Fast Magnetic Reconnection

The "*severing and reconnection of lines of force*," Parker and Krook (1956)

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Mathematics and Maxwell's equations give a characteristic time scale for magnetic reconnection [1].

The reconnection time is the ideal evolution time multiplied a term that depends logarithmically on non-ideal effects. $\tau_{rec} = \tau_{ev} \ln\left(R_m\right).$ *R_m is magnetic Reynolds number.*

This result follows from $\vec{B}$ depending non-trivially on three spatial coordinates and mathematics and physics concepts traditionally ignored in reconnection theory.

Related mathematics and physics explain [1] why temperature equilibrates in a room in of order ten minutes instead of weeks. See Aref's 1984 paper "Stirring by chaotic advection" [2].

Will explain required concepts, traditionally ignored in reconnection theory, and give simple illustrative examples.

Central Mathematics Concept: Chaotic Flows

A flow is chaotic when neighboring pairs of streamlines $d\vec{x}/dt = \vec{v}(\vec{x},t)$ in a non-zero volume of space separate exponentially.

A flow is not chaotic when the only exponentiation of streamlines is due to an X-point.

Chaotic flows are by definition deterministic and can be simple and smooth. Essentially all natural flows are chaotic.

A divergence-free flow in two dimensions, $v_x = -\partial h/\partial y$ and $v_y = \partial h/\partial x$, is generally chaotic when the Hamiltonian (streamfunction) $h(x,y,t)$ has a non-trivial dependence on all three variables.

Although chaos is defined by infinitesimally separated pairs of streamlines, streamlines also exponentiate apart when separated by a distance less than a, the characteristic spatial scale of $\vec{v}(\vec{x},t)$.

When streamlines are separated by a distance greater than a they fold back on themselves and their separation increases only diffusively—in simple cases as $\sqrt{t}$.

Chaotic Flow $\vec{v} = \hat{z} \times \vec{\nabla} h$ in a Circular Disk [3]

$$\boldsymbol{h(x,y,t) = \left(1 - \frac{r^2}{a^2}\right)^3 \tilde{h}(x,y,t)};$$

$$\tilde{h} = \frac{a^2}{\tau}\left(c_0 \cos\left(\omega_0 \frac{t}{\tau}\right) + c_1 x \cos\left(\omega_1 \frac{t}{\tau}\right) + c_2 y \sin\left(\omega_2 \frac{t}{\tau}\right) + c_3 xy \cos\left(\omega_3 \frac{t}{\tau}\right)\right).$$

Term depending on $r^2 \equiv x^2 + y^2$ keeps flow confined to interior of disk.

Chose $\boldsymbol{c_0 = 0}$, $\boldsymbol{c_1 = c_2 = c_3 = 1/4}$ and $\boldsymbol{\omega_1 = 6\pi, \omega_2 = 4\pi, \omega_3 = 0}$; the constant τ is the periodicity or transit time.

Textbook cases of chaos have the circular flow term c_0 large and $\omega_0 = 0$, but this term cannot be strong in the drive for coronal loops for it tends to make them kink.

Observed coronal loops must have a footpoint drive that is consistent with their existence and include terms that are slowly varying in (x, y) to have the large scale exponentiation that leads to fast reconnection.

Frobenius Norm as the Measure of Exponentiation

A streamline started at $\boldsymbol{x_0, y_0}$ is located at $\vec{\boldsymbol{x}} = \boldsymbol{x(x_0, y_0, t)}\hat{\boldsymbol{x}} + \boldsymbol{y(x_0, y_0, t)}\hat{\boldsymbol{y}}$ at time $\boldsymbol{t}$. The Frobenius norm is

$$\left\| \begin{matrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{matrix} \right\| \equiv \sqrt{\left(\frac{\partial x}{\partial x_0}\right)^2 + \left(\frac{\partial x}{\partial y_0}\right)^2 + \left(\frac{\partial y}{\partial x_0}\right)^2 + \left(\frac{\partial y}{\partial y_0}\right)^2}.$$

Deterministic chaos in math means the Frobenius norm increases exponentially in a finite volume of space—not just on the separatrix of an X-point.

The Frobenius norm of a divergence-free flow equals $\sqrt{\Lambda_u^2 + 1/\Lambda_u^2}$ *of a Singular Value Decomposition (SVD) of the Jacobian matrix using two unitary matrices* $\overleftrightarrow{U}$ *and* $\overleftrightarrow{V}$ *with* $\Lambda_u \geq \Lambda_s$*,*

$$\begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{pmatrix} = \overleftrightarrow{U} \cdot \begin{pmatrix} \Lambda_u & 0 \\ 0 & \Lambda_s \end{pmatrix} \cdot \overleftrightarrow{V}^\dagger.$$

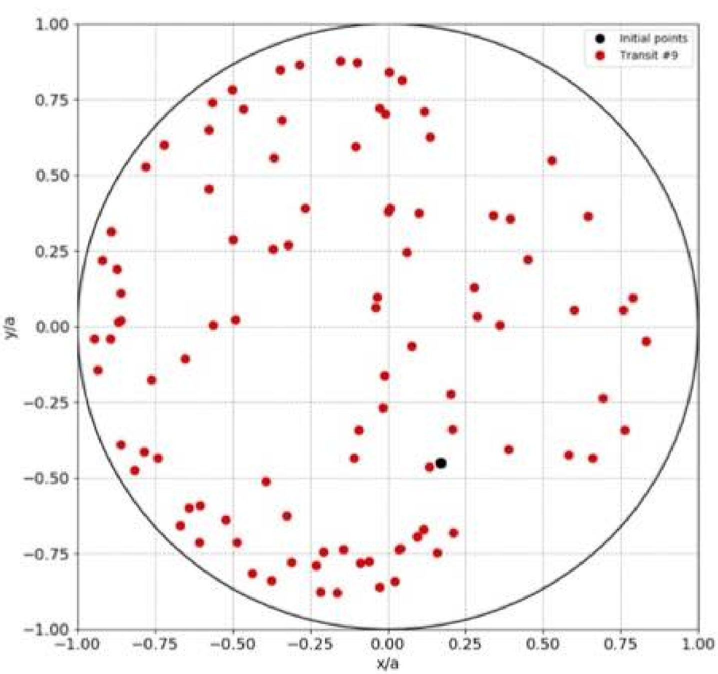
Chaos is implied when $\Lambda_u > c_u \exp(t/\tau_L)$ *as* $t \to \infty$ *with* c_u *and* τ_L *constants. When flow is divergence free,* $\Lambda_s = 1/\Lambda_u$*. SVD analysis is more difficult and less accurate but gives more information.*

