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Existence of solution for elliptic equations with supercritical Trudinger–Moser growth

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This paper is concerned with the existence of solutions for a class of elliptic equations on the unit ball with zero Dirichlet boundary condition. The nonlinearity is supercritical in the sense of Trudinger–Moser. Using a suitable approximating scheme we obtain the existence of at least one positive solution.

Keywords: Elliptic equation; positive solution; supercritical exponential growth

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a smooth and bounded domain. Yudovič [30], Pohožaev [26] and Trudinger [29] proved, in an independent way, that

$$u \in W_0^{1,N}(\Omega) \text{ implies } \int_\Omega \mathrm{e}^{|u|^{N'}}\, \mathrm{d}x < \infty, \tag{1.1}$$

where $N' = N/(N-1)$. Moreover, for any higher growth, the corresponding integral can be infinite for a suitable choice of u. After that, Moser [23] improved this assertion, showing that if $u \in W_0^{1,N}(\Omega)$, then

$$\sup_{\|\nabla u\|_{W_0^{1,N}(\Omega)} \leqslant 1} \int_\Omega \mathrm{e}^{\alpha|u|^{N'}}\, \mathrm{d}x \begin{cases} \leqslant c|\Omega|, & \text{if } \alpha \leqslant \alpha_N \\ = \infty, & \text{if } \alpha > \alpha_N \end{cases} \tag{1.2}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, c is a constant which depends on N, and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. Inequality (1.2) is now called Trudinger–Moser inequality and the term $e^{\alpha_N |u|^{N'}}$ is known as critical Trudinger–Moser growth. Several generalizations, extensions and applications of the Trudinger–Moser inequality have been given in recent years, we quote for instance [**1–3**, **7**, **13**, **14**, **24**, **25**, **27**]. An equation where (1.2) plays a role in dimension $N = 2$ is

$$\begin{cases} -\Delta u = h(x, u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $h(x, u) = \lambda u e^{u^2}$ and $\lambda > 0$ is a free parameter. Existence of solution for equation (1.3) has been considered in $\Omega \subset \mathbb{R}^2$ in many papers with h in a more general form, where $h(x, u)$ is continuous and behaves like $\exp(\alpha|u|^2)$ as $|u| \to \infty$, see [**2**, **4**, **10**, **13–15**, **22**, **27**]. The Trudinger–Moser inequality combined with the variational approach is a powerful tool to obtain existence of solution. This is the reason why most of papers treat problem (1.3) by means of variational methods, and then usually it is assumed that h has subcritical or critical growth. In [**10**], the authors considered the subcritical problem (1.3) with a small sublinear perturbation on the nonlinearity, without imposing any extra hypotheses like Ambrosetti–Rabinowitz conditions (or some additional conditions) to obtain Palais–Smale or Cerami compactness condition. For elliptic systems, we quote [**11**]. For a related problem in higher dimensions, consult [**8**]. The goal of this paper is to study problems where the function h has supercritical growth, meaning that for every $\sigma > 0$,

$$\lim_{s \to \infty} \frac{|h(x, s)|}{e^{\sigma s^2}} = \infty \quad \text{uniformly in } x. \tag{1.4}$$

More precisely we consider the following problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)-1} + f(x, u) & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{1.5}$$

where $B \subset \mathbb{R}^2$ denotes the open unit ball centred at the origin, $\lambda > 0$ is a parameter,

$q \in C(\overline{B})$ is radially symmetric and such that

$$\times\, 1 < q_- \leqslant q(x) \leqslant q_+ < 2, q_-, q_+ \in \mathbb{R}, \tag{1.6}$$

$f : B \times \mathbb{R} \to \mathbb{R}$ is a continuous function radially symmetric in the first variable (1.7)

satisfying the following condition

$$0 \leqslant s f(x, s) \leqslant a_1 |s|^{p(x)} \exp(\beta |s|^{2+g(x)}), \tag{1.8}$$

where

$\beta > 0$, $p, g \in C(\overline{B})$ are radially symmetric functions and $a_1 > 0$ is a constant (1.9)

such that

$$2 < p_- \leqslant p(x) \leqslant p_+ < \infty,\ p_-, p_+ \in \mathbb{R}, \tag{1.10}$$

and either $g \equiv 0$ or g verifies the following two conditions

(g_1) $g(0) = 0$ and $g(x) > 0$ for $x \neq 0$,

(g_2) there exists some $c > 0$ and some $\gamma > 2$ such that $g(x) \leqslant c/(-\log|x|)^{\gamma}$ for $|x|$ near 0.

From now on, when a function defined in B is radial, for convenience, we will use the same notation to represent the function on x or $r = |x|$.
Notice that the function $g(r) = r^{\alpha}$, with $\alpha > 0$, satisfies conditions (g_1) and (g_2).
We are able to state our main result.

THEOREM 1.1. *Suppose* (1.6)–(1.10) *and that either* $g \equiv 0$ *or satisfy* (g_1) *and* (g_2)*. Then there exists* $\lambda^* > 0$ *such that for every* $\lambda \in (0, \lambda^*)$ *problem* (1.5) *possesses at least one positive radially symmetric solution* $u_\lambda \in H_0^1(B)$*. Furthermore,* $\|u_\lambda\|_{H_0^1(B)} \to 0$ *as* $\lambda \to 0$*.*

In some particular cases the solution does not exist for $\lambda > 0$ large.

THEOREM 1.2. *Assume the hypotheses of theorem* 1.1*. If* $f(r,t) = t^{p(r)-1} \exp(\beta t^{2+g(r)})$ *and* $\lambda > 0$ *is sufficiently large, then problem* (1.5) *has no positive radially symmetric solution* $u_\lambda \in H_0^1(B)$*.*

REMARK 1.3. In fact, since p, q and g are bounded from above and from below, by the method of the proof of theorem 1.2 we observe that the nonexistence result is valid for nonradial solutions too.

Notice that the exponential growth in (1.8) goes beyond the usual Trudinger–Moser critical behaviour, since $2 + g(r) \geqslant 2$. Notice also that f in (1.7)–(1.8) behaves like (1.4). We point out that Ngô and Nguyen [**24**] studied problem (1.5) for $\lambda = 0$, but imposing the Ambrosetti–Rabinowitz condition on the function f. We can solve (1.5) under weaker assumptions on f using the Galerkin method, which consists of studying approximate problems. For the existence result in a similar supercritical regime, we cite [**7**]. In our approach, it is important to mention that it is necessary to verify regularity up to the boundary for the approximate solutions u_n in order to apply the comparison principle and guarantee that the approximate solutions are bounded away from zero. Thus, we can take the limit and ensure that the limit solution does not vanish. To this matter, we also use an approximation scheme in the nonlinearity, where f is replaced by an approximating sequence of Lipschitz functions due to Strauss [**28**].

The paper is organized as follows. Section 2 contains some preliminaries, results concerning to Lebesgue spaces with variable exponents and a comparison principle due to [**19**]. Section 3 is devoted to prove the existence of a solution to the approximate problem. Thus, we present the approximating sequence of Lipschitz functions and some important properties. Section 4 is devoted to the proof of the main results.

