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Hardy spaces on $\mathbb{Z}^N$

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The work of Coifman and Weiss concerning Hardy spaces on spaces of homogeneous type gives, as a particular case, a definition of $H^p(\mathbb{Z}^N)$ in terms of an atomic decomposition.

Other characterizations of these spaces have been studied by other authors, but it was an open question to see if they can be defined, as it happens in the classical case, in terms of a maximal function or via the discrete Riesz transforms.

In this paper, we give a positive answer to this question.

1. Introduction

There are several works related to the general study of Hardy spaces in spaces of homogeneous type. Let us mention the original definition given in terms of atoms by Coifman and Weiss in [4], the work of Macías and Segovia in [9], characterizing these spaces via a grand maximal function, a maximal characterization given by Uchiyama in [12] and the atomic decomposition given by Han in [7] in the setting of Triebel–Lizorkin spaces. When we look at the discrete case $\mathbb{Z}^N$, we must exclude the last two mentioned papers, because they work under the restriction of considering spaces of homogeneous type with no points of positive measure.

In the one-dimensional case, some work has been done in order to study some other characterizations of Hardy spaces on $\mathbb{Z}$. In [3], the authors studied the equivalence of several definitions for $H^p(\mathbb{Z})$, the classical one in terms of atoms introduced in [4], a second one in terms of the discrete Hilbert transform introduced by Eoff in [5], and finally via maximal and square functions. The case $N > 1$ presented some technical problems and remained open until now. In this paper we prove the equivalence with the original atomic definition of $H^p(\mathbb{Z}^N)$, with others in terms of a discrete maximal function or the discrete Riesz transforms.

In § 2, we extend the maximal definition of $H^p(\mathbb{Z})$ in terms of the discrete Poisson kernel given in [3] to an arbitrary dimension and prove, in the range $(N-1)/N < p \leqslant 1$, its equivalence with a definition in terms of the boundedness in $\ell^p(\mathbb{Z}^N)$ of the discrete Riesz transforms, which are the natural substitute in several variables of the discrete Hilbert transform.

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In § 3, we shall show that the maximal definition of $H^p(\mathbb{Z}^N)$ is equivalent with the atomic characterization, proving in this way that these new spaces agree with the original one of [4].

We shall use the standard notation about multi-indexes due to Schwartz.

We shall write $f \approx g$ to indicate the existence of two positive universal constants A and B, so that $Af \leqslant g \leqslant Bf$ and constants such as C may change from one occurrence to the next.

Also, for a function F defined in $\mathbb{R}^N$, we shall use the notation F^{d} to indicate the sequence $\{F(n)\}_{n\in\mathbb{Z}^N}$ whenever a different definition is not explicitly written.

We shall write $\star$ to indicate the convolution between two sequences.

Let E_R be the set of slowly increasing C^∞ functions f with $\operatorname{supp}\hat{f} \subset [-R,R]^N$. The elements of E_R are functions of exponential type. It is a well-known fact that if a function is in $L^p(\mathbb{R}^N)$ and its Fourier transform has compact support on, say, the cube $(-\frac{1}{2},\frac{1}{2})^N$, then its $L^p(\mathbb{R}^N)$-norm is comparable to the $\ell^p(\mathbb{Z}^N)$-norm of its samples on the set $\mathbb{Z}^N$, that is (see [2]), if $g \in E_R$ with $R < \frac{1}{2}$, then, for $0 < p < \infty$,

$$\|g\|_{L^p(\mathbb{R}^N)} \approx \|g^{\mathrm{d}}\|_{\ell^p(\mathbb{Z}^N)}. \tag{1.1}$$

In [3] (see also [1]), some useful extensions of (1.1) were proved.

Theorem 1.1. *Let $0 < p,\ q \leqslant \infty$ and $0 < R < \frac{1}{2}$. Let $\{g_t(\cdot)\}_{t>0}$ be a family of jointly measurable functions in E_R. Then*

$$\sum_{n\in\mathbb{Z}^N}\left(\int_0^\infty |g_t(n)|^q\frac{\mathrm{d}t}{t}\right)^{p/q} \approx \int_{\mathbb{R}^N}\left(\int_0^\infty |g_t(x)|^q\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}x.$$

The inequality $\leqslant$ holds without any restriction on R.

2. Discrete Hardy spaces of several variables

As we did in [3] for one variable, we can consider the following maximal definition of Hardy spaces on $\mathbb{Z}^N$.

Definition 2.1. Let $0 < p \leqslant 1$ and let us consider the discrete Poisson kernel

$$P_t^{\mathrm{d}}(n) = C_N\frac{t}{(t^2+n^2)^{(N+1)/2}}, \quad n \neq 0, \quad P_t^{\mathrm{d}}(0) = 0,$$

where C_N is a normalized constant depending on the dimension. Then we define

$$H^p_{\max}(\mathbb{Z}^N) = \{a \in \ell^p(\mathbb{Z}^N);\ \sup_{t>0}|P_t^{\mathrm{d}} \star a| \in \ell^p(\mathbb{Z}^N)\},$$

with the p-norm

$$\|a\|_{H^p_{\max}(\mathbb{Z}^N)} = \|a\|_{\ell^p(\mathbb{Z}^N)} + \|\sup_{t>0}|P_t^{\mathrm{d}} \star a|\|_{\ell^p(\mathbb{Z}^N)}.$$

As we mentioned in § 1, we shall also consider the restriction to $\mathbb{Z}^N$ of the Riesz kernel R_j of order j in $\mathbb{R}^N$, that is,

$$R_j^{\mathrm{d}}(m) = \frac{m_j}{|m|^{N+1}}, \quad \text{para}\, 1 \leqslant j \leqslant N, \quad \text{if } m = (m_1,\dots,m_N) \in \mathbb{Z}^N \setminus \{0\},$$

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and $R_j(0) = 0$. The discrete Riesz transforms, R_j^{d}, applied to a sequence a are the convolution operators

$$(R_j^{\mathrm{d}} a)(m) = (R_j^{\mathrm{d}} \star a)(m) = \sum_{n \neq m} a(n) \frac{m_j - n_j}{|m - n|^{N+1}}.$$

In the same way as we did in dimension $N = 1$ in terms of the discrete Hilbert transform (see [3, 5]), we can give the following definition.

Definition 2.2. Let $0 < p \leqslant 1$ and let us define

$$H^p_{\mathrm{Riesz}}(\mathbb{Z}^N) = \{a \in \ell^p(\mathbb{Z}^N) : R_j^{\mathrm{d}} a \in \ell^p(\mathbb{Z}^N),\ 1 \leqslant j \leqslant N\},$$

with the p-norm

$$\|a\|_{H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)} = \|a\|_{\ell^p(\mathbb{Z}^N)} + \sum_{j=1}^{N} \|R_j^{\mathrm{d}} a\|_{\ell^p(\mathbb{Z}^N)}.$$

In order to prove the equivalence of the spaces introduced above (theorem 2.6), we need some previous results.

We shall denote by $R_j(x) = x_j/|x|^{N+1}$ the usual kernel in $\mathbb{R}^N$ that defines the Riesz transform of order j, and we also write, for a function f in $\mathbb{R}^N$,

$$R_j f = \mathrm{p.\,v.}(R_j * f),$$

whenever it makes sense.