Points Started on a Circle of Radius $a/100$

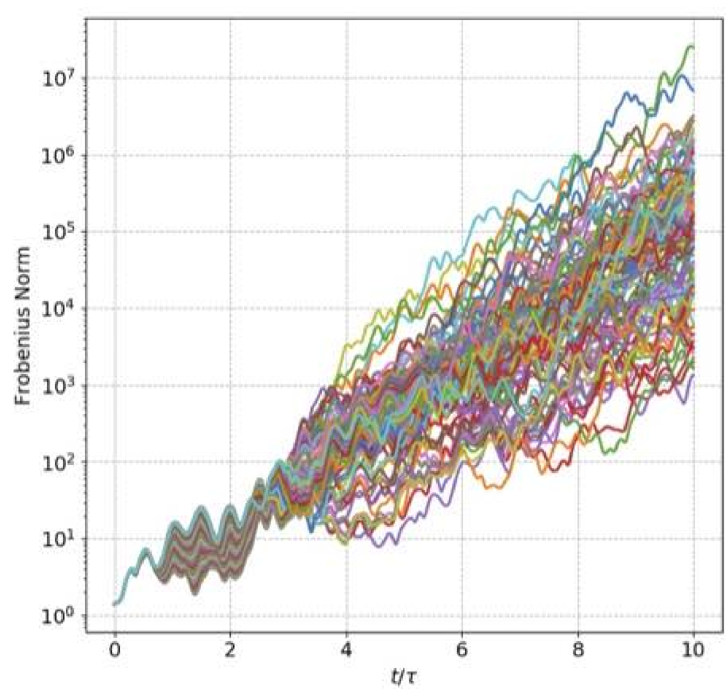
The starting points are on the perimeter of the tiny black circle centered at x/a=0.17 and y/a=-0.45. The red points are the streamline locations after ten transit times.

Note the large spread in the number exponentiations.

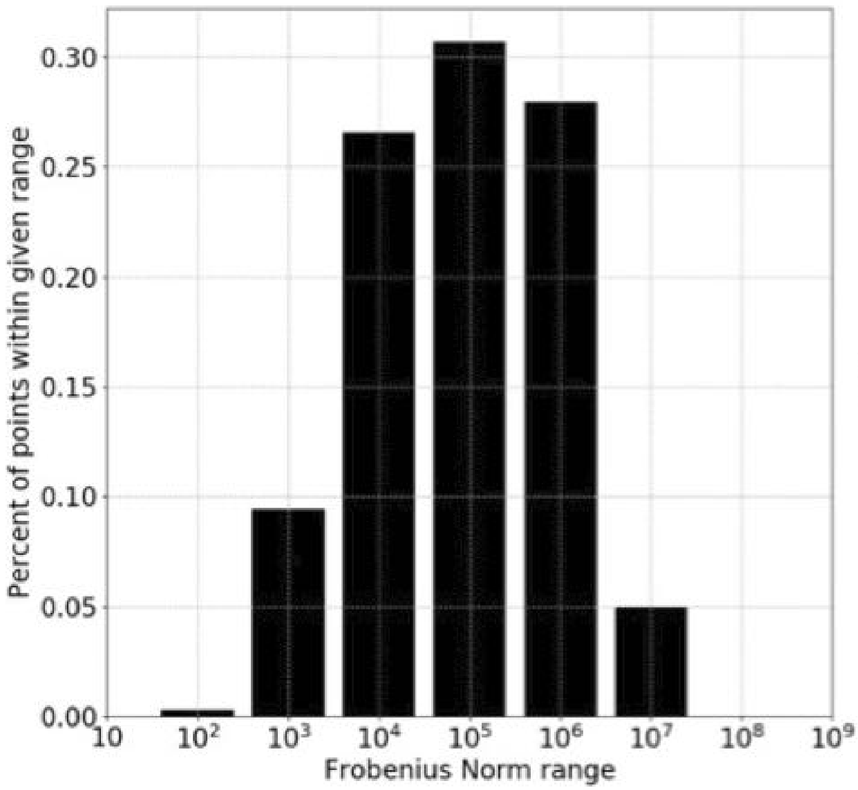
a. A Hundred Initial (black) / Final (red) Streamline Locations