2. Preliminaries

2.1. Variable exponent

We start this section presenting some results of the Lebesgue and Sobolev spaces with variable exponents (we refer to [16, chapter 3] for the definition and properties of these spaces). Set

$$L^\infty_+(B) := \Big\{y : y \in L^\infty(B), \inf_{x\in B} y(x) > 1\Big\},$$

where $B \subset \mathbb{R}^2$ is the unit ball centred at the origin. For any $y \in L^\infty_+(B)$, we define

$$y_- = y_-(B) := \inf_{x\in B} y(x), \quad y_+ = y_+(B) := \sup_{x\in B} y(x).$$

For $y \in L^\infty_+(B)$, the space

$$L^{y(x)} := \left\{u : \text{ is real measurable}, \int_B |u(x)|^{y(x)}\mathrm{d}x \leqslant \infty\right\}$$

is a Banach space equipped with the norm

$$\|u\|_{y(x)} := \inf\left\{\sigma > 0, \int_B \left|\frac{u(x)}{\sigma}\right|^{y(x)} \mathrm{d}x \leqslant 1\right\}.$$

Proposition 2.1. *If* $u \in L^{y(x)}(B)$, $\|u\|_{y(x)} = \lambda$, *then*

- *if* $\lambda \geqslant 1$, *then* $\lambda^{y_-} \leqslant \int_B |u(x)|^{y(x)}\mathrm{d}x \leqslant \lambda^{y_+}$,
- *if* $\lambda \leqslant 1$, *then* $\lambda^{y_+} \leqslant \int_B |u(x)|^{y(x)}\mathrm{d}x \leqslant \lambda^{y_-}$.

The embedding $H^1_0(B) \hookrightarrow L^{y_+}(B)$ is compact and $L^{y_+}(B) \hookrightarrow L^{y(x)}(B)$ is continuous.

Proposition 2.2. *If* $y \in L^\infty_+(B)$ *then*

$$H^1_0(B) \hookrightarrow L^{y(x)}(B)$$

is compact.

We denote by $H^1_{0,r}(B)$ the subspace of radially symmetric functions of the space $H^1_0(B)$ (which is the usual Sobolev spaces of radially symmetric functions on B), i.e.

$$H^1_{0,r}(B) = \{u \in H^1_0(B) : u = u(|x|)\}.$$

The following result was proved in [24, proposition 2.1].

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Proposition 2.3. *Let $g : [0,1) \to [0,\infty)$ be a continuous function. If g satisfies conditions (g_1) and (g_2), then there holds*

$$\sup_{u\in H^1_{0,r}(B):\int_B |\nabla u|^2 \mathrm{d}x\leqslant 1} \int_B \exp(4\pi |u|^{2+g(|x|)})\mathrm{d}x < \infty. \tag{2.1}$$

The usual Trudinger–Moser inequality, introduced in [**23**, **29**], will be very important for our purpose. Namely, given $u \in H^1_0(B)$, then

$$\mathrm{e}^{\sigma|u|^2} \in L^1(B) \text{ for every } \sigma > 0, \tag{2.2}$$

and there exists a positive constant L, such that

$$\sup_{\|u\|_{H^1_0(B)}\leqslant 1} \int_B \mathrm{e}^{\sigma|u|^2}\mathrm{d}x \leqslant L|B| \text{ for every } \sigma \leqslant 4\pi, \tag{2.3}$$

where $|B| = \int_B 1\,\mathrm{d}x$.

Remark 2.4. The following inequality from [**24**] do also holds by considering suitable conditions on g,

$$\sup_{u\in H^1_{0,r}(B):\int_B |\nabla u|^2 \mathrm{d}x\leqslant 1} \int_B \exp((4\pi + g(|x|))|u|^2)\mathrm{d}x < \infty. \tag{2.4}$$

Analysing lemma 3.2 we conclude (2.4) would produce the same results of this paper, because $\exp((4\pi + g(|x|))|u|^2) \leqslant \exp(\beta|u|^2)$ for some constant $\beta > 0$. In our context, since g is bounded, (2.1) is more general than (2.4).

2.2. Comparison principle

Let Ω be a bounded domain in $\mathbb{R}^2$. Consider the problem

$$\begin{cases} -\Delta u = h(x,u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5}$$

Definition 2.5. For $u \in C(\overline{\Omega})$ with $u \geqslant 0$, we call u a *weak supersolution* of (2.5) if

$$\int_\Omega (u\Delta\phi + h(x,u))\phi \mathrm{d}x \leqslant 0 \quad \text{for } \phi \in C^\infty_0(\Omega)^+,$$

where

$$C^\infty_0(\Omega)^+ := \{\phi \in C^\infty_0(\Omega) : \phi \geqslant 0\}.$$

A *weak subsolution* is defined by the reverse inequality. We call u a weak solution if it is both a weak supersolution and a weak subsolution.

In the following, we present a comparison result due to [**19**] (see also [**6**]).

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Proposition 2.6. *Let* Ω *be a bounded open set in* $\mathbb{R}^2$ *and* $h:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Then the following are equivalent.*

(i) *For any* $x\in\Omega$, $h(x,s)/s$ *is non-increasing with respect to* $s\in(0,\infty)$. *For any* $x_0\in\Omega$, $0<a<b$ *and* $\epsilon>0$, *it holds that*

$$h(x,a)/a-h(x,b)/b>0 \quad \textit{at some } x\in B(x_0,\epsilon).$$

(ii) *Let* B_0 *be any open subset of* Ω *and* u, v *a weak subsolution and a weak supersolution in* B_0, *respectively, and* $u,v>0$ *in* B_0. *If* $u\leqslant v$ *on* $\partial\Omega$, *then* $u\leqslant v$ *in* B_0.

3. Auxiliary problem

3.1. Approximate functions

To prove theorem 1.1 we approximate f by Lipschitz functions $f_k:B\times\mathbb{R}\to\mathbb{R}$ defined by

$$f_k(r,s)=\begin{cases}-k[F(r,-k-\frac{1}{k})-F(r,-k)], & \text{if} \quad s\leqslant -k,\\ -k[F(r,s-\frac{1}{k})-F(r,s)], & \text{if} \quad -k\leqslant s\leqslant -\frac{1}{k},\\ k^2s[F(r,-\frac{2}{k})-F(r,-\frac{1}{k})], & \text{if} \quad -\frac{1}{k}\leqslant s\leqslant 0,\\ k^2s[F(r,\frac{2}{k})-F(r,\frac{1}{k})], & \text{if} \quad 0\leqslant s\leqslant \frac{1}{k},\\ k[F(r,s+\frac{1}{k})-F(r,s)], & \text{if} \quad \frac{1}{k}\leqslant s\leqslant k,\\ k[F(r,k+\frac{1}{k})-F(r,k)], & \text{if} \quad s\geqslant k,\end{cases} \tag{3.1}$$

where $F_s=f$ and $F(r,0)$=0.

The following approximation result was proved in [**28**] and uses the explicit expression of the sequence (3.1).

