

DENSE SUBALGEBRAS OF PURELY INFINITE SIMPLE GROUPOID C^* -ALGEBRAS

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(Received 19 December 2019; first published online 30 March 2020)

Abstract A simple Steinberg algebra associated to an ample Hausdorff groupoid G is algebraically purely infinite if and only if the characteristic functions of compact open subsets of the unit space are infinite idempotents. If a simple Steinberg algebra is algebraically purely infinite, then the reduced groupoid C^* -algebra $C_r^*(G)$ is simple and purely infinite. But the Steinberg algebra seems too small for the converse to hold. For this purpose we introduce an intermediate $*$ -algebra $B(G)$ constructed using corners $1_U C_r^*(G) 1_U$ for all compact open subsets U of the unit space of the groupoid. We then show that if G is minimal and effective, then $B(G)$ is algebraically properly infinite if and only if $C_r^*(G)$ is purely infinite simple. We apply our results to the algebras of higher-rank graphs.

Keywords: purely infinite ring; purely infinite C^* -algebra; ample groupoid; Steinberg algebra; Kumjian–Pask algebra; infinite projection; infinite idempotent

2010 *Mathematics Subject Classification:* 46L05; 16S60

1. Introduction

Inspired by the success of classifying purely infinite simple C^* -algebras, Ara, Goodearl and Pardo extended the notion of purely infinite C^* -algebras to unital rings, and studied their basic properties [4]. Purely infinite rings were studied further in [1, 3, 6, 19]. In particular, [6] develops a general theory of purely infinite rings by presenting an algebraic parallel of the work in [21] on purely infinite C^* -algebras. Here we investigate a question asked in [6, Problem 8.4]. Suppose that A_0 is a dense subalgebra of a C^* -algebra A . If A_0 is purely infinite in the algebraic sense, is A purely infinite in the C^* -algebraic sense? We focus on the situation where the C^* -algebra is a simple C^* -algebra associated with an ample Hausdorff groupoid, that is, an étale groupoid with a basis of compact open sets. Since every non-unital, purely infinite, simple, nuclear C^* -algebra in the UCT class is isomorphic to the reduced C^* -algebra $C_r^*(G)$ of some ample Hausdorff groupoid G (see [35, p. 367]), this is a good setting to investigate this problem and its converse.

We investigate two different dense subalgebras of $C_r^*(G)$. The first is the complex Steinberg algebra $A(G)$ associated with an ample groupoid G . Simplicity of $A(G)$ and $C_r^*(G)$ are well understood (provided G is amenable and second countable): $A(G)$ is algebraically simple if and only if $C_r^*(G)$ is C^* -algebraically simple, and this happens if and only if G is minimal and effective [8]. For the remainder of the introduction, we assume that G is a minimal and effective ample groupoid.

It was shown in [9, Theorem 4.1] that $C_r^*(G)$ is purely infinite if and only if, for every compact open set U of the unit space, the characteristic function 1_U is an infinite projection. As a direct consequence, we obtain an affirmative answer to [6, Problem 8.4] when $A_0 = A(G)$ (see Corollary 3.1). The converse remains an open question because, in a way, $A(G)$ is too small.

The second dense subalgebra $B(G)$ of $C_r^*(G)$ that we investigate contains the corners $1_U C_r^*(G) 1_U$ for all compact open subsets U of the unit space of the groupoid, and so is much larger than $A(G)$. The reason why $B(G)$ is useful is twofold. First, unlike $C_r^*(G)$, $B(G)$ is s -unital, and so we can apply the results of [6] directly to it. Second, unlike $A(G)$, $B(G)$ has the property that $B(G)x B(G)$ is contained in $B(G)$ for every $x \in C_r^*(G)$. Using these two properties, we show in Theorem 4.7 that $B(G)$ is algebraically properly infinite if and only if $C^*(G)$ is purely infinite.

Our motivation for studying these dense subalgebras is to make progress on the long-standing open question of finding necessary and sufficient conditions on a groupoid G that characterize when the $C_r^*(G)$ is purely infinite. Anantharaman-Delaroche introduced the notion of a locally contracting groupoid in [2], and then showed the reduced C^* -algebra of a locally contracting groupoid is purely infinite [2, Proposition 2.4]. It is not known if the converse holds, even for the C^* -algebras of higher-rank graphs which can be modelled using ample groupoids.

It follows from [2, Proposition 2.2] that in an ample, locally contracting groupoid, the 1_U are all infinite projections, but again it is not known if the converse holds. For the groupoids of higher-rank graphs, it would suffice to understand what graph properties ensure that all vertex projections are infinite projections in the graph C^* -algebra [9, Corollary 5.1].

Following [14], substantial progress has been made in investigating a possible dichotomy for the C^* -algebras of ample groupoids [7, 29]: that C^* -algebras of ample groupoids are all either purely infinite or stably finite. At this stage, it has been established that the dichotomy holds when a certain type of semigroup is almost unperforated. Also, [7, Corollary 4.9] gives a sufficient condition for an inner exact groupoid to have a purely infinite C^* -algebra. [29, Proposition 7.1] gives a necessary condition on G for the C^* -algebra to be purely infinite. How either of these conditions relates to a locally contracting groupoid is not immediately clear.

The outline of this paper is as follows. In §2 we give some preliminaries, including algebraic and C^* -algebraic definitions of infinite idempotents/projections, and purely infinite rings and C^* -algebras. The main focus of §3 is to prove an algebraic version of [9, Theorem 4.1] for Steinberg algebras: $A(G)$ is algebraically purely infinite if and only if, for every compact open set U of the unit space, 1_U is an infinite idempotent in $A(G)$. In §4, we introduce $B(G)$. Our main theorem, Theorem 4.7, says that $B(G)$ is algebraically properly infinite if and only if $C_r^*(G)$ is C^* -purely infinite. The forward implication is very general in that it does not require G to be minimal or effective.

Finally, in § 5, we show how our results apply to higher-rank graphs and their algebras. The Steinberg algebra of the graph groupoid is canonically isomorphic to the Kumjian–Pask algebra of the graph [5, 15], and the C^* -algebra of the graph groupoid is isomorphic to the C^* -algebra of the graph. We reconcile our results with recent results by Larki about purely infinite Kumjian–Pask algebras [23]. Finally, in Corollary 5.5, we consider the special case of an aperiodic and cofinal higher-rank graph which has a return path with an entrance, and show these graphs give rise to purely infinite algebras. Along the way, we prove Lemma 5.6 which allows us to construct return paths with entrances. This fills a gap in [16, Corollary 5.7] and [34, Proposition 8.8].

2. Preliminaries

We start by summarizing the algebraic and C^* -algebraic definitions that we use from [6] and [21], respectively.

2.1. Algebraic preliminaries

Definition 2.1. Suppose that p and q are idempotents in a ring R , $a \in R$, $x \in M_k(R)$ and $y \in M_n(R)$. Set $RaR := \text{span}\{cad : c, d \in R\}$.

- (1) R is *s-unital* if, for each $r \in R$, there exist $u, v \in R$ such that $ur = rv = r$.
- (2) $p \leq_{\text{ring}} q$ if $pq = qp = p$.
- (3) $p <_{\text{ring}} q$ if $p \leq_{\text{ring}} q$ and $p \neq q$.
- (4) $p \sim_{\text{ring}} q$ if there exist $s, t \in R$ such that $st = p$ and $ts = q$.
- (5) $x \lesssim_{\text{ring}} y$ if there exist $\alpha \in M_{k,n}(R)$ and $\beta \in M_{n,k}(R)$ such that $x = \alpha y \beta$.
- (6) p is an *infinite idempotent* if there exists an idempotent q such that $p \sim_{\text{ring}} q <_{\text{ring}} p$.
- (7) p is a *algebraically properly infinite idempotent* if there exists an idempotent q such that $p \oplus p \sim_{\text{ring}} q \leq_{\text{ring}} p$.
- (8) a is *algebraically infinite* if there exists $t \in R \setminus \{0\}$ such that $a \oplus t \lesssim_{\text{ring}} a$.
- (9) a is *algebraically properly infinite* if $a \neq 0$ and $a \oplus a \lesssim_{\text{ring}} a$.
- (10) R is *algebraically purely infinite simple* if R is algebraically simple (R has no non-trivial two-sided ideals) and every non-zero right ideal contains an infinite idempotent.
- (11) R is *algebraically properly infinite* (also called *properly purely infinite*) if every non-zero element of R is algebraically properly infinite.
- (12) R is *algebraically purely infinite* if
 - (a) no quotient of R is a division ring, and
 - (b) whenever $a \in R$ and $b \in RaR$, we have $b \lesssim_{\text{ring}} a$.

