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ON $GL_n(B)$ WHERE B IS A BOOLEAN RING

BY
H. GONSHOR

The aim of this paper is to generalize the main results of [1] to $GL_n(B)$ by means of proofs which are more conceptual and less computational. In addition, by means of the Stone space we will obtain results which are new even for the case $n=2$. Finally we shall make some remarks of a categorical nature.

The author is especially interested in the subject because of the overlap here of many areas of mathematics. Concepts from topology, model theory, and category theory are all relevant. In addition, a natural source of counter-examples arises in, of all places, complex variables.

After the paper was written it has come to the attention of the author that some of the results are related to sophisticated results found in the literature.

Results such as Theorem 1 occur in algebraic K theory. In "Algebraic K Theory" by Bass (Benjamin, 1968) generalized Euclidean rings are defined, but the results there are irrelevant for Boolean rings since generalized Euclidean rings are integral domains. More relevant is Bass's "K-theory and Stable Algebra" IHES, Publ. Math. No. 22, page 17–18 proposition 5.1(a). Theorem 1 of our paper can be obtained as a corollary by showing that $n=1$ defines a stable range for $GL(B)$ where B is a Boolean ring. The proof of the latter is essentially the same as the main part of our self-contained proof. In either proof, the critical fact used is that $(a \cup b)=(a+ba')$. (Thus b_i in the definition on page 14 may be taken to be a_i').

Theorem 3 can be obtained from a paper by Arens and Kaplansky, "Topological Representations of Algebras", Amer. Math. Soc. Trans. 63 (1948). It would be necessary to apply Theorem 2.3 on page 461 to the biregular ring A of all n by n matrices over B. In order to do so one would have to check that the structure space of A is the same as the Stone space of B and that the three conditions in the latter theorem are satisfied. Although admittedly this is straightforward, this would appear to involve more work than the self-contained proof in the paper (especially if one uses the categorical approach in Theorem 4). A similar result is proved in the monograph by Pierce, "Modules over Commutative Regular Rings", AMS Memoir No. 70 (1967). This is expressed in a sheaf-theoretic form in Theorem 11.1 on page 44.

Let B be an arbitrary Boolean ring. For convenience we regard the Boolean ring as arising from a Boolean algebra and thus use the Boolean algebra operations freely. (A certain amount of intuition and motivation is lost if one insists on using the Boolean ring operations only.)

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We summarize some elementary facts here that will be used later:
0 and 1 are the same whether B is regarded as a Boolean algebra or ring.

$$ab = a \cap b, \; a+b = ab' \cup a'b = a \cup b - a \cap b.$$

$$a^2 = a, \qquad a+a = 0.$$

In particular, $a+b \le a \cup b$ and $a+b = a \cup b$ if $ab=0$.
As a special case of the latter identity, $a+b = a \cup ba'$.
Note also that 1 is the only unit in a Boolean ring.

THEOREM 1. *Every matrix in* $\mathrm{GL}_n(B)$ *can be expressed as a product of elementary matrices. In fact, all the matrices which occur in the product are of the form* $I+aE_{ij}$ *where* $i \ne j$.

Proof. Let $M \in \mathrm{GL}_n(B)$. Suppose $(a_1, a_2, \ldots, a_n)$ is the first row. Then $\mathrm{Det}(M)=1$, i.e. $\sum_{i=1}^n a_iA_i = 1$. Now $\sum_{i=1}^n a_iA_i \le \bigcup_{i=1}^n a_iA_i \le \bigcup_{i=1}^n a_i$. Therefore $\bigcup_{i=1}^n a_i = 1$. Now $a_1 \cup a_2 = a_1 + a_2a_1$. We add a_1' times the second column to the first column thus obtaining $a_1 \cup a_2$ in the upper left hand corner. Then we add $(a_1 \cup a_2)'$ times the third column to the first column thus obtaining $a_1 \cup a_2 \cup a_3$ in the upper left hand corner. It is clear that by proceeding inductively, we obtain $a_1 \cup a_2 \cdots \cup a_n = 1$ in the upper left hand corner. For all $i \ge 2$ we next subtract a_i times the first column from the ith column. (Subtraction and addition are, of course, the same for Boolean rings. It seems preferable in this case to say "subtract" for psychological reasons.) The top row now has the form $(1, 0, \ldots, 0)$. Next, we can make all the entries in the first column 0 except for the first by the obvious row operations. We now have a block matrix and can continue in the same way on the submatrix obtained by eliminating the first row and column.

By induction we end up with the identity matrix. The result now follows from elementary matrix theory.