Lemma 3.1. *Let* $f:B\times\mathbb{R}\to\mathbb{R}$ *be a continuous function such that* $sf(r,s)\geqslant 0$ *for every* $s\in\mathbb{R}$. *Then the sequence of Lipschitz functions* (3.1) *satisfies*

(i) $sf_k(r,s)\geqslant 0$ *for every* $s\in\mathbb{R}$;

(ii) $\forall\, k\in\mathbb{N}$ *there is a continuous function* $c_k(r)$ *such that* $|f_k(r,\xi)-f_k(r,\eta)|\leqslant c_k(r)|\xi-\eta|$ *for every* $\xi,\eta\in\mathbb{R}$;

(iii) f_k *converges uniformly to* f *in bounded sets.*

The sequence f_k of the previous lemma has some additional properties, similar to the found in [**8**–**11**]. In what follows, we will denote $g_\infty=\sup_{|r|\leqslant 1} g(r)$.

Lemma 3.2. *Assume* (1.6)–(1.10) *and that either* $g\equiv 0$ *or satisfy* (g_1) *and* (g_2). *Then the sequence* f_k *of lemma 3.1 satisfies*

(i) $\forall\, k\in\mathbb{N},\ 0\leqslant sf_k(r,s)\leqslant K_1|s|^{p(r)}\exp(\beta 2^{2+g_\infty}|s|^{2+g(r)})$ *for every* $|s|\geqslant 1/k$;
(ii) $\forall\, k\in\mathbb{N}$, $0\leqslant sf_k(r,s)\leqslant K_1|s|^2\exp(\beta 2^{2+g_\infty}|s|^{2+g(r)})$ *for every* $|s|\leqslant 1/k$,

where K_1 *is a positive constant independent of* k.

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Proof. Everywhere in this proof the constant a_1 is the one of (1.8).

First Case. Suppose that $-k \leqslant s \leqslant -1/k$.

By the mean value theorem, there exists $\eta \in (s-1/k, s)$ such that

$$f_k(r,s) = -k\Big[F\Big(r, s-\frac{1}{k}\Big) - F(r,s)\Big] = -kF_s(r,\eta)\Big(s-\frac{1}{k}-s\Big) = f(r,\eta)$$

and

$$sf_k(r,s) = sf(r,\eta).$$

Since $s-1/k < \eta < s < 0$ and $f(r,\eta) < 0$, we have $sf(r,\eta) \leqslant \eta f(r,\eta)$. Therefore,

$$\begin{aligned} 0 \leqslant sf_k(r,s) &\leqslant \eta f(r,\eta) \\ &\leqslant a_1|\eta|^{p(r)}\exp(\beta|\eta|^{2+g(r)}) \\ &\leqslant a_1\Big|s-\frac{1}{k}\Big|^{p(r)}\exp\Big(\beta\Big|s-\frac{1}{k}\Big|^{2+g(r)}\Big) \\ &\leqslant a_1\Big(|s|+\frac{1}{k}\Big)^{p(r)}\exp\Big(\beta\Big(|s|+\frac{1}{k}\Big)^{2+g(r)}\Big) \\ &\leqslant a_1(2|s|)^{p(r)}\exp(\beta(2|s|)^{2+g(r)}) \\ &\leqslant a_1 2^{p_+}|s|^{p(r)}\exp(\beta 2^{2+g_\infty}|s|^{2+g(r)}). \end{aligned}$$

Second Case. Assume $1/k \leqslant s \leqslant k$.

By the mean value theorem, there exists $\eta \in (s, s+1/k)$ such that

$$f_k(r,s) = k\Big[F\Big(r, s+\frac{1}{k}\Big) - F(r,s)] = kF_s(r,\eta)\Big(s+\frac{1}{k}-s\Big) = f(r,\eta)$$

and

$$sf_k(r,s) = sf(r,\eta).$$

Since $0 < s < \eta < s+1/k$ and $f(r,\eta) > 0$, we have $sf(r,\eta) \leqslant \eta f(r,\eta)$. Therefore,

$$\begin{aligned} 0 \leqslant sf_k(r,s) &\leqslant \eta f(r,\eta) \\ &\leqslant a_1|\eta|^{p(r)}\exp(\beta|\eta|^{2+g(r)}) \\ &\leqslant a_1\Big|s+\frac{1}{k}\Big|^{p(r)}\exp\Big(\beta\Big|s+\frac{1}{k}\Big|^{2+g(r)}\Big) \\ &\leqslant a_1 2^{p_+}|s|^{p(r)}\exp(\beta 2^{2+g_\infty}|s|^{2+g(r)}). \end{aligned}$$

Third Case. Suppose that $|s| \geqslant k$, then

$$f_k(r,s) = \begin{cases} -k\Big[F\Big(r,-k-\dfrac{1}{k}\Big) - F(r,-k)\Big], & \text{if} \quad s \leqslant -k \\ k\Big[F\Big(r,k+\dfrac{1}{k}\Big) - F(r,k)\Big], & \text{if} \quad s \geqslant k. \end{cases} \tag{3.2}$$

If $s \leqslant -k$, by the mean value theorem, there exists $\eta \in (-k-1/k, -k)$ such that

$$f_k(r,s) = k\Big[F\Big(r,-k-\frac{1}{k}\Big) - F(r,-k)\Big] = -kF_s(r,\eta)\Big(-k-\frac{1}{k}-(-k)\Big) = f(r,\eta)$$

and

$$sf_k(r,s) = sf(r,\eta).$$

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Since $-k-1/k<\eta<-k<0$ and $k<|\eta|<k+1/k$, we conclude that

$$
\begin{aligned}
0 \leqslant s f_k(r,s) &= \frac{s}{\eta}\eta f(r,\eta) \\
&\leqslant \frac{|s|}{|\eta|} a_1 |\eta|^{p(r)} \exp(\beta|\eta|^{2+g(r)}) \\
&\leqslant a_1|s|\Big(k+\frac{1}{k}\Big)^{p(r)} \exp\Big(\beta\Big(k+\frac{1}{k}\Big)^{2+g(r)}\Big) \\
&\leqslant a_1|s|\Big(|s|+\frac{1}{k}\Big)^{p(r)} \exp\Big(\beta\Big(|s|+\frac{1}{k}\Big)^{2+g(r)}\Big) \\
&\leqslant a_1|s|(2|s|)^{p(r)} \exp(\beta(2|s|)^{2+g(r)}) \\
&\leqslant a_1 2^{p_+}|s|^{p(r)} \exp(\beta 2^{2+g_\infty}|s|^{2+g(r)}).
\end{aligned} \tag{3.3}
$$

If $s \geqslant k$, by the mean value theorem, there exists $\eta \in (k, k+1/k)$ such that

$$
f_k(r,s) = k\Big[F\Big(r,k+\frac{1}{k}\Big) - F(r,k)\Big] = kF_s(r,\eta)\Big(k+\frac{1}{k}-k\Big) = f(r,\eta).
$$

By using similar computations to conclude (3.3) one has

$$
0 \leqslant s f_k(r,s) = s f(r,\eta) = \frac{s}{\eta}\eta f(r,\eta) \leqslant a_1 2^{p_+}|s|^{p(r)} \exp(\beta 2^{2+g_\infty}|s|^{2+g(r)}).
$$