Remark 2.2. Here we clarify some of the relationships between the definitions given above.

- (1) If p and q are idempotents with $p \leq_{\text{ring}} q$, then $p \lesssim_{\text{ring}} q$.
- (2) By [6, Remarks 2.7], an idempotent is an infinite idempotent if and only if it is algebraically infinite and it is an algebraically properly infinite idempotent if and only if it is algebraically properly infinite.
- (3) Let R be a ring with local units, which means that for every finite set F there exists an idempotent e such that $F \subset eRe$. (The existence of local units implies that R is s -unital.) Then R is algebraically purely infinite simple if and only if (i) R is not a division ring and (ii) for all $a, b \in R$ with $a \neq 0$ we have $b \lesssim_{\text{ring}} a$ [1, Proposition 10]; in particular, R is simple and algebraically purely infinite if and only if R is algebraically purely infinite simple.

The next lemma is an algebraic version, for s -unital rings, of [21, Lemma 3.17].

Lemma 2.3. *Suppose that R is an s -unital ring. Let $p, q, a \in R$, and suppose that q is an idempotent, and that p is an infinite idempotent.*

- (1) *If $p \lesssim_{\text{ring}} a$, then a is algebraically infinite.*
- (2) *If $p \leq_{\text{ring}} q$, then q is an infinite idempotent.*
- (3) *If $p \sim_{\text{ring}} q$, then q is an infinite idempotent.*

Proof. (1) is precisely [6, Lemma 3.9(iii)]. For (2), suppose that $p \leq_{\text{ring}} q$. By Remark 2.2(1), $p \lesssim_{\text{ring}} q$. Since p is algebraically infinite, (1) gives that q is also algebraically infinite.

For (3), suppose $p \sim_{\text{ring}} q$. Then there exist $r, s \in R$ such that $rs = p$ and $sr = q$. Let $\alpha = r$ and $\beta = s$. Then

$$\alpha q \beta = rqs = r(sr)s = (rs)^2 = p^2 = p.$$

Now $p \leq_{\text{ring}} q$ and p is algebraically infinite, and so q is algebraically infinite by (1). \square

2.2. C^* -algebraic preliminaries

Let A be a C^* -algebra. We write A^+ for the positive elements of A , and then $A^+ = \{a^*a : a \in A\}$. We write $C_0(X)$ for the C^* -algebra of continuous functions from X to \mathbb{C} that vanish at infinity. The following definitions come from [21].

Definition 2.4. Let A be a C^* -algebra, let p and q be projections in A , and let $a, b \in A^+$, $x \in M_k(A)$ and $y \in M_n(A)$.

- (1) $a \leq_{C^*} b$ if $b - a \in A^+$.
- (2) $a <_{C^*} b$ if $a \leq_{C^*} b$ and $a \neq b$.
- (3) $p \sim_{C^*} q$ if there exists $t \in A$ such that $tt^* = p$ and $t^*t = q$.

- (4) $x \lesssim_{C^*} y$ if there exists a sequence $\{\alpha_i\} \subseteq M_{k,n}(A)$ such that $\alpha_i y \alpha_i^* \rightarrow x$ in norm.
- (5) p is an *infinite projection* if there exists a projection q such that $p \sim_{C^*} q <_{C^*} p$.
- (6) p is a *C^* -properly infinite projection* if there exist mutually orthogonal projections q_1 and q_2 such that $q_1 \leq p, q_2 \leq p$ and $q_1 \sim_{C^*} p \sim_{C^*} q_2$.
- (7) a is *C^* -infinite* if there exists a non-zero $t \in A^+$ such that $a \oplus t \lesssim_{C^*} a$.
- (8) a is *C^* -properly infinite* if $a \neq 0$ and $a \oplus a \lesssim_{C^*} a$.
- (9) A C^* -subalgebra $B \subseteq A$ is *hereditary* if $a \leq_{C^*} b$ and $b \in B$ implies $a \in B$.
- (10) A is *C^* -purely infinite simple* if A is simple (A has no non-trivial closed ideals) and every hereditary subalgebra contains an infinite projection.
- (11) A is *C^* -purely infinite* if
 - (a) there are no non-zero $*$ -homomorphisms from A to \mathbb{C} , and
 - (b) for every $a, b \in A^+$ we have $b \lesssim_{C^*} a$ if and only if $b \in \overline{AaA}$.

Remark 2.5. As in the algebraic situation there are relationships between these notions.

- (1) By [21, Lemma 3.1], a projection p is a C^* -infinite projection if and only if it is C^* -infinite. Similarly, p is a C^* -properly infinite projection if and only if p is C^* -properly infinite.
- (2) By Theorem 4.16 of [21], a C^* -algebra A is C^* -purely infinite if and only if every non-zero positive element of A is C^* -properly infinite. There is no algebraic analogue of [21, Theorem 4.16]; see [6, Examples 3.5] for a counter-example.
- (3) By [21, Proposition 5.4], a C^* -algebra A that is simple and C^* -purely infinite is C^* -purely infinite simple. By [21, Proposition 4.7], a C^* -algebra A being C^* -purely infinite simple implies that A is simple and A is C^* -purely infinite. Thus the two notions of C^* -purely infinite are compatible for simple C^* -algebras.

Lemma 2.6. *Let A be a C^* -algebra and A_0 a dense subalgebra. Let p be a projection in A_0 . If p is an infinite idempotent in A_0 , then p is an infinite projection in A .*

Proof. Since p is an infinite idempotent in A_0 , there exists an idempotent $e \in A_0$ such that $p \sim_{\text{ring}} e <_{\text{ring}} p$. Thus $eA \subsetneq pA$. By [33, Exercise 3.11(1)], there exists a projection $q \in A$ such that $e \sim_{\text{ring}} q$ in A and $eA = qA$. Since \sim_{ring} is transitive, we have $p \sim_{\text{ring}} q$ in A . Thus we have $p \sim_{C^*} q$ by, for example, [33, Exercise 3.11(2)]. Also, since $pA \neq eA = qA$, we have $q \neq p$. To see that $q \leq_{C^*} p$, we show that $pq = q$, which suffices by [24, Theorem 2.3.2]. Since $qA \subsetneq pA$, there exists $a \in A$ such that $q^2 = pa$, that is, $q = pa$. So $pq = p^2a = pa = q$. Thus p is an infinite projection. \square

2.3. Groupoids and groupoid algebras

Let G be a topological groupoid. We write $G^{(0)}$ for the unit space of G , and $r, s : G \rightarrow G^{(0)}$ for the range and source maps. An open subset B of G is an *open bisection* if the source and range maps restrict to homeomorphisms on B , and $r(B)$ and $s(B)$ are open subsets of $G^{(0)}$. We say that G is *ample* if G has a basis of compact open bisections. Equivalently, G is ample if G is étale and its unit space $G^{(0)}$ is totally disconnected (see [17, Proposition 4.1]). In this paper, we consider only ample Hausdorff groupoids.

Let G be an ample Hausdorff groupoid and let K be a field. The *Steinberg algebra* $A_K(G)$ is the K -algebra of functions from G to K that are both locally constant and compactly supported. Addition is defined pointwise. The multiplication is convolution: for $f, g \in A_K(G)$, we have

$$(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta).$$

When K has an involution $k \mapsto \bar{k}$, we define an involution $f \mapsto f^*$ on $A_K(G)$ by

$$(f^*)(\gamma) = \overline{f(\gamma^{-1})}.$$

We write $A(G)$ for $A_{\mathbb{C}}(G)$.