b. A Hundred Frobenius norms

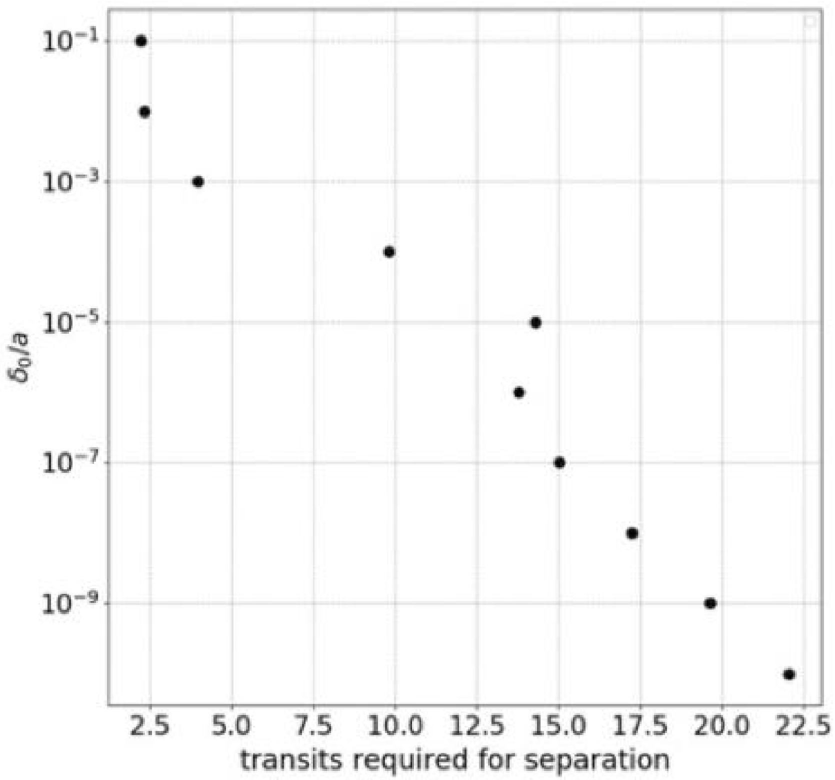


Points Started on the Perimeters of Small Circles



148< (Frobenius norm) <3X10[7]

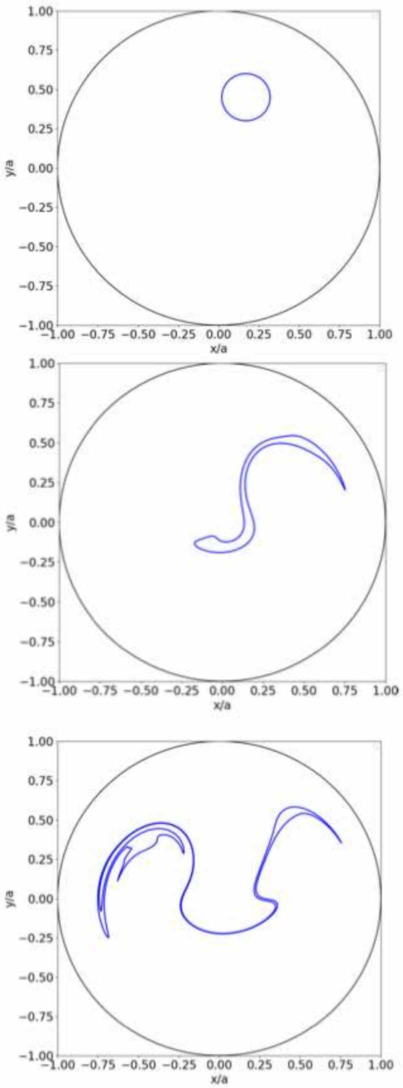
10 transits with $\frac{\delta}{a} = \frac{1}{100}$.



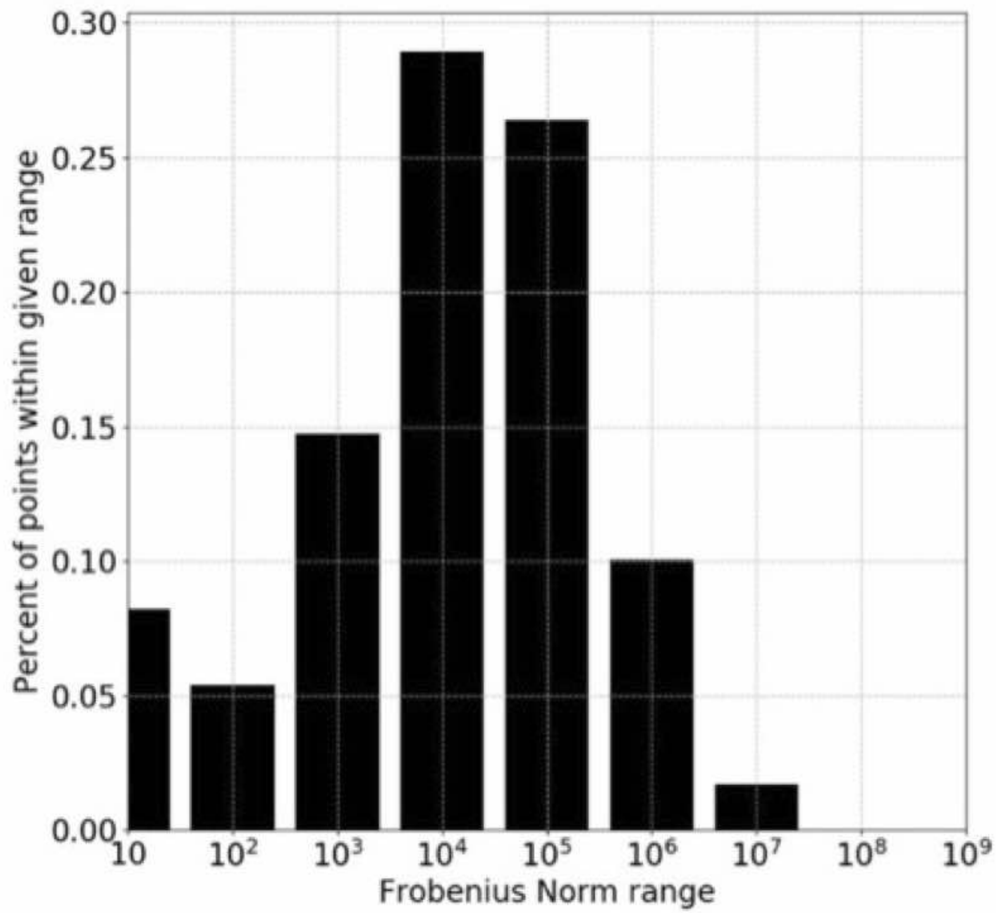
Separation of $\frac{3}{4}a$ versus $\frac{\delta_0}{a}$

