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NUMERICAL RANGE AND POSITIVE BLOCK MATRICES

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Abstract

We obtain several norm and eigenvalue inequalities for positive matrices partitioned into four blocks. The results involve the numerical range $W(X)$ of the off-diagonal block X, especially the distance d from 0 to $W(X)$. A special consequence is an estimate,

$$\operatorname{diam} W\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) - \operatorname{diam} W\left(\frac{A+B}{2}\right) \geq 2d,$$

between the diameters of the numerical ranges for the full matrix and its partial trace.

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1. The width of the numerical range

Let $\mathbb{M}_n$ denote the space of n-by-n complex matrices and let $\langle u, v\rangle = u^*v$ be the canonical inner product of $\mathbb{C}^n$, linear in the second variable. The numerical range of $X \in \mathbb{M}_n$ is defined as

$$W(X) = \{\langle h, Xh\rangle \;:\; \|h\| = 1\}.$$

The Hausdorff–Toeplitz theorem states that $W(X)$ is a compact convex set containing the spectrum of X. For a normal matrix, the numerical range is precisely the convex hull of the spectrum. The symbol $\|\cdot\|$ will also denote any symmetric norm on $\mathbb{M}_{2n}$. Such a norm is also called a unitarily invariant norm. It satisfies the unitary invariance property $\|UTV\| = \|T\|$ for all $T \in \mathbb{M}_{2n}$ and all unitary matrices $U, V \in \mathbb{M}_{2n}$, and it induces a symmetric norm on $\mathbb{M}_n$ in an obvious way, by considering $\mathbb{M}_n$ as the upper left corner of $\mathbb{M}_{2n}$ completed with some zero entries.

A positive matrix means a Hermitian positive semi-definite matrix. It has been pointed out in [7] that the width of $W(X)$ contributes to an estimate of the norm of a partitioned positive matrix $\left[\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right]$. In matrix analysis, positive matrices partitioned

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into four blocks are a fundamental tool and these matrices are also of basic importance in applications, especially in quantum information theory. The main theorem of [7] reads as follows.

THEOREM 1.1 [7]. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$. *Suppose that* $W(X)$ *has the width* ω. *Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \le \|A + B + \omega I\|.$$

Here I stands for the identity matrix and the width of $W(X)$ is the smallest distance between two parallel straight lines such that the strip between these two lines contains $W(X)$. Hence, the partial trace $A + B$ may be used to give an upper bound for the norms of the full block matrix. This note will provide a lower bound, stated in Section 2, and several consequences.

Theorem 1.1 is the first inequality involving the width of the numerical range; classical results rather deal with the numerical radius, $w(X) = \max\{|z| : z \in W(X)\}$. Our new lower bound will also have an unusual feature as it involves the distance from 0 to the numerical range, $\mathrm{dist}(0, W(X)) = \min\{|z| : z \in W(X)\}$. For a background on the numerical range we refer to [12], where the term 'field of values' is used. Some very interesting inequalities for the numerical radius can be found in [11, 13] and in the recent article [8].

For Hermitian off-diagonal blocks, Theorem 1.1 holds with $\omega = 0$. More generally, if $X = aI + bH$ for some $a, b \in \mathbb{C}$ and some Hermitian matrix H, then $\omega = 0$ as $W(X)$ is a line segment. This special case of the theorem was first shown by Mhanna [14]. In particular, if the off-diagonal blocks are normal two-by-two matrices, then we can take $\omega = 0$. This does not hold any longer for three-by-three normal matrices; a detailed study of this phenomenon is given in [10] and [9].

For Hermitian off-diagonal blocks, a stronger statement than Theorem 1.1 with $\omega = 0$ holds. The following decomposition was shown in [6, Theorem 2.2].

THEOREM 1.2 [6]. *Let* $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ *be a positive matrix partitioned into four Hermitian blocks in* $\mathbb{M}_n$. *Then, for some pair of unitary matrices* $U, V \in \mathbb{M}_{2n}$,

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} = \frac{1}{2}\left\{ U \begin{bmatrix} A + B & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A + B \end{bmatrix} V^* \right\}.$$

For decompositions of positive matrices partitioned into a larger number of blocks, see [5]. We close this section by recalling some facts on symmetric norms (classical text books such as [2, 12, 15] are good references).

