\})| > \sum_{X \subsetneq \mathcal{H}_{i\setminus j}\setminus\{m_1\}} |s_{\mathcal{H}_{i\setminus j}}(X)| + \sum_{X \subsetneq \mathcal{H}_{j\setminus i}} |s_{\mathcal{H}_{j\setminus i}}(X)|. \tag{3.3}$$

Each of the sums in inequality (3.3) has at most $2^{\log q + 1} = 2q$ monomials corresponding to the subsets of $\mathcal{H}_{i\setminus j}$ and $\mathcal{H}_{j\setminus i}$ respectively. The absolute value of each of these monomials can be bounded from above by $s \cdot |r_\tau|^{\sigma - m_1 - \Delta \log q} |r_{3-\tau}|^{m_1 + \Delta \log q}$, where s is the maximum of $|\frac{e_\tau}{e_{3-\tau}}|^\ell$ and 1. Hence, the right-hand side of (3.3) is at most

$$4q \cdot s|r_\tau|^{\sigma - m_1 - \Delta \log q} |r_{3-\tau}|^{m_1 + \Delta \log q} = 4qs\left(\frac{e_\tau}{e_{3-\tau}}\right)^{-\ell} \cdot \left|\frac{r_{3-\tau}}{r_\tau}\right|^{\Delta \log q} |s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} - \{m_1\})|.$$

There exists a number Δ' which does not depend on q such that

$$4qs\left(\frac{e_\tau}{e_{3-\tau}}\right)^{-\ell} < (\Delta')^{\log q},$$

and (iii) follows by setting Δ sufficiently large that

$$\Delta' \cdot \left|\frac{r_{3-\tau}}{r_\tau}\right|^{\Delta} < 1.$$

(iv) By Proposition 3.15, there exist $u_1, u_2 \neq 0$ and $w \notin \{-1,0,1\}$ depending on γ, δ for which it is enough to show that $h_{\mathrm{y},\mathcal{H}_i}(u_1,u_2,w) \neq h_{\mathrm{y},\mathcal{H}_j}(u_1,u_2,w)$ to get (iv). We write $h_{\mathrm{y},\mathcal{H}_i}$ for short instead of $h_{\mathrm{y},\mathcal{H}_i}(u_1,u_2,w)$ in this proof.

Since we have $h_{\mathrm{y},\mathcal{H}_i} = h_{\mathrm{y},\mathcal{H}_{i\setminus j}} \cdot h_{\mathrm{y},\mathcal{H}_i\cap\mathcal{H}_j}$, $h_{\mathrm{y},\mathcal{H}_j} = h_{\mathrm{y},\mathcal{H}_{j\setminus i}} \cdot h_{\mathrm{y},\mathcal{H}_i\cap\mathcal{H}_j}$ and $h_{\mathrm{y},\mathcal{H}_i\cap\mathcal{H}_j} \neq 0$, it is enough to show that

$$h_{\mathrm{y},\mathcal{H}_{i\setminus j}} - h_{\mathrm{y},\mathcal{H}_{j\setminus i}} \neq 0,$$

that is,

$$\prod_{h\in\mathcal{H}_{i\setminus j}} \left(1 + \frac{1}{u_1 + u_2 \cdot w^h}\right) - \prod_{h\in\mathcal{H}_{j\setminus i}} \left(1 + \frac{1}{u_1 + u_2 \cdot w^h}\right) \neq 0,$$

or equivalently,

$$\prod_{h\in\mathcal{H}_{i\setminus j}} (u_1 + u_2 \cdot w^h + 1) \prod_{h\in\mathcal{H}_{j\setminus i}} (u_1 + u_2 \cdot w^h) - \prod_{h\in\mathcal{H}_{j\setminus i}} (u_1 + u_2 \cdot w^h + 1) \prod_{h\in\mathcal{H}_{i\setminus j}} (u_1 + u_2 \cdot w^h) \neq 0. \tag{3.4}$$

Consider a product of the form found in inequality (3.4):

$$\prod_{h\in\mathcal{H}_a} (u_1 + u_2 \cdot w^h + 1) \prod_{h\in\mathcal{H}_b} (u_1 + u_2 \cdot w^h) = \sum_{X\subseteq\mathcal{H}_a\cup\mathcal{H}_b} (u_1+1)^{|\mathcal{H}_a\setminus X|} u_1^{|\mathcal{H}_b\setminus X|} w^{\sigma(X)} u_2^{|X|}.$$

Let

$$p(X) = \left((u_1+1)^{|\mathcal{H}_{i\setminus j}\setminus X|} u_1^{|\mathcal{H}_{j\setminus i}\setminus X|} - (u_1+1)^{|\mathcal{H}_{j\setminus i}\setminus X|} u_1^{|\mathcal{H}_{i\setminus j}\setminus X|}\right) \cdot u_2^{|X|} w^{\sigma(X)}.$$

It suffices to show that

$$\sum_{X\subseteq\mathcal{H}_{i\setminus j}\cup\mathcal{H}_{j\setminus i}} p(X) \neq 0. \tag{3.5}$$

We have $p(\emptyset) = p(\mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i}) = 0$, using that $|\mathcal{H}_{i\setminus j}| = |\mathcal{H}_{j\setminus i}|$. Let m_1 be the minimal element in $\mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i}$. Without loss of generality, assume $m_1 \in \mathcal{H}_{i\setminus j}$. We have

$$|p(\mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} - \{m_1\})| = |u_2^{2d-1} w^{2\sigma - m_1}|,$$
$$|p(\{m_1\})| = |((u_1+1)u_1)^{d-1} u_2 w^{m_1}|.$$

The largest exponent of w in inequality (3.5) is $w^{2\sigma-m_1}$. For all other monomials in (3.5), the power of w is smaller by at least $\Delta \log q$. Similarly, the smallest exponent of w in (3.5) is w^{m_1}. For all other monomials in (3.5), the power of w is larger by at least $\Delta \log q$.

Let $X_0 \subseteq \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i})$ be maximal with respect to $|p(X_0)|$. Since $d \leqslant \log q + 1$, we can choose Δ large enough so that we have $X_0 = \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} - \{m_1\}$ if $|w| > 1$ and $X_0 = \{m_1\}$ if $|w| < 1$.