Lemma 2.3. *Let $k \geqslant 1$ be a fixed integer, let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} \varphi \,\mathrm{d}x = 1$$

and, for $k \geqslant 2$, let

$$\int_{\mathbb{R}^N} x^\alpha \varphi(x) \,\mathrm{d}x = 0$$

for every multi-index α with $1 \leqslant |\alpha| \leqslant k - 1$. Then

$$|R_j \varphi_{1/t}(x') - R_j(x')| \leqslant C/t^k$$

uniformly in $x' \in \Sigma_{N-1}$ (the $(N-1)$-dimensional sphere).

Proof. First of all, we observe that we can write, for any $f \in \mathcal{S}(\mathbb{R}^N)$ and $x' \in \Sigma_{N-1}$,

$$R_j f(x') = \int_{|y| \leqslant 2/3} \tfrac{1}{2} R_j(y) f(x' - y) - f(x' + y) \,\mathrm{d}y + \int_{|y| > 2/3} R_j(y) f(x' - y) \,\mathrm{d}y = I_1 + I_2.$$

By the mean-value theorem, we have

$$|I_1| \leqslant \int_{|y| \leqslant 2/3} \frac{C}{|y|^N} |\nabla f(z(y))| |y| \,\mathrm{d}y,$$

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where $z(y) \in [x'-y, x'+y]$. If now $f = \varphi_{1/t}$, then $|\nabla(\varphi_{1/t})(z)| = t|\Psi_{1/t}(z)|$, where $\Psi = |\nabla\varphi|$. Since $z(y) \in [x'-y, x'+y] \subset B(x', \frac{2}{3}) \subset B(0, \frac{1}{3})^{\mathrm{c}}$, we have $|z(y)| \geqslant \frac{1}{3}$. Therefore, using the decay of φ, we have

$$|I_1| \leqslant \int_{|y|\leqslant 2/3} \frac{C}{|y|^{N-1}} O\biggl(\frac{1}{t^k}\biggr)\,\mathrm{d}y \leqslant O\biggl(\frac{1}{t^k}\biggr).$$

For I_2, we obtain that

$$I_2 = \int_{|y|>2/3,\,|x'-y|\leqslant |y|/2} R_j(y)\varphi_{1/t}(x'-y)\,\mathrm{d}y + \int_{|y|>2/3,\,|x'-y|>|y|/2} R_j(y)\varphi_{1/t}(x'-y)\,\mathrm{d}y = I_3 + I_4.$$

For I_4, we use the fact that $|R_j(y)| \leqslant C$ on $|y| > \frac{2}{3}$ and, again, the decay of φ to obtain

$$|I_4| \leqslant \int_{|y|>2/3,\,|x'-y|>|y|/2} C|\varphi_{1/t}(x'-y)|\,\mathrm{d}y \leqslant C\int_{|z|>1/3} |\varphi_{1/t}(z)|\,\mathrm{d}z = O\biggl(\frac{1}{t^k}\biggr).$$

Let $P_{k-1}[R_j, x']$ be the Taylor polynomial of degree $k-1$ of R_j in x'. Using the assumptions on φ, we have that

$$\begin{aligned} I_3 - R_j(x') &= \int_{|y|>2/3,\,|x'-y|\leqslant |y|/2} R_j(y)\varphi_{1/t}(x'-y)\,\mathrm{d}y \\ &\quad - \sum_{0\leqslant |\alpha|\leqslant k-1} \frac{1}{|\alpha|!}\frac{\partial^\alpha R_j}{\partial x^\alpha}(x') \int_{\mathbb{R}^N} (y-x')^\alpha \varphi_{1/t}(x'-y)\,\mathrm{d}y \\ &= \int_{|y|>2/3,\,|x'-y|\leqslant |y|/2} (R_j(y) - P_{k-1}[R_j, x'](y))\varphi_{1/t}(x'-y)\,\mathrm{d}y \\ &\quad - \sum_{0\leqslant |\alpha|\leqslant k-1} \frac{1}{|\alpha|!}\frac{\partial^\alpha R_j}{\partial x^\alpha}(x') \int_{|y|<2/3\cup|x'-y|>|y|/2} (y-x')^\alpha \varphi_{1/t}(x'-y)\,\mathrm{d}y \\ &= I_5 - \sum_{|\alpha|=0}^{k-1} I_6. \end{aligned}$$

For every term in I_6, we use

$$\biggl|\frac{\partial^\alpha R_j}{\partial x^\alpha}(x')\biggr| \leqslant C$$

uniformly in $x' \in \Sigma_{N-1}$, and the fact that for $|y| < \frac{2}{3}$, $\frac{1}{3} \leqslant |x'-y| \leqslant \frac{5}{3}$, to obtain

$$\int_{|y|<2/3} |y-x'|^{|\alpha|}|\varphi_{1/t}(x'-y)|\,\mathrm{d}y \leqslant C\int_{|z|\geqslant 1/3} |\varphi_{1/t}(z)|\,\mathrm{d}z = O\biggl(\frac{1}{t^k}\biggr).$$

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On the other hand, if $|x'-y|>\frac{1}{2}|y|$ and $|y|>\frac{2}{3}$, we have $|x'-y|>\frac{1}{3}$, and therefore

$$\int_{|x'-y|>|y|/2}|y-x'|^{|\alpha|}|\varphi_{1/t}(x'-y)|\,\mathrm{d}y\leqslant C\int_{|z|\geqslant 1/3}\frac{|z|^{|\alpha|}t^N}{(1+|z|t)^M}\,\mathrm{d}z=O\biggl(\frac{1}{t^k}\biggr),$$

where we have taken $M=N+|\alpha|+k$.

For I_5, we use the decay of φ to obtain

$$\begin{aligned}|I_5|&\leqslant\int_{|y|>2/3,\,|x'-y|\leqslant|y|/2}C|y-x'|^k\frac{t^N}{(1+t|x'-y|)^M}\,\mathrm{d}y\\&\leqslant\frac{1}{t^k}\int_{\mathbb{R}^N}\frac{t^N}{(1+t|x'-y|)^{M-k}}\,\mathrm{d}y\\&=O\biggl(\frac{1}{t^k}\biggr).\end{aligned}$$

□

Corollary 2.4. *If* $\varphi\in\mathcal{S}(\mathbb{R}^N)$ *and* $k\geqslant 1$ *are as in the previous lemma,*

$$(R_j\varphi)(x)=R_j(x)+O\biggl(\frac{1}{|x|^{N+k}}\biggr)$$

for every $x\in\mathbb{R}^N\setminus\{0\}$.