Exponentiation Properties

Deformed circle at
0, 1, and 2 transit times



Points started uniformly over the *r<a* region;
only ~8% of that region is chaotic



1.4< (Frobenius norm) < $8.0X10^7$

Mathematics of Vector Representations in 3D

Any vector $\vec{E}(\vec{x})$ can be represented in three-space using another vector $\vec{B}(\vec{x})$ that has no zeros in the region,

$$\vec{E} = -\vec{u} \times \vec{B} - \vec{\nabla}\Phi + \mathcal{E}\vec{\nabla}\ell.$$

Φ is a single valued potential, and ℓ is the distance along the vector $\vec{B}$. Field lines of $\vec{B}$ are given by $d\vec{x}/d\ell = \vec{B}/B$ at a given point in time. For nulls in $\vec{B}$ see [4, 5].

The component of $\vec{E}$ along $\vec{B}$ gives $\hat{b} \cdot \vec{E} = -\partial\Phi/\partial\ell + \mathcal{E}$, where $\mathcal{E}$ is a constant along $\vec{B}$, which must be chosen to make Φ single-valued. When $\vec{B}$ lies on toroidal surfaces,

$$\mathcal{E} = \lim_{L\to\infty} \frac{\int_0^L \vec{E} \cdot d\vec{\ell}}{L}.$$

The components of $\vec{E}$ perpendicular to $\vec{B}$ determine $\vec{u}_\perp$, which are the two components of $\vec{u}$ that are perpendicular to $\vec{B}$. *$\mathcal{E}$ is essentially the electromotive force.*

An implication is that Faraday's Law is equivalent to

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u}_\perp \times \vec{B} - \mathcal{E}\vec{\nabla}\ell).$$

Implications of $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u}_\perp \times \vec{B} - \mathcal{E}\vec{\nabla}\ell)$

In 1958, Newcomb proved [6]: **When $\mathcal{E} = 0$, the magnetic field lines move with the velocity $\vec{u}_\perp(\vec{x}, t)$ and do not break.**

Proven using the Clebsch representation, $\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\Theta$ *and showing* $\partial\psi/\partial t + \vec{u}_\perp \cdot \vec{\nabla}\psi = 0$ *and* $\partial\Theta/\partial t + \vec{u}_\perp \cdot \vec{\nabla}\Theta = 0$.

Reconnection occurs when $\mathcal{E} \neq 0$.

The velocity of the plasma $\vec{v}$ in which $\vec{B}$ is embedded has no direct relevance to reconnection *despite common opinion*.

The velocity of the plasma relative to the magnetic field lines, $(\vec{v} - \vec{u}_\perp)$ *determined by* $\eta_\perp \vec{j}_\perp$*, Hall terms, and Pfirsch-Schlüter currents,* which means $j_{||}$ *driven by* $\vec{\nabla} \cdot \vec{j}_\perp$.

Ideal Magnetic Energy Evolution [1]

The energy equation for ideal evolving magnetic field is,

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right)+\vec{\nabla}\cdot\left(\frac{B^2}{2\mu_0}\vec{u}_\perp\right)=-\left(\frac{B^2}{2\mu_0}\right)\left(\vec{\nabla}\cdot\vec{u}_\perp+2\vec{u}_\perp\cdot\vec{\kappa}\right).$$

Integral of left-hand side over a volume gives the change in energy in a region bounded by a perfect conductor due to the motion of that conductor.

A large exchange in energy occurs within the volume unless $\vec{\nabla}\cdot\vec{u}_\perp+2\vec{u}_\perp\cdot\vec{\kappa}=0$.

Two spatial coordinates across $\vec{B}$ are required to make $\vec{\nabla}\cdot\vec{u}_\perp+2\vec{u}_\perp\cdot\vec{\kappa}=0$.

A third coordinate along $\vec{B}$ is required for a line to change connections.

In 2D, an ideal flow $\vec{u}_\perp$ can not give an exponential enhancement of reconnection as it can in 3D.

Two-Dimensional Reconnection Theory I

Two-dimensional reconnection would be an extremely specialized topic were it not for the traditional assumption that reconnection in general could be understood using two dimensional models.

In 1988, Schindler, Hesse, and Birn gave the two requirements for reconnection to compete with an ideal evolution in two-dimensional systems [7]:

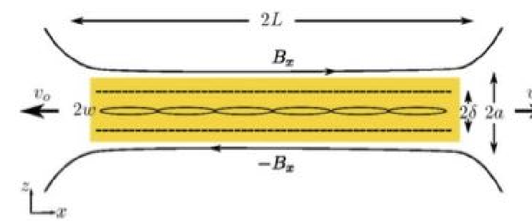
1. The reconnection must occur in a region of width $\boldsymbol{\Delta_d \equiv \eta/\mu_0 u_\perp}$, *where Δ_d is called the distinguishability distance—more later.*
2. The current density in that region must be $\boldsymbol{j \approx B_{rec}/\mu_0\Delta_d}$ with $\boldsymbol{B_{rec}}$ the reconnecting field.

***The magnetic Reynolds number $\boldsymbol{R_m \equiv a/\Delta_d}$**, where a is the system scale across $\vec{B}$, can reach 10^{12} in the solar corona.*

In 3D with $\vec{u}_\perp$ chaotic, the current density is smaller by a factor $\Big(\ln(R_m)\Big)/R_m$, when reconnection competes with evolution, and lies in many thin but wide ribbons along $\vec{B}$.