A symmetric norm on $\mathbb{M}_n$ can be defined by its restriction to the positive cone $\mathbb{M}_n^+$. Symmetric norms on $\mathbb{M}_n^+$ are characterised by three properties:

(i) $\|\lambda A\| = \lambda \|A\|$ for all $A \in \mathbb{M}_n^+$ and all $\lambda \ge 0$;
(ii) $\|UAU^*\| = \|A\|$ for all $A \in \mathbb{M}_n^+$ and all unitaries $U \in \mathbb{M}_n$;
(iii) $\|A\| \le \|A + B\| \le \|A\| + \|B\|$ for all $A, B \in \mathbb{M}_n^+$.

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Let $\lambda_1^{\downarrow}(A) \ge \cdots \ge \lambda_n^{\downarrow}(A)$ stand for the eigenvalues of $A \in \mathbb{M}_n^+$ arranged in nonincreasing order. Then the Ky Fan k-norms,

$$\|A\|_{(k)} = \sum_{j=1}^{k} \lambda_j^{\downarrow}(A),$$

are symmetric norms for $k = 1, \ldots, n$. Thus, $\|A\|_{(1)}$ is the operator norm, usually denoted by $\|A\|_\infty$, while $\|A\|_{(n)}$ is the trace norm, usually written $\|A\|_1$. For $A, B \in \mathbb{M}_n^+$, the following conditions are equivalent:

(a) $\|A\|_{(k)} \le \|B\|_{(k)}$ for $k = 1, \ldots, n$;
(b) $\|A\| \le \|B\|$ for all symmetric norms;
(c) the vector of the eigenvalues of A is dominated by a convex combination of permutations of the vector of the eigenvalues of B, equivalently,

$$A \le \sum_{i=1}^{n+1} \alpha_i U_i B U_i^*$$

for some unitary matrices U_i and some scalars $\alpha_i \ge 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$.

When these conditions hold (especially when explicitly stated as (a)), one says that A is weakly majorised by B and writes $A \prec_w B$. If furthermore in (a) one has the equality $\|A\|_{(n)} = \|B\|_{(n)}$, that is, A and B have the same trace, then A is majorised by B, written $A \prec B$. Thus, $A \prec B$ means that (c) holds with equality: A is in the convex hull of the unitary orbit of B. Theorem 1.2 is a special majorisation.

A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ is called doubly stochastic if Φ preserves positivity, identity and trace. In that case, $\Phi(A) \prec A$ for all $A \in \mathbb{M}_n^+$ (see the last section of Ando's survey [1]).

2. The distance from 0 to the numerical range

We state our main result and infer several corollaries. The proof of the theorem is postponed to Section 3.

THEOREM 2.1. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$. *Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \ge \left\| \left(\frac{A+B}{2} + dI \right) \oplus \left(\frac{A+B}{2} - dI \right) \right\|.$$

Here the direct sum is a standard notation for block-diagonal matrices

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$

Since we have equality for the trace, Theorem 2.1 is a majorisation relation. We have $(A + B)/2 \ge dI$, otherwise, the trace norm of the left-hand side would be strictly smaller than the one on the right-hand side, in contradiction with the theorem.

By a basic principle of majorisation, Theorem 2.1 is equivalent to some trace inequalities.

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Corollary 2.2. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$. *Then, for every convex function* $g : [0, \infty) \to (-\infty, \infty)$,

$$\operatorname{Tr} g\Big(\frac{A+B}{2} + dI\Big) + \operatorname{Tr} g\Big(\frac{A+B}{2} - dI\Big) \le \operatorname{Tr} g\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right).$$

Symmetric norms $\|\cdot\|$ on $\mathbb{M}_n^+$ are homogeneous, unitarily invariant, convex functionals. The concave counterpart, the symmetric anti-norms $\|\cdot\|_!$, have been introduced and studied in [3] and [4, Section 4]. We recall their basic properties, parallel to those of symmetric norms given at the end of Section 1. Symmetric anti-norms on $\mathbb{M}_n^+$ are continuous functionals characterised by three properties:

(i) $\|\lambda A\|_! = \lambda \|A\|_!$ for all $A \in \mathbb{M}_n^+$ and all $\lambda \ge 0$;
(ii) $\|UAU^*\|_! = \|A\|_!$ for all $A \in \mathbb{M}_n^+$ and all unitaries $U \in \mathbb{M}_n$;
(iii) $\|A + B\|_! \ge \|A\|_! + \|B\|_!$ for all $A, B \in \mathbb{M}_n^+$.