Proof. If $x'=x/|x|\in\Sigma_{N-1}$, we observe that

$$(R_j\varphi)(x)=|x|^{-N}R_j\varphi_{1/|x|}(x')$$

and, as a consequence of the previous lemma,

$$R_j\varphi_{1/|x|}(x')-R_j(x')=O\biggl(\frac{1}{|x|^k}\biggr).$$

Therefore,

$$(R_j\varphi)(x)-R_j(x)=|x|^{-N}(R_j\varphi_{1/|x|}(x')-R_j(x'))=O\biggl(\frac{1}{|x|^{N+k}}\biggr).$$

□

Lemma 2.5. *Let* $\varphi\in\mathcal{S}(\mathbb{R}^N)$ *be such that* $\hat{\varphi}\equiv 1$ *in a neighbourhood of zero and* $0<p<\infty$. *Then*

$$\{\sup_{t>0}|P_t(n)-(P_t*\varphi)(n)|\}_{n\in\mathbb{Z}^N\setminus\{0\}}\in\ell^p(\mathbb{Z}^N).$$

Proof. We write

$$(P_t*\varphi)(n)-P_t(n)=\int_{\mathbb{R}^N}\mathrm{e}^{-2\pi t|\xi|}(\hat{\varphi}(\xi)-1)\mathrm{e}^{2\pi\mathrm{i}n\cdot\xi}\,\mathrm{d}\xi.\tag{2.1}$$

If we denote by $H_t(\xi)=\mathrm{e}^{-2\pi t|\xi|}(\hat{\varphi}(\xi)-1)$, we observe that the hypotheses assumed on $\hat{\varphi}$ imply that $H_t\in C^\infty(\mathbb{R}^N)$, the partial derivative of H_t with respect to

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every multi-index α is uniformly bounded on $t > 0$, and

$$\lim_{|\xi|\to\infty} \frac{\partial^\alpha H_t}{\partial \xi^\alpha}(\xi) = 0.$$

Hence, if $n_{i_1}, \dots, n_{i_k}$ are the non-vanishing components not equal to zero of $n = (n_1, \dots, n_N)$, and we integrate by parts repeatedly in the integral (2.1) with respect to the corresponding variables $\xi_{i_1}, \dots, \xi_{i_k}$, we have

$$(P_t * \varphi)(n) - P_t(n) = \frac{1}{(2\pi \mathrm{i})^{|j|} n_{i_1}^{j_1} \dots n_{i_k}^{j_k}} \int_{\mathbb{R}^N} \frac{\partial^j H_t}{\partial \xi^j}(\xi) \mathrm{e}^{2\pi \mathrm{i} n\cdot\xi} \, \mathrm{d}\xi,$$

where $j_1, \dots, j_k$ are the non-vanishing components of the multi-index j that correspond to the variables $\xi_{i_1}, \dots, \xi_{i_k}$, respectively. From here, we obtain that

$$|(P_t * \varphi)(n) - P_t(n)| = O\biggl(\frac{1}{|n_{i_1}|^{j_1} \dots |n_{i_k}|^{j_k}}\biggr).$$

Taking $j_1, \dots, j_k$ big enough, depending on p, we get the result. □

Theorem 2.6. *Let $(N-1)/N < p \leqslant 1$, then*

$$H^p_{\mathrm{Riesz}}(\mathbb{Z}^N) = H^p_{\max}(\mathbb{Z}^N),$$

with equivalent H^p-norms.

Proof. Let $0 < R < \frac{1}{2}$ and let φ be a radial function of E_R such that $\hat{\varphi} \equiv 1$ in a neighbourhood of zero. Take $a \in H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)$ and set

$$g(x) = \sum_{n\in\mathbb{Z}^N} a(n)\varphi(x-n) \in L^2(\mathbb{R}^N) \cap E_R.$$

Using Fourier's inversion theorem, it follows that, for all $1 \leqslant j \leqslant N$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
(R_j g)(x) &= \biggl(-\mathrm{i}C_N \frac{\xi_j}{|\xi|}\hat{g}(\xi)\biggr)^{\vee}(x) \\
&= \int_{\mathbb{R}^N} -\mathrm{i}C_N \frac{\xi_j}{|\xi|} \sum_{n\in\mathbb{Z}^N} a(n)\mathrm{e}^{-2\pi \mathrm{i} n\cdot\xi}\hat{\varphi}(\xi)\mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\,\mathrm{d}\xi \\
&= \sum_{n\in\mathbb{Z}^N} a(n) \int_{\mathbb{R}^N} -\mathrm{i}C_N \frac{\xi_j}{|\xi|}\hat{\varphi}(\xi)\mathrm{e}^{2\pi \mathrm{i}(x-n)\cdot\xi}\,\mathrm{d}\xi \\
&= \sum_{n\in\mathbb{Z}^N} a(n)(R_j\varphi)(x-n). \qquad (2.2)
\end{aligned}$$

Corollary 2.4 tell us that, for every natural number $k \geqslant 1$, if $m \in \mathbb{Z}^N \setminus \{0\}$,

$$(R_j\varphi)(m) = R_j(m) + O\biggl(\frac{1}{|m|^{N+k}}\biggr).$$

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Since φ is radial, $R_j\varphi(0) = 0$. Then, from (2.2) and taking into account the above formula $k > N(1/p - 1)$, we obtain that

$$\begin{aligned}\|(R_j g)^{\mathrm{d}} - R_j^{\mathrm{d}} a\|_{\ell^p(\mathbb{Z}^N)}^p &\leqslant C \sum_m \biggl(\sum_{n \neq m} |a(n)| \frac{1}{|m-n|^{N+k}} \biggr)^p \\ &\leqslant C \sum_m \biggl(\sum_{n \neq m} |a(n)|^p \frac{1}{|m-n|^{(N+k)p}} \biggr) \\ &\leqslant C \|a\|_{\ell^p(\mathbb{Z}^N)}^p.\end{aligned}$$

As a consequence of this estimate and (1.1) applied to the function $R_j g$ of exponential type, we can deduce that

$$\begin{aligned}\sum_{j=1}^N \|R_j g\|_{L^p(\mathbb{R}^N)} &\leqslant C \sum_{j=1}^N \|(R_j g)^{\mathrm{d}}\|_{\ell^p(\mathbb{Z}^N)} \\ &\leqslant C \biggl(\|a\|_{\ell^p(\mathbb{Z}^N)} + \sum_{j=1}^N \|R_j^{\mathrm{d}} a\|_{\ell^p(\mathbb{Z}^N)} \biggr) \\ &= C \|a\|_{H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)}. \end{aligned} \tag{2.3}$$

Since $g \in L^2(\mathbb{R}^N)$, we can use the known characterization of $H^p(\mathbb{R}^N)$ in terms of Riesz transforms (see [10]) for the range $(N-1)/N < p \leqslant 1$ to deduce that $g \in H^p(\mathbb{R}^N)$. Moreover, from (2.3), we obtain that

$$\|\sup_{t>0} |P_t * g|\|_{L^p(\mathbb{R}^N)} \leqslant C \biggl(\|g\|_{L^p(\mathbb{R}^N)} + \sum_{j=1}^N \|R_j g\|_{L^p(\mathbb{R}^N)} \biggr) \leqslant C \|a\|_{H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)}, \tag{2.4}$$

where we have also used the fact that, for $0 < p \leqslant 1$,

$$\|g\|_{L^p(\mathbb{R}^N)} \leqslant C \|a\|_{\ell^p(\mathbb{Z}^N)} \leqslant C \|a\|_{H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)}.$$