Fourth Case. Assume $-1/k \leqslant s \leqslant 1/k$, then

$$
f_k(r,s) = \begin{cases} k^2 s\Big[F\Big(r,-\dfrac{2}{k}\Big) - F\Big(r,-\dfrac{1}{k}\Big)\Big], & \text{if} \quad -\dfrac{1}{k} \leqslant s \leqslant 0, \\ k^2 s\Big[F\Big(r,\dfrac{2}{k}\Big) - F\Big(r,\dfrac{1}{k}\Big)\Big], & \text{if} \quad 0 \leqslant s \leqslant \dfrac{1}{k}. \end{cases} \tag{3.4}
$$

If $-1/k \leqslant s \leqslant 0$, by the mean value theorem, there exists $\eta \in (-2/k, -1/k)$ such that

$$
\begin{aligned}
f_k(r,s) &= k^2 s\Big[F\Big(r,-\frac{2}{k}\Big) - F\Big(r,-\frac{1}{k}\Big)\Big] \\
&= k^2 s F_s(r,\eta)\Big(-\frac{2}{k} - \Big(-\frac{1}{k}\Big)\Big) = -ks f(r,\eta).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 \leqslant s f_k(r,s) = -ks^2 f(r,\eta) &= -k\frac{s^2}{\eta}\eta f(r,\eta) \\
&\leqslant k\frac{s^2}{|\eta|}\eta f(r,\eta) \\
&\leqslant a_1 k|s|^2|\eta|^{p(r)-1} \exp(\beta|\eta|^{2+g(r)}) \\
&\leqslant a_1 k|s|^2\Big(\frac{2}{k}\Big)^{p(r)-1} \exp\Big(\beta\Big|\frac{2}{k}\Big|^{2+g(r)}\Big) \\
&\leqslant a_1 2^{p(r)}|s|^2 \\
&\leqslant a_1 2^{p_+} \exp(\beta 2^{2+g_\infty})|s|^2 \exp(\beta|s|^{2+g(r)}).
\end{aligned} \tag{3.5}
$$

If $0 \leqslant s \leqslant 1/k$, by the mean value theorem, there exists $\eta \in (1/k, 2/k)$ such that

$$
f_k(r,s) = k^2 s\Big[F\Big(r,\frac{2}{k}\Big) - F\Big(r,\frac{1}{k}\Big)\Big] = k^2 s F_s(r,\eta)\Big(\frac{2}{k} - \frac{1}{k}\Big) = ksf(r,\eta).
$$

By using similar computations to conclude (3.5) one obtains

$$\begin{aligned}0 \leqslant s f_k(r,s) &= k s^2 f(r,\eta) = k\frac{s^2}{|\eta|}\eta f(r,\eta)\\ &\leqslant a_1 2^{p_+}\exp(\beta 2^{2+g_\infty})|s|^2\exp(\beta|s|^{2+g(r)}).\end{aligned}$$

The proof of the lemma follows by taking $K_1 = a_1 2^{p_+}\exp(\beta 2^{2+g_\infty})$, where a_1 is given in (1.8). □

The following result presents a growth behaviour to the sequence of functions $c_k(\cdot)$, and the proof can be found in [**9**, proposition 5].

PROPOSITION 3.3. *One can choose the sequence of Lipschitz constants $c_k(\cdot)$, defined in lemma 3.1, satisfying the following estimates*

$$c_k(r) \leqslant Ck\sup_t\left\{|f(r,t)|; t\in\left[-k-\frac{1}{k}, k+\frac{1}{k}\right]\right\}, \forall r\in[0,1], \tag{3.6}$$

with C a constant independent of r and k.

3.2. Approximate equations

To prove theorem 1.1 we first show the existence of a solution to the following auxiliary problem

$$\begin{cases} -\Delta v = \lambda(v_+)^{q(r)-1} + f_n(r, v_+) + \dfrac{1}{n} & \text{in} \quad B,\\ v = 0 & \text{on} \quad \partial B,\end{cases} \tag{3.7}$$

where f_n, $n\in\mathbb{N}$, are given by lemmas 3.1 and 3.2, $v_+ = \max\{v, 0\}$ and $v_- = v_+ - v$. In this section, for simplicity of notation we will omit the index n in the solution v. We will use the Galerkin method together with lemma 3.4, which is a consequence of the Brouwer fixed-point theorem (see [**20**, theorem 5.2.5]).

LEMMA 3.4. *Let $\Phi:\mathbb{R}^d\to\mathbb{R}^d$ be a continuous function such that $\langle\Phi(\xi),\xi\rangle\geqslant 0$ for every $\xi\in\mathbb{R}^d$ with $|\xi| = \varrho$ for some $\varrho > 0$. Then, there exists z_0 in the closed ball $\overline{B}_\varrho(0)$ such that $\Phi(z_0) = 0$.*

The main result in this section is the following.

LEMMA 3.5. *Suppose (1.6)–(1.10) and that either $g\equiv 0$ or satisfy (g_1) and (g_2). There exists $\lambda^* > 0$ and $n^*\in\mathbb{N}$ such that for every $\lambda\in(0,\lambda^*)$ and $n\geqslant n^*$ problem (3.7) has a weak positive solution $v\in H_0^1(B)\cap C^{1,\beta}(\overline{B})$ for some $\beta\in(0,1)$.*

Proof. Let $\mathcal{B} = \{w_1, w_2, \ldots, w_m, \ldots\}$ be an orthonormal basis of $H^1_{0,r}(B)$ and define

$$W_m = [w_1, w_2, \ldots, w_m],$$

to be the space generated by $\{w_1, w_2, \ldots, w_m\}$. Define the function $F:\mathbb{R}^m\to\mathbb{R}^m$ such that

$$I(\eta) = (I_1(\eta), I_2(\eta), \ldots, I_m(\eta))$$

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where $\eta = (\eta_1, \eta_2, ..., \eta_m) \in \mathbb{R}^m$,

$$I_j(\eta) = \int_B \nabla v \nabla w_j \mathrm{d}x - \lambda \int_B (v_+)^{q(r)-1} w_j \mathrm{d}x - \int_B f_n(r, v_+) w_j \mathrm{d}x - \frac{1}{n}\int_B w_j \mathrm{d}x,$$

$j = 1, 2, \ldots, m$, and

$$v = \sum_{i=1}^{m} \eta_i w_i \in W_m.$$

Therefore

$$(I(\eta), \eta) = \int_B |\nabla v|^2 \mathrm{d}x - \lambda \int_B (v_+)^{q(r)} \mathrm{d}x - \int_B f_n(r, v_+) v_+ \mathrm{d}x - \frac{1}{n}\int_B v \mathrm{d}x. \quad (3.8)$$