Let B be a compact open bisection, and write 1_B for the characteristic function from B to K . We will often use the fact that every f in $A_K(G)$ can be expressed as a sum

$$f = \sum_{B \in F} c_B 1_B,$$

where F is a finite set of disjoint compact open bisections and $0 \neq c_B \in K$ for each $B \in F$. For compact open bisections B and C , we have

$$1_B * 1_C = 1_{BC} \quad \text{and} \quad 1_B^* = 1_{B^{-1}}.$$

Indeed, the compact open bisections in an ample groupoid form an inverse semigroup under the operations

$$BC = \{\gamma\eta : r(\eta) = s(\gamma), \gamma \in B, \eta \in C\} \quad \text{and} \quad B^{-1} = \{\gamma^{-1} : \gamma \in B\},$$

and the multiplication in $A_K(G)$ agrees with these operations. Every Steinberg algebra is s -unital by [11, Lemma 2.6]. See [12, 36] for more information about Steinberg algebras.

The *reduced groupoid C^* -algebra* $C_r^*(G)$ was introduced in [30], and for ample groupoids the definition reduces as follows. With convolution and involution as above, the set $C_c(G)$ of continuous compactly supported functions from G to \mathbb{C} is a $*$ -algebra. Let $u \in G^{(0)}$. Set $G_u = \{\gamma \in G : s(\gamma) = u\}$. Write δ_γ for the basis of point masses in $\ell^2(G_u)$. There is a $*$ -representation $R_u : C_c(G) \rightarrow \mathcal{B}(\ell^2(G_u))$ such that, for $f \in C_c(G)$ and $\gamma \in G_u$,

$$R_u(f)\delta_\gamma = \sum_{\{\alpha \in G : s(\alpha) = r(\gamma)\}} f(\alpha)\delta_{\alpha\gamma}.$$

The reduced C^* -algebra $C_r^*(G)$ is the completion of the image of $C_c(G)$ under the direct sum $\bigoplus_{u \in G^{(0)}} R_u$. By construction, the complex Steinberg algebra $A(G)$ is a dense $*$ -subalgebra of $C_r^*(G)$.

We say that G is *effective* if, for every compact open bisection $B \subseteq G \setminus G^{(0)}$, there exists $\gamma \in B$ such that $s(\gamma) \neq r(\gamma)$. (See [8, Lemma 3.1] and [31, Corollary 3.3] for some equivalent characterizations of effective groupoids.) We say that G is *minimal* if $G^{(0)}$ has no non-trivial open invariant subsets. By [10, Theorem 4.1], $A_K(G)$ is algebraically simple if and only if G is minimal and effective. If G is second-countable, minimal and effective, then $C_r^*(G)$ is C^* -simple, and the converse holds if G is amenable (see, for example, [8, Theorem 5.1]).

Since G is ample, the unit space $G^{(0)}$ is open in G , and we may view $C_0(G^{(0)})$ as a subalgebra of $C_r^*(G)$. For $f, g \in C_0(G^{(0)})$, we have

$$f \leq_{C^*} g \text{ if and only if } f(x) \leq g(x), \quad \text{for all } x \in G^{(0)}.$$

3. Algebraically purely infinite simple Steinberg algebras

By combining [9, Theorem 4.1] and Lemma 2.6, we get the following interesting result.

Corollary 3.1. *Let G be a second-countable ample Hausdorff groupoid. Suppose that G is minimal and effective. If $A(G)$ is algebraically purely infinite, then $C_r^*(G)$ is C^* -purely infinite.*

Proof. Since G is minimal and effective, by [9, Theorem 4.1] it suffices to show that 1_U is an infinite projection for every compact open $U \subseteq G^{(0)}$. Since $A(G)$ is algebraically purely infinite simple, every idempotent is infinite [4, Proposition 1.5]. Thus every 1_U with $U \subseteq G^{(0)}$ is an infinite idempotent in $A(G)$ and hence the result follows from Lemma 2.6. □

We now prove the algebraic analogue of [9, Theorem 4.1].

Theorem 3.2. *Let K be a field and let G be an ample Hausdorff groupoid. Suppose that G is minimal and effective. Then $A_K(G)$ is algebraically purely infinite simple if and only if 1_U is an infinite idempotent for every compact open $U \subseteq G^{(0)}$.*

Proof. By [10, Theorem 4.1], $A_K(G)$ is algebraically simple if and only if G is minimal and effective.

First suppose that $A_K(G)$ is algebraically purely infinite simple. Then all non-zero idempotents are infinite by [4, Proposition 1.5]. In particular, 1_U is an infinite idempotent for every compact open $U \subseteq G^{(0)}$.

Conversely, assume 1_U is an infinite idempotent for every compact open $U \subseteq G^{(0)}$. Fix a non-zero right ideal $I \subseteq A_K(G)$. We need to show that I contains an infinite idempotent. Let $0 \neq b \in I$ and write

$$b = \sum_{B \in F} c_B 1_B,$$

where F is a finite set of disjoint compact open bisections and $0 \neq c_B \in K$ for each $B \in F$. Fix $B_0 \in F$. Then

$$c_{B_0}^{-1} b 1_{B_0^{-1}} = 1_{r(B_0)} + \sum_{B \in F, B \neq B_0} c_{B_0}^{-1} c_B 1_{BB_0^{-1}}.$$

Since F is a disjoint collection, for each $B \neq B_0$ we have $BB_0^{-1} \subseteq G \setminus G^{(0)}$. So

$$L := \bigcup_{B \in F, B \neq B_0} BB_0^{-1}$$

is a compact open subset of $G \setminus G^{(0)}$. Let $U := r(B_0)$. Then U is a compact open subset of $G^{(0)}$. Since G is effective, we apply [8, Lemma 3.1(4)] to get a non-empty open set $V \subseteq U$ such that $VLV = \emptyset$. Now

$$\begin{aligned} 1_V(c_{B_0}^{-1}b1_{B_0^{-1}})1_V &= 1_V\left(1_{r(B_0)} + \sum_{B \in F, B \neq B_0} c_{B_0}^{-1}c_B1_{BB_0^{-1}}\right)1_V \\ &= 1_V + \sum_{B \in F, B \neq B_0} c_{B_0}^{-1}c_B1_{VBB_0^{-1}V} \\ &= 1_V. \end{aligned}$$

Set $p := (c_{B_0}^{-1}b1_{B_0^{-1}})1_V$. Then $p \in I$ and

$$p^2 = (c_{B_0}^{-1}b1_{B_0^{-1}})(1_V(c_{B_0}^{-1}b1_{B_0^{-1}})1_V) = (c_{B_0}^{-1}b1_{B_0^{-1}})1_V = p.$$

Thus p is a non-zero idempotent in I . Further, $1_Vp = 1_V$ and $p1_V = p$, and hence $p \sim_{\text{ring}} 1_V$. □

Corollary 3.3. *Let K be a field, let G be an ample Hausdorff groupoid and let \mathcal{B} be a basis for $G^{(0)}$ consisting of compact open subsets. Suppose that G is minimal and effective. Then $A_K(G)$ is algebraically purely infinite simple if and only if 1_U is an algebraically infinite idempotent for every compact open $U \in \mathcal{B}$.*

Proof. The forward implication is immediate from Theorem 3.2. For the reverse implication, suppose that 1_U is an algebraically infinite idempotent for every compact open $U \in \mathcal{B}$. Fix a compact open subset $V \subseteq G^{(0)}$. By Theorem 3.2, it suffices to show that 1_V is an infinite idempotent. Since \mathcal{B} is a basis, there exists $W \in \mathcal{B}$ such that $W \subseteq V$. Then

$$1_W1_V = 1_W = 1_V1_W,$$

and so $1_W \leq_{\text{ring}} 1_V$. By assumption, 1_W is an algebraically infinite idempotent. Since $A_K(G)$ is s-unital, Lemma 2.3(2) applies, and 1_V is an algebraically infinite idempotent as well. □

A Hausdorff ample groupoid G is *locally contracting* [2, Definition 2.1] if, for every compact open $U \subseteq G^{(0)}$, there exists a non-empty compact open bisection $B \subseteq G$ such that

$$s(B) \subsetneq r(B) \subseteq U.$$

The following corollary to Theorem 3.2 is the algebraic analogue of [2, Proposition 2.4] which gives a useful sufficient condition for a C^* -algebra to be C^* -purely infinite simple.