Now, since $a \in H^p_{\mathrm{Riesz}}(\mathbb{Z}^N) \subset \ell^p \subset \ell^1$ and $P_t \in L^1(\mathbb{R}^N)$, we write, for each $t > 0$ and $x \in \mathbb{R}^N$,

$$(P_t * g)(x) = \sum_{n \in \mathbb{Z}^N} a(n)(P_t * \varphi)(x-n).$$

Therefore, if $P_t^{\varphi} = P_t * \varphi$, lemma 2.5 and theorem 1.1 imply that

$$\begin{aligned}\|\sup_{t>0} |P_t^{\mathrm{d}} \star a|\|_{\ell^p(\mathbb{Z}^N)} &\leqslant C(\|a\|_{\ell^p(\mathbb{Z}^N)} + \|\sup_{t>0} |P_t^{\varphi} \star a|\|_{\ell^p(\mathbb{Z}^N)}) \\ &= C(\|a\|_{\ell^p(\mathbb{Z}^N)} + \|\sup_{t>0} |(P_t * g)^{\mathrm{d}}|\|_{\ell^p(\mathbb{Z}^N)}) \\ &\leqslant C(\|a\|_{\ell^p(\mathbb{Z}^N)} + \|\sup_{t>0} |P_t * g|\|_{L^p(\mathbb{R}^N)}). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we conclude that

$$\|a\|_{H^p_{\max}(\mathbb{Z}^N)} \leqslant C \|a\|_{H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)}.$$

A similar argument proves the embedding $H^p_{\max}(\mathbb{Z}^N) \hookrightarrow H^p_{\mathrm{Riesz}}(\mathbb{Z}^N)$. □

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From now on, we shall write $H^p(\mathbb{Z}^N)$ to represent the space $H^p_{\max}(\mathbb{Z}^N)$, $0 < p \leqslant 1$.

In the same way that it was done in [3] for one variable, the discrete Poisson kernel P_t^{d} can be substituted by Φ_t^{d}, where $\Phi_t^{\mathrm{d}}(n) = t^{-N}\Phi(n/t)$ if $n \neq 0$, $\Phi_t^{\mathrm{d}}(0) = 0$, with Φ a function in the Schwartz class with

$$\int_{\mathbb{R}^N} \Phi = 1.$$

Theorem 2.7. *Let $0 < p \leqslant 1$ and let $\Phi \in \mathcal{S}$ be such that*

$$\int_{\mathbb{R}^N} \Phi = 1.$$

Then

$$\|a\|_{\ell^p(\mathbb{Z}^N)} + \|\sup_{t>0} |\Phi_t^{\mathrm{d}} \star a|\|_{\ell^p(\mathbb{Z}^N)} \approx \|a\|_{H^p(\mathbb{Z}^N)}$$

for every $a \in H^p(\mathbb{Z}^N)$.

In relation to this last theorem, we can prove the following proposition about the boundedness of some discrete maximal operators on $H^p(\mathbb{Z}^N)$, which will be useful in the next section.

Proposition 2.8. *Let $0 < p \leqslant 1$ and $\Phi \in \mathcal{S}(\mathbb{R}^N)$. The discrete maximal operator with kernel Φ_t^{d} is bounded from $H^p(\mathbb{Z}^N)$ into $\ell^p(\mathbb{Z}^N)$, that is, there exists some positive constant $C > 0$, independent of a, such that*

$$\|\sup_{t>0} |\Phi_t^{\mathrm{d}} \star a|\|_{\ell^p(\mathbb{Z}^N)} \leqslant C\|a\|_{H^p(\mathbb{Z}^N)}.$$

Proof. For $a \in H^p(\mathbb{Z}^N)$, and $\varphi \in \mathcal{S}(\mathbb{R}^N) \cap E_R$ such that $\hat{\varphi} \equiv 1$ in a zero neighbourhood, let us construct the function

$$g(x) = \sum_{n \in \mathbb{Z}^N} a(n)\varphi(x - n).$$

As in theorem 2.6, we can show that $g \in H^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Moreover,

$$\|g\|_{H^p(\mathbb{R}^N)} = \|\sup_{t>0} |P_t * g|\|_{L^p(\mathbb{R}^N)} \leqslant C\|a\|_{H^p(\mathbb{Z}^N)}. \tag{2.6}$$

On the other hand, we can write for each $t > 0$ and $m \in \mathbb{Z}^N$, that

$$(\Phi_t * g)(m) = ((\Phi_t * \varphi)^{\mathrm{d}} \star a)(m).$$

If $\Phi_t^{\varphi} = (\Phi_t * \varphi)^{\mathrm{d}}$, we deduce, as a consequence of theorem 1.1 and (2.6), that

$$\begin{aligned}
\|\sup_{t>0} |\Phi_t^{\varphi} \star a|\|_{\ell^p(\mathbb{Z}^N)} &= \|\sup_{t>0} |(\Phi_t * g)^{\mathrm{d}}|\|_{\ell^p(\mathbb{Z}^N)} \\
&\leqslant C\|\sup_{t>0} |\Phi_t * g|\|_{L^p(\mathbb{R}^N)} \\
&\leqslant C\|g\|_{H^p(\mathbb{R}^N)} \\
&\leqslant C\|a\|_{H^p(\mathbb{R}^N)},
\end{aligned} \tag{2.7}$$

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where we have used the fact that the maximal operator associated to the kernel $\{\Phi_t\}_{t>0}$ is bounded from $H^p(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$, which can be easily proved from the decay of Φ using the atomic decomposition of the space $H^p(\mathbb{R}^N)$ (see [8]).

Now, with a similar proof to that of lemma 2.5, we obtain that

$$\{(\Phi_t * \varphi)(n) - \Phi_t(n)\}_{n\in\mathbb{Z}^N\setminus\{0\}} \in \ell^p(\mathbb{Z}^N),$$

and thus, as a consequence of (2.7), we conclude that

$$\|\sup_{t>0}|\Phi_t^{\mathrm{d}} \star a|\|_{\ell^p(\mathbb{Z}^N)} \leqslant C(\|a\|_{\ell^p(\mathbb{Z}^N)} + \|\sup_{t>0}|\Phi_t^{\varphi} \star a|\|_{\ell^p(\mathbb{Z}^N)}) \leqslant C\|a\|_{H^p(\mathbb{Z}^N)}.$$

□

3. Atomic decompositions for sequences in $H^p(\mathbb{Z}^N)$

In order to prove the connection with the atomic version of the Hardy spaces on $\mathbb{Z}^N$ introduced in [4], we need the following result that relates sequences a in $H^p(\mathbb{Z}^N)$ and functions in $H^p(\mathbb{R}^N)$ constructed from a (see also [3,11]).

Theorem 3.1. *Let $0 < p \leqslant 1$ and $a \in H^p(\mathbb{Z}^N)$. If $\phi \in L^2(\mathbb{R}^N)$, with* $\operatorname{supp}\phi \subset \{|x| \leqslant A\}$, *then*

$$f(x) = \sum_{n\in\mathbb{Z}^N} a(n)\phi(x-n) \in H^p(\mathbb{R}^N).$$

Moreover, there exists a constant $C = C(p,N)$ such that

$$\|f\|_{H^p(\mathbb{R}^N)} \leqslant C\|a\|_{H^p(\mathbb{Z}^N)}.$$

Proof. To estimate $\|f\|_{H^p(\mathbb{R}^N)}$, we shall use the maximal characterization of the spaces $H^p(\mathbb{R}^N)$, with the kernel a function $\Psi \in \mathcal{S}$ with $\operatorname{supp}\Psi \subset B(0,1)$ (see [6]).