Given $v \in W_m$, we define

$$B_n^+ = \left\{x \in B : |v(x)| \geqslant \frac{1}{n}\right\}$$

and

$$B_n^- = \left\{x \in B : |v(x)| < \frac{1}{n}\right\}.$$

Thus, we rewrite (3.8) as

$$(I(\eta), \eta) = (I(\eta), \eta)_P + (I(\eta), \eta)_N,$$

where

$$(I(\eta), \eta)_P = \int_{B_n^+} |\nabla v|^2 \mathrm{d}x - \lambda \int_{B_n^+} (v_+)^{q(r)} \mathrm{d}x - \int_{B_n^+} f_n(r, v_+) v_+ \mathrm{d}x - \frac{1}{n}\int_{B_n^+} v \mathrm{d}x$$

and

$$(I(\eta), \eta)_N = \int_{B_n^-} |\nabla v|^2 \mathrm{d}x - \lambda \int_{B_n^-} (v_+)^{q(r)} \mathrm{d}x - \int_{B_n^-} f_n(r, v_+) v_+ \mathrm{d}x - \frac{1}{n}\int_{B_n^-} v \mathrm{d}x.$$

In what follows, we consider

$$\|v\|_{H_0^1(B)} = \varrho \quad (3.9)$$

for some $0 < \varrho \leqslant 1$ to be chose later. The proof will be given in several steps. In what follows, C will denote a generic constant.

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Step 1. Since $1 < q_- \leqslant q(r) \leqslant q_+ < 2$, by propositions 2.1 and 2.2 we obtain

$$\int_{B_n^+} (v_+)^{q(r)} \mathrm{d}x \leqslant \int_B (v_+)^{q(r)} \mathrm{d}x \leqslant C_1 \|v\|^{q_-}_{H_0^1(B)}. \tag{3.10}$$

By virtue of lemma 3.2(i), we get

$$\begin{aligned}\int_{B_n^+} f_n(x, v_+) v_+ \mathrm{d}x &\leqslant C \int_{B_n^+} |v_+|^{p(r)} \exp(\beta 2^{2+g_\infty} |v|^{2+g(r)}) \mathrm{d}x \\ &\leqslant C \left(\int_{B_n^+} |v_+|^{2p(r)} \mathrm{d}x \right)^{1/2} \left(\int_{B_n^+} \exp(\beta 2^{3+g_\infty} |v|^{2+g(r)} \mathrm{d}x) \right)^{1/2} \\ &\leqslant C_2 \|v\|^{p_-}_{H_0^1(B)} \left(\int_{B_n^+} \exp(\beta 2^{3+g_\infty} |v|^{2+g(r)} \mathrm{d}x) \right)^{1/2}.\end{aligned} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}(I(\eta), \eta)_P \geqslant& \int_{B_n^+} |\nabla v|^2 \mathrm{d}x - \lambda C_1 \|v\|^{q_-}_{H_0^1(B)} \\ &- C_2 \|v\|^{p_-}_{H_0^1(B)} \left(\int_{B_n^+} \exp(\beta 2^{3+g_\infty} |v|^{2+g(r)} \mathrm{d}x) \right)^{1/2} - \frac{C_3}{n} \|v\|_{H_0^1(B)} \\ \geqslant& \int_{B_n^+} |\nabla v|^2 \mathrm{d}x - \lambda C_1 \|v\|^{q_-}_{H_0^1(B)} \\ &- C_2 \|v\|^{p_-}_{H_0^1(B)} \left(\int_{B_n^+} \exp(\beta 2^{3+g_\infty} |v|^{2+g(r)} \mathrm{d}x) \right)^{1/2} - \frac{C_3}{n},\end{aligned} \tag{3.12}$$

where the constant C_i, $i = 1, 2, 3$ does not depend on n and m.

Step 2. We estimate the other integral,

$$\int_{B_n^-} (v_+)^{q(r)} \mathrm{d}x \leqslant \int_{B_n^-} |v|^{q(r)} \mathrm{d}x \leqslant |B| \frac{1}{n^{q_-}}. \tag{3.13}$$

By virtue of lemma 2.2(ii) we get

$$\int_{B_n^-} f_n(r, v_+) v_+ \mathrm{d}x \leqslant \int_{B_n^-} C_4 |v_+|^2 \exp(\beta 2^{2+g_\infty} |v_+|^{2+g(r)}) \mathrm{d}x \leqslant C_4 |B| \frac{1}{n^2}, \tag{3.14}$$

where $C_4 = a_1$ (a_1 is the one of (1.8)). It follows from (3.13) and (3.14) that

$$(I(\eta), \eta)_N \geqslant \int_{B_n^-} |\nabla v|^2 \mathrm{d}x - \lambda |B| \frac{1}{n^{q_-}} - C_4 |B| \frac{1}{n^2} - \frac{|B|}{n^2}. \tag{3.15}$$

Notice that

$$\int_{B_n} \exp(\beta 2^{3+g_\infty} |v|^{2+g(r)}) \mathrm{d}x \leqslant \int_{B_n} \exp\left(\beta 2^{3+g_\infty} \varrho^2 \left(\frac{|v|}{\varrho} \right)^{2+g(r)} \right) \mathrm{d}x. \tag{3.16}$$

By choosing

$$\varrho \leqslant \left(\frac{4\pi}{\beta 2^{3+g_\infty}} \right)^{1/2},$$

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we deduce, from (2.1) and (2.3), that

$$\sup_{v\in H^1_{0,r}(B):\int_B |\nabla v|^2 \mathrm{d}x\leqslant 1}\int_B \exp\left(\beta 2^{3+g_\infty}\varrho^2\left(\frac{|v|}{\varrho}\right)^{2+g(r)}\right)\mathrm{d}x < D. \tag{3.17}$$

Since

$$\int_{B_n^+} |\nabla v|^2 \mathrm{d}x + \int_{B_n^-} |\nabla v|^2 \mathrm{d}x = \int_B |\nabla v|^2 \mathrm{d}x,$$

it follows by (3.12) and (3.15) that

$$(I(\eta),\eta) \;\geqslant\; \varrho^2 - \lambda C_1 \varrho^{q_-} - C_2 C_5 \varrho^{p_-} - \frac{C_3}{n} - \lambda\,|B|\frac{1}{n^{q_-}} - C_6|B|\frac{1}{n^2}, \tag{3.18}$$

where

$$C_5 = D^{1/2}. \tag{3.19}$$

For

$$\varrho \leqslant \left(\frac{1}{2C_2C_5}\right)^{(1/p_- - 2)}, \tag{3.20}$$

we have

$$\varrho^2 - C_2 C_5 \varrho^{p_-} \geqslant \frac{\varrho^2}{2}.$$

Let $\varrho := \min\left\{1, (4\pi/\beta 2^{3+g_\infty})^{1/2}, (1/2C_2C_5)^{1/p_- - 2}\right\}$, and hence

$$(I(\eta),\eta) \;\geqslant\; \tfrac{\varrho^2}{2} - \lambda(C_1\varrho^{q_-} + |B|) - \frac{C_3}{n} - C_6|B|\frac{1}{n^2}. \tag{3.21}$$

