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Uniqueness of nodal radial solutions of superlinear elliptic equations in a ball

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The Dirichlet problem

$$\left.\begin{aligned} \Delta u + K(|x|)|u|^{p-1}u = 0 \quad &\text{in } B, \\ u = 0 \quad &\text{on } \partial B \end{aligned}\right\} \qquad (*)$$

is considered, where $B = \{x \in \mathbb{R}^N : |x| < 1\}$, $N \geqslant 3$, $p > 1$, $K \in C^2[0,1]$ and $K(r) > 0$ for $0 \leqslant r \leqslant 1$. A sufficient condition is derived for the uniqueness of radial solutions of $(*)$ possessing exactly $k-1$ nodes, where $k \in \mathbb{N}$. It is also shown that there exists $K \in C^\infty[0,1]$ such that $(*)$ has at least three radial solutions possessing exactly $k-1$ nodes, in the case $1 < p < (N+2)/(N-2)$.

1. Introduction

We consider the Dirichlet problem

$$\left.\begin{aligned} \Delta u + K(|x|)|u|^{p-1}u = 0 \quad &\text{in } B, \\ u = 0 \quad &\text{on } \partial B, \end{aligned}\right\} \tag{1.1}$$

where $B = \{x \in \mathbb{R}^N : |x| < 1\}$, $N \geqslant 3$, $p > 1$, $K \in C^2[0,1]$ and $K(r) > 0$ for $0 \leqslant r \leqslant 1$.

According to the well-known result of Gidas *et al*. [6], every positive solution of (1.1) is radially symmetric if $K'(r) \leqslant 0$ for $0 \leqslant r \leqslant 1$. On the other hand, there exist non-radial nodal solutions of (1.1) under somewhat hypotheses (see, for example, [1, 2]). As pointed out by Seok [17], if $K(r) \equiv 1$ on $[0,1]$, u is a solution of (1.1), the nodal set of u is spherical and $u(0) \neq 0$, then u is radial symmetric. In this paper we investigate the nodal radial solutions $u = u(|x|)$ of (1.1).

Let $u(r)$ be a radial solution of (1.1), where $r = |x|$. Then $u(r)$ satisfies the second-order ordinary differential equation

$$u'' + \frac{N-1}{r}u' + K(r)|u|^{p-1}u = 0 \tag{1.2}$$

for $0 < r < 1$, and the boundary condition

$$u'(0) = u(1) = 0. \tag{1.3}$$

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We consider solutions u of problem (1.2)–(1.3) satisfying $u(0) > 0$ only, since if u is a solution of (1.2)–(1.3), so is $-u$.

In this paper we study the uniqueness of solutions of the problem (1.2)–(1.3) having exactly $k-1$ zeros in $(0,1)$, where $k \in \mathbb{N}$. Hence, we consider the following problem:

$$\left.\begin{aligned} u'' + \frac{N-1}{r}u' + K(r)|u|^{p-1}u = 0, \quad 0 < r < 1,\\ u'(0) = u(1) = 0, \quad u(0) > 0,\\ u \text{ has exactly } k-1 \text{ zeros in } (0,1). \end{aligned}\right\} \tag{1.4}$$

We define the constant λ and the function $V(r)$ as follows:

$$\lambda = \frac{(N-2)p-(N+2)}{2}, \qquad V(r) = \frac{rK'(r)}{K(r)}.$$

It is known [9, 10] that if

$$V(r) \leqslant \lambda, \quad 0 \leqslant r \leqslant 1, \tag{1.5}$$

then problem (1.4) has no solution for every $k \in \mathbb{N}$. For example, (1.5) is satisfied in the case where $p \geqslant (N+2)/(N-2)$ and $K'(r) \leqslant 0$ on $[0,1]$. We can find more precise conditions for the non-existence of solutions of (1.4) in [12, 22].

The existence results for (1.4) have been obtained by Castro and Kurepa [3], Dambrosio [4], Esteban [5], Naito [14] and Struwe [18]. The following theorem has been established by Naito [14, theorem 3].

THEOREM 1.1 (Naito). *Let $k \in \mathbb{N}$. If $1 < p < (N+2)/(N-2)$, then there exists at least one solution of (1.4).*

Now we consider the case where $K(r) \equiv 1$ on $[0,1]$. Then we easily see that problem (1.4) has at most one solution, since if u is solution of (1.2), so is $v(r) := \alpha u(\alpha^{(p-1)/2}r)$ for $\alpha > 0$. Hence, from theorem 1.1 it follows that if $1 < p < (N+2)/(N-2)$, then (1.4) has a unique solution for every $k \in \mathbb{N}$. Moreover, if $p \geqslant (N+2)/(N-2)$, then (1.4) has no solution, since (1.5) holds.

Ni [15] and Ni and Nussbaum [16] considered problems of the form

$$\left.\begin{aligned} u'' + \frac{N-1}{r}u' + f(r,u) = 0, \quad 0 < r < 1,\\ u'(0) = u(1) = 0, \end{aligned}\right\} \tag{1.6}$$

and derived the sufficient conditions for the uniqueness of positive solutions of (1.6). Applying the results in [15, theorem 3.19] and [16, theorem 2.47], we conclude that problem (1.4) with $k = 1$ has at most one solution if either

$$\frac{p(N-1)-(N+3)}{2} \leqslant V(r) \leqslant \frac{(N-1)(p-1)}{2}, \qquad 0 < r < 1 \tag{1.7}$$

or

$$(N-2)p - N \leqslant V(r) \leqslant (N-2)p + N - 4, \quad 0 < r < 1. \tag{1.8}$$

In the case when $K(r) = r^l$, $l \geqslant 0$, Nagasaki [13] showed that if $1 < p < (N+2+2l)/(N-2)$, then (1.4) has a unique solution for every $k \in \mathbb{N}$, and that if $p \geqslant (N+2+2l)/(N-2)$, then (1.4) has no solution. In the case when $l > 0$, we have $K(0) = 0$. On the other hand, we assume that $K(0) > 0$ in this paper. Yanagida [21] proved that, for each $k \in \mathbb{N}$, problem (1.4) has at most one solution if $V(r)$ is non-increasing. We can apply his result independently of whether $K(0) = 0$ or $K(0) > 0$. However, very little is known about the uniqueness of solutions of (1.4) for the case where $V(r)$ is not non-increasing.