We have

$$\left| \sum_{X\subseteq\mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}: X\neq X_0} p(X) \right| < |p(X_0)|, \tag{3.6}$$

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implying that inequality (3.5) holds. To see that inequality (3.6) holds, note that

$$\Bigg|\sum_{\substack{X\subseteq \mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}:\\ X\neq X_0}} p(X)\Bigg| \leqslant 2^{2\log q+2} \max_{X\subseteq \mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}:X\neq X_0} |p(X)|.$$

Let

$$k(d) = \max\big(|(u_1+1)^d|,1\big)\cdot\max\big(|u_1^d|,1\big)\cdot\max\big(|u_2^d|,1\big).$$

Then

$$\Bigg|\sum_{\substack{X\subseteq \mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}:\\ X\neq X_0}} p(X)\Bigg| \leqslant \begin{cases} 4\cdot 2^{\log q+1}\cdot k(d)\cdot|w|^{2\sigma-m_1-\Delta\log q} & |w|>1,\\ 4\cdot 2^{\log q+1}\cdot k(d)\cdot|w|^{m_1+\Delta\log q} & |w|<1.\end{cases}$$

So, there is a constant $c>0$ depending on u_1,u_2,w such that (for large enough values of q)

$$\Bigg|\sum_{\substack{X\subseteq \mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}:\\ X\neq X_0}} p(X)\Bigg| \leqslant \begin{cases} c^{\log q}|w|^{2\sigma-m_1-\Delta\log q} & |w|>1,\\ c^{\log q}|w|^{m_1+\Delta\log q} & |w|<1.\end{cases}$$

It remains to choose Δ sufficiently large that

$$\begin{cases} \frac{c^{\log q}}{u_2^{2d-1}} < |w|^{\Delta\log q} & |w|>1,\\ |w|^{\Delta\log q} < \frac{((u_1+1)u_1)^{d-1}}{c^{\log q}} & |w|<1.\end{cases}$$ □

We are now ready to prove Theorem 1.1.

Theorem 3.18. *Let* $(\gamma,\delta)\in\mathbb{Q}^2$. *If* $\gamma\notin\{-1,0,1\}$ *and* $\delta\neq 0$, *then*

(i) *computing* $Z(G;\gamma,\delta)$ *is* #**P**-*hard, and*

(ii) *unless* #**ETH** *fails, computing* $Z(G;\gamma,\delta)$ *requires exponential time in* $\frac{n_G}{\log^6 n_G}$.

Otherwise, $Z(G;\gamma,\delta)$ *is polynomial-time computable.*

Proof. We set $\mathsf{t}=\gamma$ and $\mathsf{y}=\delta$ with $\gamma\notin\{-1,0,1\}$ and $\delta\neq 0$. By abuse of notation we refer to $c_1,c_2,d_1,d_2,\lambda_1,\lambda_2$ from Lemma 3.3 as the values they obtain when $\mathsf{t}=\gamma$. Since $\gamma\neq\{-1,0,1\}$, it is easy to verify that the following hold:

(a) $c_1+d_1, c_2+d_2\neq 0$,

(b) $\lambda_1,\lambda_2\neq 0$,

(c) $\lambda_1,\lambda_2\neq\pm(\gamma^2+\gamma)$, and

(d) $\lambda_1\neq\pm\lambda_2$.

Let $e_i=\frac{c_i+d_i}{2\gamma}$ and $r_i=\frac{\lambda_i}{\gamma^2+\gamma}$ for $i=1,2$. Let $q'=n_G^2$. Let $\mathcal{H}_0,\dots,\mathcal{H}_{\hat q}$ be the sets guaranteed in Lemma 3.17 with respect to $q',\gamma,\delta,e_1,e_2,r_1,r_2$.

First we deal with the case when $\gamma\neq-\delta$. We return to $\gamma=-\delta$ later.

We want to compute the $\hat q+1$ values $Z(G\otimes\mathcal{H}_k;\gamma,1)$. If $\delta=1$ we simply do it using the oracle to Z at $(\gamma,1)$. If $\delta=-1$ we use Lemma 3.9. Otherwise we proceed as follows.

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By Proposition 3.14, for each $0 \leqslant i,k \leqslant \hat{q}$,

$$Z(S_{\mathcal{H}_i}(G \otimes \mathcal{H}_k);\gamma,\delta) = g_{p,\mathcal{H}_i}(\gamma,\delta) \cdot Z(G \otimes \mathcal{H}_k;\gamma, g_{\mathsf{y},\mathcal{H}_i}(\gamma,\delta)). \tag{3.7}$$

It is guaranteed in Lemma 3.17 that for $i \neq j$, $g_{\mathsf{y},\mathcal{H}_i}(\gamma,\delta) \neq g_{\mathsf{y},\mathcal{H}_j}(\gamma,\delta)$.

We want to use equation (3.7) to interpolate, for each $0 \leqslant k \leqslant m_G$, the univariate polynomials $Z(G \otimes \mathcal{H}_k;\gamma,\mathsf{y})$. We use the fact that the sizes of $G \otimes \mathcal{H}_k$, and therefore the y-degrees of $Z(G \otimes \mathcal{H}_k;\gamma,\mathsf{y})$, are at most $O(n_G \log^3 n_G)$, Since $g_{p,\mathcal{H}_i}(\gamma,\delta)$ is non-zero, we can interpolate in polynomial time, for each $0 \leqslant k \leqslant m_G$, the m_G+1 polynomials $Z(G \otimes \mathcal{H}_k;\gamma,\mathsf{y})$.

So, we computed $Z(G \otimes \mathcal{H}_k;\gamma,1)$ for $0 \leqslant k \leqslant \hat{q}$. Now we use these values to interpolate t and get the univariate polynomial $Z(G \otimes \mathcal{H}_k;\mathsf{t},1)$. By Lemma 3.8,

$$Z(G; f_{\mathsf{t},\mathcal{H}_k}(e_1,e_2,r_1,r_2),1) = Z(G \otimes \mathcal{H}_k;\gamma,1) \cdot \left(f_{p,\mathcal{H}_k}(\gamma)\right)^{-1}.$$

Since $\gamma \notin \{-1,0,1\}$, $f_{p,\mathcal{H}_k}(\gamma) \neq 0$. By Lemma 3.17, $f_{\mathsf{t},\mathcal{H}_k}(e_1,e_2,r_1,r_2)$ are distinct and polynomial-time computable. Hence, the univariate polynomial $Z(G;\mathsf{t},1)$ can be interpolated. We get (i) by Proposition 2.5. Since $Z(-;\gamma,\delta)$ is only queried on graphs $S_{\mathcal{H}_i}(G \otimes \mathcal{H}_k)$ of sizes at most $O(n_G \log^6 n_G)$, (ii) holds by Proposition 2.7.

