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PRESENTING INFINITESIMAL q-SCHUR ALGEBRAS

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Abstract. Let $\mathcal{K}$ be a commutative ring containing a primitive l'th root ε of 1. The infinitesimal q-Schur algebras $\mathcal{S}_{\mathcal{K}}(n,d)_r$ over $\mathcal{K}$ form an ascending chain of subalgebras of the q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)$, which are useful in studying representations of the Frobenius kernel of the associated quantum linear group. Let $U_{\mathcal{K}}(\mathfrak{gl}_n)$ be the quantized enveloping algebra of $\mathfrak{gl}_n$ over $\mathcal{K}$. There is a natural surjective algebra homomorphism ζ_d from $U_{\mathcal{K}}(\mathfrak{gl}_n)$ to $\mathcal{S}_{\mathcal{K}}(n,d)$. The map ζ_d restricts to a surjective algebra homomorphism $\zeta_{d,r}$ from $U_{\mathcal{K}}(G_rT)$ to $\mathcal{S}_{\mathcal{K}}(n,d)_r$, where $U_{\mathcal{K}}(G_rT)$ is a certain Hopf subalgebra of $U_{\mathcal{K}}(\mathfrak{gl}_n)$, which is closely related to Frobenius–Lusztig kernels of $U_{\mathcal{K}}(\mathfrak{gl}_n)$. We give the extra defining relations needed to define the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$ as a quotient of $U_{\mathcal{K}}(G_rT)$. The map $\zeta_{d,r}$ induces a surjective algebra homomorphism $\dot{\zeta}_{d,r}:\dot{U}_{\mathcal{K}}(G_rT)\to\mathcal{S}_{\mathcal{K}}(n,d)_r$, where $\dot{U}_{\mathcal{K}}(G_rT)$ is the modified quantum algebra associated with $U_{\mathcal{K}}(G_rT)$. We also give a generating set for the kernel of $\dot{\zeta}_{d,r}$. These results can be used to give a classification of irreducible $\mathcal{S}_{\mathcal{K}}(n,d)_r$-modules over a field of characteristic p.

§1. Introduction

Let $\mathbf{U}(\mathfrak{gl}_n)$ be the quantized enveloping algebra of $\mathfrak{gl}_n$ over $\mathbb{Q}(v)$ (v an indeterminate) with Chevalley type generators E_i, F_i, and $K_j^{\pm1}$ for $1\leqslant i\leqslant n-1$ and $1\leqslant j\leqslant n$. Beilinson, Lusztig, and MacPherson (BLM) [3] constructed a realization for the quantum group $\mathbf{U}(\mathfrak{gl}_n)$ via a geometric setting of q-Schur algebras. A presentation of the q-Schur algebra $\boldsymbol{\mathcal{S}}(n,d)$ was given by Doty–Giaquinto [8]. Du–Parshall [15] provided an approach to the $\mathfrak{sl}_n$ type presentation of the q-Schur algebra $\boldsymbol{\mathcal{S}}(n,d)$ using the Beilinson–Lusztig–MacPherson's construction of $\mathbf{U}(\mathfrak{gl}_n)$. The problem of describing the defining relations of a generalized q-Schur algebra as a quotient of a quantized enveloping algebra was investigated by Doty [7], Doty–Giaquinto–Sullivan [9], [10].

Infinitesimal Schur algebras are certain important subalgebras of Schur algebras (cf. [11]). The polynomial representations of the group scheme G_rT of degree d are equivalent to the representation theory of the infinitesimal Schur algebras $\bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$. Here, G_r is the r-th Frobenius kernel of the general linear group G over $\mathcal{K}$, and T is the subscheme of G arising from diagonal elements. A theory of the infinitesimal q-Schur algebra was studied by Cox [4], [5].

Let $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$ and $\mathcal{K}$ be a commutative ring of characteristic p. Let $\varepsilon\in\mathcal{K}$ be a primitive l'th root of 1. We will regard $\mathcal{K}$ as a $\mathcal{Z}$-module by specializing v to ε. Let $U_{\mathcal{K}}(\mathfrak{gl}_n)=U_{\mathcal{Z}}(\mathfrak{gl}_n)\otimes_{\mathcal{Z}}\mathcal{K}$, where $U_{\mathcal{Z}}(\mathfrak{gl}_n)$ is the $\mathcal{Z}$-subalgebra of $\mathbf{U}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm1}$, and $\left[{K_j;0 \atop t}\right]$ for $1\leqslant i\leqslant n-1$, $1\leqslant j\leqslant n$ and $m,t\in\mathbb{N}$. For $r\geqslant1$, let $U_{\mathcal{K}}(G_r)$ be the $\mathcal{K}$-subalgebra of $U_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm1}$, and $\left[{K_j;0 \atop t}\right]$ for

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$1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, $t \in \mathbb{N}$ and $0 \leqslant m,t < lp^{r-1}$, where $l = l'$ if l' is odd, and $l = l'/2$ otherwise. Furthermore, let $U_{\mathcal{K}}(G_rT) = U_{\mathcal{K}}(G_r)U^0_{\mathcal{K}}(\mathfrak{gl}_n)$, where $U^0_{\mathcal{K}}(\mathfrak{gl}_n)$ is the zero part of $U_{\mathcal{K}}(\mathfrak{gl}_n)$. Then, we have

$$U_{\mathcal{K}}(G_1T) \subseteq \cdots \subseteq U_{\mathcal{K}}(G_rT) \subseteq U_{\mathcal{K}}(G_{r+1}T) \subseteq \cdots \subseteq U_{\mathcal{K}}(\mathfrak{gl}_n),$$

and $U_{\mathcal{K}}(\mathfrak{gl}_n) = \lim_r U_{\mathcal{K}}(G_rT)$. In the case where $l' = l$ is an odd number, let

$$\widetilde{U}_{\mathcal{K}}(G_r) = U_{\mathcal{K}}(G_r)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n\rangle,\ \widetilde{U}_{\mathcal{K}}(G_rT) = U_{\mathcal{K}}(G_rT)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n\rangle.$$

The algebra $\widetilde{U}_{\mathcal{K}}(G_1)$ is the Lusztig's small quantum group, and $\widetilde{U}_{\mathcal{K}}(G_r)$ is called Frobenius–Lusztig kernels of $U_{\mathcal{K}}(\mathfrak{gl}_n)$ (cf. [12], [22]). The representation theory of $\widetilde{U}_{\mathcal{K}}(G_r)$ and $\widetilde{U}_{\mathcal{K}}(G_rT)$ was studied in [12].

