Appendix B

Projects

B.1 Shallow Flow Over Topography

B.1.1 Background

In this project you will figure out how the instability of a parallel shear flow is modified by topography. Examples include the flow of a river over a submerged bar or island (B.1) and an alongshore ocean current encountering a seamount or a sand bar. We'll start by developing the basic theory; read carefully and fill in any algebraic steps that aren't obvious.

Consider an inviscid, homogeneous fluid flowing over a non-horizontal bottom $z = -H(x, y)$ and allowing a free surface deflection $z = \eta(x, y, t)$. In making the shallow water approximation, we assume that

- the pressure is entirely hydrostatic:

$$\pi = g(\eta - z)$$

- the horizontal velocity is independent of depth:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0.$$

With these assumptions, the horizontal momentum equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x}; \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y} \qquad \text{(B.1)}$$

To complete the set, we integrate the divergence equation (1.17) over the vertical extent of the fluid, $-H(x, y) \le z \le \eta(x, y, t)$. After combination with (B.1), the result is

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(H + \eta)u\right] + \frac{\partial}{\partial y}\left[(H + \eta)v\right] = 0. \qquad \text{(B.2)}$$

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Figure B.1 Braided drainage pattern near the confluence of the Yukon River and Koyukuk River, Alaska. The Koyukuk River (dark) joins the silt-laden Yukon River (lighter) at the right (Image by U.S. Army Air Corps, 1941).

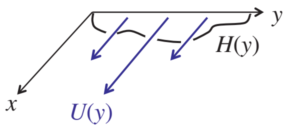

Figure B.2 Shallow flow over variable topography.

We have assumed that the background flow has no vertical shear, but it can be horizontally sheared. We therefore look for perturbations about a background flow $\vec{u} = U(y)\hat{e}^{(x)}$, $\eta = 0$. This flow represents an equilibrium state provided that the further condition

$$H = H(y) \tag{B.3}$$

is satisfied.

Accordingly, we seek a perturbation solution of the form

$$\begin{aligned} u &= U(y) + \epsilon u'(x, y, t); \\ v &= \epsilon v'(x, y, t); \\ \eta &= \epsilon \eta'(x, y, t). \end{aligned} \tag{B.4}$$

Linearizing as usual, we obtain

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -v'\frac{dU}{dy} - g\frac{\partial \eta'}{\partial x}; \tag{B.5}$$

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$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -g\frac{\partial \eta'}{\partial y}; \tag{B.6}$$

$$\frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x} = -\frac{\partial}{\partial x}\left[Hu'\right] - \frac{\partial}{\partial y}\left[Hv'\right]. \tag{B.7}$$

The perturbation vertical vorticity is given by

$$\xi' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \tag{B.8}$$

We can now derive an evolution equation for ξ' from the x and y components of the perturbation momentum equation. Subtracting the y derivative of (B.5) from the x derivative of (B.6) yields

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\xi' = \underbrace{U_{yy}v'}_{advection} + \underbrace{U_y\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)}_{stretching}. \tag{B.9}$$

Note that the total derivative U_y is (minus) the vertical vorticity of the background flow. Therefore, the first term on the right-hand side represents advection of that background vorticity by the cross-stream velocity perturbation. The second represents vortex stretching: changes in fluid depth accompanying convergence (divergence) of the horizontal flow can add to (subtract from) the background vorticity.

Here we'll look at the special case of *zero surface deflection*, $\eta' = 0$, for which the linearized equations collapse to a single equation. With $\eta' = 0$, the mass conservation equation becomes

$$\frac{\partial}{\partial x}\left[Hu'\right] + \frac{\partial}{\partial y}\left[Hv'\right] = 0. \tag{B.10}$$

This condition can be satisfied by defining a streamfunction ψ' such that

$$Hu' = -\frac{\partial \psi'}{\partial y}; \quad Hv' = \frac{\partial \psi'}{\partial x}. \tag{B.11}$$

In terms of this streamfunction, the perturbation vorticity becomes

$$\xi' = \frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial \psi'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{H}\frac{\partial \psi'}{\partial y}\right) \tag{B.12}$$

and its evolution equation can be written as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left\{\frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial \psi'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{H}\frac{\partial \psi'}{\partial y}\right)\right\} = \left(\frac{U_y}{H}\right)_y\frac{\partial \psi'}{\partial x}. \tag{B.13}$$

B.1.2 Project

You are now ready to explore instability of a parallel shear flow that is modified by topography. You'll do this in six sub-projects, two analytical and four numerical.