Consider the case $\gamma = -\delta$. By Proposition 3.14, for every G we have

$$Z(S_{\{1\}}(G);\gamma,\delta) = (\delta \cdot (1-\delta^2))^{n_G} \cdot Z(G;\gamma,-\delta^2),$$

and the desired hardness results follow by the corresponding for $Z(G;\gamma,-\delta^2)$ (using that $\gamma \neq -(-\delta^2)$, $-\delta^2 \notin \{-1,0,1\}$ and that $(\delta \cdot (1-\delta^2))^{n_G}$ is non-zero).

Now we consider the cases where $\gamma \in \{-1,0,1\}$ or $\delta = 0$. Two cases are easily computed, namely $Z(G;1,\delta) = (1+\delta)^{n_G}$ and $Z(G;\gamma,0) = 1$.

The other two cases follow, for example, from Lemma 6.3 in [13]. In that lemma it is shown in particular that partition functions $Z_{A,D}(G)$ with a matrix A of edge-weights and a diagonal matrix D of vertex weights can be computed in polynomial time if A has rank 1 or is bipartite with rank 2. For $\gamma = 0$ we have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix},$$

so A is bipartite with rank 2. For $\gamma = -1$ we have

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix},$$

so A has rank 1. Note that Lemma 6.3 in [13] extends to negative values of δ. We refer the reader to [13] for details. □

4. Simple bipartite planar graphs

In this section we show that the evaluations of $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ are generally #**P**-hard to compute, even when restricted to simple graphs which are both bipartite and planar. To do so, we use that for 3-regular graphs, $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is essentially equivalent to $Z(G;\mathsf{t},\mathsf{y})$. We use a two-dimensional graph transformation $R^{\ell,q}(G)$, which is applied to simple

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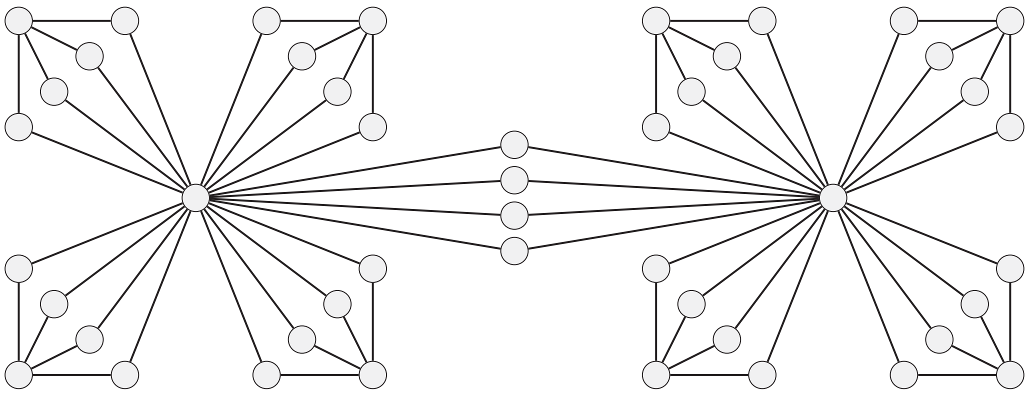

Figure 4. The construction of the graph $R^{\ell,q}(P_2)$ for $\ell = 1$ and $q = 2$, where P_2 is the path with two vertices and one edge.

3-regular bipartite planar graphs and emits simple bipartite planar graphs in order to interpolate $Z(G; \mathsf{t}, \mathsf{y})$.

4.1. Definitions

The following is a variation of *k*-*thickening* for simple graphs.

Definition 4.1 (*k*-simple thickening). Given $\ell \in \mathbb{N}^+$ and a graph H, we define a graph $STh^{\ell}(H)$ as follows. For every edge $e = (u, w)$ in $E(H)$, we add 4ℓ new vertices $v_{e,1}, \ldots, v_{e,4\ell}$ to H. For each $v_{e,i}$, we add two new edges $(u, v_{e,i})$ and $(w, v_{e,i})$. Finally, we remove the edge e from the graph. Let $N_{\ell}(e)^+$ denote the subgraph of $STh^{\ell}(H)$ induced by the set of vertices $\{v_{e,1}, \ldots, v_{e,4\ell}, u, v\}$.

The graph transformation used in the hardness proof is as follows.

Definition 4.2 ($R^{\ell,q}(G)$). Let G be a graph. For each $w \in V(G)$, let $G_w^q = (V_w^q, E_w^q)$ be a new copy of the star with $2q$ leaves. Denote by c_w the centre of the star G_w^q. Let $R^{\ell,q}(G) = (V_R^{\ell,q}, E_R^{\ell,q})$ be the graph obtained from the disjoint union of $STh^{\ell}(G)$ and $STh^{\ell}(G_w^q)$ for all $w \in V(G)$ by identifying w and c_w for all $w \in V(G)$.

Remarks.

(i) The construction of $R^{\ell,q}(G)$ can also be described as follows. Given G, we attach $2q$ new vertices to each vertex v of $V(G)$ to obtain a new simple graph G'. Then, $R^{\ell,q}(G) = STh^{\ell}(G')$.

(ii) For every simple planar graph G and $\ell, q \in \mathbb{N}^+$, $R^{\ell,q}(G)$ is a simple bipartite planar bipartite graph with n_R vertices and m_R edges, where $n_R = n_G(1 + 2q(1 + 4\ell)) + 4\ell m_G$ and and $m_R = 8\ell m_G + 16\ell q n_G$.

Figure 4 shows the graph $R^{1,2}(P_2)$.

In the following it is convenient to consider a multivariate version of $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ denoted $Z(G;\bar{\mathsf{x}},\bar{\mathsf{y}}.\bar{\mathsf{z}})$. This approach was introduced for the Tutte polynomial by Sokal [26]. $Z(G;\bar{\mathsf{x}},\bar{\mathsf{y}}.\bar{\mathsf{z}})$ has indeterminates which correspond to every $v \in V(G)$ and every $e \in E(G)$.