Jimbo [20] proved that there is a natural surjective algebra homomorphism ζ_d from $\mathbf{U}(\mathfrak{gl}_n)$ to the q-Schur algebra $\boldsymbol{\mathcal{S}}(n,d)$. The map $\zeta_d : \mathbf{U}(\mathfrak{gl}_n) \to \boldsymbol{\mathcal{S}}(n,d)$ induces a surjective algebra homomorphism

$$\zeta_{d,r} : U_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r,$$

where $\mathcal{S}_{\mathcal{K}}(n,d)_r$ is the infinitesimal q-Schur algebra over $\mathcal{K}$ (cf. [18, Prop. 6.1]). Note that $\mathcal{S}_{\mathcal{K}}(n,d)_r$ is a quotient algebra of $\widetilde{U}_{\mathcal{K}}(G_rT)$ in the case where $l' = l$ is odd. We prove in Theorem 4.10 that $\ker \zeta_{d,r}$ is generated by the elements $1 - \sum_{\mu \in \Lambda(n,d)} K_\mu$, $K_iK_\lambda - \varepsilon^{\lambda_i}K_\lambda$, $\left[{K_i;0 \atop t}\right]K_\lambda - \left[{\lambda_i \atop t}\right]_\varepsilon K_\lambda$ for $1 \leqslant i \leqslant n$, $t \in \mathbb{N}$ and $\lambda \in \Lambda(n,d)$, where $\Lambda(n,d)$ is the set of all compositions of d into n parts.

Let $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ be the modified quantum group with generators E_i1_λ, $1_\lambda F_i$, and 1_λ for $1 \leqslant i \leqslant n-1$ and $\lambda \in \mathbb{Z}^n$. The map $\zeta_d : \mathbf{U}(\mathfrak{gl}_n) \to \boldsymbol{\mathcal{S}}(n,d)$ induces a surjective algebra homomorphism

$$\dot{\zeta}_d : \dot{\mathbf{U}}(\mathfrak{gl}_n) \to \boldsymbol{\mathcal{S}}(n,d).$$

Let $\dot{U}_{\mathcal{K}}(\mathfrak{gl}_n) = \dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n) \otimes_{\mathcal{Z}} \mathcal{K}$, where $\dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n)$ is the $\mathcal{Z}$-subalgebra of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}1_\lambda$, $1_\lambda F_i^{(m)}$ for $1 \leqslant i \leqslant n-1$, $m \in \mathbb{N}$ and $\lambda \in \mathbb{Z}^n$. Let $\dot{U}_{\mathcal{K}}(G_rT)$ be the $\mathcal{K}$-subalgebra of $\dot{U}_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}1_\lambda$ and $1_\lambda F_i^{(m)}$ for $1 \leqslant i \leqslant n-1$, $\lambda \in \mathbb{Z}^n$ and $0 \leqslant m < lp^{r-1}$. The map $\dot{\zeta}_d : \dot{\mathbf{U}}(\mathfrak{gl}_n) \to \boldsymbol{\mathcal{S}}(n,d)$ induces a surjective algebra homomorphism

$$\dot{\zeta}_{d,r} : \dot{U}_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r.$$

We prove in Theorem 5.5 that $\ker \dot{\zeta}_{d,r}$ is generated by the elements 1_λ for $\lambda \notin \Lambda(n,d)$.

The organization of the paper is as follows. We recall the BLM construction of the quantum group $\mathbf{U}(\mathfrak{gl}_n)$ in Section 2. In Section 3, we introduce the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$. A generating set for the kernel of the epimorphism $\zeta_{d,r} : U_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r$ is obtained in Section 4. In Section 5, we investigate the kernel of the epimorphism $\dot{\zeta}_{d,r} : \dot{U}_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r$. In Section 6, we discuss the classical case. In Section 7, we investigate Borel subalgebras of the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$. As an application, we give a classification of irreducible $\mathcal{S}_{\mathcal{K}}(n,d)_r$-modules over a field of characteristic p in Section 8.

Throughout this paper, let $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$ where v is an indeterminate. For $i\in\mathbb{Z}$ let $[i]=\frac{v^i-v^{-i}}{v-v^{-1}}$. For integers N,t with $t\geqslant 0$, let

$$\begin{bmatrix} N \\ t \end{bmatrix}=\frac{[N][N-1]\cdots[N-t+1]}{[t]^!}\in\mathcal{Z},$$

where $[t]^!=[1][2]\cdots[t]$.

Let $\mathcal{K}$ be a commutative ring containing a primitive l'th root ε of 1 with $l'\geqslant 1$. Let $l\geqslant 1$ be defined by

$$l=\begin{cases} l' & \text{if } l' \text{ is odd,}\\ l'/2 & \text{if } l' \text{ is even.}\end{cases}$$

Let p be the characteristic of $\mathcal{K}$. The commutative ring $\mathcal{K}$ will be viewed as a $\mathcal{Z}$-module by specializing v to ε. For $c\in\mathbb{Z}$ and $t\in\mathbb{N}$, we will denote the image of $\left[\begin{smallmatrix} c \\ t\end{smallmatrix}\right]\in\mathcal{Z}$ in $\mathcal{K}$ by $\left[\begin{smallmatrix} c \\ t\end{smallmatrix}\right]_\varepsilon$. For $\mu\in\mathbb{Z}^n$ and $\lambda\in\mathbb{N}^n$ let $\left[\begin{smallmatrix} \mu \\ \lambda\end{smallmatrix}\right]_\varepsilon=\left[\begin{smallmatrix} \mu_1 \\ \lambda_1\end{smallmatrix}\right]_\varepsilon\cdots\left[\begin{smallmatrix} \mu_n \\ \lambda_n\end{smallmatrix}\right]_\varepsilon$.

§2. The BLM construction of $\mathbf{U}(\mathfrak{gl}_n)$

Following [20], we define the quantized enveloping algebra $\mathbf{U}(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$ to be the $\mathbb{Q}(v)$ algebra with generators

$$E_i,\ F_i\quad (1\leqslant i\leqslant n-1),\ K_j,\ K_j^{-1}\quad (1\leqslant j\leqslant n),$$

and relations

(a) $K_iK_j=K_jK_i,\ K_iK_i^{-1}=1$;
(b) $K_iE_j=v^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i$;
(c) $K_iF_j=v^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i$;
(d) $E_iE_j=E_jE_i,\ F_iF_j=F_jF_i$ *when* $|i-j|>1$;
(e) $E_iF_j-F_jE_i=\delta_{i,j}\frac{\widetilde{K}_i-\widetilde{K}_i^{-1}}{v-v^{-1}}$, *where* $\widetilde{K}_i=K_iK_{i+1}^{-1}$;
(f) $E_i^2E_j-(v+v^{-1})E_iE_jE_i+E_jE_i^2=0$ *when* $|i-j|=1$;
(g) $F_i^2F_j-(v+v^{-1})F_iF_jF_i+F_jF_i^2=0$ *when* $|i-j|=1$.