Corollary 3.4. *Let K be a field and let G be an ample Hausdorff groupoid. Suppose that G is minimal, effective and locally contracting. Then $A_K(G)$ is algebraically purely infinite simple.*

Proof. Fix $U \subseteq G^{(0)}$. By Theorem 3.2, it suffices to show that 1_U is an infinite idempotent. Since G is locally contracting, there exists a compact open bisection $B \subseteq G$ such that $s(B) \subsetneq r(B) \subseteq U$. Then

$$1_{s(B)}1_{r(B)} = 1_{s(B)} = 1_{r(B)}1_{s(B)}$$

gives $1_{s(B)} \leq_{\text{ring}} 1_{r(B)}$, and $s(B) \subsetneq r(B)$ gives $1_{s(B)} <_{\text{ring}} 1_{r(B)}$. Further,

$$1_{B^{-1}}1_B = 1_{B^{-1}B} = 1_{s(B)} \quad \text{and} \quad 1_B1_{B^{-1}} = 1_{BB^{-1}} = 1_{r(B)}$$

and so $1_{r(B)} \sim_{\text{ring}} 1_{s(B)}$. Thus

$$1_{r(B)} \sim_{\text{ring}} 1_{s(B)} <_{\text{ring}} 1_{r(B)}.$$

Therefore $1_{r(B)}$ is an infinite idempotent. Since $1_{r(B)} \leq_{\text{ring}} 1_U$, 1_U is algebraically infinite by Lemma 2.3(2). By [6, Remarks 2.7], 1_U is an algebraically infinite idempotent as required. \square

4. A larger dense subalgebra

We want to connect the notion of C^* -purely infinite simplicity in the C^* -algebra with some kind of algebraic pure infiniteness. Corollary 3.1 is a start, but we seek an *if and only if* characterization and $A(G)$ seems too small. Even with the assumption that G is minimal and effective, if $C_r^*(G)$ does not have an identity, then it is not s-unital and it is not algebraically simple: for example, the Pedersen ideal (see [27, Theorem 5.6.1]) is always a non-trivial dense non-closed ideal.

We now introduce a large s-unital subalgebra of $C_r^*(G)$. Define

$$B(G) := \{1_U x 1_U : x \in C_r^*(G) \text{ and } U \subseteq G^{(0)} \text{ is a compact open set}\}.$$

It is straightforward to check that $A(G)$ is a subset of $B(G)$. Since the Pedersen ideal contains all the projections of $C_r^*(G)$ (see [27, 5.6.3]), it contains all elements of the form 1_U for compact open $U \subseteq G^{(0)}$. Thus $B(G)$ is contained in the Pedersen ideal.

We apply techniques from [6, Theorem 3.17] to see that when G is minimal and effective, $B(G)$ is algebraically properly infinite if and only if $C_r^*(G)$ is C^* -purely infinite.

Lemma 4.1. *Let G be an ample Hausdorff groupoid. Then $B(G)$ is a dense s-unital subalgebra of $C_r^*(G)$.*

Proof. That $B(G)$ is dense in $C_r^*(G)$ follows because $A(G) \subseteq B(G)$ and $A(G)$ is dense in $C_r^*(G)$ by [12, Proposition 4.2]. For any $1_U x 1_U \in B(G)$, we have $1_U(1_U x 1_U)1_U = 1_U x 1_U$, and so $B(G)$ is s-unital. To see that $B(G)$ is a subalgebra, fix $b_1 := 1_{U_1} x 1_{U_1}$

and $b_2 := 1_{U_2}x1_{U_2}$ in $B(G)$ and $r \in \mathbb{C}$. Let $U := U_1 \cup U_2$. Then U is compact and open because the U_i are. Then

$$b_1 - b_2 = 1_U(1_{U_1}x1_{U_1} - 1_{U_2}x1_{U_2})1_U \in B(G).$$

Similarly,

$$b_1b_2 = 1_U(1_{U_1}x1_{U_1}1_{U_2}x1_{U_2})1_U \in B(G) \quad \text{and} \quad rb_1 = r1_{U_1}x1_{U_1} = 1_{U_1}(rx)1_{U_1} \in B(G). \quad \square$$

Remark 4.2. If $G^{(0)}$ is compact, then $C_r^*(G)$ is unital with unit $1_{G^{(0)}}$. Thus, for any $x \in C_r^*(G)$, $x = 1_{G^{(0)}}x1_{G^{(0)}}$ and hence $B(G) = C_r^*(G)$.

Proposition 4.3. *Let G be a second-countable ample Hausdorff groupoid. Suppose that G is minimal and effective. If $B(G)$ is algebraically properly infinite, then $C_r^*(G)$ is C^* -purely infinite.*

Proof. We show that $C_r^*(G)$ is C^* -purely infinite by verifying the two conditions of Definition 2.4(11). First, we have to show that there are no non-zero C^* -homomorphisms from $C_r^*(G)$ to \mathbb{C} . Aiming for a contradiction, suppose that there is a non-zero C^* -homomorphism $\pi : C_r^*(G) \rightarrow \mathbb{C}$. Since G is minimal and effective, $C_r^*(G)$ is simple [30, Proposition II.4.6], so π is injective, and $C_r^*(G)$ is C^* -isomorphic to \mathbb{C} . In particular, $A(G)$ is isomorphic to \mathbb{C} . But $A(G)$ is algebraically purely infinite, and hence contains an infinite idempotent. But then so does \mathbb{C} , a contradiction. Hence, there are no non-zero homomorphisms from $C_r^*(G)$ to \mathbb{C} .

Second, we have to show that, for $x, y \in C_r^*(G)^+$, we have $y \lesssim_{C^*} x$ if and only if $y \in \overline{C_r^*(G)xC_r^*(G)}$. Since $C_r^*(G)$ is simple, $\overline{C_r^*(G)xC_r^*(G)} = C_r^*(G)$; so it suffices to show that $y \lesssim_{C^*} x$ for all $x, y \in C_r^*(G)^+$. Fix $x, y \in C_r^*(G)^+$. Since $B(G)$ is dense in $C_r^*(G)$, we have

$$\overline{B(G)xB(G)} = \overline{C_r^*(G)xC_r^*(G)} = C_r^*(G).$$

Fix $\epsilon > 0$. Since y is a positive element, so is \sqrt{y} . Since $\sqrt{y} \in \overline{B(G)xB(G)}$, there exists a sequence $b_n \in B(G)xB(G)$ such that $b_n \rightarrow \sqrt{y}$. Since $B(G)B(G)xB(G) \subseteq B(G)$ we have

$$b_n b_n^* \in B(G)xB(G)B(G)xB(G) \subseteq B(G)xB(G)$$

and $b_n b_n^* \rightarrow y$. Thus there exists a positive $b \in B(G)xB(G)$ such that $\|b - y\| < \epsilon$. Now [21, Lemma 2.5(ii)] gives $(y - \epsilon)_+ \lesssim_{C^*} b$. Since ϵ was fixed, [21, Proposition 2.6] gives $y \lesssim_{C^*} b$.

We claim that $b \lesssim_{C^*} x$. By the definition of $B(G)xB(G)$, there exist $c_i, d_i \in B(G)$ such that $b = \sum_{i=1}^k c_i x d_i$, and hence there exists a compact open subset W of $G^{(0)}$ such that $b = \sum_{i=1}^k 1_W c_i 1_W x 1_W d_i 1_W$. Thus $b \in B(G)1_W x 1_W B(G)$. Since $B(G)$ is s -unital (by Lemma 4.1) and is algebraically properly infinite (by Proposition 4.6), it follows from [6, Lemma 3.4(i)] that $B(G)$ is algebraically purely infinite. Thus by Definition 2.1(12b), $b \in B(G) \cap B(G)1_W x 1_W B(G)$ implies $b \lesssim_{\text{ring}} 1_W x 1_W$ in $B(G)$, that is, there exist $c, d \in B(G)$ such that

$$b = c(1_W x 1_W)d = (c1_W)x(1_W d).$$

Now applying [32, Proposition 2.4 (iii) \Rightarrow (ii)] to the positive elements b, x of $C_r^*(G)$ and the constant sequences $c1_W, 1_W d$ gives a sequence $\{r_n\} \subseteq C_r^*(G)$ such that $r_n x r_n^* \rightarrow b$. Thus $b \lesssim_{C^*} x$. Since \lesssim_{C^*} is transitive, we get $y \lesssim_{C^*} x$. \square

Lemma 4.4. *If 1_U is a C^* -properly infinite projection in $C_r^*(G)$, then 1_U is an algebraically properly infinite idempotent in $B(G)$.*