Therefore, we write

$$\begin{aligned}
&\|\sup_{t>0}|\Psi_t * f|\|^p_{L^p(\mathbb{R}^N)}\\
&\quad= \int_{\mathbb{R}^N} \sup_{t>0}\left|\int_{\mathbb{R}^N} \sum_{n\in\mathbb{Z}^N} a(n)\Psi_t(x-y)\phi(y-n)\,\mathrm{d}y\right|^p \mathrm{d}x\\
&\quad= \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t>0}\left|\int_{\mathbb{R}^N} \Psi_t(x-y)\biggl(\sum_{|n-m|\leqslant C_0} + \sum_{|n-m|>C_0}\biggr)\right.\\
&\qquad\qquad\left.\times (a(n)\phi(y-n))\,\mathrm{d}y\right|^p \mathrm{d}x\\
&\quad\leqslant \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t>0}\left|\sum_{|n-m|\leqslant C_0} a(n)\int_{\mathbb{R}^N} \Psi_t(x-y)\phi(y-n))\,\mathrm{d}y\right|^p \mathrm{d}x\\
&\qquad+ \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t>0}\left|\int_{\mathbb{R}^N} \sum_{|n-m|>C_0} a(n)\phi(y-n)(\Psi_t(x-y)\right.\\
&\qquad\qquad\left.- P_{N_0}[\Psi_t, m-n](x-y))\,\mathrm{d}y\right|^p \mathrm{d}x
\end{aligned}$$

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$$+ \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t>0} \biggl| \int_{\mathbb{R}^N} \sum_{|n-m|>C_0} a(n)\phi(y-n) \times P_{N_0}[\Psi_t, m-n](x-y)\,\mathrm{d}y \biggr|^p \mathrm{d}x$$

$$= \text{(I)} + \text{(II)} + \text{(III)},$$

where $P_{N_0}[\Psi_t, m-n](x-y)$ denotes the Taylor polynomial of degree

$$N_0 = [N(1/p-1)]$$

of Ψ_t in $m-n$ at the value $x-y$.

Being $\phi \in L^2(\mathbb{R}^N)$, the maximal function $\sup_{t>0} |\Psi_t * \phi|$ is locally integrable and we can estimate (I) as follows:

$$\begin{aligned}\text{(I)} &= \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t>0} \biggl| \sum_{|n-m|\leqslant C_0} a(n)(\Psi_t * \phi)(x-n) \biggr|^p \mathrm{d}x \\ &\leqslant \sum_{m\in\mathbb{Z}^N} \sum_{|m-n|\leqslant C_0} |a(n)|^p \int_{m-n+[0,1)^N} \sup_{t>0} |\Psi_t * \phi|^p(x)\,\mathrm{d}x \leqslant C\|a\|^p_{\ell^p(\mathbb{Z}^N)}.\end{aligned}$$

To estimate (II), we observe that, for every multi-index α such that $|\alpha| = N_0+1$, for all $M>0$ and $0<\theta<1$,

$$\begin{aligned}&|D^\alpha \Psi_t(m-n-\theta(x-y-m+n))(x-y-m+n)^\alpha| \\ &\qquad \leqslant Ct^{-N-N_0-1}\biggl(1+\frac{|m-n-\theta(x-y-m+n)|}{t}\biggr)^{-M} \\ &\qquad = Ct^{M-N-N_0-1}(t+|m-n-\theta(x-y-m+n)|)^{-M}. \qquad (3.1)\end{aligned}$$

Since $x-m\in[0,1)^N$, $|y-n|\leqslant A$ and $|m-n|>C_0$, taking C_0 large enough, we get

$$t+|n-m-\theta(x-y-m+n)| \geqslant C|n-m|.$$

Therefore, if we take $M = N+N_0+1$ in (3.1), we obtain that

$$\begin{aligned}\text{(II)} &= \sum_{m\in\mathbb{Z}^N} \biggl(\int_{m+[0,1)^N} \sup_{t>0} \biggl| \int_{\mathbb{R}^N} \sum_{|n-m|>C_0} a(n)\phi(y-n) \times \sum_{|\alpha|=N_0+1} \frac{1}{|\alpha|!} D^\alpha \Psi_t \\ &\qquad \times (m-n-\theta(x-y-m+n)) \times (x-y-m+n)^\alpha\,\mathrm{d}y \biggr|^p \biggr) \mathrm{d}x \\ &\leqslant C \sum_{m\in\mathbb{Z}^N} \biggl(\sum_{|n-m|>C_0} \frac{|a(n)|}{|m-n|^{N+N_0+1}} \int_{\mathbb{R}^N} |\phi(y)|\,\mathrm{d}y \biggr)^p \\ &\leqslant C\|a\|^p_{H^p(\mathbb{Z}^N)}.\end{aligned}$$

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Since $\operatorname{supp}\Psi \subset B(0,1)$, (III) can be bounded by means of

$$\begin{aligned}
\text{(III)} &= \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t\geqslant 1} \bigg| \sum_{|n-m|>C_0} a(n) \sum_{|\alpha|=0}^{N_0} \frac{1}{|\alpha|!}\frac{1}{t^{N+|\alpha|}} (D^\alpha \Psi) \\
&\qquad\qquad \times \left(\frac{m-n}{t}\right) \int_{\mathbb{R}^N} \phi(y)(x-m-y)^\alpha \, \mathrm{d}y \bigg|^p \, \mathrm{d}x \\
&\leqslant \sum_{|\alpha|=0}^{N_0} \sum_{m\in\mathbb{Z}^N} \int_{m+[0,1)^N} \sup_{t\geqslant 1} \bigg| \sum_{|n-m|>C_0} a(n) \frac{1}{|\alpha|! t^{|\alpha|}} (D^\alpha \Psi)_t \\
&\qquad\qquad \times (m-n) \int_{\mathbb{R}^N} \phi(y)(x-m-y)^\alpha \, \mathrm{d}y \bigg|^p \, \mathrm{d}x \\
&\leqslant \sum_{|\alpha|=0}^{N_0} \sum_{m\in\mathbb{Z}^N} \sup_{t\geqslant 1} \bigg| \sum_{|n-m|>C_0} a(n) \frac{1}{|\alpha|!} (D^\alpha \Psi)_t (m-n) \bigg|^p \\
&\qquad\qquad \times \left(\int_{[0,1)^N} \bigg| \int_{\mathbb{R}^N} \phi(y)(x-y)^\alpha \, \mathrm{d}y \bigg|^p \, \mathrm{d}x \right) \\
&\leqslant \|a\|^p_{H^p(\mathbb{Z}^N)},
\end{aligned}$$

where we have used the following application of proposition 2.8:

$$\| \sup_{t\geqslant 1} |(D^\alpha \Psi)^{\mathrm{d}}_t \star a| \|_{\ell^p(\mathbb{Z}^N)} \leqslant C \|a\|_{H^p(\mathbb{Z}^N)}.$$

Therefore, we have obtained

$$\| \sup_{t>0} |\Psi_t * f| \|_{L^p(\mathbb{R}^N)} \leqslant C \|a\|_{H^p(\mathbb{R}^N)}.$$

□

Let us now recall the definition of an atom in $H^p(\mathbb{Z}^N)$ (see [4]).

Definition 3.2. Let $0 < p \leqslant 1$. We say that a sequence $a = \{a(n)\}_{n\in\mathbb{Z}^N}$ is an H^p-atom in $\mathbb{Z}^N$ if the following conditions hold.

(i) There exists a cube Q in $\mathbb{Z}^N$ such that $\operatorname{supp} a \subset Q$.