Following [22], let $U_{\mathcal{Z}}(\mathfrak{gl}_n)$ be the Lusztig integral form of $\mathbf{U}(\mathfrak{gl}_n)$ generated by $E_i^{(m)},F_i^{(m)}$, $K_j^{\pm1}$, and $\left[\begin{smallmatrix} K_j;c \\ t\end{smallmatrix}\right](1\leqslant i\leqslant n-1,1\leqslant j\leqslant n,m,t\in\mathbb{N},c\in\mathbb{Z})$, where

$$E_i^{(m)}=\frac{E_i^m}{[m]^!},\quad F_i^{(m)}=\frac{F_i^m}{[m]^!}\quad\text{and}\quad \begin{bmatrix} K_j;c \\ t\end{bmatrix}=\prod_{s=1}^{t}\frac{K_jv^{c-s+1}-K_j^{-1}v^{-c+s-1}}{v^s-v^{-s}},$$

with $[m]^!=[1][2]\cdots[m]$ and $[i]=\frac{v^i-v^{-i}}{v-v^{-1}}$. The following result is given by Lusztig [21].

Lemma 2.1. *The following formulas hold in* $U_{\mathcal{Z}}(\mathfrak{gl}_n)$:

(1) $E_i^{(m)}\left[\begin{smallmatrix} K_j;c \\ t\end{smallmatrix}\right]=\left[\begin{smallmatrix} K_j;c+m(-\delta_{i,j}+\delta_{i+1,j}) \\ t\end{smallmatrix}\right]E_i^{(m)}$;
(2) $F_i^{(m)}\left[\begin{smallmatrix} K_j;c \\ t\end{smallmatrix}\right]=\left[\begin{smallmatrix} K_j;c-m(-\delta_{i,j}+\delta_{i+1,j}) \\ t\end{smallmatrix}\right]F_i^{(m)}$;
(3) *For* $k,l\in\mathbb{N}$, *we have*

$$E_i^{(k)}F_i^{(l)}=\sum_{\substack{0\leqslant t\leqslant k\\ t\leqslant l}}F_i^{(l-t)}\begin{bmatrix} \widetilde{K}_i;2t-k-l \\ t\end{bmatrix}E_i^{(k-t)},$$

where $\left[\begin{smallmatrix} \widetilde{K}_i;c \\ t\end{smallmatrix}\right]=\prod_{s=1}^{t}\frac{\widetilde{K}_iv^{c-s+1}-\widetilde{K}_i^{-1}v^{-c+s-1}}{v^s-v^{-s}}$.

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Let $\Pi(n)=\{\alpha_i \mid 1\leqslant i\leqslant n-1\}$, where $\alpha_i=\boldsymbol{e}_i-\boldsymbol{e}_{i+1}$ with $\boldsymbol{e}_i=(0,\ldots,0,\underset{i}{1},0\cdots,0)\in\mathbb{Z}^n$. We have the following direct sum decomposition:

$$\mathbf{U}(\mathfrak{gl}_n)=\bigoplus_{\nu\in\mathbb{Z}\Pi(n)}\mathbf{U}(\mathfrak{gl}_n)_\nu,$$

where $\mathbf{U}(\mathfrak{gl}_n)_\nu$ is defined by the conditions $\mathbf{U}(\mathfrak{gl}_n)_{\nu'}\mathbf{U}(\mathfrak{gl}_n)_{\nu''}\subseteq\mathbf{U}(\mathfrak{gl}_n)_{\nu'+\nu''}$, $K_j^{\pm1}\in\mathbf{U}(\mathfrak{gl}_n)_0$, $E_i\in\mathbf{U}(\mathfrak{gl}_n)_{\alpha_i}$, $F_i\in\mathbf{U}(\mathfrak{gl}_n)_{-\alpha_i}$ for all $\nu',\nu''\in\mathbb{Z}\Pi(n)$, $1\leqslant i\leqslant n-1$ and $1\leqslant j\leqslant n$.

Following [23, 23.1], we introduce the modified quantum group $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ associated with $\mathbf{U}(\mathfrak{gl}_n)$ as follows. Let

$$\dot{\mathbf{U}}(\mathfrak{gl}_n)=\bigoplus_{\lambda,\mu\in\mathbb{Z}^n}{}_\lambda\mathbf{U}(\mathfrak{gl}_n)_\mu,$$

where

$${}_\lambda\mathbf{U}(\mathfrak{gl}_n)_\mu=\mathbf{U}(\mathfrak{gl}_n)/\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}(K^{\mathbf{j}}-v^{\lambda\cdot\mathbf{j}})\mathbf{U}(\mathfrak{gl}_n)+\sum_{\mathbf{j}\in\mathbb{Z}^n}\mathbf{U}(\mathfrak{gl}_n)(K^{\mathbf{j}}-v^{\mu\cdot\mathbf{j}})\right),$$

and $\lambda\cdot\mathbf{j}=\sum_{1\leqslant i\leqslant n}\lambda_i j_i$. Let $\pi_{\lambda,\mu}:\mathbf{U}(\mathfrak{gl}_n)\to{}_\lambda\mathbf{U}(\mathfrak{gl}_n)_\mu$ be the canonical projection.

We define the product in $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ as follows. For $\lambda',\mu',\lambda'',\mu''\in\mathbb{Z}^n$ with $\lambda'-\mu'$, $\lambda''-\mu''\in\mathbb{Z}\Pi(n)$ and any $t\in\mathbf{U}(\mathfrak{gl}_n)_{\lambda'-\mu'}$, $s\in\mathbf{U}(\mathfrak{gl}_n)_{\lambda''-\mu''}$, define

$$\pi_{\lambda',\mu'}(t)\pi_{\lambda'',\mu''}(s)=\begin{cases}\pi_{\lambda',\mu''}(ts), & \text{if } \mu'=\lambda'',\\ 0 & \text{otherwise.}\end{cases}$$

Then $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ becomes an associative $\mathbb{Q}(v)$-algebra with the above product. Moreover, the algebra $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ is naturally a $\mathbf{U}(\mathfrak{gl}_n)$-bimodule defined by $t'\pi_{\lambda',\lambda''}(s)t''=\pi_{\lambda'+\nu',\lambda''-\nu''}(t'st'')$ for $t'\in\mathbf{U}(\mathfrak{gl}_n)_{\nu'},s\in\mathbf{U}(\mathfrak{gl}_n),t''\in\mathbf{U}(\mathfrak{gl}_n)_{\nu''}$, and $\lambda',\lambda''\in\mathbb{Z}^n$. Let $\dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n)$ be the $\mathcal{Z}$-subalgebra of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}1_\lambda$ and $1_\lambda F_i^{(m)}$ for $1\leqslant i\leqslant n-1$ and $m\in\mathbb{N}$, where $1_\lambda=\pi_{\lambda,\lambda}(1)$.