Define $\rho = \varrho^2/8$ and

$$\lambda^* = \frac{\varrho^2}{4(C_1\varrho^{q_-} + |B|)}.$$

We choose $n^* \in \mathbb{N}$ such that

$$\frac{C_3}{n^*} + C_6|B|\frac{1}{(n^*)^2} < \frac{\varrho^2}{8}.$$

Let $\eta \in \mathbb{R}^m$, such that $|\eta| = \varrho$, then for $\lambda < \lambda^*$ and $n \geqslant n^*$ we obtain

$$(I(\eta),\eta) \geqslant \rho > 0. \tag{3.22}$$

Step 3. Since f_n is a Lipschitz function, for every $n \in \mathbb{N}$, it easy to see that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function. By lemma 3.4, for every $m \in \mathbb{N}$ there exists $y_m \in \mathbb{R}^m$ with $|y_m| \leqslant \varrho$ such that $I(y_m) = 0$, that is, there exists $v_m \in W_m$ satisfying

$$\|v_m\|_{H_0^1(B)} \leqslant \varrho \text{ for every } m \in \mathbb{N}$$

and such that

$$\int_B \nabla v_m \nabla w \mathrm{d}x = \lambda\int_B ((v_m)_+)^{q(r)-1} w\mathrm{d}x + \int_B f_n(r,(v_m)_+) w\mathrm{d}x + \frac{1}{n}\int_B w\mathrm{d}x, \tag{3.23}$$

for all $w \in W_m$.

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Since $W_m \subset H^1_{0,r}(B)$, $\forall\, m \in \mathbb{N}$ and ϱ does not depend on m, then the sequence (v_m) is bounded in $H^1_{0,r}(B)$. By taking a subsequence, if necessary, there exists $v \in H^1_{0,r}(B)$ such that

$$v_m \rightharpoonup v \text{ weakly in } H^1_{0,r}(B), \tag{3.24}$$

$$v_m \to v \text{ strongly in } L^2(B) \text{ and a.e. in } B. \tag{3.25}$$

Thus,

$$\|v\|_{H^1_0(B)} \leqslant \liminf_{m\to\infty} \|v_m\|_{H^1_0(B)} \leqslant \varrho. \tag{3.26}$$

By lemma 3.1(ii) and proposition 3.3, we obtain

$$|f_n(r,(v_m)_+) - f_n(r,v_+)| \leqslant Cn|(v_m)_+ - v_+| \leqslant Cn|v_m - v|.$$

Hence, (3.25) leads to

$$f_n(r,(v_m)_+) \to f_n(r,v_+) \text{ in } L^2(B). \tag{3.27}$$

Take $k \in \mathbb{N}$, then for every $m \geqslant k$ one has

$$\int_B \nabla v_m \nabla w_k \mathrm{d}x = \lambda \int_B (v_m)_+^{q(r)-1} w_k \mathrm{d}x + \int_B f_n(r,(v_m)_+) w_k \mathrm{d}x + \frac{1}{n}\int_B w_k \mathrm{d}x, \tag{3.28}$$

for all $w_k \in W_k$. Thus from (3.24), we obtain

$$\int_B \nabla v_m \nabla w_k \mathrm{d}x \to \int_B \nabla v \nabla w_k \mathrm{d}x. \tag{3.29}$$

We use (3.27), and the compact embedding $H^1_0(B) \hookrightarrow L^{q(r)}(B)$. Letting $m \to \infty$, by using the convergences before, it follows that

$$\begin{aligned}\lambda \int_B (v_m)_+^{q(r)-1} w_k \mathrm{d}x + \int_B f_n(r,(v_m)_+) w_k \mathrm{d}x \\ \to \lambda \int_B v_+^{q(r)-1} w_k \mathrm{d}x + \int_B f_n(r,v_+) w_k \mathrm{d}x.\end{aligned} \tag{3.30}$$

By (3.29) and (3.30)

$$\int_B \nabla v \nabla w_k \mathrm{d}x = \lambda \int_B v_+^{q(r)-1} w_k \mathrm{d}x + \int_B f_n(r,v_+) w_k \mathrm{d}x + \frac{1}{n}\int_B w_k \mathrm{d}x, \ \ \forall\, w_k \in W_k. \tag{3.31}$$

But $[W_k]_{k\in\mathbb{N}}$ is dense in $H^1_{0,r}(B)$, hence by linearity we get

$$\int_B \nabla v \nabla w \mathrm{d}x = \lambda \int_B v_+^{q(r)-1} w \mathrm{d}x + \int_B f_n(r,v_+) w \mathrm{d}x + \frac{1}{n}\int_B w \mathrm{d}x, \ \ \forall\, w \in H^1_{0,r}(B). \tag{3.32}$$

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Now, we borrow some ideas from [5, 9] to guarantee that v is a (weak) solution to problem (3.7), that is, v satisfies

$$\int_B \nabla v \nabla w \mathrm{d}x = \lambda \int_B v_+^{q(r)-1} w \mathrm{d}x + \int_B f_n(r, v_+) w \mathrm{d}x + \frac{1}{n} \int_B w \mathrm{d}x, \;\; \forall\, w \in H_0^1(B). \tag{3.33}$$

For the reader's convenience, we will keep the proof here. Define the operator $\Psi : H_0^1(B) \to (H_0^1(B))'$ by

$$\langle \Psi(v), w\rangle = \int_B \nabla v \nabla w \mathrm{d}x - \lambda \int_B v_+^{q(r)-1} w \mathrm{d}x - \int_B f_n(r, v_+) w \mathrm{d}x - \frac{1}{n} \int_B w \mathrm{d}x,$$

for all $v, w \in H_0^1(B)$, where $(H_0^1(B))'$ denote the dual space of $H_0^1(B)$ (the space of bounded linear functionals on $H_0^1(B)$). Since v satisfies equality (3.32), then

$$\langle \Psi(v), w_r\rangle = 0, \forall w_r \in H_{0,r}^1(B).$$

Notice that $H_{0,r}^1(B)$ is a closed subspace of the Hilbert space $H_0^1(B)$, so that, we can write

$$H_0^1(B) = H_{0,r}^1(B) \oplus (H_{0,r}^1(B))^{\perp}.$$

Therefore, for any $w \in H_0^1(B)$ we can split it as

$$w = w_r + w^{\perp}, \;\; \text{with} \;\; w_r \in H_{0,r}^1(B) \text{ and } \; w^{\perp} \in (H_{0,r}^1(B))^{\perp}.$$

On the other hand, since $H_{0,r}^1(B)$ is also a Hilbert space, we can identify, through the duality, $\Psi(v)$ with an element in $H_{0,r}^1(B)$. Then $\langle \Psi(v), w^{\perp}\rangle = 0$. Thus

$$\langle \Psi(v), w\rangle = \langle \Psi(v), w_r\rangle + \langle \Psi(v), w^{\perp}\rangle = 0, \;\forall\; w \in H_0^1(B),$$

and hence, v is a (weak) solution to problem (3.7).