The main result of this paper is as follows.

THEOREM 1.2 (main theorem). *Let $k \in \mathbb{N}$. Assume that*

$$[V(r) - p(N-2) - N + 4][V(r) - p(N-2) + N] - 2rV'(r) < 0, \quad 0 < r < 1. \tag{1.9}$$

Then the solution of (1.4) exists and is unique.

REMARK 1.3.

(i) Letting $r \to +0$ in (1.9), we have $p \leqslant N/(N-2)$. Hence, by theorem 1.1, we see that if (1.9) holds, then (1.4) has at least one solution for each $k \in \mathbb{N}$.

(ii) There exist $K \in C^2[0,1]$ and $p > 1$ for which (1.7) and (1.8) are not satisfied, although (1.9) is satisfied (see example 1.5, below).

We have the following corollary of theorem 1.2.

COROLLARY 1.4. *Let $k \in \mathbb{N}$. Assume that (1.8) holds and $V'(r) > 0$ for $0 < r < 1$. Then the solution of (1.4) exists and is unique.*

EXAMPLE 1.5. We consider the case where $p = N/(N-2)$ and $K(r) = \mathrm{e}^{(2N-3)r}$. Then we see that $V(r) = (2N-3)r$, so that $V(r)$ is strictly increasing, and neither (1.7) nor (1.8) is satisfied. Therefore, we cannot apply the results in [15, 16, 21]. On the other hand, since

$$\begin{aligned}
[V(r) - p(N-2) - N + 4][V(r) - p(N-2) + N] - 2rV'(r) \\
&= ((2N-3)r - 2N + 2)(2N-3)r \\
&< (2N - 3 - 2N + 2)(2N-3)r \\
&= -(2N-3)r \\
&< 0
\end{aligned}$$

for $0 < r < 1$, from theorem 1.2, it follows that the solution of (1.4) exists and is unique for each $k \in \mathbb{N}$.

In theorem 1.2 we cannot remove condition (1.9). Indeed, we have the following result. In particular, it is emphasized that the uniqueness of solutions of (1.4) is not caused by the smoothness of the function $K(r)$.

THEOREM 1.6. *Let $1 < p < (N+2)/(N-2)$. For each $k \in \mathbb{N}$, there exists $K \in C^\infty[0,1]$ such that $K(r) > 0$ for $0 \leqslant r \leqslant 1$ and such that (1.4) has at least three solutions.*

We also note that (1.1) has three positive solutions for some $K(r)$.

In the case $N = 1$, Moore and Nehari [11] proved that there exists a piecewise continuous function K such that (1.1) has at least three positive solutions. We use their idea in the proof of theorem 1.6.

2. Proof of theorem 1.2

The proof of theorem 1.2 is based on the shooting method. Namely, we consider the solution $u(r,\alpha)$ of (1.2) satisfying the initial condition

$$u(0)=\alpha>0, \qquad u'(0)=0, \tag{2.1}$$

where $\alpha>0$ is a parameter. Since $K\in C^2[0,1]$, we see that $u(r,\alpha)$ exists on $[0,1]$ and is unique, $u,u'\in C^1([0,1]\times(0,\infty))$ and $u_\alpha(r,\alpha)=\partial u(r,\alpha)/\partial\alpha$ is a solution of the linearized problem

$$\left.\begin{aligned} w''+\frac{N-1}{r}w'+pK(r)|u(r,\alpha)|^{p-1}w=0, \quad r\in(0,1],\\ w(0)=1, \qquad w'(0)=0. \end{aligned}\right\} \tag{2.2}$$

(see, for example, [20, §§ 6 and 13]).

We note that $u(r,\alpha)$ and $u'(r,\alpha)$ cannot vanish simultaneously. In fact, if, for some $r_0\in(0,1]$, $u(r_0,\alpha)=u'(r_0,\alpha)=0$, then, by the uniqueness of the initial-value problem, $u(r,\alpha)\equiv 0$ for $r\in(0,1]$, which contradicts (2.1).

We define z_i to be the ith zero of $u(r,\alpha)$, if such a z_i exists. Then we easily find that

$$(-1)^i u'(z_i,\alpha)=(-1)^i\frac{\mathrm{d}}{\mathrm{d}r}u(z_i,\alpha)>0 \quad \text{for } i=1,2,\dots. \tag{2.3}$$

To prove theorem 1.2, we need the following lemma. The proof will be given in the next section.

Lemma 2.1. *Assume that there exists the kth zero z_k of $u(r,\alpha)$ in $(0,1]$. Let w be the solution of (2.2). If (1.9) holds, then $(-1)^i w(z_i)>0$ for $i=1,2,\dots,k$.*