(A) In (B.13), assume a normal mode solution of the form

$$\psi' = \hat{\psi}(y)e^{\imath k(x-ct)}. \tag{B.14}$$

and substitute to obtain an ordinary differential equation.

Test your understanding: How do we know that $\hat{\psi}$ has to be a function of y?

(B) Following the proof of Rayleigh's theorem in section 3.15.1, show that a necessary condition for instability is

$$\left(\frac{U_y}{H}\right)_y = 0$$

for some y. You can assume that the side walls are steep, so that the impermeable boundary condition can be written as $v' = 0$, or

$$\hat{\psi} = 0. \tag{B.15}$$

Does instability require an inflection point? Suppose that U_y is constant. Show that instability is possible only if $H(y)$ has a local maximum or minimum.

(C) Write a subroutine that accepts as input the vectors y, U, and H and the scalar k, and gives as output σ (the growth rate of the FGM) and $\hat{\psi}$ (the corresponding eigenfunction). For this exercise, you will solve the equation in scaled form. The length scale will be the half-width of the channel, so that y is within the range $0 < y < 2$. The velocity will be given by a parabolic profile symmetric about the river center, scaled by the maximum velocity:

$$U = y(2-y). \tag{B.16}$$

The water depth will have a uniform value H_0 except for a bump representing the submerged bar:

$$H = H_0\left(1 - a\,\text{sech}^2\frac{y-y_0}{W}\right)$$

where $a < 1$ is the amplitude of the bar, y_0 is the location of the bar crest, and W is a width scale. These profiles are shown below for a typical parameter set.

For this parameter set, discretize using 200 points and compute the fastest growth rate for $k = [0.2 : 0.2 : 10]$. Plot the result. State the wavelength of the FGM as a multiple of the half-width of the river. State the e-folding time as a multiple of the time taken by the maximum current to traverse a distance equal to the half-width.

(D) In this part of the project, you will see how the instability varies with the cross-stream position of the bar. Repeat the stability analysis for y_0 =

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1, 1.2, 1.4, ..., 2 and plot the growth rate, phase speed, and wavenumber of the FGM as a function of y_0.

(E) Now we will see how instability depends on the amplitude of the bar. Based on our earlier discussion of shear instability, do you think there will be instability when $a = 0$? Why or why not? From part (C), identify the value of y_0 at which the growth rate is a maximum. Setting y_0 to this value, vary the amplitude a from 0 to 1, and plot the growth rate, phase speed, and wavenumber of the FGM as functions of a. Is there a minimum amplitude at which instability occurs?

(F) How do the instability characteristics change if $a < 0$ (i.e., for flow over a depression)?

B.2 Barotropic Instability on the β-plane

Consider an inviscid, homogeneous fluid in a rotating environment ($\nu = 0$, $b = 0$, $f \neq 0$). The equations of motion are

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{B.17}$$

$$\frac{D\vec{u}}{Dt} = -\vec{\nabla}\pi + \vec{u} \times f\hat{e}^{(z)}. \tag{B.18}$$

We now make three additional assumptions:

(i) The flow is purely horizontal: $\vec{u} = \{u, v\}$.
(ii) Nothing varies in the vertical: $\partial/\partial z = 0$.
(iii) Rather than setting f to a constant (the f-plane approximation), we use a first-order Taylor series approximation for the meridional dependence: $f = f_0 + \beta y$.[1]