Two-Dimensional Reconnection Theory II

It is commonly thought that a near-singular current density is a requirement for reconnection to be significant.



Huang, Comisso, and Bhattacharjee (2019)

Little research has been done on the time scale for the formation of a current density $j \propto R_m$. The natural time scale is $\sim R_m \tau_A$ with τ_A the time scale for shear Alfvén to propagate along the field lines [3, 8].

Most research has been on the maintenance of a near-singular current density, which requires methods for getting plasma out of the way of field lines approaching the reconnection layer.

This gives ion dynamics an importance in 2D that it doesn't have in 3D.

Modern reconnection theory has focused on plasmoids [9] *as a method of quickly removing plasma from a two-dimensional reconnection region.*

Distinguishability Distance, Δ_d [3]

Physics implies that when two magnetic field lines are closer than Δ_d at any point on their trajectories, then they are indistinguishable. *That is, they have reconnected.*

A simple parallel Ohm's Law has $E_{||} = \eta j_{||} + \left(\frac{c}{\omega_{pe}}\right)^2 \mu_0 \frac{\partial j_{||}}{\partial t}$, which gives

$$\frac{\partial}{\partial t}\left(\vec{B} - \left(\frac{c}{\omega_{pe}}\right)^2 \nabla^2 \vec{B}\right) = \vec{\nabla} \times \left(\vec{u}_\perp \times \vec{B}\right) - \frac{\eta}{\mu_0} \vec{\nabla} \times \vec{j}_{||}.$$

In 3D reconnection, *the flow $u_\perp$ varies on the scale a and $j_{||} \sim B/\mu_0 a$. Resistive reconnection competes with evolution when $|\vec{\nabla} \times \vec{j}_{||}| \sim j_{||}/\Delta_d$ with*

$$\boldsymbol{\Delta_d = \frac{\eta}{\mu_0 u_\perp}},$$

similar in effect to numerical diffusion in a code. The current density $j_{||}$ lies in many thin, $\sim \Delta_d$, ribbons along the magnetic field.

The left-hand side implies $\Delta_d \geq \frac{c}{\omega_{pe}}$, *similar in effect to a finite grid in a code.*

Simplified Model of Coronal Loops [3]

A perfectly conducting cylinder of height L and radius a enclosing an ideal pressureless plasma.

Bottom and sides of cylinder are stationary. The top flows with a velocity $\vec{v}_t = \hat{z} \times \vec{\nabla} h_t(x, y, t)$. The initial magnetic field is $\vec{B}_0 = B_0 \hat{z}$. Use the stream function discussed earlier for $h_t(x, y, t)$.

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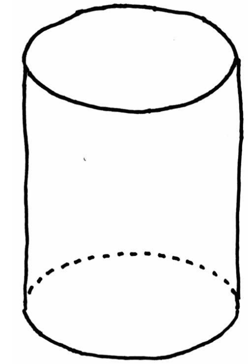


$\Delta_{max}/\Delta_{min}$ is the ratio of the maximum to the minimum separation of a neighboring pair of magnetic field lines.

Because lines of B *are fixed at the bottom but exponentially separate at the top,* $\Delta_{max}/\Delta_{min}$ *increases exponentially.*

When $\Delta_{max}/\Delta_{min} \approx a/\Delta_d$, lines will lose their separate identities and reconnect when the time equals

(Evolution-Time)$\times \ln(a/\Delta_d)$, or $\tau_{rec} = \tau_{ev} \ln(R_m)$.

Parallel Current $K \equiv \frac{\mu_0 j_{||}}{B}$ [3]

Ampere's law implies [3, 10] *spatial average of* $|KL| \gtrsim$ *# of e-folds in a distance* L.

$$\vec{\nabla}\cdot\vec{j} = 0 \text{ is equivalent to } \vec{B}\cdot\vec{\nabla}K = \vec{B}\cdot\vec{\nabla}\times\left(\frac{\mu_0 \vec{f}_L}{B^2}\right) \quad \text{where} \quad \vec{f}_L \equiv \vec{j}\times\vec{B}.$$

Assuming the plasma is pressureless, the only way to balance the Lorentz force $\vec{f}_L$ is plasma inertia, which means the shear Alfvén wave.

When the evolution is slow compared to the transit time of shear Alfvén waves along the magnetic field lines, K is a constant along each magnetic field line.

$$\text{An ideal evolution implies } \frac{\partial K}{\partial t} = \frac{\partial \Omega}{\partial \ell}, \quad \text{where} \quad \Omega \equiv \hat{z}\cdot\vec{\nabla}\times\vec{u}_\perp \text{ and } \frac{\partial}{\partial \ell} \equiv \frac{\vec{B}}{B}\cdot\vec{\nabla}.$$

$$\text{When } \frac{\partial K}{\partial \ell} = 0, \quad \Omega = \Omega_t(x_0, y_0, t)\frac{\ell}{L} \text{ and } \frac{\partial K(x_0, y_0, t)}{\partial t} = \frac{\Omega_t}{L},$$

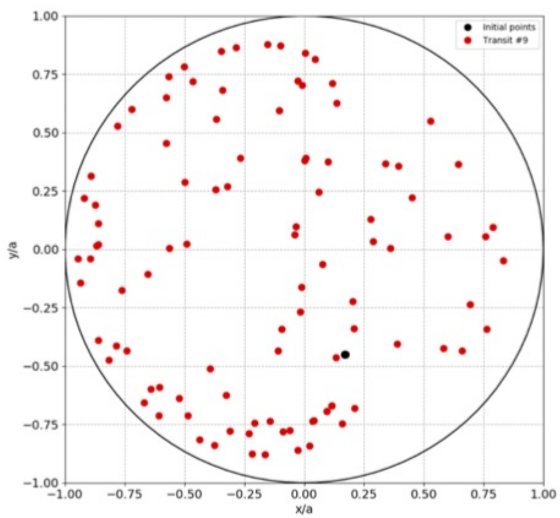
K can increase no faster than linearly in time.