Definition 4.3. Let $\bar{\mathsf{x}} = (\mathsf{x}_e : e \in E(G))$, $\bar{\mathsf{y}} = (\mathsf{y}_u : u \in V(G))$ and $\bar{\mathsf{z}} = (\mathsf{z}_e : e \in E(G))$ be tuples of distinct indeterminates. Let

$$Z(G;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}) = \sum_{S\subseteq V(G)} \Bigg(\prod_{e\in E_G(S)} \mathsf{x}_e\Bigg)\Bigg(\prod_{u\in S} \mathsf{y}_u\Bigg)\Bigg(\prod_{e\in E_G(\bar{S})} \mathsf{z}_e\Bigg).$$

We may write $\mathsf{x}_{w,v}$ and $\mathsf{z}_{w,v}$ instead of x_e and z_e for an edge $e = (w,v)$. Clearly, by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every $e \in E(G)$, and $\mathsf{y}_u = \mathsf{y}$ for every $u \in V(G)$, we get $Z(G;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}) = Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$.

We furthermore define a variation of $Z(G;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$ obtained by restricting the range of the summation variable as follows.

Definition 4.4. Given a graph H and $B, C \subseteq V(H)$ with B and C disjoint, let

$$Z(H,B,C;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}) = \sum_{A:B\subseteq A\subseteq V(H),\, A\cap C=\emptyset} \Bigg(\prod_{e\in E_G(A)} \mathsf{x}_e\Bigg)\Bigg(\prod_{u\in A\setminus B} \mathsf{y}_u\Bigg)\Bigg(\prod_{e\in E_G(\bar{A})} \mathsf{z}_e\Bigg), \tag{4.1}$$

where the summation is over all $A \subseteq V(H)$, such that A contains B and is disjoint from C.

We have $Z(H,\emptyset,\emptyset;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}) = Z(H;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$.

4.2. Lemmas, statement of Theorem 1.2 and its proof

For every edge $e \in E(G)$ between u and v, let

$$\omega_1(e,S) = Z(N_\ell(e)^+, S\cap\{u,v\}, \{u,v\}\setminus S;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}),$$

and for every vertex $w \in V$, let

$$\omega_2(w,S) = Z(STh^\ell(G_w^q), S\cap\{w\}, \{w\}\setminus S;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}).$$

Let

$$\omega_1(S) = \prod_{e\in E(G)} \omega_1(e,S) \quad \text{and} \quad \omega_2(S) = \prod_{w\in V(G)} \omega_2(w,S).$$

Let $\omega_{i,\mathrm{triv}}(S)$ for $i = 1,2$ be the polynomials in x, y and z obtained from $\omega_i(S)$ by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every $e \in E^{\ell,q}$ and $\mathsf{y}_v = \mathsf{y}$ for every $v \in V_R^{\ell,q}$.

Lemma 4.5.

$$Z(R^{\ell,q}(G);\mathsf{x}.\mathsf{y}.\mathsf{z}) = \sum_{S\subseteq V(G)} \omega_{1,\mathrm{triv}}(S)\cdot\omega_{2,\mathrm{triv}}(S)\cdot \mathsf{y}^{|S|}.$$

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Proof. Each edge of $R^{\ell,q}(G)$ is either contained in some $N_\ell(e)^+$ for $e \in E(G)$ or in some $STh^\ell(G^q_w)$ for $w \in V(G)$. Hence, by the definitions of $Z(R^{\ell,q}(G); \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$, $\omega_1(S)$ and $\omega_2(S)$,

$$Z(R^{\ell,q}(G); \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = \sum_{S \subseteq V(G)} \omega_1(S) \cdot \omega_2(S) \cdot \prod_{w \in S} \mathsf{y}_w$$

holds and the lemma follows. □

Lemma 4.6. *Let $e = (u, w)$ be an edge of G. Then*

$$\omega_{1,\mathrm{triv}}(e, S) = \begin{cases} (\mathsf{y} + \mathsf{z}^2)^{4\ell} & |\{u, v\} \cap S| = 0, \\ (\mathsf{xy} + \mathsf{z})^{4\ell} & |\{u, v\} \cap S| = 1, \\ (\mathsf{yx}^2 + 1)^{4\ell} & |\{u, v\} \cap S| = 2. \end{cases}$$

Proof. The value of $\omega_1(e, S)$ depends only on whether $u, w \in S$. Consider $A \subseteq V(N_\ell(e)^+)$, which satisfies the summation conditions in equation (4.1) for $Z(N_\ell(e)^+, S \cap \{u, w\}, \{u, w\} \setminus S; \mathsf{x}, \mathsf{y}, \mathsf{z})$.

(i) The case $w \in S$ and $u \notin S$. Exactly one edge e' incident to $v_{e,i}$ crosses the cut $[A, \bar{A}]_{N_\ell(e)^+}$. The other edge e'' incident to $v_{e,i}$ belongs to $E(A)$ or $E(\bar{A})$, depending on whether $v_{e,i} \in A$. We get

$$\omega_1(e, S) = \prod_{i=1}^{4\ell} (\mathsf{x}_{v_{e,i},w} \mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},u}).$$

(ii) The case $w \notin S$ and $u \in S$. This case is similar to the previous case, and we get

$$\omega_1(e, S) = \prod_{i=1}^{4\ell} (\mathsf{x}_{v_{e,i},u} \mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w}).$$

(iii) The case $w, u \in S$. For each $v_{e,i}$, either $v_{e,i} \in A$, in which case both edges $(v_{e,i}, w)$ and $(v_{e,i}, u)$ are in $E(A)$, or $v_{e,i} \notin S$, and both edges $(v_{e,i}, w)$ and $(v_{e,i}, u)$ cross the cut. We get

$$\omega_1(e, S) = \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},u} \mathsf{x}_{v_{e,i},w} + 1).$$

(iv) The case $w, u \notin S$. For each $v_{e,i}$, either $v_{e,i} \in S$ and then both edges incident to $v_{e,i}$ cross the cut, or $v_{e,i} \notin S$ and neither of the two edges crosses the cut. We get

$$\omega_1(e, S) = \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w} \mathsf{z}_{v_{e,i},u}).$$

The lemma follows by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every edge e and $\mathsf{y}_u = \mathsf{y}$ for every vertex u. □

Lemma 4.7. *Let*

$$g_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z}) = \mathsf{y} \cdot \left(\mathsf{yx}^2 + 1\right)^{4\ell} + \left(\mathsf{yx} + \mathsf{z}\right)^{4\ell},$$
$$h_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z}) = \left(\mathsf{y} + \mathsf{z}^2\right)^{4\ell} + \mathsf{y} \cdot \left(\mathsf{yx} + \mathsf{z}\right)^{4\ell}.$$