The equations can now be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{B.19}$$

$$\frac{Du}{Dt} - fv = -\frac{\partial \pi}{\partial x} \tag{B.20}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \pi}{\partial y}, \tag{B.21}$$

where

[1] More exactly, $f = 2\Omega \sin\varphi$ where Ω is the angular velocity of the planet, $\varphi = y/R$ is the latitude, and R is the planet's radius. For Earth, $\Omega = 7.2921 \times 10^5 \mathrm{s}^{-1}$ and $R = 6371$ km. Near a given latitude φ_0, $f_0 = 2\Omega \sin\varphi_0$ and $\beta = 2\Omega R \cos\varphi_0$. On Earth, $2\Omega R = 2.3 \times 10^{-11} \mathrm{m}^{-1}\mathrm{s}^{-1}$. At the equator ($\varphi_0 = 0$), f_0 vanishes and β is a maximum. At the poles ($\varphi_0 = \pm\pi/2$), the opposite is true. Therefore the f-plane approximation is most accurate in polar regions, while the β-plane approximation is valid in any sufficiently small band of latitudes.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}. \tag{B.22}$$

We now consider perturbations to a parallel shear flow that is purely zonal and varies only in the meridional:

$$u = U(y) + \epsilon u'; \quad v = \epsilon v'; \quad \pi = \Pi + \epsilon \pi'. \tag{B.23}$$

(A) Substitute this into the equations of motion and show that

- At $O(\epsilon^0)$, the background velocity and pressure fields must be in geostrophic balance: $\partial\Pi/\partial y = -fU$.
- At $O(\epsilon^1)$, obtain the perturbation velocity equations

$$u'_t + Uu'_x + (U_y - f)v' = -\pi'_x \tag{B.24}$$

$$v'_t + Uv'_x + fu' = -\pi'_y. \tag{B.25}$$

(B) Cross-differentiate (B.24, B.25) to obtain the equation for the perturbation vertical vorticity $\zeta' = v'_x - u'_y$:

$$\zeta'_t + U\zeta'_x = -(f_0 - U_y)_y v' = -(\beta - U_{yy})v'. \tag{B.26}$$

The right-hand side represents advection of the background vorticity gradient by the meridional velocity perturbation. The background vorticity has two parts: f_0 is due to the rotation of the planet and $-U_y$ is due to the parallel shear flow.

(C) Because the flow is 2D, we can represent it in terms of a streamfunction:

$$u' = \psi'_y\,; \quad v' = \psi'_x\,; \quad \zeta' = \nabla^2\psi'. \tag{B.27}$$

Substitute this into (B.26). Now assume that ζ' has the normal mode form

$$\zeta' = \hat{\zeta}(y)e^{\iota k(x-ct)}, \tag{B.28}$$

where k is the zonal wavenumber and c the zonal phase speed. With that assumption, derive:

$$\hat{\psi}_{yy} = \left(\frac{\beta - U_{yy}}{U - c} - k^2\right)\hat{\psi}. \tag{B.29}$$

Note that this is just the 2D form of the Rayleigh equation (3.19) with an extra term to account for planetary rotation.

(D) Setting $U = 0$, derive the dispersion relation for barotropic Rossby waves:

$$c = \frac{-\beta}{k^2 + \ell^2}. \tag{B.30}$$

(Be sure to state any additional assumptions you make.) Rossby waves are unique in that they travel only to the west, as you can see from the fact that c is negative definite. Rossby waves are closely analogous to the vortical waves described in section 3.12.2, with planetary rotation playing the same role as the kinks in the velocity profile.

(E) Following the classical proof of Rayleigh's theorem (section 3.15.1), show that (B.29) can have unstable normal mode solutions only if the background vorticity $f - U_y$ changes sign somewhere in the range of y.

B.3 Bioconvection

Review the introduction to bioconvection in section 9.3. Complete the derivation of the equilibrium state and the perturbation equations (9.21) and (9.22). Convert these into in normal mode form, then write a subroutine to solve them using the method of your choice. Scale the equations based on the layer thickness V_s and the diffusivity κ. Show that the solution depends only on the Prandtl number ν/κ, the scaled swimming velocity $\alpha = V_s H/\kappa$, and the Rayleigh number $Ra = g' H^3 \bar{n}/(\nu\kappa)$.