We now follow [6] to recall the definition of q-Schur algebras as follows. The Hecke algebra $\mathcal{H}_{\mathcal{Z}}(d)$ associated with $\mathfrak{S}_d$ is the $\mathcal{Z}$-algebra generated by T_i $(1\leqslant i\leqslant d-1)$, with the following relations:

$$(T_i+1)(T_i-q)=0,\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\ T_iT_j=T_jT_i\ (|i-j|>1).$$

where $q=v^2$. Let $\boldsymbol{\mathcal{H}}(d)=\mathcal{H}_{\mathcal{Z}}(d)\otimes_{\mathcal{Z}}\mathbb{Q}(v)$. If $w=s_{i_1}s_{i_2}\cdots s_{i_m}$ is reduced let $T_w=T_{i_1}T_{i_2}\cdots T_{i_m}$. Then the set $\{T_w\mid w\in\mathfrak{S}_d\}$ forms a $\mathcal{Z}$-basis for $\mathcal{H}_{\mathcal{Z}}(d)$. Let $\Lambda(n,d)=\{\lambda\in\mathbb{N}^n\mid\sigma(\lambda)=d\}$, where $\sigma(\lambda)=\sum_{1\leqslant i\leqslant n}\lambda_i$. For $\lambda\in\Lambda(n,d)$, let $x_\lambda=\sum_{w\in\mathfrak{S}_\lambda}T_w$, where $\mathfrak{S}_\lambda$ is the Young subgroup of $\mathfrak{S}_d$. The endomorphism algebras

$$\mathcal{S}_{\mathcal{Z}}(n,d):=\mathrm{End}_{\mathcal{H}_{\mathcal{Z}}(d)}\left(\bigoplus_{\lambda\in\Lambda(n,d)}x_\lambda\mathcal{H}_{\mathcal{Z}}(d)\right),\quad \boldsymbol{\mathcal{S}}(n,d):=\mathrm{End}_{\boldsymbol{\mathcal{H}}(d)}\left(\bigoplus_{\lambda\in\Lambda(n,d)}x_\lambda\boldsymbol{\mathcal{H}}(d)\right),$$

are called q-Schur algebras over $\mathcal{Z}$ and over $\mathbb{Q}(v)$, respectively.

We now recall the BLM construction of $\mathbf{U}(\mathfrak{gl}_n)$. Let $\widetilde{\Theta}(n)$ be the set of all $n\times n$ matrices over $\mathbb{Z}$ with all off diagonal entries in $\mathbb{N}$. Let $\Theta(n)$ be the set of all $n\times n$ matrices over $\mathbb{N}$. Let $\Theta(n,d)$ be the set of all $n\times n$ matrices A over $\mathbb{N}$ such that $\sigma(A)=d$, where $\sigma(A)=\sum_{1\leqslant i,j\leqslant n} a_{i,j}$. For $A\in\widetilde{\Theta}(n)$, let $\mathrm{ro}(A)=(\sum_j a_{1,j},\ldots,\sum_j a_{n,j})$ and $\mathrm{co}(A)=(\sum_i a_{i,1},\ldots,\sum_i a_{i,n})$.

The q-Schur algebra $\mathcal{S}_{\mathcal{Z}}(n,d)$ was reconstructed using the geometry of pairs of n-step filtrations on a d-dimensional vector space in [3]. In particular, a normalized $\mathcal{Z}$-basis $\{[A]\}_{A\in\Theta(n,d)}$ for $\mathcal{S}_{\mathcal{Z}}(n,d)$ was constructed. Using the stabilization property of multiplication for q-Schur algebra, an important $\mathcal{Z}$-algebra $K_{\mathcal{Z}}(n)$ (without 1), with basis $\{[A]\}_{A\in\widetilde{\Theta}(n)}$, was constructed in [1, §4]. Let $\mathbf{K}(n)=K_{\mathcal{Z}}(n)\otimes_{\mathcal{Z}}\mathbb{Q}(v)$. Following [3, 5.1], we define $\widehat{\mathbf{K}}(n)$ to be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A\in\widetilde{\Theta}(n)}\beta_A[A]$ satisfying the following property: for any $\mathbf{x}\in\mathbb{Z}^n$, the sets $\{A\in\widetilde{\Theta}(n)\mid\beta_A\neq 0,\mathrm{ro}(A)=\mathbf{x}\}$ and $\{A\in\widetilde{\Theta}(n)\mid\beta_A\neq 0,\mathrm{co}(A)=\mathbf{x}\}$ are finite. The product of two elements $\sum_{A\in\widetilde{\Theta}(n)}\beta_A[A]$, $\sum_{B\in\widetilde{\Theta}(n)}\gamma_B[B]$ in $\widehat{\mathbf{K}}(n)$ is defined to be $\sum_{A,B}\beta_A\gamma_B[A]\cdot[B]$, where $[A]\cdot[B]$ is the product in $K_{\mathcal{Z}}(n)$. Then $\widehat{\mathbf{K}}(n)$ is an associative algebra.

Let $\Theta^{\pm}(n)$ be the set of all $A\in\Theta(n)$ such that all diagonal entries are zero. For $A\in\Theta^{\pm}(n)$ and $\mathbf{j}\in\mathbb{Z}^n$, let

$$A(\mathbf{j},d)=\sum_{\lambda\in\Lambda(n,d-\sigma(A))} v^{\lambda\cdot\mathbf{j}}[A+\mathrm{diag}(\lambda)]\in\boldsymbol{\mathcal{S}}(n,d),$$
$$A(\mathbf{j})=\sum_{\lambda\in\mathbb{Z}^n} v^{\lambda\cdot\mathbf{j}}[A+\mathrm{diag}(\lambda)]\in\widehat{\mathbf{K}}(n),$$

where $\lambda\cdot\mathbf{j}=\sum_{1\leqslant i\leqslant n}\lambda_i j_i$.

We shall denote by $\mathbf{V}(n)$ the subspace of $\widehat{\mathbf{K}}(n)$ spanned by the elements $A(\mathbf{j})$ for $A\in\Theta^{\pm}(n)$ and $\mathbf{j}\in\mathbb{Z}^n$. For $1\leqslant i,j\leqslant n$, let $E_{i,j}\in\Theta(n)$ be the matrix whose (i,j)-entry is 1 and the other entries are 0. The following result was given by Beilinson–Lusztig–MacPherson [3].