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Let w be a vertex of G. Then

$$\omega_{2,\mathrm{triv}}(w,S) = \begin{cases} \left(g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})\right)^{2q} & w \in S, \\ \left(h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})\right)^{2q} & w \notin S. \end{cases}$$

Proof. Consider A which satisfies the summation conditions in equation (4.1) for

$$Z(STh^{\ell}(G_w^q), S \cap \{w\}, \{w\} \setminus S; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}).$$

(i) The case $w \in S$ (or, equivalently, $c_w \in A$). Let $u \in V_w^q \setminus \{c_w\}$ and $e = \{u, c_w\}$. If $u \in A$, then the vertices u and $v_{e,1}, \dots, v_{e,4\ell}$ contribute

$$\mathsf{y}_u \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},w} \mathsf{x}_{v_{e,i},u} + 1).$$

Otherwise, if $u \notin A$, then the vertices u and $v_{e,1}, \dots, v_{e,4\ell}$ contribute

$$\prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},w} + \mathsf{z}_{v_{e,i},u}).$$

Hence, $\omega_2(w,S)$ equals in this case

$$\prod_{u \in V_w^q} \left(\mathsf{y}_u \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},w} \mathsf{x}_{v_{e,i},u} + 1) + \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},w} + \mathsf{z}_{v_{e,i},u}) \right).$$

(ii) The case $w \notin S$ (or, equivalently, $c_w \notin A$). Let $u \in V_w^q \setminus \{c_w\}$ and $e = \{u, c_w\}$. If $u \in A$, then the vertices u and $v_{e,1}, \dots, v_{e,4\ell}$ contribute

$$\mathsf{y}_u \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},u} + \mathsf{z}_{v_{e,i},w}).$$

Otherwise, if $u \notin A$, then the vertices u and $v_{e,1}, \dots, v_{e,4\ell}$ contribute

$$\prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w} \mathsf{z}_{v_{e,i},u}).$$

Hence, $\omega_2(w,S)$ equals in this case

$$\prod_{u \in V_w^q} \left(\prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},u} + \mathsf{z}_{v_{e,i},w}) + \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w} \mathsf{z}_{v_{e,i},u}) \right).$$

The lemma follows by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every edge e and $\mathsf{y}_u = \mathsf{y}$ for every vertex u. □

Lemma 4.8. *If G is d-regular, then*

$$f_{p,R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell,q) \cdot Z(G; f_{\mathsf{t},R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell), f_{\mathsf{y},R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell,q)) = Z(R^{\ell,q}(G); \mathsf{x},\mathsf{y},\mathsf{z}),$$

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where

$$f_{p,R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell,q) = \big(h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})\big)^{2qn_G}(\mathsf{y}+\mathsf{z}^2)^{2\ell d n_G},$$
$$f_{\mathrm{t},R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell) = \left(\frac{(\mathsf{yx}+\mathsf{z})^2}{(\mathsf{yx}^2+1)(\mathsf{y}+\mathsf{z}^2)}\right)^{2\ell},$$
$$f_{\mathrm{y},R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell,q) = \mathsf{y}\cdot\left(\frac{\mathsf{yx}^2+1}{\mathsf{y}+\mathsf{z}^2}\right)^{2\ell d}\left(\frac{g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})}{h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})}\right)^{2q}.$$

Proof. We want to rewrite $Z(R^{\ell,q}(G);\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$ as a sum over subsets S of vertices of G. Using Lemma 4.5, in order to compute $Z(R^{\ell,q}(G);\mathsf{x},\mathsf{y},\mathsf{z})$ we first need to find $\omega_{1,\mathrm{triv}}(S)$ and $\omega_{2,\mathrm{triv}}(S)$. Using Lemma 4.7, $\omega_{2,\mathrm{triv}}(S)$ is given by

$$\omega_{2,\mathrm{triv}}(S) = \big(g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})\big)^{2q|S|}\cdot\big(h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})\big)^{2qn_G-2q|S|}.$$

In order to compute $\omega_{1,\mathrm{triv}}(S)$, consider $S\subseteq V(G)$. Since G is d-regular, the number of edges contained in S is $\frac{1}{2}(d\cdot|S|-|[S,\bar{S}]_G|)$, and the number of edges contained in $\bar{S}$ is $\frac{1}{2}(dn_G-d\cdot|S|-|[S,\bar{S}]_G|)$. Hence, by Lemma 4.6, $\omega_{1,\mathrm{triv}}(S)$ is given by

$$\omega_{1,\mathrm{triv}}(S) = (\mathsf{xy}+\mathsf{z})^{4\ell|[S,\bar{S}]_G|}(\mathsf{yx}^2+1)^{4\ell\cdot\frac{d\cdot|S|-|[S,\bar{S}]_G|}{2}}(\mathsf{y}+\mathsf{z}^2)^{4\ell\cdot\frac{dn_G-d\cdot|S|-|[S,\bar{S}]_G|}{2}}.$$

Using Lemma 4.5,

$$Z(R^{\ell,q}(G);\mathsf{x},\mathsf{y},\mathsf{z}) = \sum_{S\subseteq V(G)}\omega_{1,\mathrm{triv}}(S)\cdot\omega_{2,\mathrm{triv}}(S)\cdot\mathsf{y}^{|S|},$$

which is equal to $(\mathsf{y}+\mathsf{z}^2)^{4\ell\cdot\frac{dn_G}{2}}$ times

$$\sum_{S\subseteq V(G)}\left(\frac{(\mathsf{yx}+\mathsf{z})^2}{(\mathsf{yx}^2+1)(\mathsf{y}+\mathsf{z}^2)}\right)^{2\ell|[S,\bar{S}]_G|}\left(\mathsf{y}\cdot\left(\frac{\mathsf{yx}^2+1}{\mathsf{y}+\mathsf{z}^2}\right)^{2\ell d}\right)^{|S|}\cdot\omega_{2,\mathrm{triv}}(S). \tag{4.2}$$

Plugging the expression for $\omega_{2,\mathrm{triv}}(S)$ into equation (4.2), we get that $Z(R^{\ell,q}(G);\mathsf{x},\mathsf{y},\mathsf{z})$ equals $f_{p,R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell,q)$ times

$$\sum_{S\subseteq V(G)}\left(\frac{(\mathsf{yx}+\mathsf{z})^2}{(\mathsf{yx}^2+1)(\mathsf{y}+\mathsf{z}^2)}\right)^{2\ell|[S,\bar{S}]_G|}\left(\mathsf{y}\cdot\left(\frac{\mathsf{yx}^2+1}{\mathsf{y}+\mathsf{z}^2}\right)^{2\ell d}\left(\frac{g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})}{h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})}\right)^{2q}\right)^{|S|},$$

and the lemma follows. □

Lemma 4.9. *Let* $e\in\mathbb{Q}\backslash\{-1,0,1\}$ *and let* $a,b,c>0$ *and* $b\neq c$. *Then there is* $c_1\in\mathbb{N}$ *for which the sequence*

$$h(\ell) = \frac{e\cdot b^\ell + a^\ell}{c^\ell + e\cdot a^\ell} \tag{4.3}$$

is strictly monotone increasing or decreasing for $\ell\geqslant c_1$.