Now we employ the Prüfer transformation for the solution $u(r,\alpha)$ of problem (1.2) and (2.1). For the solution $u(r,\alpha)$ with $\alpha>0$, we define the functions $\rho(r,\alpha)$ and $\theta(r,\alpha)$ by

$$u(r,\alpha)=\rho(r,\alpha)\sin\theta(r,\alpha),$$
$$r^{N-1}u'(r,\alpha)=\rho(r,\alpha)\cos\theta(r,\alpha),$$

where $'=\mathrm{d}/\mathrm{d}r$. Since $u(r,\alpha)$ and $u'(r,\alpha)$ cannot vanish simultaneously, we see that $\rho(r,\alpha)$ and $\theta(r,\alpha)$ are written in the form

$$\rho(r,\alpha)=([u(r,\alpha)]^2+r^{2(N-1)}[u'(r,\alpha)]^2)^{1/2}>0$$

and

$$\theta(r,\alpha)=\arctan\frac{u(r,\alpha)}{r^{N-1}u'(r,\alpha)},$$

respectively. From $u,u'\in C^1([0,1]\times(0,\infty))$, it follows that $\rho,\theta\in C^1((0,1]\times(0,\infty))$. By a simple calculation we find that

$$\theta'(r,\alpha)=r^{-N+1}\cos^2\theta(r,\alpha)+r^{N-1}K(r)|\rho(r,\alpha)|^{p-1}|\sin\theta(r,\alpha)|^{p+1}>0,$$

for $r\in(0,1]$, which shows that $\theta(r,\alpha)$ is strictly increasing in $r\in(0,1]$ for each fixed $\alpha>0$. From (2.1) it follows that $\rho(0,\alpha)=\alpha$ and $\theta(0,\alpha)\equiv\frac{1}{2}\pi(\operatorname{mod}2\pi)$. For simplicity we take $\theta(0,\alpha)=\frac{1}{2}\pi$. It is easy to see that $u(r,\alpha)$ is a solution of (1.4) if and only if

$$\theta(1,\alpha)=k\pi. \tag{2.4}$$

Hence, the number of solutions of (1.4) is equal to the number of roots $\alpha>0$ of (2.4).

LEMMA 2.2. *Let $k\in\mathbb{N}$ and let $u(r,\alpha_0)$ be a solution of (1.4) for some $\alpha_0>0$. Suppose that (1.9) holds. Then $\theta_\alpha(1,\alpha_0)>0$.*

Proof. Observe that

$$\theta_\alpha(r,\alpha)=\frac{u_\alpha(r,\alpha)r^{N-1}u'(r,\alpha)-u(r,\alpha)r^{N-1}u'_\alpha(r,\alpha)}{[u(r,\alpha)]^2+[r^{N-1}u'(r,\alpha)]^2}.$$

Since $z_k=1$ and $u(1,\alpha_0)=0$, we obtain

$$\theta_\alpha(1,\alpha_0)=\frac{u_\alpha(z_k,\alpha_0)}{u'(z_k,\alpha_0)}.$$

Note that $(-1)^k u'(z_k,\alpha_0)>0$, by (2.3). It follows from lemma 2.1 that

$$(-1)^k u_\alpha(z_k,\alpha_0)>0,$$

which implies that $\theta_\alpha(1,\alpha_0)>0$. The proof is complete. □

Proof of theorem 1.2. Recalling remark 1.3, we see that (1.4) has at least one solution. We show that the solution of (1.4) is unique. Assume to the contrary that there exist numbers α_1 and α_2 such that $u(r,\alpha_1)$ and $u(r,\alpha_2)$ are solutions of (1.4) and $0<\alpha_1<\alpha_2$. Then $\theta(1,\alpha_1)=\theta(1,\alpha_2)=k\pi$. Lemma 2.2 implies that $\theta_\alpha(1,\alpha_1)>0$ and $\theta_\alpha(1,\alpha_2)>0$. Hence, we see that $\theta(1,\alpha_0)=k\pi$ and $\theta_\alpha(1,\alpha_0)\leqslant 0$ for some $\alpha_0\in(\alpha_1,\alpha_2)$. This contradicts lemma 2.2. Consequently, (1.4) has at most one solution. The proof of theorem 1.2 is complete. □

3. Proof of lemma 2.1

In this section we prove lemma 2.1. Henceforth we assume that there exists the kth zero z_k of $u(r,\alpha)$ in $(0,1]$. For the solutions $u(r,\alpha)$ and $w(r)$ of (1.2), (2.1) and (2.2), respectively, we set

$$U(t,\alpha)=tu(t^{1/(N-2)},\alpha),\qquad W(t)=tw(t^{1/(N-2)}).$$

Then $U=U(t,\alpha)$ and $W=W(t)$ satisfy

$$U''+M(t)|U|^{p-1}U=0,\quad 0<t\leqslant 1, \tag{3.1}$$

$$U(0,\alpha)=0,\qquad U'(0,\alpha)=\alpha, \tag{3.2}$$

$$W''+pM(t)|U(t,\alpha)|^{p-1}W=0,\quad 0<t\leqslant 1, \tag{3.3}$$

$$W(0)=0,\qquad W'(0)=1, \tag{3.4}$$

where $'=\mathrm{d}/\mathrm{d}t$ and

$$M(t)=(N-2)^{-2}t^{-p-(N-4)/(N-2)}K(t^{1/(N-2)}).$$

Set $Z_i = z_i^{N-2}$, $i = 1, 2, \ldots, k$ and $Z_0 = 0$. Then we see that

$$U(Z_i, \alpha) = 0, \quad i = 0, 1, 2, \ldots, k,$$
$$(-1)^{i-1} U(t, \alpha) > 0 \quad \text{for } t \in (Z_{i-1}, Z_i), \quad i = 1, 2, \ldots, k,$$

By (3.1), there exist S_i, $i = 1, 2, \ldots, k$ such that

$$\begin{gathered} S_i \in (Z_{i-1}, Z_i), \\ U'(S_i, \alpha) = 0, \quad i = 1, 2, \ldots, k, \\ U'(t, \alpha) > 0 \quad \text{for } t \in (0, S_1), \\ (-1)^i U'(t, \alpha) > 0 \quad \text{for } t \in (S_i, S_{i+1}), \quad i = 1, 2, \ldots, k-1, \\ (-1)^k U'(t, \alpha) > 0 \quad \text{for } t \in (S_k, Z_k]. \end{gathered} \tag{3.5}$$

Lemma 3.1. *Let W be a solution of (3.3), (3.4). Then, for each $i \in \{1, 2, \ldots, k\}$, W has at least one zero in (Z_{i-1}, Z_i).*