Points Started on a Circle of Radius $a/100$

The starting points are on the perimeter of the tiny black circle centered at x/a=0.17 and y/a=-0.45. The red points are the streamline locations after ten transit times.

Note the large spread in both the number exponentiations and in the current density K.

a. A Hundred Initial (black) / Final (red) Streamline Locations



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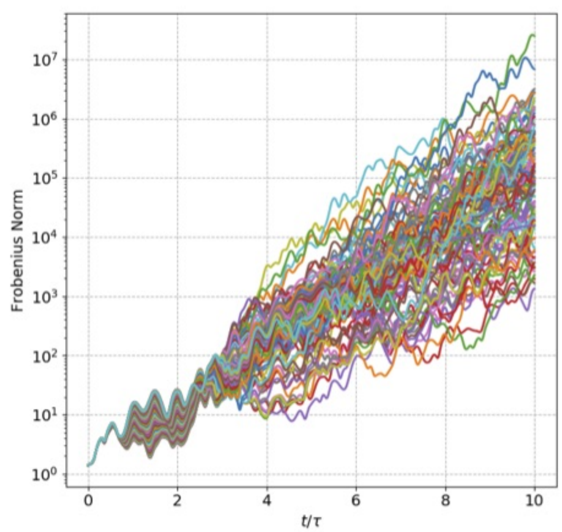
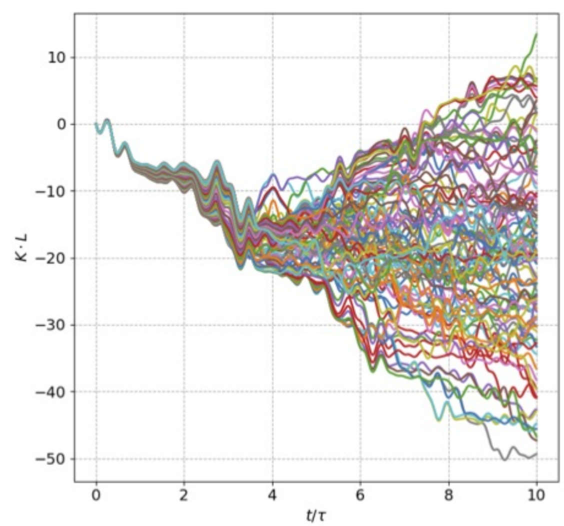
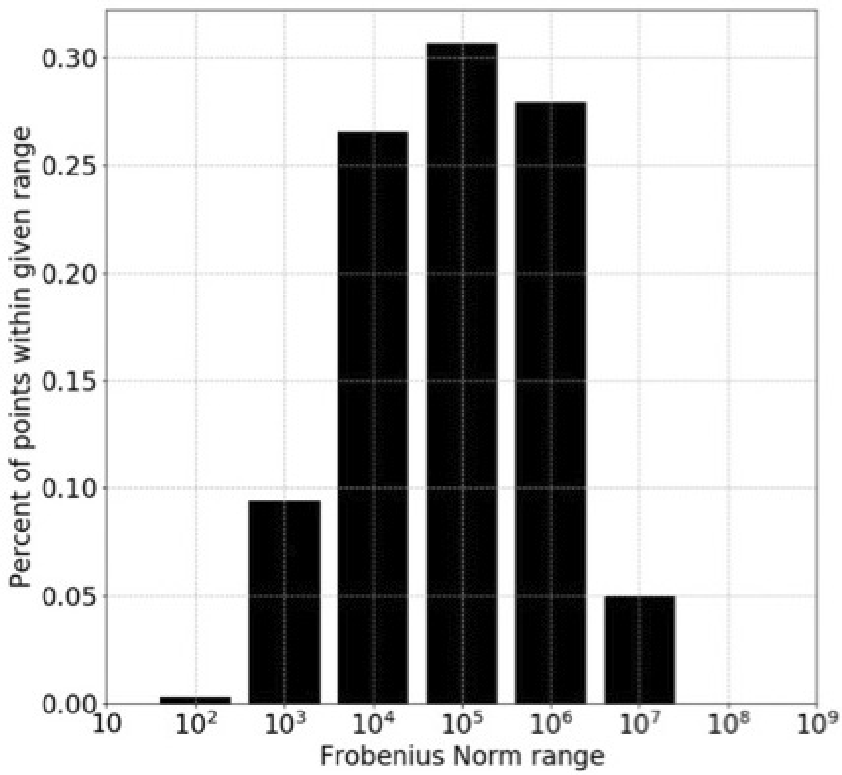
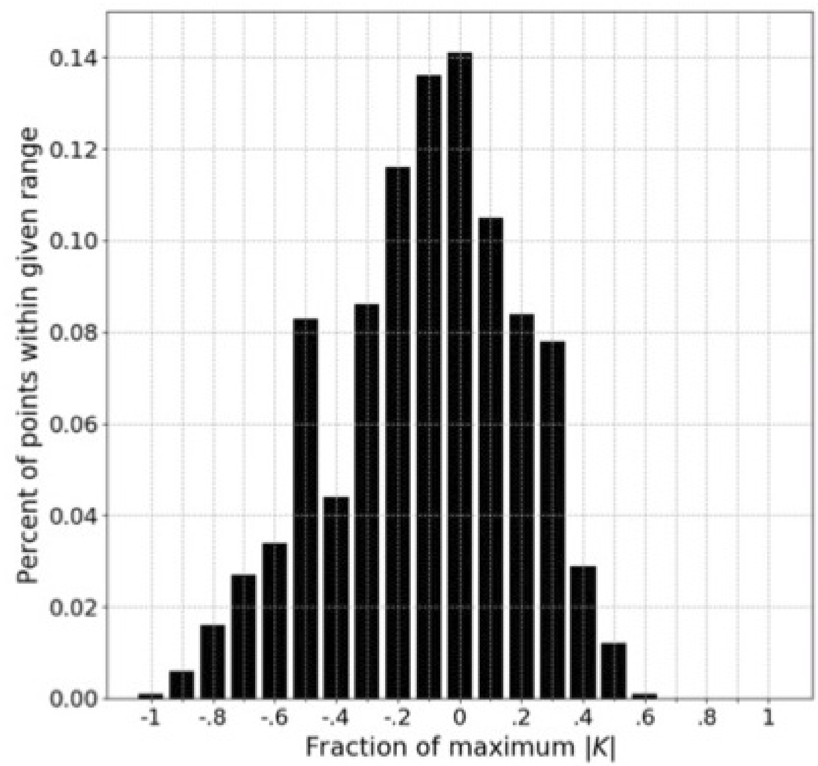