Furthermore, $v \geqslant 0$ in B. In fact, using as a test function in (3.33) the function v_- we obtain

$$\int_B \nabla v \nabla v_- \mathrm{d}x \;\;=\;\; \lambda \textstyle\int_B v_+^{q(r)-1} v_- \mathrm{d}x + \int_B f_n(r, v_+) v_- \mathrm{d}x + \displaystyle\frac{1}{n} \int_B v_- \mathrm{d}x. \tag{3.34}$$

Hence

$$-\|v_-\|^2_{H_0^1(B)} = \lambda \int_B v_+^{q(r)-1} v_- \mathrm{d}x + \int_B f_n(r, v_+) v_- \mathrm{d}x + \frac{1}{n} \int_B v_- \mathrm{d}x \geqslant 0.$$

Therefore, $v_- \equiv 0$ a.e. in B.

The first inequality in hypothesis (1.8) and the equation in (3.7) guarantee that $v \neq 0$. Here the presence of $1/n > 0$ is needed. Hence the equation (3.7) has a weak solution $v \in H_0^1(B)$. Next, we observe that hypothesis (1.8), lemma 3.1 and proposition 3.3 allow us to refer to [**21**, theorem 7.1, chapter 7] from which we infer that $v \in L^\infty(B)$. Indeed, notice that the function

$$F_n(r,v) = \lambda(v_+)^{q(r)-1} + f_n(r,v) + \frac{1}{n}$$

satisfies

$$\begin{aligned} 0 \leqslant \operatorname{sign}(v) F_n(r,v) &\leqslant \lambda |v|^{q(r)-1} + c_n(r)|v| + \frac{1}{n} \\ &\leqslant \widetilde{C}(n)(1+|v|). \end{aligned}$$

Since $C(n) \in L^\infty(B)$, it is possible to choose $\alpha_2 = 1$ in condition (7.2) of [**21**]. Thus, we can apply [**21**, theorem 7.1, chapter 7] to obtain $v \in L^\infty(B)$. Now, from $0 < q_- - 1 \leqslant q(r) - 1 \leqslant q_+ - 1 < 1$ and f_n is Lipschitz, we infer that $\lambda v^{q(r)-1} + f_n(r,v) + 1/n \in L^s(B)$ for all $s > 2$. Hence $v \in C^{1,\beta}(\overline{B})$ with $0 < \beta < 1$, see [**17**]. Therefore, $v \in H_0^1(B) \cap C^{1,\beta}(\overline{B})$. Applying the strong maximum principle and Hopf boundary point lemma [**17**, pages 34 and 35], entails $v > 0$ in B and $\partial v/\partial \nu < 0$ on ∂B holds. □

REMARK 3.6. To apply [**21**, theorem 7.1] and infer that $v \in L^\infty(B)$, notice that it is necessary to consider the approximating functions f_n (given by lemma 3.1) instead of f.

4. Proof of theorem 1.1

The next result was proven in [**9**, lemma 4.1].

LEMMA 4.1. *For any constant $b > 0$ the problem*

$$\begin{cases} -\Delta u = b u^{q(r)-1} & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{4.1}$$

admits a solution $u_0 \in C^1(\overline{B})$.

REMARK 4.2. Notice that the Hopf boundary point lemma [**17**], page 34, ensures that $\partial u_0/\partial \nu < 0$ on ∂B holds. □

Now, for each $\lambda \in (0, \lambda^*)$, we are able to prove theorem 1.1. For each $n \in \mathbb{N}$ we know (by lemma 3.5) that equation (3.7) has a (weak) solution $u_n \in H_0^1(B) \cap C^{1,\beta}(\overline{B})$, for some $\beta \in (0,1)$.

By (3.26) we have that

$$\|u_n\|_{H_0^1(B)} \leqslant \varrho, \ \forall\, n \in \mathbb{N},$$

and ϱ does not depend on n. Thus, by passing to a subsequence, there exists $u_\lambda \in H_{0,r}^1(B)$ such that

$$u_n \rightharpoonup u_\lambda \ \text{ weakly in } \ H_0^1(B) \text{ as } n \to \infty. \tag{4.2}$$

By Sobolev compact imbedding we get

$$u_n \to u_\lambda \quad \text{strongly in} \quad L^2(B) \quad \text{and a.e. in} \quad B. \tag{4.3}$$

Note that

$$\begin{cases} -\Delta u_n \geqslant \lambda u_n^{q(r)-1} & \text{in} \quad B, \\ u_n > 0 & \text{in} \quad B, \\ u_n = 0 & \text{on} \quad \partial B. \end{cases} \tag{4.4}$$

By lemma 4.1 and proposition 2.6 with $b = \lambda$, it follows that

$$u_n \geqslant u_0 \text{ in } B, \ \forall\, n \in \mathbb{N}. \tag{4.5}$$

Letting $n \to \infty$ in (4.5) we obtain

$$u_\lambda \geqslant u_0 > 0 \text{ a.e. in } B,$$

showing that $u_\lambda > 0$ in B.

Next we prove that u_λ is a solution of (1.5). Since

$$u_n \to u_\lambda \text{ a.e. in } B,$$

we have

$$f_n(\cdot, u_n(\cdot)) \to f(\cdot, u_\lambda(\cdot)) \text{ a.e. in } B, \tag{4.6}$$

by the uniform convergence of lemma 3.1(iii).

Since $f_n(r, \cdot)$ is continuous, we obtain

$$f_n(r, u_n)^2 \to f_n(r, u_\lambda)^2 \quad \text{a.e. in} \quad B. \tag{4.7}$$

Since $\|u_n\|_{H_0^1(B)} \leqslant \varrho$, by (1.8), (3.17), and Hölder inequality, for $(u_n)_+ \neq 0$, we get

$$\int_B |f_n(r, u_n)|^2 \mathrm{d}x \leqslant C \|u_n\|_{H_0^1(B)}^{2(p_- -1)} \times \left(\int_B \exp\left(\beta 2^{3+g_\infty} \varrho^2 \left(\frac{|(u_n)_+|}{\|(u_n)_+\|_{H_0^1(B)}} \right)^{2+g(r)} \right) \mathrm{d}x \right)^{1/2^{2+g_\infty}} < C, \tag{4.8}$$

for each n. Thus, by (4.8) $(\|f_n(\cdot, u_n(\cdot))\|_{L^2(B)})_{n=1}^\infty$ is a bounded sequence of numbers, and by (4.7) we have $f_n(r, u_n) \to f(r, u_\lambda)$ a.e. in B, then [**18**, theorem 13.44] leads to

$$f_n(r, u_n) \rightharpoonup f(r, u_\lambda) \quad \text{weakly in} \quad L^2(B). \tag{4.9}$$

Recall from (3.33) that

$$\int_B \nabla u_n \nabla w \mathrm{d}x = \lambda \int_B u_n^{q(r)-1} w \mathrm{d}x + \int_B f_n(r,u_n) w \mathrm{d}x + \frac{1}{n}\int_B w \mathrm{d}x, \ \ \forall\, w \in H_0^1(B). \tag{4.10}$$

By (4.9), (4.10) and the compact embedding $H_0^1(B) \hookrightarrow L^{q(r)}(B)$, taking $n \to \infty$, we have

$$\int_B \nabla u_\lambda \nabla w \mathrm{d}x = \lambda \int_B u_\lambda^{q(r)-1} w \mathrm{d}x + \int_B f(r,u_\lambda) w \mathrm{d}x, \ \ \forall\, w \in H_0^1(B). \tag{4.11}$$