Let $\lambda_1^{\uparrow}(A) \le \cdots \le \lambda_n^{\uparrow}(A)$ stand for the eigenvalues of $A \in \mathbb{M}_n^+$ arranged in nondecreasing order. Then the Ky Fan k-anti-norms,

$$\|A\|_{(k)!} = \sum_{j=1}^{k} \lambda_j^{\uparrow}(A),$$

are symmetric anti-norms for $k = 1, \ldots, n$. The following conditions are equivalent:

(a) $\|A\|_{(k)!} \ge \|B\|_{(k)!}$ for $k = 1, \ldots, n$;
(b) $\|A\|_! \ge \|B\|_!$ for all symmetric anti-norms;
(c) the vector of the eigenvalues of A dominates some convex combination of permutations of the vector of the eigenvalues of B, equivalently,

$$A \ge \sum_{i=1}^{n+1} \alpha_i U_i B U_i^*$$

for some unitary matrices U_i and some scalars $\alpha_i \ge 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$.

The continuity assumption is not essential, but deleting it would lead to rather strange functionals which are not continuous on the boundary of $\mathbb{M}_n^+$, such as $\|A\|_! := \operatorname{Tr} A$ if A is invertible and $\|A\|_! := 0$ if A is not invertible.

Note that the trace norm is both a symmetric norm and a symmetric anti-norm and that the majorisation $A \prec B$ in $\mathbb{M}_n^+$ also entails that $\|A\|_! \ge \|B\|_!$ for all symmetric anti-norms. Thus, Theorem 2.1 is equivalent to the following statement.

Corollary 2.3. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$. *Then, for all symmetric anti-norms,*

$$\left\|\Big(\frac{A+B}{2} + dI\Big) \oplus \Big(\frac{A+B}{2} - dI\Big)\right\|_! \ge \left\|\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right\|_!.$$

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COROLLARY 2.4. *Let* $\left[\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right]$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \mathrm{dist}(0, W(X))$. *Then*

$$\lambda_1^{\downarrow}\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) - \lambda_1^{\downarrow}\left(\frac{A+B}{2}\right) \geq d$$

and

$$\lambda_1^{\uparrow}\left(\frac{A+B}{2}\right) - \lambda_1^{\uparrow}\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) \geq d.$$

PROOF. The first inequality follows from Theorem 2.1 applied to the symmetric norm $A \mapsto \lambda_1^{\downarrow}(A)$ (the operator norm on the positive cone), while the second inequality follows from Corollary 2.3 applied to the anti-norm $A \mapsto \lambda_1^{\uparrow}(A)$. □

By adding these two inequalities we get an estimate for the spread of the matrices, that is, for the diameter of the numerical ranges.

COROLLARY 2.5. *For every positive matrix partitioned into four blocks of the same size,*

$$\mathrm{diam}\, W\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) - \mathrm{diam}\, W\left(\frac{A+B}{2}\right) \geq 2d,$$

where d *is the distance from* 0 *to* $W(X)$.

Of course,

$$\mathrm{diam}\, W\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) \geq \mathrm{diam}\, W\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \geq \mathrm{diam}\, W\left(\frac{A+B}{2}\right),$$

but the ratio

$$\rho = \frac{1}{2d}\left\{\mathrm{diam}\, W\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) - \mathrm{diam}\, W\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right\}$$

can be arbitrarily small as shown by the following example with blocks in $\mathbb{M}_2$:

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \end{bmatrix},$$

where we note that ρ then takes the value α^{-1}, which tends to 0 as $\alpha \to \infty$.

The Minkowski inequality for positive m-by-m matrices,

$$\det{}^{1/m}(A+B) \geq \det{}^{1/m}(A) + \det{}^{1/m}(B),$$

shows that the functional $A \mapsto \det^{1/m}(A)$ is a symmetric anti-norm on $\mathbb{M}_m^+$. For this anti-norm Theorem 2.1 reads as follows.

COROLLARY 2.6. *Let* $\left[\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right]$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \mathrm{dist}(0, W(X))$. *Then*

$$\det\left\{\left(\frac{A+B}{2}\right)^2 - d^2 I\right\} \geq \det\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right).$$

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Letting $X = 0$, we recapture a basic property: the determinant is a log-concave map on the positive cone of $\mathbb{M}_n$. Hence, Corollary 2.6 refines this property.

By a basic principle of majorisation, Corollary 2.3 is equivalent to the following seemingly more general statement.

Corollary 2.7. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$, *let* $d = \operatorname{dist}(0, W(X))$ *and let* $f(t)$ *be a nonnegative concave function on* $[0, \infty)$. *Then*

$$\left\| f\Big(\frac{A+B}{2} + dI\Big) \oplus f\Big(\frac{A+B}{2} - dI\Big) \right\|_! \ge \left\| f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) \right\|_!$$

for all symmetric anti-norms.