Proof. Let $i \in \{1, 2, \ldots, k\}$. Assume to the contrary that $W(t) \neq 0$ for $t \in (Z_{i-1}, Z_i)$. Let U be a solution of (3.1), (3.2). We may assume without loss of generality that $W(t) > 0$ and $U(t) > 0$ for $t \in (Z_{i-1}, Z_i)$, since another case can be treated similarly. Then we see that $U'(Z_i) < 0$ and $U'(Z_{i-1}) > 0$, and hence

$$W(Z_i)U'(Z_i) - W(Z_{i-1})U'(Z_{i-1}) \leqslant 0.$$

An easy computation shows that

$$(WU' - W'U)' = (p-1)M(t)|U|^{p-1}UW, \quad 0 < t \leqslant 1. \tag{3.6}$$

Note that $(WU' - W'U)'$ is integrable on $[0, 1]$, because of (3.2) and (3.4). Integrating (3.6) over (Z_{i-1}, Z_i), we have

$$W(Z_i)U'(Z_i) - W(Z_{i-1})U'(Z_{i-1}) > 0.$$

This is a contradiction. Consequently, W has at least one zero in (Z_{i-1}, Z_i). $\square$

The following identity plays a crucial part in the proof of lemma 2.1. This identity has been obtained in [19], by using the idea due to Korman and Ouyang [8]. (See also [7, lemma 4.1].)

Lemma 3.2. *Let U be a solution of (3.1), (3.2), and let W be a solution of (3.3), (3.4). Then*

$$[[M(t)]^{-1/2}[W'U' - WU''] - ([M(t)]^{-1/2})'WU']' = -([M(t)]^{-1/2})''WU' \tag{3.7}$$

for $0 < t \leqslant 1$.

Proof. By (3.1), we note that $U''' = -M'(t)|U|^{p-1}U - pM(t)|U|^{p-1}U'$ for $0 < t \leqslant 1$. A direct calculation shows that (3.7) follows immediately. $\square$

Lemma 3.3. *Let U be a solution of (3.1), (3.2), and let W be a solution of (3.3), (3.4). Then*

$$\lim_{t \to +0} [[M(t)]^{-1/2}[W'U' - WU''] - ([M(t)]^{-1/2})'WU'] = 0.$$

Proof. In view of (3.2) and (3.4), it is sufficient to show that

$$\lim_{t\to+0}[M(t)]^{-1/2}U''(t) = 0, \tag{3.8}$$

$$\lim_{t\to+0}([M(t)]^{-1/2})'W(t) = 0. \tag{3.9}$$

Since

$$\lim_{t\to+0}\frac{U(t)}{t} = \lim_{t\to+0}U'(t) = \alpha \quad \text{and} \quad p-1+\frac{2}{N-2} > 0,$$

we see that

$$\begin{aligned}
|[M(t)]^{-1/2}U''(t)| &= |[M(t)]^{-1/2}M(t)|U(t)|^{p-1}U(t)| \\
&= |M(t)|^{1/2}|U(t)|^{p} \\
&= \frac{1}{N-2}t^{(p-1+[2/(N-2)])/2}[K(t^{1/(N-2)})]^{1/2}\left|\frac{U(t)}{t}\right|^{p},
\end{aligned}$$

so (3.8) holds.

We observe that

$$\begin{aligned}
([M(t)]^{-1/2})' = \sigma(N-2)t^{\sigma-1}[K(t^{1/(N-2)})]^{-1/2} \\
- \tfrac{1}{2}t^{\sigma+(1/(N-2))-1}[K(t^{1/(N-2)})]^{-3/2}K'(t^{1/(N-2)}),
\end{aligned} \tag{3.10}$$

where

$$\sigma = \frac{1}{2}\left(p+\frac{N-4}{N-2}\right) > 0.$$

Since $\lim_{t\to+0}W(t)/t = W'(0) = 1$, we have

$$\lim_{t\to+0}t^{\sigma-1}W(t) = \lim_{t\to+0}t^{\sigma}\frac{W(t)}{t} = 0 \tag{3.11}$$

and hence

$$\lim_{t\to+0}t^{\sigma+(1/(N-2))-1}W(t) = 0. \tag{3.12}$$

Combining (3.10)–(3.12), we conclude that (3.9) holds. The proof is complete. □

LEMMA 3.4. *Inequality (1.9) holds if and only if* $([M(t)]^{-1/2})'' < 0$ *for* $0 < t < 1$.

Proof. Set $t = r^{N-2}$. Then $[M(t)]^{-1/2} = (N-2)r^{\rho}[K(r)]^{-1/2}$, where $\rho = \frac{1}{2}(p(N-2)+N-4)$. Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[M(t)]^{-1/2} = \rho r^{\rho-N+2}K^{-1/2} - \tfrac{1}{2}r^{\rho-N+3}K^{-3/2}K'$$

and

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}[M(t)]^{-1/2} \\
&\quad = \frac{r^{\rho-2N+4}}{(N-2)K^{1/2}}\left[\rho(\rho-N+2) - \frac{2\rho-N+3}{2}\frac{rK'}{K} + \frac{3}{4}\left(\frac{rK'}{K}\right)^2 - \frac{1}{2}\frac{r^2K''}{K}\right].
\end{aligned}$$

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Recall that $V(r) = rK'(r)/K(r)$. Since $r^2K''/K = rV' - V + V^2$, we have

$$\begin{aligned}\frac{4(N-2)K^{1/2}}{r^{\rho-2N+4}} \frac{\mathrm{d}^2}{\mathrm{d}t^2}[M(t)]^{-1/2} \\ &= 4\rho(\rho - N + 2) - 2(2\rho - N + 2)V + V^2 - 2rV' \\ &= (V - 2\rho)(V - 2(\rho - N + 2)) - 2rV' \\ &= (V - p(N-2) - N + 4)(V - p(N-2) + N) - 2rV'.\end{aligned}$$

Therefore, (1.9) holds if and only if $([M(t)]^{-1/2})'' < 0$ for $0 < t < 1$. □

LEMMA 3.5. *Assume that (1.9) holds. Let W be a solution of (3.3), (3.4). Then the following hold:*