Now we will deduce that $\|u_\lambda\|_{H_0^1(B)} \to 0$ as $\lambda \to 0$. Using $w = u_n$ as a test function in (4.10), we obtain

$$\begin{aligned}\int_B |\nabla u_n|^2 \mathrm{d}x &= \lambda \int_{B_n^+ \cup B_n^-} u_n^{q(r)} \mathrm{d}x + \int_{B_n^+ \cup B_n^-} f_n(r,u_n) u_n \mathrm{d}x + \frac{1}{n}\int_{B_n^+ \cup B_n^-} u_n \mathrm{d}x \\ &\leqslant \lambda C_1 \|u_n\|_{H_0^1(B)}^{q_-} + C_2 \|u_n\|_{H_0^1(B)}^{p_-} \\ &\quad \times \left(\int_B \exp\left(\beta 2^{3+g_\infty} \|u_n\|_{H_0^1(B)}^2 \left(\frac{|u_n|}{\|u_n\|_{H_0^1(B)}}\right)^{2+g(r)}\right)\mathrm{d}x\right)^{1/2} \\ &\quad + \frac{C_3}{n}\|u_n\|_{H_0^1(B)} + \lambda\,|B|\frac{1}{n^{q_-}} + C_6|B|\frac{1}{n^2} \\ &\leqslant \lambda C_1 + C_2 C_5 \|u_n\|_{H_0^1(B)}^{p_-} + \frac{C_3}{n} + \lambda\,|B|\frac{1}{n^{q_-}} + C_6|B|\frac{1}{n^2},\end{aligned} \tag{4.12}$$

where C_1, C_2, C_3, C_5 and C_6 are given in (3.10)–(3.12), (3.19) and (3.18), respectively. Since $u_n \neq 0$, from (4.12), we have the following estimate

$$\|u_n\|_{H_0^1(B)}^2 (1 - C_2 C_5 \|u_n\|_{H_0^1(B)}^{p_- - 2}) \ \leqslant \ \lambda C_1 + \frac{C_3}{n} + \lambda\,|B|\frac{1}{n^{q_-}} + C_6|B|\frac{1}{n^2}. \tag{4.13}$$

By (3.20), we obtain

$$\|\tilde{u}_n\|_{H_0^1(B)}^{p_- - 2} \leqslant \frac{1}{2C_2C_5}.$$

Thus,

$$\|u_n\|_{H_0^1(B)} \ \leqslant \ \left[2\left(\lambda C_1 + \frac{C_3}{n} + \lambda\,|B|\frac{1}{n^{q_-}} + C_6|B|\frac{1}{n^2}\right)\right]^{1/2}. \tag{4.14}$$

By (4.2) and (4.14), we obtain

$$\|u_\lambda\|_{H_0^1(B)} \leqslant \liminf_{n\to\infty} \|u_n\|_{H_0^1(B)} \leqslant [2\lambda C_1]^{1/2}. \tag{4.15}$$

Thus, we conclude the proof. □

We proceed to prove theorem 1.2, we use an idea inspired in [**12**].

Proof. Assume by contradiction that that $\lambda^* = \infty$. Then there is a sequence $\lambda_n \to \infty$ and solutions $u_n \in C^1(\overline{B})$, $u_n > 0$ in B. Define

$$P(r,t) = \lambda t^{q(r)-1} + t^{p(r)-1} e^{\beta t^{2+g(r)}},$$

$$P_1(t) = \lambda t^{q_- -1} + t^{p_+ -1} e^{\beta t^2}$$

and

$$P_2(t) = \lambda t^{q_+ -1} + t^{p_- -1} e^{\beta t^2}.$$

We will show that there is a constant $C_\lambda > 0$ such that

$$P(r,t) \geqslant \min\{P_1(t), P_2(t)\} \geqslant C_\lambda t \quad for \quad t > 0.$$

In fact there exist constants $C_{1,\lambda} > 0$ and $C_{2,\lambda} > 0$ such that $P_1(t) \geqslant C_{1,\lambda} t$ and $P_2(t) \geqslant C_{2,\lambda} t$ for $t > 0$. And the asserted constant is $C_\lambda = \min\{C_{1,\lambda}, C_{2,\lambda}\}$. We will make the calculations for $P_1(t)$, the reasoning for $P_2(t)$ is analogue. Define the function $Q_1(t) = P_1(t)t^{-1}$. Then $Q_1(t) \to \infty$ as $t \to 0^+$ and as $t \to \infty$. The minimum value is $Q_1(t_1) = C_{1,\lambda}$, where $t_1 > 0$ is the unique root of

$$e^{\beta t^2} t^{p_+ - q_-} [2\beta t^2 + p_+ - 2] = \lambda(2 - q_-).$$

Notice that t_1 increases as λ increases. And the constant $C_{1,\lambda}$ has the same behaviour with respect to λ. Here we are considering λ sufficiently large. Let $\sigma_1 > 0$ the first eigenvalue of the Laplacian and $\varphi_1 > 0$ the associated first eigenfunction satisfying

$$\begin{cases} -\Delta\varphi_1 = \sigma_1\varphi_1 & \text{in} \quad B \\ \varphi_1 = 0 & \text{on} \quad \partial B. \end{cases}$$

Since $C_{1,\lambda_n} \to \infty$ as $\lambda_n \to \infty$, for each given $\delta > 0$, there is λ_{n_0} such that $C_{1,\lambda_{n_0}} \geqslant \sigma_1 + \delta + 1$. Hence the solution $u_{n_0} > 0$ of (1.5) corresponding to λ_{n_0} satisfies

$$\begin{cases} -\Delta u_{n_0} \geqslant (C_{1,\lambda_0} - 1)u_{n_0} \geqslant (\sigma_1 + \delta)u_{n_0} & \text{in} \quad B \\ u_{n_0} = 0 & \text{on} \quad \partial B. \end{cases}$$

On the other hand, taking $\varepsilon > 0$ small enough we obtain $\varepsilon\varphi_1 < u_{n_0}$ in B, this is possible because $u_{n_0} \geqslant u_0$ and $\partial u_0/\partial\nu < 0$ on ∂B, see remark 4.2. Furthermore, we have

$$\begin{cases} -\Delta(\varepsilon\varphi_1) = (\varepsilon\sigma_1)\varphi_1 \leqslant (\sigma_1 + \delta)(\varepsilon\varphi_1) & \text{in} \quad B, \\ \varphi_1 = 0 & \text{on} \quad \partial B, \end{cases}$$

and hence $\varepsilon\varphi_1$ is a sub-solution. By the sub-super-solution method, there is a solution $\varepsilon\varphi_1 < \zeta < u_{n_0}$ in B of

$$\begin{cases} -\Delta\zeta = (\sigma_1 + \delta)\zeta & \text{in} \quad B \\ \zeta = 0 & \text{on} \quad \partial B. \end{cases}$$

We thus have a contradiction to the fact that σ_1 is isolated, so we really have $\lambda^* < \infty$. □

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