c. A Hundred currents



Thousand Points Started on the Perimeter of the Small Circle





148< (Frobenius norm) <$3X10^7$

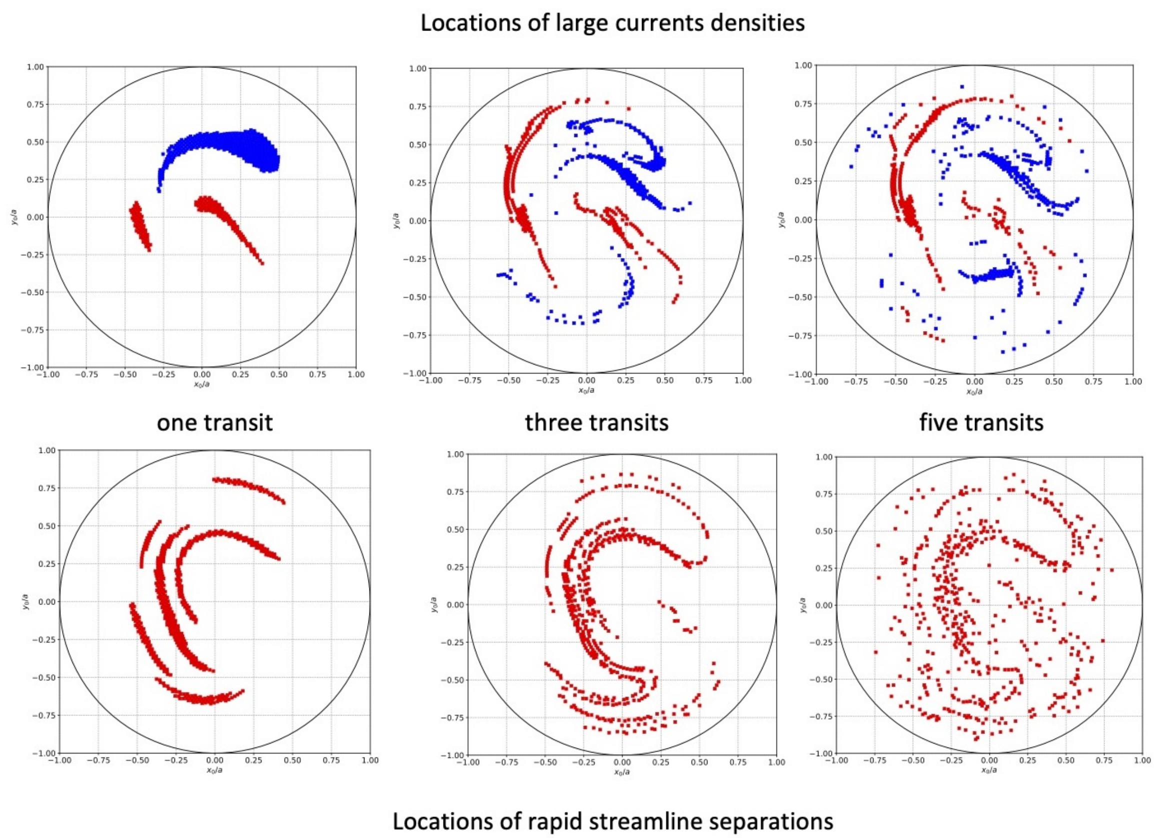
Max |KL|=36.7

Ten transits $\frac{\delta_0}{a} = \frac{1}{100}$

Points Started Uniformly Over the Region $r < a$

Large Frobenius norms are associated with large current densities, but the spatial correlation is not high.