Proof. Let $p := 1_U$ be an algebraically properly infinite projection in $C_r^*(G)$. By the definition of algebraically properly infinite projection, there exist $q_1, q_2, x, y \in C_r^*(G)$ such that

$$x^*x = p, \quad xx^* = q_1 \leq_{C^*} p, \quad y^*y = p, \quad \text{and} \quad yy^* = q_2 \leq_{C^*} p - q_1.$$

Thus x and y are partial isometries in $C_r^*(G)$ with range projections q_1 and q_2 , respectively, and source projection p . Thus $xp = x$ and $q_1x = x$, and similarly for y . Take $x' := pxp$ and $y' := pyy$. Then $x', y' \in B(G)$. Now let z be the 1×2 matrix $[x' \ y']$. It is straightforward to check that

$$z^*z = p \oplus p \quad \text{and} \quad zz^* = q_1 + q_2 \leq_{C^*} p.$$

Thus p is an algebraically properly infinite idempotent in $B(G)$. □

Lemma 4.5. *Let G be a minimal and effective groupoid. Suppose that $U, V \subseteq G^{(0)}$ are non-empty compact open sets such that 1_U is an algebraically properly infinite idempotent in $B(G)$. Then $1_V \lesssim_{\text{ring}} 1_U$ in $B(G)$.*

Proof. Since G is minimal and effective, $A(G)$ is simple. Thus the ideal generated by 1_U in $A(G)$ is all of $A(G)$. Therefore, there exist $a_i, b_i \in A(G)$ such that

$$\begin{aligned} 1_V &= \sum_{i=1}^n a_i 1_U b_i \\ &\lesssim_{\text{ring}} \bigoplus_{i=1}^n a_i 1_U b_i \quad \text{by [6, Lemma 2.2(vi)] because } A(G) \text{ is s-unital} \\ &\lesssim_{\text{ring}} \bigoplus_{i=1}^n 1_U \quad \text{by [6, Lemma 2.2(ii)]} \\ &\lesssim_{\text{ring}} 1_U \text{ in } B(G) \end{aligned}$$

because 1_U is algebraically properly infinite in $B(G)$. Finally, since \lesssim_{ring} is transitive by [6, Lemma 2.2(i)], we have $1_V \lesssim_{\text{ring}} 1_U$ in $B(G)$. □

Proposition 4.6. *Let G be a second-countable ample Hausdorff groupoid. Suppose that G is minimal and effective, and that $C_r^*(G)$ is C^* -purely infinite. Then the algebra*

$$B(G) = \{1_U x 1_U : x \in C_r^*(G) \text{ and } U \subseteq G^{(0)} \text{ is a compact open set}\}$$

of Lemma 4.1 is algebraically properly infinite.

Proof. Since G is minimal and effective, $C_r^*(G)$ is C^* -purely infinite simple. Then [21, Theorem 4.16] says that every positive element of $C_r^*(G)$ is C^* -properly infinite. In particular, for every compact open $U \subseteq G^{(0)}$, 1_U is a C^* -properly infinite projection in

$C_r^*(G)$. So Lemma 4.4 says every 1_U is an algebraically properly infinite idempotent in $B(G)$.

Fix a non-zero $a \in B(G)$. We need to show that a is algebraically properly infinite, that is, that $a \oplus a \lesssim_{\text{ring}} a$. Since $a \in B(G)$, there exist a compact open $V \subseteq G^{(0)}$ and $x \in C_r^*(G)$ such that $a = 1_V x 1_V$.

We claim that there exists a compact open $U \subseteq G^{(0)}$ such that $1_U \lesssim_{\text{ring}} a^* a$ in $B(G)$. Since $a^* a$ is positive, by the proof of [9, Lemma 3.2] there exists a non-zero $h \in C_0(G^{(0)})^+$ such that $h \lesssim_{\text{ring}} a^* a$ in $C_r^*(G)$. (The statement of [9, Lemma 3.2] gives h such that $h \lesssim_{C^*} a^* a$, but a look at the proof shows $h \lesssim_{\text{ring}} a^* a$.) Since $h \in C_0(G^{(0)})^+ \setminus \{0\}$, there exists a compact open subset U of $G^{(0)}$ such that $h(u) > 0$ for all $u \in U$. Define g by

$$g(u) = \begin{cases} \frac{1}{\sqrt{h(u)}} & \text{if } u \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in C_0(G^{(0)})$, and $ghg = 1_U$. Thus $1_U \lesssim_{\text{ring}} h$ in $C_r^*(G)$. Now the transitivity of \lesssim_{ring} gives $1_U \lesssim_{\text{ring}} a^* a$ in $C_r^*(G)$. In particular, there exists $y, z \in C_r^*(G)$ such that $za^* ay = 1_U$.

Let $W := U \cup V$. Then $1_U = 1_W 1_U 1_W$ and $a^* a = 1_V x^* 1_V 1_V x 1_V = 1_W a^* a 1_W$, and

$$1_W z 1_W a^* a 1_W y 1_W = 1_W z a^* a y 1_W = 1_W 1_U 1_W = 1_U.$$

Thus $1_U \lesssim_{\text{ring}} a^* a$ in $B(G)$, as claimed.

We have $a^* a = a^* a 1_V$ and $a^* = 1_V x^* 1_V \in B(G)$, and hence

$$a^* a = a^* a 1_V = (1_V x^* 1_V) a (1_V).$$

Thus $a^* a \lesssim_{\text{ring}} a$ in $B(G)$. Now $1_U \lesssim_{\text{ring}} a$ in $B(G)$ by transitivity of \lesssim_{ring} .

Since $B(G)$ is s-unital and 1_U an algebraically properly infinite idempotent in $B(G)$, we apply [6, Lemma 3.9(ii)] to get that

$$a \oplus 1_U \lesssim_{\text{ring}} a$$

in $B(G)$. We have $a \lesssim_{\text{ring}} 1_V$ because $a = (1_V x 1_V) 1_V (1_V)$ and Lemma 4.5 implies $1_V \lesssim_{\text{ring}} 1_U$ in $B(G)$. The transitivity of \lesssim_{ring} implies $a \lesssim_{\text{ring}} 1_U$ in $B(G)$. We also have $a \lesssim_{\text{ring}} a$ because $B(G)$ is s-unital, and so

$$a \oplus a \lesssim_{\text{ring}} a \oplus 1_U$$

by [6, Lemma 2.2(ii)]. Finally, transitivity gives $a \oplus a \lesssim_{\text{ring}} a$ in $B(G)$. □

Combining Propositions 4.3 and 4.6, we get our main theorem.

Theorem 4.7. *Let G be a second-countable ample Hausdorff groupoid. Suppose that G is minimal and effective. Then $B(G)$ is algebraically properly infinite if and only if $C_r^*(G)$ is C^* -purely infinite simple.*

Theorem 4.7 together with Corollary 3.1 give the following corollary.

Corollary 4.8. *Let G be an ample Hausdorff groupoid. Suppose that G is effective and minimal. If $A(G)$ is algebraically purely infinite, then $B(G)$ is algebraically properly infinite.*

We conclude this section with some open questions.

Questions 4.9.

- (1) Is there a groupoid G such that $A(G)$ is algebraically purely infinite simple and $C_r^*(G)$ is not algebraically purely infinite?
- (2) Is there a groupoid G such that $C_r^*(G)$ is C^* -purely infinite simple and $A(G)$ is not algebraically purely infinite simple? (In other words, does the converse of Corollary 3.1 fail?)
- (3) Let U be a compact open subset of $G^{(0)}$. If 1_U is an infinite projection in $C_r^*(G)$, is it an infinite idempotent in $A(G)$? This would be a kind of converse to Lemma 2.6.

5. Kumjian–Pask algebras

In this section we show how our results apply to the subclass of algebras associated with higher-rank graphs. C^* -purely infinite simple C^* -algebras of higher-rank graphs are considered in [9, 14, 16, 25, 26, 34], but, in general, a condition on the graph that characterizes pure infiniteness has remained elusive.

Kumjian–Pask algebras were defined in [5] for higher-rank graphs without sources, and were then generalized to locally convex graphs in [13] and to finitely aligned graphs in [15]. The groupoids built from k -graphs are described, for example, in [18, 20, 22, 38]. To make our exposition as self-contained as possible we will include some of this construction below.