(ii) $\|a\|_\infty \leqslant 1/(\#Q)^{1/p}$, where $\#Q$ represents the cardinality of Q.

(iii) $\sum n^\alpha a(n) = 0$ for every multi-index $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant N(p^{-1}-1)$.

The atomic space $H^p_{\mathrm{at}}(\mathbb{Z}^N)$ consists of the space of sequences a such that

$$a = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where a_j are H^p-atoms and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover,

$$\|a\|_{H^p_{\mathrm{at}}(\mathbb{Z}^N)} = \inf\left\{ \left(\sum_j |\lambda_i|^p \right)^{1/p} \right\},$$

where the infimum is taking over all possible representations of a as above.

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The standard proof in the setting of homogeneous-type spaces shows the following.

THEOREM 3.3. *Let $0 < p \leqslant 1$. Then the space $H^p_{\mathrm{at}}(\mathbb{Z}^N)$ is continuously embedded in $H^p(\mathbb{Z}^N)$.*

With the aim of getting an atomic decomposition for sequences in $H^p(\mathbb{Z}^N)$, we must construct, for each $k = [N(1/p-1)]$, some auxiliary functions in $\mathbb{R}^N$, say C_k, with some special conditions (see [3] for the case $N = 1$). The required properties for C_k are the following.

(i) $\operatorname{supp} C_k \subset [-\frac{1}{2}(k+1), \frac{1}{2}(k+1)]^N = Q_k$.

(ii) If k is even and $m \in \mathbb{Z}^N$, C_k must be a polynomial of degree less than or equal to k over any cube of the form $m + [-\frac{1}{2}, \frac{1}{2}]^N$. (From now on, we shall restrict ourselves to the case of k an even integer, in the remark that follows theorem 3.7 we shall hint the necessary changes for k odd integer.)

(iii) For every multi-index $j \in \mathbb{N}^N$ such that $0 \leqslant |j| \leqslant k$,

$$\sum_{m\in\mathbb{Z}^N} m^j C_k(x-m) = P_j(x), \quad x \in \mathbb{R}^N,$$

where P_j is a polynomial of degree $|j|$ fixed by the following condition

$$\int_{n+[-1/4,1/4]^N} P_j(x)\,\mathrm{d}x = n^j \quad \text{for each } n \in \mathbb{Z}^N \tag{3.2}$$

(if $n = j = 0$, we shall understand that $n^j = 1$).

Let us now prove that (3.2) determines in a unique way the polynomials P_j.

LEMMA 3.4. *For each $j \in \mathbb{N}^N$, there exists a unique polynomial $P_j(x)$ of degree $|j|$ that verifies (3.2).*

Proof. Let

$$P_j(x) = \sum_{|s|\leqslant |j|} B^s_j x^s,$$

and let us check that the coefficients $\{B^s_j\}_{|s|\leqslant|j|}$ are determined in a unique way. To do this, we observe that

$$\begin{aligned}
\int_{n+[-1/4,1/4]^N} P_j(x)\,\mathrm{d}x &= \sum_{|s|\leqslant |j|} B^s_j \int_{[-1/4,1/4]^N} (x+n)^s\,\mathrm{d}x \\
&= \sum_{|s|\leqslant |j|} B^s_j \biggl[\int_{[-1/4,1/4]^N} x^{s-\beta} \sum_{\beta\leqslant s} \binom{s}{\beta} n^\beta\,\mathrm{d}x\biggr] \\
&= \sum_{|s|\leqslant |j|} B^s_j \sum_{\beta\leqslant s} \binom{s}{\beta} n^\beta I_{s-\beta},
\end{aligned}$$

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where

$$I_\alpha = \int_{[-1/4,1/4]^N} x^\alpha \, dx.$$

Therefore, the coefficients B_j^s have to satisfy the following system:

$$\sum_{|\beta|\leqslant |j|} n^\beta \biggl[\sum_{|s|\leqslant |j|,\, s\geqslant \beta} \binom{s}{\beta} I_{s-\beta} B_j^s \biggr] = n^j \quad \forall |j| \leqslant k.$$

If $\beta = j$, we obtain

$$I_0 B_j^j = 1,$$

from which the coefficient B_j^j is uniquely determined. Now, if $\beta \neq j$, we have that

$$\sum_{|s|\leqslant |j|,\, s\geqslant \beta} \binom{s}{\beta} I_{s-\beta} B_j^s = 0. \tag{3.3}$$

On the other hand, if $s \geqslant \beta$ and $|s| \leqslant |\beta|$ implies $s = \beta$, then if $\beta \neq j$, but $|\beta| = |j|$, equation (3.3) reduces to $B_j^\beta = 0$.

With this remark, we have fixed all coefficients B_j^β such that $|\beta| = |j|$. Let us suppose that we have solved the system of equations given by (3.3), up to find coefficients B_j^β with $|j| - r \leqslant |\beta| \leqslant |j|$, $r \geqslant 0$ an integer. Now, from (3.3), we obtain that for $|\beta| = |j| - (r+1)$,

$$\begin{aligned} 0 &= \sum_{|s|\leqslant |j|,\, s\geqslant \beta} \binom{s}{\beta} I_{s-\beta} B_j^s \\ &= \sum_{|j|-(r+1)\leqslant |s|\leqslant |j|,\, s\geqslant \beta} \binom{s}{\beta} I_{s-\beta} B_j^s \\ &= I_0 B_j^\beta + \sum_{|j|-r\leqslant |s|\leqslant |j|,\, s\geqslant \beta} \binom{s}{\beta} I_{s-\beta} B_j^s, \end{aligned}$$

and therefore, B_j^β is uniquely determined. □

LEMMA 3.5. *Fixed* P_j *as in the previous lemma, for every* j *multi-index such that* $0 \leqslant |j| \leqslant k$, *there exist piecewise polynomial functions* C_k *verifying conditions (i), (ii) and (iii).*

Proof. We represent by

$$J = \{ j = (j_1, \dots, j_N) \in \mathbb{Z}^N : |j_i| \leqslant \tfrac{1}{2}k,\ 1 \leqslant i \leqslant N \}.$$

We observe that $m \in J$ if and only if $m + [-\frac{1}{2}, \frac{1}{2}]^N \subset Q_k$. For each $m \in J$, let C_k^m be a polynomial of degree less than or equal to k, to be found, such that

$$C_k(x) = C_k^m(x), \quad x \in -m + [-\tfrac{1}{2}, \tfrac{1}{2}]^N,$$

and write

$$C_k^m(x) = \sum_{|j|\leqslant k} A_j^m (x+m)^j, \quad x \in -m + [-\tfrac{1}{2}, \tfrac{1}{2}]^N, \tag{3.4}$$

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with unknown coefficients $\{A_j^m\}_{|j|\leqslant k}$. In order to verify condition (iii) we impose, first of all, the equations

$$\sum_{m\in\mathbb{Z}^N} m^j C_k(x-m) = P_j(x), \quad x\in[-\tfrac{1}{2},\tfrac{1}{2}]^N. \tag{3.5}$$

Now, if $x\in[-\frac{1}{2},\frac{1}{2}]^N$, $x-m\in -m+[-\frac{1}{2},\frac{1}{2}]^N$ and, from (3.4), we get that (3.5) is equivalent to the fact that, for every $0\leqslant|j|\leqslant k$,

$$P_j(x) = \sum_{m\in J} m^j\biggl(\sum_{|s|\leqslant k} A_s^m x^s\biggr) = \sum_{|s|\leqslant k} x^s\biggl(\sum_{m\in J} m^j A_s^m\biggr), \quad x\in[-\tfrac{1}{2},\tfrac{1}{2}]^N.$$

Writing, as in the previous lemma,

$$P_j(x) = \sum_{|s|\leqslant|j|} B_j^s x^s,$$

if we equal coefficients, we observe that for a fixed multi-index s such that $|s|\leqslant|j|$, the system

$$\sum_{m\in J} m^j A_s^m = B_j^s, \quad 0\leqslant|j|\leqslant k, \tag{3.6}$$

holds. And for s such that $|s|>|j|$, $B_j^s=0$.