3. Proof of Theorem 2.1

We want to show the majorisation in $\mathbb{M}_{2n}^+$:

$$\begin{bmatrix} \frac{1}{2}(A+B) + dI & 0 \\ 0 & \frac{1}{2}(A+B) - dI \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}, \tag{3.1}$$

where $d = \operatorname{dist}(0, W(X))$. We use two lemmas; the first one might belong to folklore.

Lemma 3.1. *Let* $\{A_k\}_{k=1}^m$ *and* $\{B_k\}_{k=1}^m$ *be two families of* r*-by-*r *positive matrices such that* $A_k \prec B_k$ *for each* k. *Then*

$$\oplus_{k=1}^m A_k \prec \oplus_{k=1}^m B_k.$$

Proof. Let p_k denote any integer such that $0 \le p_k \le m$, $k = 1, \dots, m$. Then, for each integer $p = 1, \dots, mr$,

$$\begin{aligned} \sum_{j=1}^{p} \lambda_j^{\downarrow}\Big(\bigoplus_{k=1}^{m} A_k\Big) &= \max_{p_1+p_2+\cdots+p_m=p} \sum_{k=1}^{m} \sum_{j=1}^{p_k} \lambda_j^{\downarrow}(A_k) \\ &\le \max_{p_1+p_2+\cdots+p_m=p} \sum_{k=1}^{m} \sum_{j=1}^{p_k} \lambda_j^{\downarrow}(B_k) \\ &= \sum_{j=1}^{p} \lambda_j^{\downarrow}\Big(\bigoplus_{k=1}^{m} B_k\Big) \end{aligned}$$

with equality for $p = mr$. □

Lemma 3.2. *Let* $X, Y \in \mathbb{M}_n^+$ *and let* $\delta > 0$ *be such that* $X \ge Y \ge \delta I$. *Then*

$$\begin{bmatrix} X + \delta I & 0 \\ 0 & X - \delta I \end{bmatrix} \prec \begin{bmatrix} X + Y & 0 \\ 0 & X - Y \end{bmatrix}.$$

Proof. Let $\{e_k\}_{k=1}^n$ be an orthonormal basis of $\mathbb{C}^n$ and define two n-by-n diagonal positive matrices

$$D_+ = \operatorname{diag}(\langle e_1, (X+Y)e_1\rangle, \dots, \langle e_n, (X+Y)e_n\rangle)$$

and

$$D_- = \mathrm{diag}(\langle e_1, (X-Y)e_1\rangle, \ldots, \langle e_n, (X-Y)e_n\rangle).$$

Since extracting a diagonal is a doubly stochastic map (a pinching),

$$\begin{bmatrix} D_+ & 0 \\ 0 & D_- \end{bmatrix} \prec \begin{bmatrix} X+Y & 0 \\ 0 & X-Y \end{bmatrix}. \tag{3.2}$$

Now choose the basis $\{e_k\}_{k=1}^n$ as a basis of eigenvectors for X, so that $\lambda_k^{\downarrow}(X) = \langle e_k, Xe_k\rangle$, and observe that the majorisation in $\mathbb{M}_2^+$,

$$\begin{pmatrix} \lambda_k^{\downarrow}(X)+\delta & 0 \\ 0 & \lambda_k^{\downarrow}(X)-\delta \end{pmatrix} \prec \begin{pmatrix} \langle e_k, (X+Y)e_k\rangle & 0 \\ 0 & \langle e_k, (X-Y)e_k\rangle \end{pmatrix},$$

holds for every k. Applying Lemma 3.1 then shows that

$$\bigoplus_{k=1}^{n} \begin{pmatrix} \lambda_k^{\downarrow}(X)+\delta & 0 \\ 0 & \lambda_k^{\downarrow}(X)-\delta \end{pmatrix} \prec \bigoplus_{k=1}^{n} \begin{pmatrix} \langle e_k, (X+Y)e_k\rangle & 0 \\ 0 & \langle e_k, (X-Y)e_k\rangle \end{pmatrix}.$$

This means that

$$\begin{bmatrix} X+\delta I & 0 \\ 0 & X-\delta I \end{bmatrix} \prec \begin{bmatrix} D_+ & 0 \\ 0 & D_- \end{bmatrix}$$

and we may combine this majorisation with (3.2) to complete the proof. □

Proof of (3.1). Suppose first that $d = 0$, that is, $0 \in W(X)$. Note that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \tag{3.3}$$

as the operation of taking the block diagonal is doubly stochastic.