(i) *$W(t) > 0$ for $t \in (0, S_1]$;*

(ii) *W has at most one zero in $(S_i, S_{i+1}]$ for each $i \in \{1, 2, \dots, k-1\}$;*

(iii) *W has at most one zero in $(S_k, Z_k]$.*

Proof of lemma 3.5. First we prove (i). Suppose that there exists $t_2 \in (0, S_1]$ such that $W(t_2) = 0$ and $W(t) > 0$ for $t \in (0, t_2)$. Then we have $W'(t_2) < 0$. Since $t_2 \in (0, S_1]$, we see that $U'(t_2) \geqslant 0$, so that $W'(t_2)U'(t_2) \leqslant 0$. Integrating (3.7) over $(0, t_2]$ and using lemmas 3.3 and 3.4, we have $W'(t_2)U'(t_2) > 0$. This is a contradiction. The proof of (i) is complete.

We now prove (ii) only, as we can prove (iii) in exactly the same way.

Assume that there exist t_1 and t_2 such that $S_i < t_1 < t_2 \leqslant S_{i+1}$, $W(t_1) = W(t_2) = 0$ and $W(t) \neq 0$ for $t \in (t_1, t_2)$. We may suppose that $W(t) > 0$ for $t \in (t_1, t_2)$, since the case where $W(t) < 0$ for $t \in (t_1, t_2)$ can be treated similarly. Then we have $W'(t_1) > 0$ and $W'(t_2) < 0$. Let U be a solution of (3.1), (3.2). Integrating of (3.7) over $[t_1, t_2]$ and then multiplying it by $(-1)^i$ and using lemma 3.4 and (3.5), we obtain

$$[M(t_2)]^{-1/2}W'(t_2)(-1)^iU'(t_2) - [M(t_1)]^{-1/2}W'(t_1)(-1)^iU'(t_1) > 0.$$

This contradicts (3.5), $W'(t_1) > 0$ and $W'(t_2) < 0$. The proof is complete. □

Proof of lemma 2.1. By lemmas 3.1 and 3.5, there exists a number $C_1 \in (S_1, Z_1)$ such that $W(t) > 0$ for $t \in (0, C_1)$, $W(C_1) = 0$ and $W(t) < 0$ for $t \in (C_1, S_2]$. In particular, we have $W(Z_1) < 0$. Also from lemmas 3.1 and 3.5 we see that there exists a number $C_2 \in (S_2, Z_2)$ such that $W(t) < 0$ for $t \in (S_2, C_2)$, $W(C_2) = 0$, and $W(t) > 0$ for $t \in (C_2, S_3]$, so that $W(Z_2) > 0$. By continuing this process, we conclude that $(-1)^iW(Z_i) > 0$ for $i = 1, 2, \dots, k$. This means that $(-1)^iw(z_i) > 0$ for $i = 1, 2, \dots, k$. The proof is complete. □

4. Proof of theorem 1.6

In order to prove theorem 1.6 we need the following lemma.

LEMMA 4.1. *Assume that $k \in \mathbb{N}$ and $1 < p < (N+2)/(N-2)$. Then there exist $R > 0$, $L \in C^\infty[0, R]$ and solutions v_0 and v_1 of*

$$v'' + \frac{N-1}{r}v' + L(r)|v|^{p-1}v = 0 \tag{4.1}$$

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such that $L(r) > 0$ for $0 \leqslant r \leqslant R$, v_0 has at least k zeros in $[0, R)$, v_1 has at most $k-1$ zeros in $[0, R]$ and $v_1(0) > v_0(0) > 0$.

The proof of lemma 4.1 is given in the next section. By the well-known fact [21, lemma 2.1(a)], we note that the solutions v_i in lemma 4.1 satisfy $v_i'(0) = 0$.

We denote by $v(r, \alpha)$ the solution of (4.1) with $v(0) = \alpha$ and $v'(0) = 0$.

By using the results of Naito [14, lemmas 6.2 and 6.3], we obtain the following lemma.

LEMMA 4.2. *Let $R > 0$ and $1 < p < (N+2)/(N-2)$. Suppose that $L \in C^1[0, R]$ and $L(r) > 0$ for $0 \leqslant r \leqslant R$. Then the following hold:*

(i) *for sufficiently small $\alpha > 0$, $v(r, \alpha) > 0$ on $[0, R]$;*

(ii) *the number of zeros of $v(r, \alpha)$ in $[0, R]$ tends to ∞ as $\alpha \to \infty$.*

Proof of theorem 1.6. Let $R > 0$ and $L \in C^\infty[0, R]$ as in lemma 4.1. Then there exist α_0 and α_1 such that $v(r, \alpha_0)$ has at least k zeros in $[0, R)$, $v(r, \alpha_1)$ has at most $k-1$ zeros in $[0, R]$, and $\alpha_1 > \alpha_0 > 0$. We use the Prüfer transformation for the solution $v(r, \alpha)$, that is, we define the functions $\rho(r, \alpha)$ and $\theta(r, \alpha)$ by

$$v(r, \alpha) = \rho(r, \alpha) \sin \theta(r, \alpha),$$
$$r^{N-1} v'(r, \alpha) = \rho(r, \alpha) \cos \theta(r, \alpha),$$

where $' = \mathrm{d}/\mathrm{d}r$. We see that $\theta(R, \alpha_0) > k\pi$ and $\theta(R, \alpha_1) < k\pi$. By lemma 4.2, there exist α_*, α^* such that $0 < \alpha_* < \alpha_0 < \alpha_1 < \alpha^*$, and the following (i) and (ii) are satisfied:

(i) $\theta(R, \alpha) < \pi$ for $\alpha \in (0, \alpha_*]$;

(ii) $\theta(R, \alpha) > k\pi$ for $\alpha \geqslant \alpha^*$.