Reproduce the results shown in Figure 9.14.

(i) How does the stability boundary depend on the Prandtl number? Could you have anticipated this?
(ii) As α increases from zero, the critical Rayleigh number drops rapidly. Is this what you would expect? Why?
(iii) Identify the value of α at which the critical Ra smallest. As α increases above that value, the critical Rayleigh number increases. Is that what you would expect? Why?

B.4 Hydrostatic Normal Modes of a Density Profile.

Hydrostatic normal modes are described by the Taylor-Goldstein equation (4.18) with no mean flow ($U = 0$) and in the long-wave limit $k \to 0$:

$$\hat{w}_{zz} + \frac{B_z}{c^2}\hat{w} = 0.$$

(a) Find or invent a stable density profile typical of the ocean or atmosphere and investigate its normal modes, both the phase speeds and the vertical structures.

Use the numerical method described in section 6.2 and coded in homework problem 18. If you set ν and κ to very small values, say $10^{-6}\text{m}^2/\text{s}$, the viscous and diffusive terms will improve your resolution without significantly affecting the results.

(b) Find or invent a realistic mean velocity profile $U(z)$ to go with your density profile. Repeat the calculations above with $U(z)$ included and see what difference it makes.

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B.5 Stability Boundaries for Double Diffusive Instability

Here you will determine the stability boundaries for salt fingering and diffusive convection by considering the algebraic properties of cubic equations.

Introduction

Consider the cubic polynomial

$$\sigma^3 + A_2\sigma^2 + A_1\sigma + A_0 = 0, \tag{B.31}$$

in which the coefficients A_2, A_1, and A_0 are real numbers. The stability equation (9.8) for double diffusive instability is a cubic of this type, with the additional properties $A_1 > 0$ and $A_2 > 0$.

Roots are in general complex: $\sigma = \sigma_r + \iota\sigma_i$. Roots with $\sigma_i = 0$ and $\sigma_i \neq 0$ describe stationary and oscillatory normal modes, respectively. A stability boundary is a curve that divides a region in which the flow is unstable (i.e., at least one root has a positive real part) from a region that is stable (all roots have negative real part).

We can simplify the analysis of (B.31) with a change of variables:

$$\boxed{s^3 + s^2 + \alpha_1 s + \alpha_0 = 0,} \tag{B.32}$$

where

$$s = \frac{\sigma}{A_2}; \quad \alpha_1 = \frac{A_1}{A_2^2}; \quad \text{and } \alpha_0 = \frac{A_0}{A_2^3}$$

Because A_2 is real and positive, this change of variables does not affect whether any of these quantities is real or complex, positive or negative. In particular, $\alpha_1 > 0$ in gravitationally stable stratification $B_z > 0$.

The cubic equation (B.32) has three roots, each of which is a function of α_0 and α_1. These can be either

- all real, or
- one real and two complex conjugates.

The three roots obey the following relations:

$$s_1 + s_2 + s_3 = -1 \tag{B.33}$$

$$s_1s_2 + s_2s_3 + s_1s_3 = \alpha_1 \tag{B.34}$$

$$s_1s_2s_3 = -\alpha_0. \tag{B.35}$$

Using this information, you will find the stability boundaries for stationary and oscillatory roots, then identify the side of each boundary on which the flow is unstable.

(A) Three real roots

Assume that all three roots of (B.32) are real.

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(i) Show that, if one or more roots are zero, then α_0 must be zero. Now show the converse, i.e., if $\alpha_0 = 0$, at least one root is zero.

(ii) Now, what about the other two roots? If one of them is positive, then $\alpha_0 = 0$ is not a stability boundary. Show that this is not the case (i.e., both of the other two roots are negative). [Hint: Keep in mind that $\alpha_1 > 0$.]

(iii) On which side of the line $\alpha_0 = 0$ is the flow unstable? To answer this, differentiate (B.32) to show that $\partial s/\partial\alpha_0$, evaluated at $\alpha_0 = 0$, is $-1/\alpha_1$, and is therefore negative.