REMARK 1. This technique can be generalized to arbitrary matrices. In general we can guarantee to end up with a diagonal matrix of the form

$$\begin{bmatrix} a_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & a_n \end{bmatrix}$$

where $(\forall i)(a_i \ge a_{i+1})$. Since we are interested primarily in invertible matrices in this paper we leave this as an exercise to the reader.

REMARK 2. The proof seems similar to the standard proof for Euclidean rings. Actually, although Boolean rings look so different from Euclidean rings in many respects they have something in common, so that a unified proof is possible for both cases. They both have the property that any ideal (a, b) can be written in the form (c) where c can be obtained in a finite number of steps from a and b by linear combinations in which one of the coefficients is always 1. For Euclidean rings this is, of course, the Euclidean algorithm. For Boolean rings we have $(a, b)=(a \cup b)$ and $a \cup b=a+ba'=b+ab'$. Hence a single step suffices and one can even choose at will which element is to have coefficient 1. This makes the details of the proof easier for Boolean rings than it is for Euclidean rings.

Our main process for splitting matrices will be independent of Theorem 1. We shall deal with finite partitions of B, i.e., sets of the form $\{e_1, e_2, \ldots, e_n\}$ with $e_ie_j=0$ for $i\neq j$ and $\cup_{i=1}^n e_i=1$. First we have a general result about idempotents valid in any commutative ring R.

LEMMA 1. *Let e be idempotent. Then the map* $A\to 1+e(A-1)$ *is an endomorphism of* $\mathrm{GL}_n(R)$.

REMARK. It is convenient to express the map as $T_e: 1+A\to 1+eA$.

Proof. $T[(1+A)(1+B)]=T[1+A+B+AB]=1+eA+eB+eAB$. $T(1+A)T(1+B)=(1+eA)(1+eB)=1+eA+eB+e^2AB$. Since e is idempotent both results are the same. Clearly $T(1)=1$. If $1+A$ has $1+B$ as inverse then $1+eA$ has $1+eB$ as inverse. Hence T maps $\mathrm{GL}_n(R)$ into $\mathrm{GL}_n(R)$.

LEMMA 2. *If* $ef=0$ *then* $(I+eA)(I+fA)=I+(e+f)A$.

Proof. Trivial.

LEMMA 3. *If* $ef=0$ *then* $(I+eA)(I+fB)=(I+fB)(I+eA)$.

Proof. Both sides equal $I+eA+fB$.

LEMMA 4. $T_eT_f=T_{ef}$. *In particular* $T_e^2=T_e$ *and if* $ef=0$ *then* T_eT_f *maps everything into I.*

Proof. Trivial.

LEMMA 5. T_e *restricted to* $\mathrm{GL}_n(0, 1)$ *is an isomorphism onto the set of all invertible matrices of the form* $I+A$ *where the entries in A are either* 0 *or e.* (*It is understood that* $e\neq 0$.)

REMARK. The diagonal entries of $I+A$ are, of course, either 1 or e'. Note again the preference in expressing matrices in the form $I+A$.

Proof. Since 0 entries remain 0 and 1 entries become e the map is clearly 1-1. Now every invertible matrix of the required form is clearly of the form $I+eA$ where $I+A$ is a matrix of 0's and 1's. Hence to complete the proof it suffices to show that if $I+A$ is not invertible then neither is $I+eA$. Now if $I+A$ is not invertible then $\mathrm{Det}(I+A)\neq 1$. Since the only entries are 0 and 1, $\mathrm{Det}(I+A)=0$. Hence

$$e\,\mathrm{Det}(I+eA) = \mathrm{Det}\, e(I+eA) = \mathrm{Det}(eI+eA) = e\,\mathrm{Det}(I+A) = 0.$$

Therefore $\mathrm{Det}(I+eA)\leq e'<1$. Thus $I+eA$ is not invertible.

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We are now ready to discuss the splitting process. First let $e_1, e_2, \ldots, e_n$ be a partition.

LEMMA 6. *The map* $(x_1, x_2, \ldots, x_n) \to \prod_{i=1}^n T_{e_i}(x_i)$ *is an isomorphism from the n-fold direct product of* $\mathrm{GL}_n(0, 1)$ *with itself into* $\mathrm{GL}_n(B)$.

Proof. By Lemmas 1 and 3 it is a homomorphism. By Lemma 4

$$T_{e_i}\left[\prod_{i=1}^{n} T_{e_i}(x_i)\right] = T_{e_i}(x_i).$$

Combining this with Lemma 5 shows that the map is one-one.

We now show how every matrix in $\mathrm{GL}_n(B)$ can be expressed in the form $\prod_{i=1}^n T_{e_i}(x_i)$ for a suitable partition. For a finite Boolean algebra we shall use the partition by the atoms. In the general case the partition will depend on the matrix.