We view \mathbb{N}^k as a category with one object and composition given by addition. A *higher-rank graph of rank k* or *k -graph* is a countable category Λ with a functor $d : \Lambda \rightarrow \mathbb{N}^k$, called the *degree map*, that satisfies a *unique factorization property*: if $\lambda \in \Lambda$ with $d(\lambda) = m + n$, then there exist paths $\mu, \nu \in \Lambda$ such that $d(\mu) = m, d(\nu) = n$ and $\lambda = \mu\nu$. We call the domain and codomain of a morphism λ the source and range of λ , respectively, and write $s(\lambda)$ and $r(\lambda)$ for them. We call the objects and morphisms in Λ vertices and paths, respectively; we denote the set of vertices by Λ^0 and the set of paths by Λ . Thus $s, r : \Lambda \rightarrow \Lambda^0$. For $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we set $v\Lambda := r^{-1}(v)$, $\Lambda^n := d^{-1}(n)$ and $v\Lambda^n := v\Lambda \cap \Lambda^n$.

For $m = (m_1, \dots, m_k)$ and $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, we define

$$m \wedge n := (\min\{m_1, n_1\}, \dots, \min\{m_k, n_k\}),$$

$$m \vee n := (\max\{m_1, n_1\}, \dots, \max\{m_k, n_k\}).$$

Let $\lambda, \mu \in \Lambda$. If $d(\lambda) = m + n$, we write $\lambda(0, m)$ and $\lambda(m, m + n)$ for the unique paths of degree m and n such that $\lambda = \lambda(0, m)\lambda(m, m + n)$. Then $\tau \in \Lambda$ is a *minimal common*

extension of λ and μ if

$$d(\tau) = d(\lambda) \vee d(\mu), \quad \tau(0, d(\lambda)) = \lambda \quad \text{and} \quad \tau(0, d(\mu)) = \mu.$$

We define

$$\Lambda^{\min}(\lambda, \mu) := \{(\rho, \tau) \in \Lambda \times \Lambda : \lambda\rho = \mu\tau \text{ is a minimal common extension of } \lambda, \mu\}.$$

Then Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$. This class of k -graphs was introduced in [28] and is the most general type of k -graph considered in the literature.

A set $E \subseteq v\Lambda$ is *exhaustive* if, for every $\lambda \in v\Lambda$, there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. The set of finite exhaustive sets is then

$$\text{FE}(\Lambda) = \bigcup_{v \in \Lambda^0} \{E \subseteq v\Lambda \setminus \{v\} : E \text{ is finite and exhaustive}\}.$$

The unit space of the groupoid of a finitely aligned higher-rank graph consists of the set of boundary paths, defined as follows. First, for $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, let $\Omega_{k,m}$ be the category with objects $\{p \in \mathbb{N}^k : p \leq m\}$, morphisms

$$\{(p, q) : p, q \in \mathbb{N}^k, p \leq q \leq m\},$$

$r(p, q) = p$ and $s(p, q) = q$. With degree functor $d(p, q) = q - p$, $\Omega_{k,m}$ is a k -graph. A degree-preserving functor $x : \Omega_{k,m} \rightarrow \Lambda$ is a *boundary path* of Λ if, for all $n \in \mathbb{N}^k$ with $n \leq m$ and all $E \in x(n, n)\text{FE}(\Lambda)$, there exists $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$. We write $\partial\Lambda$ for the set of all boundary paths.

For $p \in \mathbb{N}^k$ and $\lambda \in \Lambda^p$, we define $\sigma^p(\lambda) = \lambda(p, d(\lambda))$. As a set, the groupoid of a finitely aligned higher-rank graph is

$$G_\Lambda := \{(x, m, y) \in \partial\Lambda \times \mathbb{Z}^k \times \partial\Lambda : \text{there exists } p, q \in \mathbb{N}^k \text{ such that } p \leq d(x), q \leq d(y), \\ p - q = m \text{ and } \sigma^p(x) = \sigma^q(y)\}.$$

The range of $(x, m, y) \in G_\Lambda$ is x and its source is y . Composition is given by

$$(x, m, y)(y, n, z) = (x, m + n, z),$$

and a computation shows that this is well defined. We identify $\partial\Lambda$ with $G_\Lambda^{(0)}$ by $x \mapsto (x, 0, x)$.

When endowed with the topology described in [38], G_Λ is a second-countable, Hausdorff and ample groupoid. For $\lambda \in \Lambda$ and finite subsets F of $s(\lambda)\Lambda$, define

$$Z(\lambda) := \{x \in \partial\Lambda : x(0, d(\lambda)) = \lambda\} \quad \text{and} \quad Z(\lambda \setminus F) := Z(\lambda) \setminus \left(\bigcup_{\nu \in F} Z(\lambda\nu) \right).$$

Then the unit space $G_\Lambda^{(0)}$ has a basis of compact open sets of the form $Z(\lambda \setminus F)$. For each $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, define

$$Z(\lambda, \mu) := \{(\lambda x, d(\lambda) - d(\mu), \mu x) : x \in \partial\Lambda \text{ and } s(\lambda) = s(\mu) = r(x)\}.$$

Then the basis for the topology on G_Λ consists of sets of the form

$$Z(\lambda, \mu \setminus F) := Z(\lambda, \mu) \setminus \left(\bigcup_{\nu \in F} Z(\lambda\nu, \mu\nu) \right),$$

where again F is a finite subset of $s(\lambda)\Lambda$. The subspace topology of $G_\Lambda^{(0)}$ is the one described above.

Given a field K , we define the Kumjian–Pask algebra and the C^* -algebra of Λ by

$$\text{KP}_K(\Lambda) := A_K(G_\Lambda) \quad \text{and} \quad C^*(\Lambda) := C_r^*(G_\Lambda),$$

respectively. These definitions are consistent with previous definitions in the literature via the isomorphism characterized by $s_\mu \mapsto 1_{Z(\mu, s(\mu))}$ (see [15, Proposition 5.4] for $\text{KP}_K(\Lambda)$ and [18, Theorem 6.9] for $C^*(\Lambda)$). To be consistent with the literature we write $s_v := 1_{Z(v, v)}$ and $s_\mu := 1_{Z(\mu, s(\mu))}$. We write $\text{KP}(\Lambda)$ for $\text{KP}_\mathbb{C}(\Lambda)$.

A k -graph Λ is *cofinal* if, for all $v \in \Lambda^0$ and $x \in \partial\Lambda$, there exists $n \leq d(x)$ such that $v\Lambda x(n)$ is non-empty. A k -graph is *aperiodic* if, for all $v \in \Lambda^0$, there exists $x \in v\partial\Lambda$ such that $\alpha x = \beta x$ implies $\alpha = \beta$. By [15, Proposition 7.1], G_Λ is minimal if and only if Λ is cofinal; and by [15, Proposition 6.3], G_Λ is effective if and only if Λ is aperiodic.

Corollary 5.1. *Let Λ be a finitely aligned higher-rank graph. Suppose that Λ is cofinal and aperiodic. Then $\text{KP}(\Lambda)$ is algebraically purely infinite simple if and only if s_v is an infinite idempotent for all $v \in \Lambda^0$.*

Proof. Suppose that $\text{KP}(\Lambda)$ is algebraically purely infinite simple. Fix $v \in \Lambda^0$. Then $Z(v)$ is a compact open subset of $\partial\Lambda = G_\Lambda^{(0)}$. Thus $s_v = 1_{Z(v)}$ is an infinite idempotent by Theorem 3.2.

Conversely, suppose that s_v is an infinite idempotent for all $v \in \Lambda^0$. We consider the basis $\mathcal{B} = \{Z(\lambda \setminus F) : \lambda \in \Lambda, \text{ finite } F \subseteq s(\lambda)\Lambda\}$ of compact open sets for $G_\Lambda^{(0)}$, and seek to apply Corollary 3.3. First, consider a non-empty $Z(\lambda)$. Then we have

$$1_{Z(\lambda)} = s_\lambda s_{\lambda^*} \sim_{\text{ring}} s_{\lambda^*} s_\lambda = s_{s(\lambda)}.$$

By assumption, $s_{s(\lambda)}$ is an infinite idempotent. Since $\text{KP}(\Lambda)$ is s -unital, $s_\lambda s_{\lambda^*}$ is an infinite idempotent by Lemma 2.3(3). It follows that $1_{Z(\lambda)}$ is an infinite idempotent in $\text{KP}(\Lambda)$.