Proof. $h(\ell)$ can be rewritten as

$$h(\ell) = \frac{e\cdot\tilde{b}^\ell+1}{\tilde{c}^\ell+e}$$

by dividing both the numerator and the denominator of the right-hand side of equation (4.3) by a^ℓ and setting $\tilde{b} = \frac{b}{a}$ and $\tilde{c} = \frac{c}{a}$. We have $\tilde{b} \neq \tilde{c}$ and $\tilde{b}, \tilde{c} > 0$.

Let $h(x) = \frac{e \cdot \tilde{b}^x + 1}{\tilde{c}^x + e}$. The derivative of $h(x)$ is given by

$$h'(x) = \frac{e \ln \tilde{b} \cdot \tilde{b}^x(\tilde{c}^x + e) - \ln \tilde{c} \cdot \tilde{c}^x(e \cdot \tilde{b}^x + 1)}{(\tilde{c}^x + e)^2} = \frac{e^2 \ln \tilde{b} \cdot \tilde{b}^x - \ln \tilde{c} \cdot \tilde{c}^x + e(\ln \tilde{b} - \ln \tilde{c})\tilde{b}^x\tilde{c}^x}{(\tilde{c}^x + e)^2}. \tag{4.4}$$

The denominator of $h'(x)$ is non-zero for large enough x. Therefore, there exists x_0 such that $h'(x)$ is continuous on $[x_0, \infty)$, so it is enough to show that $h'(x) \neq 0$ for all large enough x to get the desired result.

If $\tilde{b} = 1$ then $(\tilde{c}^x + e)^2 h'(x) = -(1 + e) \ln \tilde{c} \cdot \tilde{c}^x$, and if $\tilde{c} = 1$ then $(\tilde{c}^x + e)^2 h'(x) = (e^2 + e) \ln \tilde{b} \cdot \tilde{b}^x$. In both cases $h'(x)$ is non-zero, using that $\tilde{b} \neq \tilde{c}$ and $\tilde{b}, \tilde{c} > 0$.

Otherwise, $\tilde{b}$, $\tilde{c}$ and $\tilde{b}\tilde{c}$ are distinct. Let $A_1 = \{\tilde{b}^x, \tilde{c}^x, \tilde{b}^x\tilde{c}^x\}$. Let A_2 be the subset of A_1 which contains the functions of A_1 which have non-zero coefficients in equation (4.4). Note that $\tilde{b}^x\tilde{c}^x$ belongs of A_2. There is a function in A_2 which dominates the other functions of A_2. This implies that $h'(x)$ is non-zero for large enough values of x. □

Theorem 1.2 is now given precisely and proved as follows.

Theorem 4.10. *For all* $(\gamma, \delta, \varepsilon) \in \mathbb{Q}^3$ *such that*

(i) $\delta \neq \{-1, 0, 1\}$,
(ii) $\delta + \varepsilon^2 \notin \{-1, 0, 1\}$,
(iii) $\delta + \varepsilon^2 \neq \pm(\delta\gamma^2 + 1)$,
(iv) $\delta\gamma^2 + 1 \neq 0$,
(v) $\gamma\delta + \varepsilon \neq 0$, *and*
(vi) $(\gamma\delta + \varepsilon)^4 \neq (\delta\gamma^2 + 1)^2(\delta + \varepsilon^2)^2$.

$Z(-; \gamma, \delta, \varepsilon)$ *is* #**P**-*hard on simple bipartite planar graphs.*

Proof. We will show that, on 3-regular bipartite planar graphs G, the polynomial $Z(G; \mathsf{t}, \mathsf{y})$ is polynomial-time computable using oracle calls to $Z(-; \gamma, \delta, \varepsilon)$. The oracle is only queried with input of simple bipartite planar graphs. Using Proposition 2.5, computing $Z(G; \mathsf{t}, \mathsf{y})$ is #**P**-hard on 3-regular bipartite planar graphs.

Using (i) and (ii), it can be verified that there exists $c_0 \in \mathbb{N}^+$ such that for all $\ell \geqslant c_0$ and $q \in \mathbb{N}^+$, $f_{p,R}(\gamma, \delta, \varepsilon, 2\ell, q) \neq 0$. We can use Lemma 4.8 to manufacture, in polynomial time, evaluations of $Z(G; \mathsf{t}, \mathsf{y})$ that will be used to interpolate $Z(G; \mathsf{t}, \mathsf{y})$.

Let $\ell \geqslant c_0$ and let

$$E_{\mathsf{y},1} = \frac{\delta\gamma^2 + 1}{\delta + \varepsilon^2} \quad \text{and} \quad E_{\mathsf{y},2,\ell} = \frac{\delta(\delta\gamma^2 + 1)^{4\ell} + (\gamma\delta + \varepsilon)^{4\ell}}{(\delta + \varepsilon^2)^{4\ell} + \delta(\gamma\delta + \varepsilon)^{4\ell}}.$$

We have that $f_{\mathsf{y},R}(\gamma, \delta, \varepsilon, \ell, q) = \delta(E_{\mathsf{y},1})^{2d\ell}(E_{\mathsf{y},2,\ell})^{2q}$. Using (iv) we have $E_{\mathsf{y},1} \neq 0$.

Consider $E_{\mathsf{y},2,\ell}$ as a function of ℓ. Using (i), (ii), (iii) and (iv) and Lemma 4.9 with $a = (\gamma\delta + \varepsilon)^4$, $b = (\delta\gamma^2 + 1)^4$, $c = (\delta + \varepsilon^2)^4$ and $e = \delta$, there exists c_1 such that $E_{\mathsf{y},2,\ell}$ is strictly monotone increasing or decreasing. Hence, there exists $c_2 \geqslant c_1$ such that, for every $\ell \geqslant c_2$, $E_{\mathsf{y},2,\ell} \notin \{-1, 0, 1\}$. Moreover, $c_2 = c_2(\gamma, \delta, \varepsilon)$ is a function of γ, δ and ε.