Hence, there exist β_1, β_2 and β_3 such that $\alpha_* < \beta_1 < \alpha_0 < \beta_2 < \alpha_1 < \beta_3 < \alpha^*$, and $\theta(R, \beta_i) = k\pi$ for $i = 1, 2, 3$. Consequently, the problem (4.1) with $v'(0) = v(R) = 0$ and $v(0) > 0$ has three solutions v_1, v_2 and v_3 possessing exactly $k-1$ zeros in $(0, R)$. We find that $v_1(Rr)$, $v_2(Rr)$ and $v_3(Rr)$ are solutions of the problem

$$u'' + \frac{N-1}{r} u' + R^2 L(Rr) |u|^{p-1} u = 0, \quad 0 < r < 1,$$
$$u'(0) = u(1) = 0.$$

This completes the proof of theorem 1.6. □

5. Proof of lemma 4.1

We assume that $k \in \mathbb{N}$ and $1 < p < (N+2)/(N-2)$. Let φ_1 be a solution of

$$u'' + \frac{N-1}{r} u' + |u|^{p-1} u = 0, \quad r > 0,$$
$$u(0) = 1, \qquad u'(0) = 0.$$

From $1 < p < (N+2)/(N-2)$ it follows that φ_1 has infinity many zeros in $(0,\infty)$ (see, for example, [9, 10]). Set $\Phi_1(t) = t\varphi_1(t^{1/(N-2)})$. Then Φ_1 has infinity many zeros in $(0,\infty)$ and is a solution of

$$\begin{gathered} U'' + (N-2)^{-2}t^{-p-(N-4)/(N-2)}|U|^{p-1}U = 0, \quad t > 0, \\ U(0) = 0, \qquad U'(0) = 1. \end{gathered} \tag{5.1}$$

Let $T_1 > 0$ such that $\Phi_1(t)$ has exactly $k-1$ zeros in $(0, T_1)$ and $\Phi_1'(T_1) = 0$ and $(-1)^{k-1}\Phi_1(T_1) > 0$. We set

$$\Phi(t,c) = c^{2/[(p-1)(N-2)]-1}\Phi_1(ct), \quad c > 0.$$

Then $\Phi(t,c)$ is the solution of (5.1) with $U(0) = 0$ and $U'(0) = c^{2/[(p-1)(N-2)]}$. There exists $\varepsilon > 0$ so small that $\Phi(t, 1-\varepsilon)$ has exactly $k-1$ zeros in $(0, T_1)$ and $(-1)^{k-1}\partial\Phi(T_1, 1-\varepsilon)/\partial t > 0$ and $(-1)^{k-1}\Phi(T_1, 1-\varepsilon) > 0$. We set $\Phi_0(t) = \Phi(t, 1-\varepsilon)$.

Let y_1 be a solution of

$$\begin{gathered} y'' + |y|^{p-1}y = 0, \\ y(0) = \Phi_1(T_1), \qquad y'(0) = 0. \end{gathered} \tag{5.2}$$

Then there exists $\rho > 0$ such that $y_1(\rho) = 0$ and $y_1(t) \neq 0$ for $t \in [0, \rho)$. Take $T_2 > T_1$ to be so large that

$$|\Phi_0(T_1) + \Phi_0'(T_1)(T_2 - T_1)| \geqslant 2\left(\frac{4\pi}{\rho}\right)^{2/(p-1)}. \tag{5.3}$$

Lemma 5.1. *There exists a function $M_\delta \in C^\infty(0,\infty)$ for each sufficiently small $\delta > 0$, such that $M_\delta(t) = (N-2)^{-2}t^{-p-(N-4)/(N-2)}$ for $0 < t \leqslant T_1$, $M_\delta(t) > 0$ for $T_1 < t < T_2$, $M_\delta(t) = 1$ for $t \geqslant T_2$, and*

$$\lim_{\delta\to+0}\int_{T_1}^{T_2} M_\delta(t)\,\mathrm{d}t = 0. \tag{5.4}$$

Let U_0 and U_1 be solutions of

$$U'' + M_\delta(t)|U|^{p-1}U = 0$$

with the initial conditions $U_0(T_1) = \Phi_0(T_1)$, $U_0'(T_1) = \Phi_0'(T_1)$ and $U_1(T_1) = \Phi_1(T_1)$, $U_1'(T_1) = 0$, respectively. We see that $U_0(t) = \Phi_0(t)$ and $U_1(t) = \Phi_1(t)$ for $0 < t \leqslant T_1$.

Lemma 5.2. *There exist $\delta_0 > 0$ and $T_3 > T_2$ such that if $0 < \delta < \delta_0$, then U_0 has at least one zero in $[T_1, T_3)$, and $U_1(t) \neq 0$ for $t \in [T_1, T_3]$.*

Let δ_0 and T_3 be numbers in lemma 5.2. We assume that $0 < \delta < \delta_0$. Then U_0 has at least k zeros in $(0, T_3)$ and U_1 has exactly $k-1$ zeros in $(0, T_3]$. We set

$$R = T_3^{1/(N-2)}, \qquad L(r) = (N-2)^2 r^{p(N-2)+(N-4)} M_\delta(r^{N-2})$$

and

$$v_i(r) = r^{-(N-2)}U_i(r^{N-2}), \quad i = 1, 2.$$

Then we see that $v_0(r)$ and $v_1(r)$ are solutions of (4.1) and that v_0 has at least k zeros in $[0, R]$ and v_1 has exactly $k-1$ zeros in $[0, R]$. From

$$v_i(0) = \lim_{r \to +0} \frac{U_i(r^{N-2})}{r^{N-2}} = \lim_{t \to +0} \frac{U_i(t)}{t} = U_i'(0), \quad i = 1, 2,$$

it follows that $v_1(0) > v_0(0) > 0$. Since $M_\delta(t) = (N-2)^{-2} t^{-p-(N-4)/(N-2)}$ for $0 < t \leqslant T_1$, we find that $L(r) = 1$ for $0 < r \leqslant T_1^{1/(N-2)}$. Hence, setting $L(0) = 1$, we conclude that $L \in C^\infty[0, R]$ and $L(r) > 0$ for $0 \leqslant r \leqslant R$. This completes the proof of lemma 4.1.