Second, consider a non-empty $Z(\lambda \setminus F)$. Then there exists $\lambda' \in s(\lambda)\Lambda$ such that $Z(\lambda\lambda') \subseteq Z(\lambda \setminus F)$. Then

$$1_{Z(\lambda\lambda')} \leq_{\text{ring}} 1_{Z(\lambda \setminus F)},$$

and $1_{Z(\lambda\lambda')}$ is infinite by Lemma 2.3(2). Thus $\text{KP}(\Lambda)$ algebraically purely infinite simple by Corollary 3.3. □

Corollary 5.2. *Let Λ be a finitely aligned higher-rank graph. Suppose that Λ is cofinal and aperiodic.*

- (1) *If $\text{KP}(\Lambda)$ is algebraically purely infinite simple, then $C^*(\Lambda)$ is C^* -purely infinite simple.*

- (2) Suppose that s_v is an infinite idempotent in $KP(\Lambda)$ for all $v \in \Lambda^0$. Then each s_v is an infinite projection in $C^*(\Lambda)$.

Proof. (1) is immediate from Theorem 4.7.

(2) Since s_v is an infinite idempotent in $KP(\Lambda)$ for all $v \in \Lambda^0$, $KP(\Lambda)$ is algebraically purely infinite simple by Corollary 5.1. Since $KP(\Lambda)$ and $A(G_\Lambda)$ are isomorphic, Theorem 4.7 implies that $C^*(\Lambda)$ is C^* -purely infinite simple. Then every projection in $C^*(\Lambda)$ is an infinite projection by [9, Corollary 5.1]. □

We now connect our work with that of Larki in [23]. Recall from [16, Lemma 3.2] that a pair (μ, ν) is called a *generalized cycle* if $Z(\mu) \subseteq Z(\nu)$. This generalized cycle has an *entrance* if the containment is strict.

Let Λ be a finitely aligned, aperiodic and cofinal k -graph such that every vertex of Λ can be reached from a generalized cycle with an entrance. Then [23, Theorem 5.4] implies that $KP(\Lambda)$ is algebraically purely infinite simple. The next lemma shows that this also follows from Corollary 3.4.

Lemma 5.3. *Let Λ be a finitely aligned higher-rank graph. Suppose that every vertex of Λ can be reached from a generalized cycle with an entrance. Then G_Λ is locally contracting.*

Proof. Fix a non-empty compact open $U \subseteq G^{(0)}$. Then there exist $\lambda \in \Lambda$ and a finite subset F of $s(\lambda)\Lambda$ such that $\emptyset \neq Z(\lambda \setminus F) \subseteq U$. Further, since every vertex can be reached from a generalized cycle, $Z(\lambda \setminus F) \neq \{\lambda\}$. Thus there exists $\lambda' \in s(\lambda)\Lambda$ such that $d(\lambda') \neq 0$ and $Z(\lambda\lambda') \subseteq Z(\lambda \setminus F)$. By assumption, $s(\lambda')$ can be reached from a generalized cycle (μ, ν) with an entrance. Suppose $\lambda\lambda'\beta\mu \in \Lambda$. Then $(\lambda\lambda'\beta\mu, \lambda\lambda'\beta\nu)$ is a generalized cycle with an entrance. Then

$$B = Z(\lambda\lambda'\beta\mu, \lambda\lambda'\beta\nu)$$

is a compact open bisection with

$$s(B) = Z(\lambda\lambda'\beta\nu) \subsetneq Z(\lambda\lambda'\beta\mu) = r(B) \subseteq U.$$

Thus G_Λ is locally contracting. □

We now restrict to row-finite locally convex k -graphs. Let Λ be a k -graph. Then Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$; Λ is *locally convex* if, for every distinct $i, j \in \{1, \dots, k\}$, $v \in \Lambda^0$ and paths $\lambda \in v\Lambda^{e_i}, \mu \in v\Lambda^{e_j}$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty. Since our graphs might have sources, for $n \in \mathbb{N}$ we define

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \leq n\}.$$

When Λ is locally convex, by [37, Proposition 2.12] the boundary path space $\partial\Lambda$ equals

$$\Lambda^{\leq \infty} := \{x : \Omega_{k,m} \rightarrow \Lambda : x \text{ is a functor such that if } v \in \Omega_{k,m}^0 \text{ and } v\Omega_{k,m}^{\leq e_i} = \{v\}, \\ \text{then } x(v)\Lambda^{\leq e_i} = \{x(v)\}\}.$$

A path $\alpha \in \Lambda \setminus \Lambda^0$ is a *return path* if $r(\alpha) = s(\alpha)$; a return path α has an *entrance* if there exists a $\delta \in s(\alpha)\Lambda$ with $d(\delta) \leq d(\alpha)$ and $\alpha(0, d(\delta)) \neq \delta$. Our aim is to prove the following proposition.

Proposition 5.4. *Let Λ be a row-finite locally convex higher-rank graph. Suppose that Λ is cofinal and aperiodic, and that Λ contains a return path with an entrance. Then each vertex in Λ can be reached from a return path with an entrance.*

We then obtain the following corollary.

Corollary 5.5. *Let Λ be a row-finite locally convex higher-rank graph. Suppose that Λ is cofinal and aperiodic, and that Λ contains a return path with an entrance. Then $KP(\Lambda)$ and $C^*(\Lambda)$ are algebraically and C^* -purely infinite simple, respectively.*

Proof. Let μ be a return path with an entrance. Then $(\mu\mu, \mu)$ is a generalized cycle with an entrance. Then Proposition 5.4 implies that every vertex can be reached from a generalized cycle. By [23, Theorem 5.4], $KP(\Lambda)$ is algebraically purely infinite simple. Then $C^*(\Lambda)$ is C^* -purely infinite simple by Theorem 4.7. \square

For 1-graphs, Proposition 5.4 follows directly from cofinality: if α is a return path with an entrance, then $x = \alpha\alpha\cdots$ is a boundary path, and cofinality implies that, for every $v \in \Lambda^0$, there exists $\mu_v \in v\Lambda$ such that $s(\mu_v)$ is a vertex on x and hence on α . When we try to apply this idea to a general k -graph it fails: $x = \alpha\alpha\cdots$ may not be a boundary path. In fact, x is not a boundary path whenever $d(\alpha) \wedge (1, 1, \dots, 1) \neq (1, 1, \dots, 1)$. The keys to proving Proposition 5.4 are Lemma 5.6 and Corollary 5.7, where we construct a boundary path x that has enough return paths with entrances.

Lemma 5.6. *Let Λ be a row-finite and locally convex higher-rank graph. Suppose that Λ has a return path α with an entrance and that there exists $\mu \in s(\alpha)\Lambda \setminus \{s(\alpha)\}$ with $d(\mu) \wedge d(\alpha) = 0$. Then there exist $P \in \mathbb{N}$, a return path β with an entrance such that $d(\beta) = Pd(\alpha)$, and a path from $s(\beta)$ to $s(\alpha)$ of degree at least $d(\mu)$.*

Proof. Let α be a return path with an entrance δ . We may extend δ until $\delta \in s(\alpha)\Lambda^{\leq d(\alpha)}$. By assumption, there exists $\mu \in s(\alpha)\Lambda \setminus \{s(\alpha)\}$ with $d(\mu) \wedge d(\alpha) = 0$.

By local convexity applied to the pair (μ, δ) , there exists $\mu_0 \in s(\delta)\Lambda^{d(\mu)}$. If $s(\delta) = r(\delta)$ we can take $\mu_0 = \mu$. We will now use the unique factorization property to construct the path β .

Consider $\delta\mu_0$. Applying the unique factorization property to the path $\delta\mu_0$ gives $\mu_1 \in s(\alpha)\Lambda^{d(\mu)}$ and $\gamma_1 \in \Lambda^{\leq d(\alpha)}s(\mu_0)$ such that

$$\delta\mu_0 = \mu_1\gamma_1.$$

By the unique factorization property and induction, for $i \geq 2$ there exist $\mu_i \in s(\alpha)\Lambda^{d(\mu)}$ and $\gamma_i \in \Lambda^{d(\alpha)}s(\mu_{i-1})$ such that

$$\alpha\mu_{i-1} = \mu_i\gamma_i. \tag{5.1}$$

Notice that $\{\mu_i : i \geq 0\} \subseteq s(\alpha)\Lambda^{d(\mu)}$ is finite because Λ is row-finite. So there exists $i \neq j$ such that $\mu_i = \mu_j$, and in particular $s(\mu_i) = s(\mu_j)$. Let $m, n \in \mathbb{N}$ with $m < n$ be such that (m, n) is smallest with respect to the dictionary order and such that $\mu_m = \mu_n$. Then

$$\theta := \gamma_n\gamma_{n-1} \cdots \gamma_{m+1}$$

is a return path.