We get that for $q_1 \neq q_2 \in [n_G+1]$ and $\ell > c_2$, $(E_{\mathsf{y},2,\ell})^{2q_1} \neq (E_{\mathsf{y},2,\ell})^{2q_2}$. Since $\delta(E_{y,1})^{2d\ell}$ is not equal to 0 and does not depend on q, we get that for $q_1 \neq q_2 \in [n_G+1]$, $f_{\mathsf{y},R}(\gamma,\delta,\varepsilon,\ell,q_1) \neq f_{\mathsf{y},R}(\gamma,\delta,\varepsilon,\ell,q_2)$.

For every $\ell \in [m_G+c_2+1] \setminus [c_2]$, we can interpolate in polynomial time the univariate polynomial $Z(G;f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell),\mathsf{y})$. Then, we can use the polynomial $Z(G;f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell),\mathsf{y})$ to compute $Z(G;f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell),j)$ for every $\ell \in [m_G+c_2+1] \setminus [c_2]$ and every $j \in [n_G+1]$. Let

$$E_{\mathsf{t}} = \left(\frac{(\gamma\delta+\varepsilon)^2}{(\delta\gamma^2+1)(\delta+\varepsilon^2)}\right)^2,$$

and it holds that $f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell,q) = (E_{\mathsf{t}})^{\ell}$. Clearly, $E_{\mathsf{t}} \neq -1$ and, by (v) and (vi), $E_{\mathsf{t}} \notin \{0,1\}$. Hence, for every $\ell_1 \neq \ell_2 \in \mathbb{N}^+$ we have $f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell_1) \neq f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell_2)$. Therefore, we can compute the value of the bivariate polynomial $Z(G;\mathsf{t},\mathsf{y})$ on a grid of points of size $(m_G+1)\times(n_G+1)$ in polynomial time using the oracle, and use them to interpolate $Z(G;\mathsf{t},\mathsf{y})$. □

5. Computation on graphs of bounded clique-width

In this section we prove Theorem 1.3. Let G be a graph and let $cw(G)$ be its clique-width. As discussed in Section 2.3, a k-expression $t(G)$ for G with $k \leqslant 2^{3cw(G)+2}-1$ can be computed in **FPT**-time. Let $\bar{c} = (c_v : v \in V(G))$ be the labels from $[k]$ associated with the vertices of G by $t(G)$. We will show how to compute a multivariate polynomial $Z_{\text{labelled}}(G,\bar{c};\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$ with indeterminate set

$$\{\mathsf{x}_{\{i,j\}},\mathsf{y}_i,\mathsf{z}_{\{i,j\}} \mid i,j \in [k]\},$$

to be defined below. Note that it is not the same multivariate polynomial as in Section 4. For simplicity of notation we write, for example, $\mathsf{x}_{i,j}$ or $\mathsf{x}_{j,i}$ for $\mathsf{x}_{\{i,j\}}$. The multivariate polynomial $Z_{\text{labelled}}(G,\bar{c};\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$ is defined by

$$\sum_{S\subseteq V(G)}\left(\prod_{v\in S}\mathsf{y}_{c_v}\right)\left(\prod_{(u,v)\in E_G(S)}\mathsf{x}_{c_u,c_v}\right)\left(\prod_{(u,v)\in E_G(\bar{S})}\mathsf{z}_{c_u,c_v}\right). \tag{5.1}$$

The leftmost product in equation (5.1) is over all vertices v in S. The two other products are over all edges in $E_G(S)$ and $E_G(\bar{S})$ respectively. It is not hard to see that $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is obtained from $Z_{\text{labelled}}(G,\bar{c};\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$ by substituting all the indeterminates $\mathsf{x}_{i,j}$, y_i and $\mathsf{z}_{i,j}$ by three indeterminates, x, y and z, respectively.

Given tuples of natural numbers $\bar{a} = (a_i : i \in [k])$, $\bar{b} = (b_{i,j} : i,j \in [k])$ and $\bar{c} = (c_{i,j} : i,j \in [k])$, we denote by $t_{\bar{a},\bar{b},\bar{c}}(G)$ the coefficient of the monomial

$$\prod_{i\in[k]}\mathsf{y}_i^{a_i}\prod_{i,j\in k}\mathsf{x}_{i,j}^{b_{i,j}}\mathsf{z}_{i,j}^{c_{i,j}}$$

in $Z_{\text{labelled}}(G;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$. We call a triple $(\bar{a},\bar{b},\bar{c})$ *valid* if $a_1+\ldots+a_k \leqslant n_G$ and, for all $i,j \in [k]$, $b_{i,j},c_{i,j} \leqslant m_G$. If $(\bar{a},\bar{b},\bar{c})$ is not valid, then $t_{\bar{a},\bar{b},\bar{c}}(G) = 0$. Therefore, to determine the polynomial $Z_{\text{labelled}}(G;\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$ we need only find $t_{\bar{a},\bar{b},\bar{c}}(G)$ for all valid triples $(\bar{a},\bar{b},\bar{c})$.

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The $t_{\bar{a},\bar{b},\bar{c}}(G)$ form an $(k+2k^2)$-dimensional array with $(\max\{n_G, m_G\})^{k+2k^2}$ integer entries. Each entry in this table can be bounded from above by 2^{n_G} and thus can be written in polynomial space, so the size of the table is of the form ${n_G}^{p_1(cw(G))}$, where p_1 is a function of $cw(G)$ which does not depend on n_G.

We compute $Z_{\text{labelled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ of G by dynamic programming on the structure of the k-expression of G.

Algorithm 5.1.

1 If (G, i) is a singleton of any colour i, $Z_{\text{labelled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = 1 + \mathsf{y}_i$.

2 If $(G, \bar{c})$ is the disjoint union of $(H, \bar{c}_{H_1})$ and $(H_2, \bar{c}_{H_2})$, then

$$Z_{\text{labelled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = Z_{\text{labelled}}(H_1, \bar{c}_{H_1}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{z}) \cdot Z_{\text{labelled}}(H_2, \bar{c}_{H_2}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{z}).$$

3 The case $(G, \bar{c}) = \eta_{p,r}(H, \bar{c}_H)$. Let d_r and d_p be the number of vertices of colours r and p in H, respectively.