THEOREM 2.2. (1) *$\mathbf{V}(n)$ is a subalgebra of $\widehat{\mathbf{K}}(n)$ and there is an algebra isomorphism $\mathbf{U}(\mathfrak{gl}_n)\xrightarrow{\sim}\mathbf{V}(n)$ satisfying*

$$E_h\mapsto E_{h,h+1}(\mathbf{0}),\ K_1^{j_1}K_2^{j_2}\cdots K_n^{j_n}\mapsto 0(\mathbf{j}),\ F_h\mapsto E_{h+1,h}(\mathbf{0}).$$

(2) *There is an algebra epimorphism $\zeta_d:\mathbf{U}(\mathfrak{gl}_n)\to\boldsymbol{\mathcal{S}}(n,d)$ satisfying*

$$E_h\mapsto E_{h,h+1}(\mathbf{0},d),\ K_1^{j_1}K_2^{j_2}\cdots K_n^{j_n}\mapsto 0(\mathbf{j},d),\ F_h\mapsto E_{h+1,h}(\mathbf{0},d).$$

We shall identify $\mathbf{U}(\mathfrak{gl}_n)$ with $\mathbf{V}(n)$. By [14] we have the following result (cf. [17]).

LEMMA 2.3. (1) *There is an algebra isomorphism $\varphi:\dot{\mathbf{U}}(\mathfrak{gl}_n)\to\mathbf{K}(n)$ satisfying*

$$\pi_{\lambda\mu}(u)\mapsto[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)],$$

for all $u\in\mathbf{U}(\mathfrak{gl}_n)$ and $\lambda,\mu\in\mathbb{Z}^n$. Furthermore, we have $\varphi(\dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n))=K_{\mathcal{Z}}(n)$.

(2) *There is a surjective algebra homomorphism $\dot{\zeta}_d:\mathbf{K}(n)\to\boldsymbol{\mathcal{S}}(n,d)$ such that*

$$\dot{\zeta}_d([A])=\begin{cases}[A], & \text{if } A\in\Theta(n,d);\\ 0, & \text{otherwise.}\end{cases}$$

We shall identify $\dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n)$ with $K_{\mathcal{Z}}(n)$.

§3. The infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$

Let $U_{\mathcal{K}}(\mathfrak{gl}_n)=U_{\mathcal{Z}}(\mathfrak{gl}_n)\otimes_{\mathcal{Z}}\mathcal{K}$. We shall denote the images of $E_i^{(m)},F_i^{(m)}$, etc. in $U_{\mathcal{K}}(\mathfrak{gl}_n)$ by the same letters. Let $U_{\mathcal{K}}^+(\mathfrak{gl}_n)$ (resp. $U_{\mathcal{K}}^-(\mathfrak{gl}_n)$) be the subalgebra of $U_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}$ (resp. $F_i^{(m)}$) for $1\leqslant i\leqslant n-1$ and $m\in\mathbb{N}$. Let $U_{\mathcal{K}}^0(\mathfrak{gl}_n)$ be the subalgebra of $U_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $K_j^{\pm 1}$ and $\left[{K_j;0 \atop t}\right]$ for $1\leqslant j\leqslant n$ and $t\in\mathbb{N}$. Then we have $U_{\mathcal{K}}(\mathfrak{gl}_n)\cong U_{\mathcal{K}}^+(\mathfrak{gl}_n)\otimes U_{\mathcal{K}}^0(\mathfrak{gl}_n)\otimes U_{\mathcal{K}}^-(\mathfrak{gl}_n)$. The algebras $U_{\mathcal{K}}^+(\mathfrak{gl}_n)$ and $U_{\mathcal{K}}^-(\mathfrak{gl}_n)$ are both $\mathbb{N}$-graded in terms of the degrees of monomials in the $E_i^{(m)}$ and $F_i^{(m)}$.

For $r\geqslant 1$, let $U_{\mathcal{K}}(G_r)$ be the $\mathcal{K}$-subalgebra of $U_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, and $\left[{K_j;0 \atop t}\right]$ for $1\leqslant i\leqslant n-1$, $1\leqslant j\leqslant n$, $t\in\mathbb{N}$ and $0\leqslant m,t<lp^{r-1}$. Furthermore, let

$$U_{\mathcal{K}}(G_rT)=U_{\mathcal{K}}(G_r)U_{\mathcal{K}}^0(\mathfrak{gl}_n).$$

Clearly, the algebra $U_{\mathcal{K}}(G_rT)$ is a Hopf subalgebra of $U_{\mathcal{K}}(\mathfrak{gl}_n)$. Let $U_{\mathcal{K}}^+(G_rT)$ (resp. $U_{\mathcal{K}}^-(G_rT)$) be the subalgebra of $U_{\mathcal{K}}(G_rT)$ generated by the elements $E_i^{(m)}$ (resp. $F_i^{(m)}$) for $1\leqslant i\leqslant n-1$ and $0\leqslant m<lp^{r-1}$. Then we have $U_{\mathcal{K}}(G_rT)\cong U_{\mathcal{K}}^+(G_rT)\otimes U_{\mathcal{K}}^0(\mathfrak{gl}_n)\otimes U_{\mathcal{K}}^-(G_rT)$.

Let $\Theta^+(n)=\{A\in\Theta(n)\mid a_{i,j}=0,\forall i\geqslant j\}$ and $\Theta^-(n)=\{A\in\Theta(n)\mid a_{i,j}=0,\forall i\leqslant j\}$. For $A\in\widetilde{\Theta}(n)$, write $A=A^++\mathrm{diag}(\lambda)+A^-$ with $A^+\in\Theta^+(n)$, $A^-\in\Theta^-(n)$ and $\lambda\in\mathbb{Z}^n$. Let

$$\Theta^{\pm}(n)_r=\{A\in\Theta^{\pm}(n)\mid a_{i,j}<lp^{r-1},\forall i\neq j\}.$$

Let

$$\Theta^+(n)_r=\Theta^{\pm}(n)_r\cap\Theta^+(n),\ \Theta^-(n)_r=\Theta^{\pm}(n)_r\cap\Theta^-(n). \tag{3.1}$$

For $A\in\Theta^{\pm}(n)_r$, let

$$E^{(A^+)}=M_nM_{n-1}\cdots M_2\in U_{\mathcal{K}}^+(G_rT)\qquad\text{and}\qquad F^{(A^-)}=M_2'M_3'\cdots M_n'\in U_{\mathcal{K}}^-(G_rT),$$

where

$$M_j=M_j(A^+)=E_{j-1}^{(a_{j-1,j})}(E_{j-2}^{(a_{j-2,j})}E_{j-1}^{(a_{j-2,j})})\cdots(E_1^{(a_{1,j})}E_2^{(a_{1,j})}\cdots E_{j-1}^{(a_{1,j})}),$$

and

$$M_j'=M_j'(A^-)=(F_{j-1}^{(a_{j,1})}\cdots F_2^{(a_{j,1})}F_1^{(a_{j,1})})\cdots(F_{j-1}^{(a_{j,j-2})}F_{j-2}^{(a_{j,j-2})})F_{j-1}^{(a_{j,j-1})}.$$