Using the unitary congruence with

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}, \tag{3.4}$$

we observe that

$$J \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} J^* = \begin{bmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} \frac{1}{2}(A+B) & 0 \\ 0 & \frac{1}{2}(A+B) \end{bmatrix} \prec \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and combining with (3.3) establishes (3.1) for the case $d = 0$.

Now assume that $d > 0$, that is, $0 \notin W(X)$. Using the unitary congruence implemented by

$$\begin{bmatrix} I & 0 \\ 0 & e^{-i\theta} I \end{bmatrix},$$

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we may replace the right-hand side of (3.1) with

$$\begin{bmatrix} A & e^{i\theta}X \\ e^{-i\theta}X^* & B \end{bmatrix}.$$

Thanks to the rotation property $W(e^{i\theta}X) = e^{i\theta}W(X)$, by choosing the appropriate θ, we may and do assume that $W(X)$ lies in the half-plane of $\mathbb{C}$ consisting of complex numbers with real parts greater than or equal to d:

$$W(X) \subset \{z = x + iy \ : \ x \geq d\}.$$

The projection property for the real part of the numerical range, $\mathrm{Re}\, W(X) = W(\mathrm{Re}\, X)$ with $\mathrm{Re}\, X = \frac{1}{2}(X + X^*)$, then ensures that

$$\mathrm{Re}\, X \geq dI.$$

Now, using again a unitary congruence with (3.4),

$$J \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} J^* = \begin{bmatrix} \frac{1}{2}(A+B) - \mathrm{Re}X & * \\ * & \frac{1}{2}(A+B) + \mathrm{Re}X \end{bmatrix},$$

where $*$ stands for unspecified entries. Hence,

$$\begin{bmatrix} \frac{1}{2}(A+B) - \mathrm{Re}X & 0 \\ 0 & \frac{1}{2}(A+B) + \mathrm{Re}X \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \frac{1}{2}(A+B) + \mathrm{Re}X & 0 \\ 0 & \frac{1}{2}(A+B) - \mathrm{Re}X \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

and applying Lemma 3.2 then yields (3.1). □

References

[1] T. Ando, 'Majorization, doubly stochastic matrices, and comparison of eigenvalues', *Linear Algebra Appl.* **118** (1989), 163–248.
[2] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, 169 (Springer, New York, 1996).
[3] J.-C. Bourin and F. Hiai, 'Norm and anti-norm inequalities for positive semi-definite matrices', *Int. J. Math.* **22** (2011), 1121–1138.
[4] J.-C. Bourin and F. Hiai, 'Jensen and Minkowski inequalities for operator means and anti-norms', *Linear Algebra Appl.* **456** (2014), 22–53.
[5] J.-C. Bourin and E.-Y. Lee, 'Decomposition and partial trace of positive matrices with Hermitian blocks', *Int. J. Math.* **24** (2013), 1350010.
[6] J.-C. Bourin, E.-Y. Lee and M. Lin, 'Positive matrices partitioned into a small number of Hermitian blocks', *Linear Algebra Appl.* **438** (2013), 2591–2598.
[7] J.-C. Bourin and A. Mhanna, 'Positive block matrices and numerical ranges', *C. R. Acad. Sci. Paris* **355**(10) (2017), 1077–1081.
[8] B. E. Cain, 'Improved inequalities for the numerical radius: when inverse commutes with the norm', *Bull. Aust. Math. Soc.* **97**(2) (2018), 293–296.
[9] M. Gumus, J. Liu, S. Raouafi and T.-Y. Tam, 'Positive semi-definite 2×2 block matrices and norm inequalities', *Linear Algebra Appl.* **551** (2018), 83–91.

https://doi.org/10.1017/S0004972720000520 Published online by Cambridge University Press

[10] T. Hayashi, ‘On a norm inequality for a positive block-matrix’, *Linear Algebra Appl.* **566** (2019), 86–97.
[11] J. A. R. Holbrook, ‘Multiplicative properties of the numerical radius in operator theory’, *J. reine angew. Math.* **237** (1969), 166–174.
[12] R. Horn and C. R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, 1994), corrected reprint of the 1991 original.
[13] F. Kittaneh, ‘Numerical radius inequalities for Hilbert space operators’, *Studia Math.* **168**(1) (2005), 73–80.
[14] A. Mhanna, ‘On symmetric norm inequalities and positive definite block-matrices’, *Math. Inequal. Appl.* **21**(1) (2018), 133–138.
[15] B. Simon, *Trace Ideals and Their Applications*, 2nd edn, Mathematical Surveys and Monographs, 120 (American Mathematical Society, Providence, RI, 2005).

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