Proof of lemma 5.1. Take $h \in C^\infty(\mathbb{R})$ such that $h(x) = 0$ for $x \leqslant 0$, $0 < h(x) < 1$ for $x \in (0, 1)$, and $h(x) = 1$ for $x \geqslant 1$. For example,

$$h(x) = \frac{\int_{-1}^{2x-1} g(t)\,\mathrm{d}t}{\int_{-1}^{1} g(t)\,\mathrm{d}t},$$

where

$$g(t) = \begin{cases} \exp\left(-\dfrac{1}{1-t^2}\right), & |t| < 1, \\ 0, & |t| \geqslant 1. \end{cases}$$

Note that $h^{(i)}(0) = h^{(i)}(1) = 0$, $i = 1, 2, \dots$. We define the function $M_\delta(t)$ by

$$M_\delta(t) = \begin{cases} m(t), & t \in (0, T_1], \\ h\left(1 - \dfrac{t - T_1}{\delta}\right) m(t) + \delta h\left(\dfrac{t - T_1}{\delta}\right), & t \in (T_1, T_1 + \delta), \\ \delta, & t \in [T_1 + \delta, T_2 - \delta], \\ (1-\delta) h\left(\dfrac{t - T_2 + \delta}{\delta}\right) + \delta, & t \in (T_2 - \delta, T_2), \\ 1, & t \geqslant T_2, \end{cases}$$

where $m(t) = (N-2)^{-2} t^{-p-(N-4)/(N-2)}$. Then we find that $M_\delta \in C^\infty(0, \infty)$, $M_\delta(t) > 0$ for $t > 0$ and $M_\delta(t)$ satisfies (5.4). □

To prove lemma 5.2 we need the following two lemmas.

Lemma 5.3. *Let $T \in \mathbb{R}$ and let $c > 0$. Let y be a solution of (5.2) with $|y(T)| > 2(2\pi/c)^{2/(p-1)}$. Then y has at least one zero in $(T, T + c)$.*

Proof. We may assume without loss of generality that $T = 0$ and $y(0) > 0$, since if y is a solution of (5.2), then $\pm y(t + T)$ is also a solution of (5.2). Assume to the contrary that $y(t) > 0$ for $t \in (0, c)$. Then we see that $y''(t) < 0$ for $t \in (0, c)$, so that $y(t)$ is concave on $(0, c)$. Hence, we have

$$y(t) \geqslant \frac{y(0)}{c}(c - t) \geqslant \tfrac{1}{2} y(0) > \mu^{2/(p-1)}, \quad 0 \leqslant t \leqslant \tfrac{1}{2}c, \tag{5.5}$$

where $\mu = 2\pi/c$. Set $v(t) = \sin(\mu t)$. Then v is a solution of

$$v'' + \mu^2 v = 0, \qquad v(0) = v(\tfrac{1}{2}c) = 0.$$

From (5.5) it follows that $|y(t)|^{p-1} > \mu^2$ for $0 \leqslant t \leqslant \frac{1}{2}c$. By applying Sturm's comparison theorem, we conclude that y has at least one zero in $(0, \frac{1}{2}c)$. This is a contradiction. The proof is complete. □

LEMMA 5.4. *Let c_1 and c_2 be constants with $c_1 \neq 0$ and $c_1 c_2 \geqslant 0$. For every sufficiently small $\delta > 0$, let v_δ be the solution of*

$$v'' + M_\delta(t)|v|^{p-1}v = 0, \qquad v(T_1) = c_1, \qquad v'(T_1) = c_2,$$

where $M_\delta(t)$ is the function in lemma 5.1. Then $v_\delta(t)$ and $v'_\delta(t)$ converge to $c_1 + c_2(t - T_1)$ and c_2 uniformly on $[T_1, T_2]$ as $\delta \to +0$, respectively.

Proof. We may assume without loss of generality that $c_1 > 0$. We first show that $v_\delta(t) > 0$ on $[T_1, T_2]$ for all sufficiently small $\delta > 0$. Note that $v_\delta(T_1) = c_1 > 0$. Assume to the contrary that there exist sequences $\{\delta_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \delta_n = 0$, $z_n \in (T_1, T_2]$, $v_{\delta_n}(z_n) = 0$ and $v_{\delta_n}(t) > 0$ on $[T_1, z_n)$ for $n = 1, 2, \dots$. We see that

$$v'_\delta(t) = c_2 - \int_{T_1}^t M_\delta(s)|v_\delta(s)|^{p-1}v_\delta(s)\,\mathrm{d}s \tag{5.6}$$

and

$$v_\delta(t) = c_1 + c_2(t - T_1) - \int_{T_1}^t (t-s)M_\delta(s)|v_\delta(s)|^{p-1}v_\delta(s)\,\mathrm{d}s \tag{5.7}$$

for $t \in [T_1, T_2]$, so that $0 \leqslant v_{\delta_n}(t) \leqslant C$ for $t \in [T_1, z_n]$, where $C = c_1 + c_2(T_2 - T_1)$. Therefore, we find that

$$v_{\delta_n}(z_n) \geqslant c_1 - C^p \int_{T_1}^{z_n} (z_n - s)M_{\delta_n}(s)\,\mathrm{d}s \geqslant c_1 - C^p T_2 \int_{T_1}^{T_2} M_{\delta_n}(s)\,\mathrm{d}s.$$

From (5.4) it follows that $v_{\delta_N}(z_N) > 0$ for some large N. This is a contradiction.