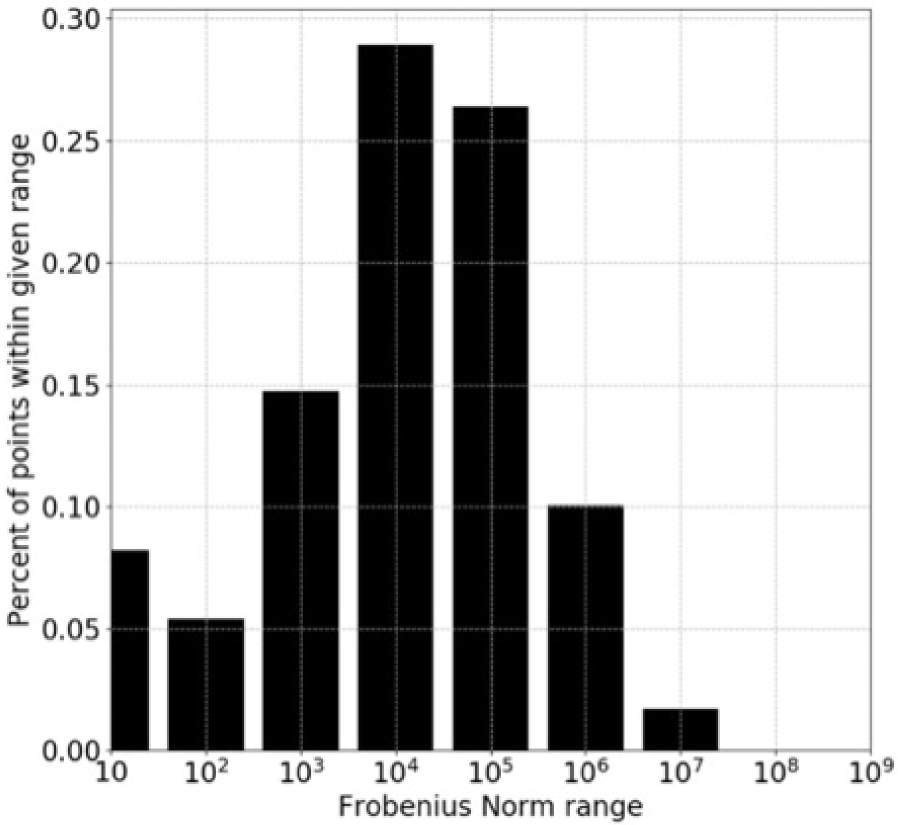
Red implies K is negative and black positive.



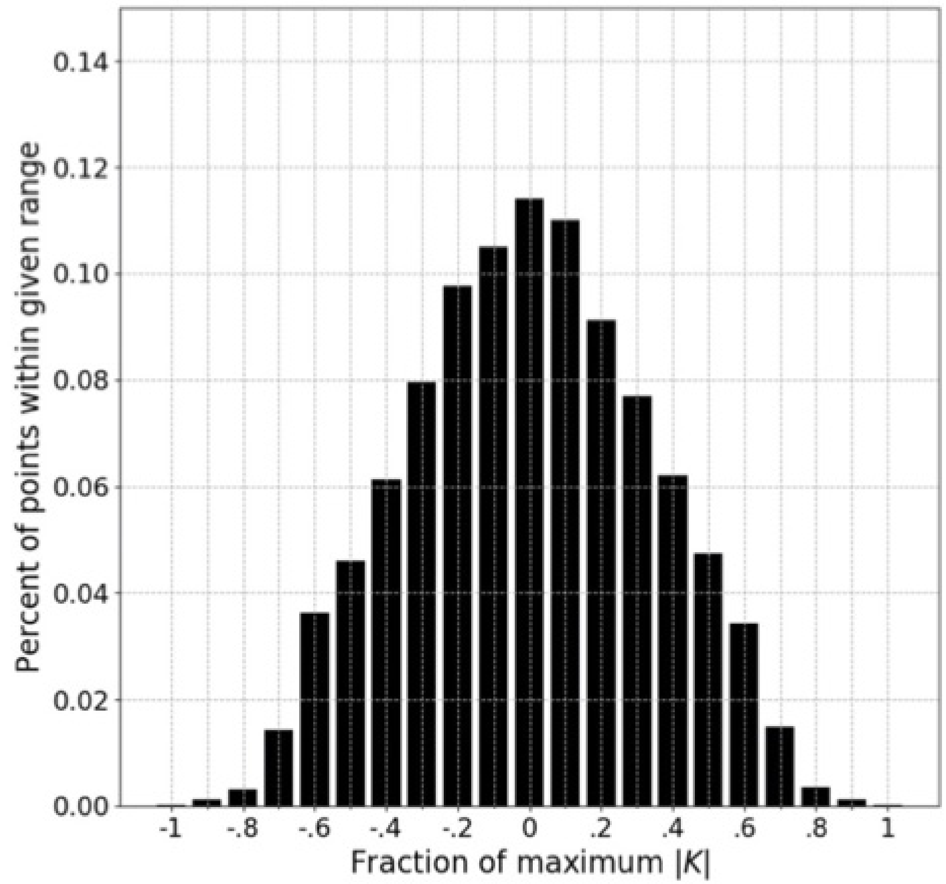
Need typical $|KL| \gtrsim \ln($*Frobenius norm*$)$ *for consistency with Ampere's law* [3, 10].

Points Started Uniformly Over Region $r < a$

Ten Thousand Uniformly Spread Points on r<a Circle after Ten Transits



1.4< (Frobenius norm) < $8.0X10^7$



Max |KL|=41.8

Discussion

1. When the ideal magnetic field line evolution velocity $\vec{u}_\perp$ is chaotic, magnetic reconnection will occur on a time scale $\sim 10a/u_\perp$ *with the current density lying in thin but broad ribbons with a magnitude only logarithmically,* ~ 10*, bigger than its nominal value,* $B_{rec}/\mu_0 a$.

2. Once reconnection starts, static force balance is frequently lost, and $\boldsymbol{u_\perp} \to \boldsymbol{V_A}$, where V_A is the Alfvén speed. This explains Parker's observation [11] that the typical reconnection speed is $\sim 0.1V_A$. Would expect $\sim V_A/\ln(R_m)$ with $\ln(R_m) \sim 20$.

3. A chaotic $\vec{u}_\perp$ is not energetically possible in two coordinate models, *which makes two-dimensional theory of little relevance for understanding reconnection in nature and the laboratory.*

4. When tokamak magnetic surfaces respond to ideal perturbations they can have exponential increases in $\Delta_{max}/\Delta_{min}$ separations between magnetic surfaces when the ideal displacement is comparable to the distance between low order rational surfaces [12]. *The separation between two magnetic field lines in a surface is bounded, which makes the Lyapunov exponent zero.*

References

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