Given a multi-index s, the system above has $(k+1)^N$ unknowns, $\{A_s^m\}_{m\in J}$, and $\binom{N+k}{k}$ equations. Since the range of the system (3.6) coincides with the number of equations, we can choose a subset $I\subset J\setminus\{0\}$ having cardinality

$$(k+1)^N - \binom{N+k}{k}$$

such that

$$A_s^m = 0 \quad \text{for every } m\in I \text{ and all } |s|\leqslant k,$$

and determine the remaining unknowns in a unique way. Then, for $m\in I$, $C_k^m\equiv 0$.

In such a way, we have constructed a function C_k satisfying conditions (i), (ii) and equation (3.5). We must now check that (3.5) is also true for all $x\in\mathbb{R}^N$. If $x\in\mathbb{R}^N$ and $n\in\mathbb{Z}^N$ is such that $x\in n+[-\frac{1}{2},\frac{1}{2}]^N$, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^N} m^j C_k(x-m) &= \sum_{m\in\mathbb{Z}^N} (m+n)^j C_k(x-n-m)\\ &= \sum_{m\in\mathbb{Z}^N}\biggl(\sum_{\alpha\leqslant j}\binom{j}{\alpha} m^\alpha n^{j-\alpha}\biggr) C_k(x-n-m)\\ &= \sum_{\alpha\leqslant j}\binom{j}{\alpha} n^{j-\alpha} P_\alpha(x-n),\end{aligned}$$

where the last equality follows from (3.5), since $x-n\in[-\frac{1}{2},\frac{1}{2}]^N$. It is easy to prove that this last polynomial is equal to P_j, checking that it also verifies condition (3.2), which, following lemma 3.4, determines P_j. This fact shows us that condition (iii) is true for all $x\in\mathbb{R}^N$, and so we are done. □

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REMARK 3.6. From the construction of the functions C_k, we observe that, due to (3.2) and property (iii),

$$\sum_{m\in\mathbb{Z}^N}\int_{[-1/4,1/4]^N} C_k(x-m)\,\mathrm{d}x = \int_{[-1/4,1/4]^N} P_0(x)\,\mathrm{d}x = 1,$$

and for $1 \leqslant |j| \leqslant k$,

$$\sum_{m\in\mathbb{Z}^N} m^j \int_{[-1/4,1/4]^N} C_k(x-m)\,\mathrm{d}x = \int_{[-1/4,1/4]^N} P_j(x)\,\mathrm{d}x = 0.$$

These equations imply that

$$\int_{[-1/4,1/4]^N} C_k(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{n+[-1/4,1/4]^N} C_k(x)\,\mathrm{d}x = 0, \quad n \in \mathbb{Z}^N \setminus \{0\}. \tag{3.7}$$

THEOREM 3.7. *Let* $0 < p \leqslant 1$ *and* $a \in H^p(\mathbb{Z}^N)$. *Then there exists a sequence* $\{\lambda_i\}_{i=0}^{\infty}$ *such that*

$$\sum_{i=0}^{\infty} |\lambda_i|^p < \infty$$

and a set of H^p*-atoms in* $\mathbb{Z}^N$, $\{a_i\}_{i=0}^{\infty}$, *verifying*

$$a(n) = \sum_{i=0}^{\infty} \lambda_i a_i(n), \quad n \in \mathbb{Z}^N.$$

Moreover, for a positive constant $C > 0$, *independent of* a, *we have that*

$$\left(\sum_{i=0}^{\infty} |\lambda_i|^p\right)^{1/p} \leqslant C\|a\|_{H^p(\mathbb{Z}^N)}.$$

Proof. First, let us assume that $k = [N(1/p-1)]$ is even. Let $a \in H^p(\mathbb{Z}^N)$, and set

$$f(x) = \sum_{n\in\mathbb{Z}^N} a(n)\chi_{[-1/4,1/4)^N}(x-n).$$

By theorem 3.1, $f \in H^p(\mathbb{R}^N)$ and

$$\|f\|_{H^p(\mathbb{R}^N)} \leqslant C\|a\|_{H^p(\mathbb{Z}^N)}.$$

Since f is also in $L^2(\mathbb{R}^N)$, it can be decomposed in terms of H^p-atoms $\{b_i\}_{i\geqslant 0}$, that is,

$$f(x) = \sum_{i=0}^{\infty} \lambda_i b_i(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$

where

$$\sum_{i=0}^{\infty} |\lambda_i|^p < C\|f\|^p_{H^p(\mathbb{R}^N)}.$$

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Let us consider, for every $i \in \mathbb{N}$, Q_i, the smallest cube containing the support of an atom b_i and write

$$J_1 = \{i \in \mathbb{N};\ |Q_i| > 1/8^N\} \quad \text{and} \quad J_2 = \{i \in \mathbb{N};\ |Q_i| \leqslant 1/8^N\}.$$

If $i \in J_1$, we have that

$$\|b_i\|_\infty \leqslant \frac{1}{|Q_i|^{1/p}} \leqslant 8^{N/p},$$

and hence the series

$$\sum_{i \in J_1} \lambda_i b_i(y)$$

converges for a.e. $x \in \mathbb{R}^N$ and in the distributions sense to a function in $L^2(\mathbb{R}^N)$.

Thus, for each $m \in \mathbb{Z}^N$, and C_k the function constructed in the previous lemma, we get

$$\begin{aligned}(f * C_k)(m) &= \biggl[\biggl(\sum_{i=0}^{\infty} \lambda_i b_i(\cdot)\biggr) \star C_k\biggr](m) \\ &= \int_{[-(k+1)/2,(k+1)/2]^N} \biggl(\sum_{i=0}^{\infty} \lambda_i b_i(m-y)\biggr) C_k(y)\,\mathrm{d}y \\ &= \int_{[-(k+1)/2,(k+1)/2]^N} \biggl(\sum_{i \in J_1} \lambda_i b_i(m-y)\biggr) C_k(y)\,\mathrm{d}y \\ &\quad + \int_{[-(k+1)/2,(k+1)/2]^N} \biggl(\sum_{i \in J_2} \lambda_i b_i(m-y)\biggr) C_k(y)\,\mathrm{d}y. \end{aligned} \tag{3.8}$$

For the first term, using the dominated convergence theorem, we have

$$\int_{[-(k+1)/2,(k+1)/2]^N} \biggl(\sum_{i \in J_1} \lambda_i b_i(m-y)\biggr) C_k(y)\,\mathrm{d}y = \sum_{i \in J_1} \lambda_i (b_i * C_k)(m).$$