In fact, we will choose the atoms in the Boolean algebra generated by the entries of the matrix. Specifically, if the entries are $a_1, \ldots, a_m$ then the atoms are the non-zero elements of the form $\prod_{i=1}^n b_i$ where $b_i = a_i$ or a_i'. Each atom e has the important property that for every entry a, $ea=0$ or $ea=e$.

Let $e_1, e_2, \ldots, e_n$ be such a partition and suppose $I+A \in \mathrm{GL}_n(R)$. Then by Lemma 2, $I+A=\prod_{i=1}^n (I+e_iA)$. Each $I+e_iA$ is invertible. Because of the above property and Lemma 5, $I+e_iA$ has the form $T_{e_i}(x_i)$. This proves the following lemma.

LEMMA 7. *Every element in* $\mathrm{GL}_n(B)$ *can be expressed in the form* $\prod_{i=1}^n T_{e_i}(x_i)$ *for some partition* $(e), e_2, \ldots, e_n)$.

REMARK. Clearly, if a partition works for a given A so does any refinement. Hence a common partition can always be found for a finite set of matrices.

Lemmas 6 and 7 are enough to definitively settle the finite case.

THEOREM 2. *If B is the finite Boolean algebra with n atoms, then* $\mathrm{GL}_n(B)$ *is isomorphic to the direct product of* $\mathrm{GL}_n(0, 1)$ *with itself n times.*

Proof. Lemma 7 says that the map in Lemma 6 is onto in the special case where the e's are the atoms.

The general case is more complicated since the partitions are not fixed. However, we can at least amalgamate the e_i's corresponding to the same x if we desire.

Many of the results of [1] follow immediately from what we have so far; e.g., as a consequence of Theorem 2 we have [1, Theorem 5]. The decomposition described here is obtained in a different way in the special case $n=2$ for finite B in the proof of [1, Theorem 6]. It is of interest to compare the two styles. In [1] the explicit structure of $\mathrm{GL}_2(0, 1)$, i.e., as S_3 plays an important part whereas in our approach the specific properties of $\mathrm{GL}_n(0, 1)$ never arise. We also note that the form listed in the statement of [1, Theorem 1] is essentially the same as the expression on the top of page 269 in [1]. An expression such as the latter which is nothing but $\prod T_{e_i}(x_i)$ in our notation can be obtained by our procedure for an *arbitrary* Boolean algebra using amalgamation and the explicit structure of $GL_2(0, 1)$. In [1] a counting argument shows that every matrix has the required form in the finite case. One can get from the expression on top of page 269 to [1, Theorem 1] by changing the parametrization. This is easy to see if the form in Theorem 1 is expressed in the language of Boolean algebras. We let $a^0=(a^1 \pm a^2 \pm a^3 \pm a^4 \pm a^5)'$ and then make the substitution:

$$\begin{aligned} a^0 \cup a^1 \cup a^2 \cup a^4 &= a \\ a^1 \cup a^4 &= x \\ a^2 \cup a^4 &= x' \\ a^5 &= w \end{aligned}$$

Lemmas 6 and 7 heuristically suggest that the matrices of $GL_n(B)$ may be regarded as finite functions from B into $GL_n(0, 1)$ where the domain changes with the function. This situation can be handled very nicely by means of the Stone space. Let S be the Stone space corresponding to B and let E be the clopen set which corresponds to $e \in B$. We can now state the main theorem.

THEOREM 3. (*The main theorem*). $GL_n(B)$ *is isomorphic to the set of all continuous functions from* S *to* $GL_n(0, 1)$ *where* $GL_n(0, 1)$ *is given the discrete topology and multiplication is defined in a pointwise manner. Specifically the isomorphism* V *takes* $\prod T_{e_i}(x_i)$ *into the function which is* x_i *on* E_i *for all* i.

Proof. By Lemma 7 every matrix can be expressed in the required form. We first show that the mapping V is well-defined. Suppose a matrix has two representations $\prod T_{e_i}(x_i)$ and $\prod T_{f_i}(y_i)$. Then the partitions $\{e_i\}$ and $\{f_i\}$ has a common refinement $\{g_i\}$. Each product can be expressed as a product with respect to the partition $\{g_i\}$ by means of Lemma 2. In fact, we can do this in a finite number of steps of the following type: Let $e=f+g$ with $fg=0$, then change $T_e(x)$ to $T_f(x)T_g(x)$. The uniqueness result in Lemma 6 says that the representations in terms of $\{g_i\}$ are necessarily identical in both cases. It therefore suffices to show that each step mentioned above leaves the function unaltered. But this is obvious since $e=f+g$ with $fg=0$ implies that $e=f+g$ with $fg=0$ and hence being x on E is the same as being x on f and x on G.