Let $\delta > 0$ be sufficiently small. Since $v_\delta(t) > 0$ for $t \in [T_1, T_2]$, by (5.7), we see that $0 < v_\delta(t) \leqslant C$ for $t \in [T_1, T_2]$. From (5.6) and (5.7) it follows that

$$0 \leqslant c_2 - v'_\delta(t) \leqslant C^p \int_{T_1}^{T_2} M_\delta(s)\,\mathrm{d}s$$

and

$$0 \leqslant c_1 + c_2(t - T_1) - v_\delta(t) \leqslant C^p T_2 \int_{T_1}^{T_2} M_\delta(s)\,\mathrm{d}s$$

for $t \in [T_1, T_2]$. Hence, (5.4) implies that $v_\delta(t)$ and $v'_\delta(t)$ converge to $c_1 + c_2(t - T_1)$ and c_2 uniformly on $[T_1, T_2]$, respectively. The proof is complete. □

Proof of lemma 5.2. Let y_0 be the solution of (5.2) with

$$y_0(T_2) = \Phi_0(T_1) + \Phi'_0(T_1)(T_2 - T_1), \qquad y'_0(T_2) = \Phi'_0(T_1).$$

In view of (5.3), lemma 5.3 implies that y_0 has at least one zero in $(T_2, T_2 + \frac{1}{2}\rho)$. Lemma 5.4 implies that $\lim_{\delta\to+0} U_0(T_2) = y_0(T_2)$ and $\lim_{\delta\to+0} U'_0(T_2) = y'_0(T_2)$. Therefore, by a general theory on the continuous dependence of solutions on initial conditions (see, for example, [20, § 13]), we see that U_0 has at least one zero in $(T_2, T_2 + \frac{3}{4}\rho)$ for all sufficiently small $\delta > 0$.

Again by lemma 5.4 we find that $U_1(t) \neq 0$ on $[T_1, T_2]$ for all sufficiently small $\delta > 0$, $\lim_{\delta \to +0} U_1(T_2) = \Phi_1(T_1)$ and $\lim_{\delta \to +0} U_1'(T_2) = 0$. We note that $y_1(t - T_2)$ is the unique solution of (5.2) with $y(T_2) = \Phi(T_1)$ and $y'(T_2) = 0$ and $y_1(t - T_2) \neq 0$ for $t \in [T_2, T_2 + \rho)$. Hence, $U_1(t) \neq 0$ on $[T_1, T_2 + \frac{3}{4}\rho]$ for all sufficiently small $\delta > 0$, by a general theory on the continuous dependence of solutions on initial conditions. The proof is complete. □

References

1 T. Bartsch, T. Weth and M. Willem. Partial symmetry of least energy nodal solutions to some variational problems. *J. Analysis Math.* **96** (2005), 1–18.

2 A. Castro and M. Clapp. Upper estimates for the energy of solutions of nonhomogeneous boundary-value problems. *Proc. Am. Math. Soc.* **134** (2006), 167–175.

3 A. Castro and A. Kurepa. Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball. *Proc. Am. Math. Soc.* **101** (1987), 57–64.

4 W. Dambrosio. On the multiplicity of radial solutions to superlinear Dirichlet problems in bounded domains. *J. Diff. Eqns* **196** (2004), 91–118.

5 M. J. Esteban. Multiple solutions of semilinear elliptic problems in a ball. *J. Diff. Eqns* **57** (1985), 112–137.

6 B. Gidas, W.-M. Ni and L. Nirenberg. Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **68** (1979), 209–243.

7 P. Korman. Global solution branches and exact multiplicity of solutions for two point boundary-value problems. In *Handbook of differential equations: ordinary differential equations*, vol. III (ed. A. Canada, P. Drabek and A. Fonda), pp. 547–606 (Elsevier, 2006).

8 P. Korman and T. Ouyang. Solution curves for two classes of boundary-value problems. *Nonlin. Analysis* **27** (1996), 1031–1047.

9 T. Kusano and M. Naito. Positive entire solutions of superlinear elliptic equations. *Hiroshima Math. J.* **16** (1986), 361–366.

10 T. Kusano and M. Naito. Oscillation theory of entire solutions of second-order superlinear elliptic equations. *Funkcial. Ekvac.* **30** (1987), 269–282.

11 R. A. Moore and Z. Nehari. Nonoscillation theorems for a class of nonlinear differential equations. *Trans. Am. Math. Soc.* **93** (1959), 30–52.

12 H. Myogahara, E. Yanagida and S. Yotsutani. Structure of positive radial solutions for semilinear Dirichlet problems on a ball. *Funkcial. Ekvac.* **45** (2002), 1–21.

13 K. Nagasaki. Radial solutions for $\Delta u + |x|^l |u|^{p-1} u = 0$ on the unit ball in $\mathbb{R}^n$. *J. Fac. Sci. Univ. Tokyo (1)* A **36** (1989), 211–232.

14 Y. Naito. Bounded solutions with prescribed numbers of zeros for the Emden–Fowler differential equation. *Hiroshima Math. J.* **24** (1994), 177–220.

15 W.-M. Ni. Uniqueness of solutions of nonlinear Dirichlet problems. *J. Diff. Eqns* **50** (1983), 289–304.

16 W.-M. Ni and R. D. Nussbaum. Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$. *Commun. Pure Appl. Math.* **38** (1985), 67–108.

17 Y. J. Seok. A note on the radial symmetry of solutions with spherical nodal sets. *J. Math. Analysis Applic.* **296** (2004), 719–721.

18 M. Struwe. Superlinear elliptic boundary-value problems with rotational symmetry. *Arch. Math.* **39** (1982), 233–240.

19 S. Tanaka. On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary-value problem. *Diff. Integ. Eqns* **20** (2007), 93–104.

20 W. Walter. *Ordinary differential equations* (Springer, 1998).

21 E. Yanagida. Structure of radial solutions to $\Delta u + K(|x|)|u|^{p-1}u = 0$ in $\mathbb{R}^n$. *SIAM J. Math. Analysis* **27** (1996), 997–1014.

22 E. Yanagida and S. Yotsutani. Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$. *Arch. Ration. Mech. Analysis* **124** (1993), 239–259.

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