3(a) *For every valid $(\bar{a}, \bar{b}, \bar{c})$, if*

$$b_{p,r} = \begin{cases} a_p \cdot a_r & p \neq r, \\ \binom{a_p}{2} & p = r, \end{cases} \quad \text{and} \quad c_{p,r} = \begin{cases} (d_p - a_p)\cdot(d_r - a_r) & p \neq r, \\ \binom{d_p - a_p}{2} & p = r, \end{cases} \tag{5.2}$$

set

$$t_{\bar{a},\bar{b},\bar{c}}(G) = \sum_{\bar{b}', \bar{c}'} t_{\bar{a},\bar{b}',\bar{c}'}(H),$$

where the summation is over all valid tuples $\bar{b}' = (b'_{i,j} : i, j \in [k])$ and $\bar{c}' = (c'_{i,j} : i, j \in [k])$ such that $b'_{i,j} = b_{i,j}$ and $c'_{i,j} = c_{i,j}$ if $\{i, j\} \neq \{p, r\}$.

3(b) *For every valid $(\bar{a}, \bar{b}, \bar{c})$, if equation (5.2) does not hold, set $t_{\bar{a},\bar{b},\bar{c}}(G) = 0$.*

4 The case $(G, \bar{c}) = \rho_{p\to r}(H, \bar{c}_H)$.

4(a) *For every valid $(\bar{a}, \bar{b}, \bar{c})$, if $a_p = 0$, set*

$$t_{\bar{a},\bar{b},\bar{c}}(G) = \sum_{\bar{a}', \bar{b}', \bar{c}'} t_{\bar{a}',\bar{b}',\bar{c}'}(H),$$

where the summation is over all valid tuples $\bar{a}' = (a'_i : i \in [k])$, $\bar{b}' = (b'_{i,j} : i, j \in [k])$ and $\bar{c}' = (c'_{i,j} : i, j \in [k])$ such that

- $a_r = a'_p + a'_r$,
- $a_i = a'_i$ *for all* $i \notin \{p, r\}$,
- *for all* $j \in [k] \setminus \{p\}$,

$$b_{j,r} = \begin{cases} b'_{j,p} + b'_{j,r} & \text{if } j \neq r, \\ b'_{r,r} + b'_{p,r} + b'_{p,p} & \text{if } j = r, \end{cases} \quad \text{and} \quad c_{j,r} = \begin{cases} c'_{j,p} + c'_{j,r} & \text{if } j \neq r, \\ c'_{r,r} + c'_{p,r} + c'_{p,p} & \text{if } j = r, \end{cases}$$

and

- *for all* $i, j \in [k] \setminus \{p, r\}$, $b_{i,j} = b'_{i,j}$ *and* $c_{i,j} = c'_{i,j}$.

4(b) *For every valid $(\bar{a}, \bar{b}, \bar{c})$, if $a_p \neq 0$, set $t_{\bar{a},\bar{b},\bar{c}}(G) = 0$.*

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Correctness.

1 By direct computation.
2 Proved in [1] for $Z(G;\mathsf{t},\mathsf{y})$. The trivariate case is similar.
3 The case $G=\eta_{p,r}(H)$. Let S be a subset of vertices of $V(G)=V(H)$ with a_p and a_r vertices of colours p and r respectively. After adding all possible edges between vertices of colour p and colour r in S, the number of edges between such vertices in $E_G(S)$ is $a_p\cdot a_r$ if $r\neq p$ and $\binom{a_p}{2}$ if $p=r$. Similarly, the number of edges between vertices coloured p and r in $E_G(\bar{S})$ is $(d_p-a_p)\cdot(d_r-a_r)$ if $r\neq p$ and $\binom{d_p-a_p}{2}$ if $p=r$.
4 The case $G=\rho_{p\to r}(H)$. Let S be a subset of vertices of $V(G)=V(H)$. After recolouring every vertex of colour p in S to colour r, we have $a_p=0$. Every edge between a vertex coloured p and any other vertex lies after the recolouring between a vertex coloured r and another vertex. There is one special case, which is the edges that lie between vertices coloured r after the recolouring. Before the recolouring these edges were incident to vertices coloured any combination of p and r.

Running time. The size of the $(2^{3cw(G)+2}-1)$-expression is bounded by $n^c\cdot f_1(k)$ for some constant c, which does not depend on $cw(G)$, and for some function f_1 of $cw(G)$. Now we look at the possible operations performed by Algorithm 5.1.

1 The time does not depend on n since G is of size $O(1)$.
2 The time can be bounded by the size of the table $t_{\bar{a},\bar{b},\bar{c}}$ to the power of 3, *i.e.*, $n^{3p_1(cw(G))}$.
3 For $\mu_{p,r}$, the algorithm loops over all the values in the table $t_{\bar{a},\bar{b},\bar{c}}$, and for each entry possibly computes a sum over at most m_G elements. Then, the algorithm loops over all the values again and performs $O(1)$ operations.
4 For $\rho_{p\to r}$, the algorithm loops over all the values in the table $t_{\bar{a},\bar{b},\bar{c}}$, and for each entry possibly computes a sum over elements of the table $t_{\bar{a},\bar{b},\bar{c}}$. Then, the algorithm loops over all the values again and performs $O(1)$ operations.

Hence, Algorithm 5.1 runs in time $O(n_G^{f(cw(G))})$ for some function f.[4]

6. Conclusion and open problems

Applying the reductions used in the proof of Theorem 1.1 to planar graphs gives again planar graphs. Combining Theorem 1.1 and its proof with Lemma 4.8, a hardness result for the trivariate Ising polynomial on planar graphs analogous to Theorem 1.2 follows. However, neither Theorem 1.2 nor the analogue for planar graphs are dichotomy theorems, since each of them leaves an exceptional set of low dimension unresolved. Theorem 1.2 serves mainly to suggest the existence of a dichotomy theorem for $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ on bipartite planar graphs.

Another open problem which arises from the paper is whether $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ requires exponential time to compute in general under **#ETH**. One approach to the latter problem would be to prove that, say, the permanent or the number of maximum cuts require exponential time under **#ETH** even when restricted to regular graphs.

[4] Running times of this kind are referred to as *fixed-parameter polynomial time* (**FPPT**) in [20], where the computation of various graph polynomials of graphs of bounded clique-width is treated.

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