First, suppose that $m \neq 0$. By the minimality of (m, n) , $\gamma_m \neq \gamma_i$ for $m + 1 \leq i \leq n$, and hence γ_m is an entrance to θ . Notice that $\delta\mu_0\gamma_1 \dots \gamma_m$ is a path of degree more than $d(\mu)$ from $s(\theta)$ to $s(\alpha)$. Since $d(\gamma_i) = d(\alpha)$ for $i \geq 2$, we get that $\beta := \theta$ with $P = m - n$ satisfies the lemma.

Second, suppose that $m = 0$. Then

$$\theta = \gamma_n \cdots \gamma_1.$$

Here θ is a return path, but it may not have an entrance. We will construct another return path with an entrance as required. Since $\mu_n = \mu_0$, we have $s(\alpha) = r(\delta) = s(\delta)$. In particular, $d(\delta) = d(\alpha)$ (for otherwise we could extend $\delta \in \Lambda^{\leq d(\alpha)}$ further using α).

We claim that there exist $q \in \mathbb{N}$ and $1 \leq i \leq n$ such that $\gamma_{i+qn} \neq \gamma_i$. Aiming for a contradiction, we suppose that $\gamma_{i+qn} = \gamma_i$ for all $q \in \mathbb{N}$ and $1 \leq i \leq n$. We will show by induction on q that

$$\Gamma_q := \{\mu_1, \mu_{1+n}, \dots, \mu_{1+qn}\}$$

has $q + 1$ distinct elements. Then $\Gamma_q \subseteq s(\alpha)\Lambda^{d(\mu)}$ for every q , and this contradicts that $s(\alpha)\Lambda^{d(\mu)}$ is finite. When $q = 0$ we have $\Gamma_q = \{\mu_1\}$ and there is nothing to prove. For the inductive step, we will use the following two observations.

- (1) $\mu_1 \neq \mu_{1+tn}$ for any $t \in \mathbb{N}$ (for otherwise (5.1) gives $\alpha\mu_{tn} = \mu_{1+tn}\gamma_{1+tn} = \mu_1\gamma_1 = \delta\mu_0$, contradicting that $\alpha \neq \delta$).
- (2) Let $1 \leq i \leq n$. Then (5.1) gives

$$\mu_{i+1+tn}\gamma_{i+1} = \mu_{i+1+tn}\gamma_{i+1+tn} = \alpha\mu_{i+tn}.$$

So if $\{\mu_{i+tn} : 0 \leq t \leq q\}$ has $q + 1$ elements, then $\{\mu_{i+1+tn} : 0 \leq t \leq q\}$ has $q + 1$ elements.

Now suppose that $q \geq 0$ and that Γ_q has $q + 1$ elements. We apply observation (5) once to get that

$$\{\mu_2, \mu_{2+n}, \dots, \mu_{2+qn}\}$$

has $q + 1$ elements. After $n - 1$ further applications of observation (5), we get that

$$\{\mu_{1+q}, \mu_{1+2q}, \dots, \mu_{1+(q+1)n}\}$$

has $q + 1$ elements. Now we use observation (1) to see that

$$\{\mu_1, \mu_{1+q}, \mu_{1+2q}, \dots, \mu_{1+(q+1)n}\}$$

has $q + 2$ elements. This proves our claim.

Let (q_0, i_0) be smallest in the dictionary order such that $\gamma_{i_0+q_0n} \neq \gamma_{i_0}$. Since $\{\mu_i : i \geq i_0 + q_0n\}$ is finite, there exist smallest (i, j) such that $i_0 + q_0n \leq i < j$ and $s(\mu_i) = s(\mu_j)$. Then

$$\beta := \gamma_j \dots \gamma_{i+1}$$

is a return path of degree $(j - i)d(\alpha)$. If $i > i_0 + q_0n$, then γ_{i-1} is an entrance to β . If $i = i_0 + q_0n$, then γ_{i_0-1} is an entrance. In either case there is a path from $s(\beta)$ to $s(\alpha)$ of degree at least $d(\mu)$. □

Corollary 5.7. *Let Λ be a row-finite and locally convex k -graph. Suppose that Λ contains a return path with an entrance. Then there exists $x \in \Lambda^{\leq \infty}$ such that, for all $m \leq d(x)$, there exist $n \geq m$ and a $p \in \mathbb{N}^k$ such that $x(n, n + p)$ is a return path with an entrance.*

Proof. Let α be a return path with an entrance. We construct a boundary path by induction. If $\alpha\alpha \cdots \in \Lambda^{\leq \infty}$ we are done. So suppose that $\alpha\alpha \cdots \notin \Lambda^{\leq \infty}$. Let

$$t = (1, \dots, 1) - (1, \dots, 1) \wedge d(\alpha).$$

Since $\alpha\alpha \cdots \notin \Lambda^{\leq \infty}$, there exists $\mu_1 \in s(\alpha)\Lambda^{\leq t}$. Using Lemma 5.6, we find $P_1 \in \mathbb{N}$ and a return path β_1 with an entrance such that, without loss of generality, $\beta_1 \in s(\mu_1)\Lambda^{P_1 d(\alpha)}$.

Let $i \geq 1$, and suppose that we have $\mu_i \in \Lambda^{\leq t}$, P_i and $\beta_i \in s(\mu_i)\Lambda^{P_i d(\alpha)}s(\mu_i)$ such that β_i has an entrance. If $\beta_i\beta_i \cdots \in \Lambda^{\leq \infty}$, then take $x = \beta_i\beta_i \cdots$. Otherwise there exists $\mu_{i+1} \in s(\beta_i)\Lambda^{\leq t}$. Using Lemma 5.6 applied to β_i , there exist $P'_{i+1} \in \mathbb{N}$ and a return path $\beta_{i+1} \in s(\mu_{i+1})\Lambda^{P'_{i+1} P_i d(\alpha)}$ with an entrance. Take $P_{i+1} = P'_{i+1} P_i$.

If this process terminates at β_{i_0} , take $x = \beta_{i_0}\beta_{i_0} \cdots$. If this process never terminates, take

$$x = \alpha\mu_1\beta_1\mu_2\beta_2 \cdots .$$

In either case, $x \in \Lambda^{\leq \infty}$ has the desired properties. □

Remark 5.8. Corollary 5.7 fills a gap in the proof of [16, Corollary 5.7] and in the last statement of [34, Proposition 8.8]. The last statement of [34, Proposition 8.8] claims that if Λ is cofinal and contains a return path with an entrance, then every $v \in \Lambda^0$ can be reached from a return path with an entrance. Our Corollary 5.7 and cofinality ensure that this is indeed the case. For [16, Corollary 5.7], Evans and Sims construct a single return path with entrance and then apply [34, Proposition 8.8].

We are now ready to prove Proposition 5.4.

Proof of Proposition 5.4. Since Λ contains a return path with an entrance, Corollary 5.7 gives $x \in \Lambda^{\leq \infty}$ such that, for all $m \leq d(x)$, there exist $n \geq m$ and p such that $x(n, n + p)$ is a return path with an entrance. Let $v \in \Lambda^0$. Since Λ is cofinal there exists $m \leq d(x)$ such that $v\Lambda x(m)$ is non-empty. Pick $\omega_v \in v\Lambda x(m)$. Now pick $n \geq m$ and p such that $x(n, n + p)$ is a return path with an entrance. Then $\mu_v = \omega_v x(m, n)$ connects v to a return path with an entrance as desired. □

Acknowledgements. Part of this research was done while the first named author was visiting the other two authors: he would like to thank them for their hospitality during this visit. This research was supported by the Marsden grant 18-VUW-056 from the Royal Society of New Zealand and by the Seed grant IGRS03 from the University of Dayton Research Institute. We thank an anonymous referee for helpful suggestions, in particular for showing us Lemma 2.6.

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