For $A\in\Theta^{\pm}(n)$ let

$$\deg(A)=\sum_{1\leqslant i,j\leqslant n}|j-i|a_{i,j}.$$

Then we have $\deg(E^{(A^+)})=\deg(A^+)$ and $\deg(F^{(A^-)})=\deg(A^-)$ for $A\in\Theta^{\pm}(n)$. For $\lambda\in\mathbb{N}^n$ and $\mathbf{j}\in\mathbb{Z}^n$ let

$$K_\lambda=\prod_{1\leqslant i\leqslant n}\begin{bmatrix}K_i;0\\ \lambda_i\end{bmatrix},K^{\mathbf{j}}=\prod_{1\leqslant i\leqslant n}K_i^{j_i}.$$

The following result is given in [18, Lem. 6.3].

Proposition 3.1. (1) *The set* $\left\{E^{(A^+)}K^\delta K_\lambda F^{(A^-)}\mid A\in\Theta^{\pm}(n)_r,\delta,\lambda\in\mathbb{N}^n,\delta_i\in\{0,1\},\forall i\right\}$ *forms a* $\mathcal{K}$*-basis for* $U_{\mathcal{K}}(G_rT)$.

(2) *The set* $\{E^{(A^+)} \mid A \in \Theta^+(n)_r\}$ *(resp.* $\{F^{(A^-)} \mid A \in \Theta^-(n)_r\}$*) forms a* $\mathcal{K}$*-basis for* $U^+_{\mathcal{K}}(G_rT)$ *(resp.* $U^-_{\mathcal{K}}(G_rT)$*).*

For $A \in \widetilde{\Theta}(n)$ let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s\leqslant i;t\geqslant j} a_{s,t} & \text{if } i<j \\ \sum_{s\geqslant i;t\leqslant j} a_{s,t} & \text{if } i>j. \end{cases}$$

Following [3], for $A,B \in \widetilde{\Theta}(n)$, define $B \preccurlyeq A$ if and only if $\sigma_{i,j}(B) \leqslant \sigma_{i,j}(A)$ for all $i \neq j$. Put $B \prec A$ if $B \preccurlyeq A$ and $\sigma_{i,j}(B) < \sigma_{i,j}(A)$ for some $i \neq j$.

Proposition 3.2. (1) *The set* $\{A^+(\mathbf{0})K^\delta K_\lambda A^-(\mathbf{0}) \mid A \in \Theta^\pm(n)_r, \delta,\lambda \in \mathbb{N}^n, \delta_i \in \{0,1\}, \forall i\}$ *forms a -basis for* $U_{\mathcal{K}}(G_rT)$.

(2) *The set* $\{A(\mathbf{0}) \mid A \in \Theta^+(n)_r\}$ *(resp.* $\{A(\mathbf{0}) \mid A \in \Theta^-(n)_r\}$*) forms a* $\mathcal{K}$*-basis for* $U^+_{\mathcal{K}}(G_rT)$ *(resp.* $U^-_{\mathcal{K}}(G_rT)$*).*

Proof. By [3, 4.6(c)] for $A \in \Theta^\pm(n)_r$, we have

$$E^{(A^+)} = A^+(\mathbf{0}) + f, \quad F^{(A^-)} = A^-(\mathbf{0}) + g, \tag{3.2}$$

where f is a $\mathcal{K}$-linear combination of $B(\mathbf{0})$ for $B \in \Theta^+(n)$ with $B \prec A^+$ and g is a $\mathcal{K}$-linear combination of $C(\mathbf{0})$ for $C \in \Theta^-(n)$ with $C \prec A^-$. By [18, Lem. 6.3] we know that f must a $\mathcal{K}$-linear combination of $B(\mathbf{0})$ for $B \in \Theta^+(n)_r$ with $B \prec A^+$ and g is a $\mathcal{K}$-linear combination of $C(\mathbf{0})$ for $C \in \Theta^-(n)_r$ with $C \prec A^-$. Now the assertion follows from Proposition 3.1. ∎

Let $\dot{U}_{\mathcal{K}}(\mathfrak{gl}_n) = \dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n) \otimes_{\mathcal{Z}} \mathcal{K} = K_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \mathcal{K}$. We shall denote the images of $E_i^{(m)}1_\lambda$, $1_\lambda F_i^{(m)}$, $E^{(A^+)}1_\lambda$, $1_\lambda F^{(A^-)}$ in $\dot{U}_{\mathcal{K}}(\mathfrak{gl}_n)$ by the same letters. For $A \in \widetilde{\Theta}(n)$ let

$$[A]_\varepsilon = [A] \otimes 1 \in \dot{U}_{\mathcal{K}}(\mathfrak{gl}_n).$$

Let $\dot{U}_{\mathcal{K}}(G_rT)$ be the $\mathcal{K}$-subalgebra of $\dot{U}_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $E_i^{(m)}1_\lambda$ and $1_\lambda F_i^{(m)}$ for $1 \leqslant i \leqslant n-1$, $\lambda \in \mathbb{Z}^n$ and $0 \leqslant m < lp^{r-1}$.

For $A \in \widetilde{\Theta}(n)$ and $1 \leqslant i \leqslant n$, let

$$\boldsymbol{\sigma}(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)),$$

where $\sigma_i(A) = a_{i,i} + \sum_{1\leqslant j<i}(a_{ij} + a_{ji})$. Let

$$\widetilde{\Theta}(n)_r = \{A \in \widetilde{\Theta}(n) \mid a_{i,j} < lp^{r-1}, \forall i \neq j\}.$$

We have the following monomial, BLM and PBW bases of $\dot{U}_{\mathcal{K}}(G_rT)$.

Proposition 3.3. *Each of the following sets forms a* $\mathcal{K}$*-basis of* $\dot{U}_{\mathcal{K}}(G_rT)$*:*

(1) $\mathscr{M}_r := \{E^{(A^+)}1_{\boldsymbol{\sigma}(A)}F^{(A^-)} \mid A \in \widetilde{\Theta}(n)_r\}$;
(2) $\mathscr{L}_r := \{[A]_\varepsilon \mid A \in \widetilde{\Theta}(n)_r\}$;
(3) $\mathscr{P}_r := \{A^+(\mathbf{0})1_{\boldsymbol{\sigma}(A)}A^-(\mathbf{0}) \mid A \in \widetilde{\Theta}(n)_r\}$.