(B) one real, two complex roots

Assume that one root is real and the other two are complex conjugates.

(i) Show that the real root is zero if and only if $\alpha_0 = 0$, as in part A1.

(ii) Show that the complex conjugate roots have zero real part if and only if $\alpha_0 = \alpha_1$.

(iii) Substitute $s = s_r + \iota s_i$ into (B.32) and split the result into real and imaginary parts.

(iv) Show that, for the real root, the equation is the same as (B.32), and therefore that the argument of part A3 still holds.

(v) Now consider the complex conjugate roots. Assuming that $s_i \neq 0$, eliminate s_i to obtain

$$26s_r^3 - 16s_r^2 - 2(1 - 4\alpha_1)s_r + \alpha_1 - \alpha_0 = 0. \tag{B.36}$$

(vi) Differentiate this to show that $\partial s_r/\partial\alpha_0$, evaluated at $\alpha_0 = \alpha_1$, is $1/(8\alpha_1 - 2)$, and is therefore positive.

(C) Stability boundaries in terms of α_0 and α_1

Here, you will summarize your results from parts A and B with a simple sketch. On the $\alpha_0 - \alpha_1$ plane, sketch the lines $\alpha_0 = 0$ and $\alpha_0 = \alpha_1$. (Only the half-plane $\alpha_1 \geq 0$ matters.) Indicate the three regions in which

- real roots are positive,
- complex roots have positive real part,
- all roots have negative real part.

(D) Stability boundaries in terms of R_ρ, Pr, and τ

(i) Referring back to (9.8), use the foregoing results to show that stationary modes are unstable if $1 < R_\rho < \tau^{-1}$.

(ii) Show that oscillatory modes are unstable if $1 < R_\rho^* < (Pr + 1)/(Pr + \tau)$.

(iii) One of the above cases pertains to the case where cool, fresh water overlies warm, salty water, the other to the opposite case. Which is which?

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B.6 Rayleigh-Taylor Instability in Outer Space

Consider a gravitating star surrounded by interstellar gas. Set up the equations of motion in spherical coordinates (e.g., Smyth, 2017; Kundu et al., 2016). Neglect viscosity, diffusion, and the Coriolis effect. Derive the conditions for static equilibrium and the perturbation equations.

Imagine a spherical shell, surrounding the star, across which the buoyancy $B(r)$ of the stellar gas changes by an amount b_0. This is the spherical analog of the interface in section 2.2.4. Substitute the buoyancy profile into the perturbation equations and explore the solutions.

B.7 Universality of Convection-like Instabilities.

Convective, centrifugal, and inertial instabilities are mathematically similar, and for each we have established an upper bound on the growth rate. Moreover, in each case we have seen examples where the growth rate increases with increasing wavenumber and asymptotes at the proven upper bound. In section 7.8.1 we gave a rather hand-waving argument as to why this should be so. In this project you will construct an explicit proof using Sturm-Liouville theory and results from the calculus of variations.

Groen (1948) considered internal gravity waves in an inviscid, motionless, stably stratified fluid. For a buoyancy profile that is unrestricted except that B_z is everywhere positive, he proved the following two theorems.

- Wave frequency is a monotonically increasing function of wavenumber.
- The maximum frequency, found in the limit of infinite wavenumber, is $\sqrt{\max_z B_z}$.

Work through Groen's proof, only now assume that the stratification is unstable. (2.29) is a good starting point.[2] To go a step further, see if you can prove the corresponding results for centrifugal instability, where the geometry is cylindrical.

Note that this proof works only if B_z is *everywhere* negative. What if B_z is only negative for some z? Based on our discussions so far (including sections 2.2.3 and 7.8.1 and problem 16 in Appendix A) would you expect the result to hold (i.e., the growth rate bound to be achieved) in that case? If so, can you prove it?

[2] If you are willing to trust Groen's mathematics (and there is no reason not to other than the habitual skepticism of a good scientist), you can prove the results easily by showing that the instability problem is isomorphic to Groen's wave problem.

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