Let us now see that the second term in (3.8) is equal to 0. If we analyse how the atomic decomposition is obtained for our function f (see [8]), we see that we can assume that $\operatorname{supp} b_i \cap \operatorname{supp} f \neq \emptyset$. Therefore, if $i \in J_2$ and $l \in \mathbb{Z}^N$, then either $\operatorname{supp} b_i \subset l + [-\frac{3}{8}, \frac{3}{8}]^N$, or $\operatorname{supp} b_i \cap (l + [-\frac{1}{2}, \frac{1}{2}]^N) = \emptyset$ and thus

$$\sum_{i \in J_2} \lambda_i b_i(y) = 0 \quad \text{a.e. } y \in (l + [-\tfrac{1}{2}, \tfrac{1}{2}]^N) \cap (l + [-\tfrac{3}{8}, \tfrac{3}{8}]^N)^{\mathrm{c}}. \tag{3.9}$$

If, as in the previous lemma, we write

$$J = \{j = (j_1, \dots, j_N) \in \mathbb{Z}^N;\ |j_i| \leqslant \tfrac{1}{2}k,\ 1 \leqslant i \leqslant N\},$$

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we obtain from (3.9) that

$$\begin{aligned}\int_{m+[-(k+1)/2,(k+1)/2]^N} \Big(\sum_{i\in J_2} \lambda_i b_i(y)\Big) C_k(m-y)\,\mathrm{d}y \\ &= \sum_{j\in J} \int_{m+j+[-1/2,1/2]^N} \Big(\sum_{i\in J_2} \lambda_i b_i(y)\Big) C_k(m-y)\,\mathrm{d}y \\ &= \sum_{j\in J} \int_{m+j+[-3/8,3/8]^N} \Big(\sum_{i\in J_2} \lambda_i b_i(y)\Big) C_k(m-y)\,\mathrm{d}y.\end{aligned}$$

Now, given $j \in J$, let $\varphi_j \in \mathcal{S}(\mathbb{R}^N)$, with $\operatorname{supp}\varphi_j \subseteq m+j+[-\frac{1}{2},\frac{1}{2}]^N$ and $\varphi_j \equiv 1$ on $m+j+[-\frac{3}{8},\frac{3}{8}]^N$. By (3.9), we can write this last equation as

$$\sum_{j\in J}\Big(\int_{\mathbb{R}^N}\Big(\sum_{i\in J_2} \lambda_i b_i(y)\Big)\varphi_j(y) C_k(m-y)\,\mathrm{d}y\Big).$$

Since C_k is equal to a polynomial of degree less than or equal to k on the support of φ_j, we see that $\varphi_j(\cdot)C_k(m-\cdot) \in \mathcal{S}(\mathbb{R}^N)$, and hence the above expression is

$$\sum_{j\in J}\sum_{i\in J_2} \lambda_i \langle b_i(\cdot), C_k(m-\cdot)\varphi_j(\cdot)\rangle.$$

By the cancellation property of the atom b_i and the choice of φ_j, we can easily deduce that for every $i \in J_2$ and $j \in J$,

$$\langle b_i(\cdot), C_k(m-\cdot)\varphi_j(\cdot)\rangle = \langle b_i(\cdot), C_k(m-\cdot)\rangle = 0,$$

and therefore

$$\int_{[-(k+1)/2,(k+1)/2]^N} \Big(\sum_{i\in J_2} \lambda_i b_i(m-y)\Big) C_k(y)\,\mathrm{d}y = 0.$$

Consequently, from this argument and (3.7), we have proved that

$$\begin{aligned} a(m) &= \sum_{n\in\mathbb{Z}^N} a(m-n) \int_{n+[-1/4,1/4]^N} C_k(x)\,\mathrm{d}x \\ &= (f * C_k)(m) \\ &= \sum_{i\in J_1} \lambda_i (b_i * C_k)(m) \\ &= \sum_{i\in J_1} \lambda_i a_{i,k}(m). \end{aligned} \tag{3.10}$$

Let us now prove that $a_{i,k} = \{(b_i * C_k)(m)\}_m$ are H^p-atoms in $\mathbb{Z}^N$. We observe that the only atoms taking part in (3.10) are those having support in cubes Q_i with $|Q_i| > 1/8^N$ and such that $a_{i,k} \not\equiv 0$. Thus we have the following.

(i) $\operatorname{supp} a_{i,k} \subseteq \operatorname{supp}(b_i * C_k) \cap \mathbb{Z}^N \subseteq (Q_i + [-\frac{1}{2}(k+1), \frac{1}{2}(k+1)]^N) \cap \mathbb{Z}^N \subseteq B_{i,k}$, with $B_{i,k}$ being a ball in $\mathbb{Z}^N$. Since $|Q_i| > 1/8^N$, the cardinality of $B_{i,k}$ can be estimated as $\#B_{i,k} \leqslant (k+2+|Q_i|^{1/N})^N \leqslant C(k,N)|Q_i|$.

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(ii) $\|a_{i,k}\|_\infty \leqslant \|b_i\|_\infty \int_{\mathbb{R}^N} |C_k(x)|\,\mathrm{d}x \leqslant \dfrac{C(k)}{|Q_i|^{1/p}} \leqslant \dfrac{C(k,p,N)}{(\#B_{i,k})^{1/p}}$.

(iii) If $0 \leqslant |j| \leqslant k$, using property (iii) of the functions C_k and the cancellation property of the atoms b_i for polynomials of degree less than or equal to $k = [N(1/p-1)]$, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^N} m^j a_{i,k}(m) &= \sum_{m\in\mathbb{Z}^N} m^j (b_i * C_k)(m)\\ &= \sum_{m\in\mathbb{Z}^N} m^j \int_{\mathbb{R}^N} b_i(y) C_k(m-y)\,\mathrm{d}y\\ &= (-1)^{|j|} \int_{\mathbb{R}^N} b_i(y) P_j(-y)\,\mathrm{d}y\\ &= 0.\end{aligned}$$

On the other hand, we have by theorem 3.1 that

$$\sum_{i=0}^{\infty} |\lambda_i|^p \leqslant C\|f\|^p_{H^p(\mathbb{R}^N)} \leqslant C\|a\|^p_{H^p(\mathbb{Z}^N)},$$

and we get the result. □

Remark 3.8. If $k = [N(1/p-1)]$ is odd, we must replace the function f above by

$$f(x) = \sum_{n\in\mathbb{Z}^N} a(n)\chi_{[1/4,3/4]^N}(x-n), \quad a \in H^p(\mathbb{Z}^N).$$

In this case, the same proof works if we replace conditions (ii) and (iii) for functions C_k by the following ones.

(ii′) C_k must be a polynomial of degree less than or equal to k over any cube of the form $m + [0,1]^N$, $m \in \mathbb{Z}^N$.

(iii′) For every multi-index $j \in \mathbb{N}^N$ such that $0 \leqslant |j| \leqslant k$,

$$\sum_{m\in\mathbb{Z}^N} m^j C_k(x-m) = P_j(x), \quad x \in \mathbb{R}^N,$$

where P_j is a polynomial of degree $|j|$ fixed by the condition

$$\int_{n+[1/4,3/4]^N} P_j(x)\,\mathrm{d}x = n^j, \quad n \in \mathbb{Z}^N$$

(if $n = j = 0$, we understand that $n^j = 1$).

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