Proof. Let $\dot{U}_{\mathcal{K}}(G_rT)'$ be the $\mathcal{K}$-submodule of $\dot{U}_{\mathcal{K}}(\mathfrak{gl}_n)$ spanned by the elements $[A]_\varepsilon$ for $A \in \widetilde{\Theta}(n)_r$. By [3, 4.6(a)] for $1 \leqslant h \leqslant n-1$, $0 \leqslant m < lp^{r-1}$, $A \in \widetilde{\Theta}(n)_r$, we have

$$E_h^{(m)}[A]_\varepsilon = \sum_{\substack{\mathbf{t}\in\Lambda(n,m) \\ a_{h+1,u}\geqslant t_u,\,\forall u\neq h+1}} \varepsilon^{\beta(\mathbf{t})} \prod_{1\leq u\leq n} \begin{bmatrix} a_{h,u}+t_u \\ t_u \end{bmatrix}_\varepsilon \Big[A + \sum_{1\leq u\leq n} t_u(E_{h,u} - E_{h+1,u})\Big]_\varepsilon,$$

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where $\beta(\mathbf{t})=\sum_{j>u}(a_{h,j}-a_{h+1,j})t_u+\sum_{u<u'}t_ut_{u'}$. If $A+\sum_u t_u(E_{h,u}-E_{h+1,u})\notin\widetilde{\Theta}(n)_r$ for some $\mathbf{t}\in\Lambda(n,m)$, then we have $a_{h,u}+t_u\geqslant lp^{r-1}$ for some $u\neq h$. Since $A\in\widetilde{\Theta}(n)_r$, $\mathbf{t}\in\Lambda(n,m)$ and $m<lp^{r-1}$, we have $a_{h,u}<lp^{r-1}$ and $t_u<lp^{r-1}$. Hence, by [18, Cor. 3.4] we have $\left[{a_{h,u}+t_u \atop t_u}\right]_\varepsilon=0$. Therefore, we have

$$E_h^{(m)}\dot{U}_{\mathcal{K}}(G_rT)'\subseteq\dot{U}_{\mathcal{K}}(G_rT)'.$$

Similarly, we have

$$F_h^{(m)}\dot{U}_{\mathcal{K}}(G_rT)'\subseteq\dot{U}_{\mathcal{K}}(G_rT)',$$

for $1\leqslant h\leqslant n-1$ and $0\leqslant m<lp^{r-1}$. Consequently, we have

$$\dot{U}_{\mathcal{K}}(G_rT)\subseteq\dot{U}_{\mathcal{K}}(G_rT)\dot{U}_{\mathcal{K}}(G_rT)'\subseteq\dot{U}_{\mathcal{K}}(G_rT)'. \tag{3.3}$$

Furthermore, by [3, 4.6(c)] for $A\in\widetilde{\Theta}(n)_r$,

$$E^{(A^+)}1_{\boldsymbol{\sigma}(A)}F^{(A^-)}=[A]_\varepsilon+f, \tag{3.4}$$

where f is a $\mathcal{K}$-linear combination of $[B]_\varepsilon$ for $B\in\widetilde{\Theta}(n)$ with $B\prec A$. By (3.3), we see that f must be a $\mathcal{K}$-linear combination of $[B]_\varepsilon$ for $B\in\widetilde{\Theta}(n)_r$ with $B\prec A$. It follows from (3.2) that

$$A^+(\mathbf{0})1_{\boldsymbol{\sigma}(A)}A^-(\mathbf{0})=[A]_\varepsilon+g, \tag{3.5}$$

where g is a $\mathcal{K}$-linear combination of $[B]_\varepsilon$ for $B\in\widetilde{\Theta}(n)_r$ with $B\prec A$. Therefore, each of the sets $\mathscr{M}_r$, $\mathscr{L}_r$, $\mathscr{P}_r$ forms a $\mathcal{K}$-basis of $\dot{U}_{\mathcal{K}}(G_rT)'$ and

$$\dot{U}_{\mathcal{K}}(G_rT)'\subseteq\dot{U}_{\mathcal{K}}(G_rT).$$

Hence, by (3.3) we have $\dot{U}_{\mathcal{K}}(G_rT)=\dot{U}_{\mathcal{K}}(G_rT)'$. The proof is completed. □

Let $\mathcal{S}_{\mathcal{K}}(n,d)=\mathcal{S}_{\mathcal{Z}}(n,d)\otimes_{\mathcal{Z}}\mathcal{K}$. By [13] we have $\zeta_d(U_{\mathcal{Z}}(\mathfrak{gl}_n))=\mathcal{S}_{\mathcal{Z}}(n,d)$. Therefore, the map $\zeta_d:\mathbf{U}(\mathfrak{gl}_n)\to\boldsymbol{\mathcal{S}}(n,d)$ given in Theorem 2.2 restricts to a surjective algebra homomorphism

$$\zeta_d:U_{\mathcal{Z}}(\mathfrak{gl}_n)\to\mathcal{S}_{\mathcal{Z}}(n,d). \tag{3.6}$$

A generating set for the kernel of $\zeta_d:U_{\mathcal{Z}}(\mathfrak{gl}_n)\to\mathcal{S}_{\mathcal{Z}}(n,d)$ was given in [19]. The map ζ_d induces, upon tensoring with $\mathcal{K}$, a surjective algebra homomorphism

$$\zeta_d:U_{\mathcal{K}}(\mathfrak{gl}_n)\to\mathcal{S}_{\mathcal{K}}(n,d). \tag{3.7}$$

Let

$$\mathtt{e}_i=\zeta_d(E_i),\ \mathtt{f}_i=\zeta_d(F_i),\ \mathtt{k}_j=\zeta_d(K_j),$$

for $1\leqslant i\leqslant n-1$ and $1\leqslant j\leqslant n$. For $A\in\Theta(n)$ and $\lambda\in\mathbb{N}^n$, let

$$\mathtt{e}^{(A^+)}=\zeta_d(E^{(A^+)}),\ \mathtt{f}^{(A^-)}=\zeta_d(F^{(A^-)}),\ \mathtt{k}_\lambda=\zeta_d(K_\lambda).$$

For $A\in\Theta(n,d)$, let

$$[A]_\varepsilon=[A]\otimes 1\in\mathcal{S}_{\mathcal{K}}(n,d).$$

By [15, Cor. 5.3], we have

$$\mathtt{k}_\lambda=[\mathrm{diag}(\lambda)]_\varepsilon, \tag{3.8}$$

for $\lambda\in\Lambda(n,d)$.

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Let $\mathcal{S}_{\mathcal{K}}(n,d)_r$ be the infinitesimal q-Schur algebra introduced in [4]. The algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$ is a $\mathcal{K}$-subalgebra of the q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)$. Let

$$\Theta(n,d)_r = \{A \in \Theta(n,d) \mid a_{ij} < lp^{r-1} \text{ for all } i \neq j\}.$$

According to [4, 5.3.1] and the proof of [16, Th. 5.5], we have the following result.