By using a common refinement and the fact that T_e is a homomorphism it is easy to see that V is a homomorphism. Again by using a common refinement, it is clear that different matrices must differ in the corresponding x_i for some e_i, hence they give rise to different functions. Thus V is one-one.

Now every product of the form $\prod T_{e_i}(x_i)$ is invertible. Hence the image of V consists of the set of all functions f which can be obtained by means of a finite partition of S into clopen sets $\{E_i\}$ and on each E_i an arbitrary choice of an element in $GL_n(0, 1)$ for its value. It is well-known and easy to see that this gives precisely the set of continuous functions.

As an immediate corollary we have the result in [1, Theorem 2] that the sixth power of any element in $GL_2(B)$ is 1. This could of course also have been obtained earlier from Lemma 7.

More generally, since the main theorem expresses $GL_n(B)$ as a subdirect product of groups of the form $GL_n(0, 1)$ it follows that every universal Horn sentence true in $GL_n(0, 1)$ is automatically true in $GL_n(B)$. We can say even more in our situation; namely, every Horn sentence is preserved even if it is not universal. Technically speaking, this follows from the fact that Skolem functions can always be chosen to be continuous because of the nice nature of the topologies involved. This is not always true for subdirect products. The ring of analytic functions in the unit circle does not satisfy the sentence $(\forall x)(\exists y)(y^2=x)$ even though Z does. Sentences that are not Horn are not necessarily preserved. As an example, we can take $GL_2(B)$ and the sentence $(\forall x)(x^3=1 \vee x^2=1)$. This is true in $GL_2(0, 1)=S_3$ but not in $GL_2(B)$ for any other B.

Theorem 3 is also useful for the light it sheds in the case where B_I is the ring of all finite and cofinite subsets of I as discussed in [1] starting at the last line on page 269. The Stone space is the one point compactification of the integers, hence the continuous functions may be regarded as the set of all ultimately constant sequences. This makes the structure of the group quite transparent. We have a countable direct sum of copies of $GL_n(0, 1)$ together with an extra copy of $GL_n(0, 1)$ [the set of constant functions] which generate $GL_n(B)$.

Theorem 3 is not too useful for the case where B is the free Boolean algebra on countably many generators also studied in [1]. In that case the Stone space is the Cantor set. Continuous functions on such a space are not succinctly described so that the point of view in [1] appears to be the best.

We finally make some remarks of a categorical nature. The reader who is proficient in thinking categorically will notice that the main results can be regarded from a different point of view. GL_n may be regarded as a functor from rings to groups which obviously preserves monics. It follows from Theorem 1 that GL_n restricted to Boolean rings preserves epics. GL_n also preserves direct products. Many of the earlier lemmas can be proved categorically using this fact by means of suitable identifications. We preferred our approach since the direct proofs are easy, and on the other hand the identifications used in the categorical approach require some caution.

Anyway, we note that $(0, 1)$ is an injective cogenerator in the category of Boolean algebras. More important, B can be expressed in terms of $(0, 1)$ in a special way, i.e., as a direct limit of monics of algebras which are finite direct products of algebras of the form $\{0, 1\}$. In fact, let the index set be the set of partitions of B and the ordering defined by refinement. Corresponding to the index $(e_1, e_2, \ldots, e_n)$ we have the n-fold direct product of $\{0, 1\}$ with itself $\{0, 1\}_{e_1} \times \{0, 1\}_{e_2} \cdots \{0, 1\}_{e_n}$ indexed by the e's. For a simple refinement, i.e., one in which e_i is replaced by f_i and g_i we have the map induced by the maps which are fixed on $\{0, 1\}_{e_j}$ for $j \neq i$

and for which $\{0, 1\}_{e_i} \to \{0, 1\}_{f_i} \times \{0, 1\}_{g_i}$ is the diagonal map. It is easy to see that B itself is the direct limit of this family. The best way to see this is probably by regarding the elements of B as continuous characteristic functions on S. The maps then become ordinary inclusions.

We can now generalize the main theorem from GL_n to a large class of functors.

THEOREM 4. *Let F be a functor from Boolean rings to a category of algebras which is closed with respect to directed unions. Suppose F preserves products and direct limits of monics. Then $F(B)$ is isomorphic to the set of all continuous functions from S to $F(0, 1)$ where $F(0, 1)$ is given the discrete topology and the operations are defined in a pointwise manner.*

REMARK. We use "algebra" in the sense of universal algebra.

The proof follows easily from the previous remarks. Thus in GL_2 we generalized not only from 2 to n but we also generalized out the GL part!

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DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903
U.S.A.

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