LEMMA 3.4. *The set* $\mathscr{L}_{d,r} := \{[A]_\varepsilon \mid A \in \Theta(n,d)_r\}$ *forms a* $\mathcal{K}$*-basis of* $\mathcal{S}_{\mathcal{K}}(n,d)_r$.

By [18, Prop. 6.4], we have the following result.

LEMMA 3.5. *For* $d \in \mathbb{N}$ *we have* $\zeta_d(U_{\mathcal{K}}(G_rT)) = \mathcal{S}_{\mathcal{K}}(n,d)_r$.

The map $\dot{\zeta}_d : \dot{\mathbf{U}}(\mathfrak{gl}_n) = \mathbf{K}(n) \to \boldsymbol{\mathcal{S}}(n,d)$ given in Lemma 2.3 restricts to a surjective algebra homomorphism

$$\dot{\zeta}_d : \dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n) \to \mathcal{S}_{\mathcal{Z}}(n,d); \tag{3.9}$$

tensoring with $\mathcal{K}$, we obtain a surjective algebra homomorphism

$$\dot{\zeta}_d : \dot{U}_{\mathcal{K}}(\mathfrak{gl}_n) \to \mathcal{S}_{\mathcal{K}}(n,d). \tag{3.10}$$

Combining Lemma 2.3 with Proposition 3.3, we obtain the following result.

LEMMA 3.6. *For* $d \in \mathbb{N}$ *we have* $\dot{\zeta}_d(\dot{U}_{\mathcal{K}}(G_rT)) = \mathcal{S}_{\mathcal{K}}(n,d)_r$.

For $\lambda,\mu \in \mathbb{Z}^n$, write $\lambda \leqslant \mu \Leftrightarrow \lambda_i \leqslant \mu_i$ for $1 \leqslant i \leqslant n$. We have the following monomial and PBW bases of $\mathcal{S}_{\mathcal{K}}(n,d)_r$.

PROPOSITION 3.7. *Each of the following set forms a* $\mathcal{K}$*-basis of* $\mathcal{S}_{\mathcal{K}}(n,d)_r$*:*

(1) $\mathscr{M}_{d,r} = \{\mathtt{e}^{(A^+)}\mathtt{k}_\lambda \mathtt{f}^{(A^-)} \mid A \in \Theta^\pm(n)_r, \lambda \in \Lambda(n,d), \lambda \geqslant \boldsymbol{\sigma}(A)\}$;
(2) $\mathscr{P}_{d,r} = \{A^+(\mathbf{0},d)\mathtt{k}_\lambda A^-(\mathbf{0},d) \mid A \in \Theta^\pm(n)_r, \lambda \in \Lambda(n,d), \lambda \geqslant \boldsymbol{\sigma}(A)\}$.

Proof. By Lemma 2.3, (3.4), (3.5), and (3.8), for $A \in \Theta^\pm(n)_r$, $\lambda \in \Lambda(n,d)$ with $\lambda \geqslant \boldsymbol{\sigma}(A)$, we have

$$\begin{aligned}\mathtt{e}^{(A^+)}\mathtt{k}_\lambda \mathtt{f}^{(A^-)} &= \dot{\zeta}_d(E^{(A^+)}1_\lambda F^{(A^-)}) = [A+\lambda-\boldsymbol{\sigma}(A)]_\varepsilon + f,\\ A^+(\mathbf{0},d)\mathtt{k}_\lambda A^-(\mathbf{0},d) &= [A+\lambda-\boldsymbol{\sigma}(A)]_\varepsilon + g,\end{aligned}$$

where $f,g \in \mathrm{span}_{\mathcal{K}}\{[B]_\varepsilon \mid B \in \Theta(n,d)_r, B \prec A\}$. Now the assertion follows from Lemma 3.4. ∎

§4. The algebra $\mathcal{T}_{\mathcal{K}}(n,d)_r$

For $d \in \mathbb{N}$ let

$$\mathcal{T}_{\mathcal{K}}(n,d)_r = U_{\mathcal{K}}(G_rT)/I_{d,r},$$

where $I_{d,r}$ is the two-sided ideal of $U_{\mathcal{K}}(G_rT)$ generated by the elements $1-\sum_{\mu\in\Lambda(n,d)} K_\mu$, $K_iK_\lambda - \varepsilon^{\lambda_i}K_\lambda$ and $\left[{K_i;0 \atop t}\right]K_\lambda - \left[{\lambda_i \atop t}\right]_\varepsilon K_\lambda$ for $1 \leqslant i \leqslant n$, $t \in \mathbb{N}$ and $\lambda \in \Lambda(n,d)$. For $1 \leqslant i \leqslant n-1$, $t \in \mathbb{N}$ and $0 \leqslant m < lp^{r-1}$ let

$$\mathbf{e}_i^{(m)} = E_i^{(m)} + I_{d,r}, \mathbf{f}_i^{(m)} = F_i^{(m)} + I_{d,r}.$$

Furthermore, for $1 \leqslant j \leqslant n$, $c \in \mathbb{Z}$, $t \in \mathbb{N}$ and $\lambda \in \mathbb{N}^n$ let

$$\mathbf{k}_j = K_j + I_{d,r}, \begin{bmatrix}\mathbf{k}_j;c\\ t\end{bmatrix} = \begin{bmatrix}K_j;c\\ t\end{bmatrix} + I_{d,r}, \mathbf{k}_\lambda = K_\lambda + I_{d,r}.$$

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We will prove in Theorem 4.10 that the algebra $\mathcal{T}_{\mathcal{K}}(n,d)_r$ is isomorphic to the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$.

LEMMA 4.1. (1) *For* $\lambda,\mu\in\Lambda(n,d)$ *we have* $\mathbf{k}_\lambda\mathbf{k}_\mu=\delta_{\lambda,\mu}\mathbf{k}_\lambda$.
(2) *Assume* $\nu\in\mathbb{N}^n$ *is such that* $\sigma(\nu)>d$. *Then we have* $\mathbf{k}_\nu=0$.

Proof. For $1\leqslant i\leqslant n$ and $t\in\mathbb{N}$, we have $\left[\begin{smallmatrix}\mathbf{k}_i;0\\ t\end{smallmatrix}\right]\mathbf{k}_\mu=\left[\begin{smallmatrix}\mu_i\\ t\end{smallmatrix}\right]_\varepsilon\mathbf{k}_\mu$. It follows that

$$\mathbf{k}_\lambda\mathbf{k}_\mu=\begin{bmatrix}\mu\\ \lambda\end{bmatrix}_\varepsilon\mathbf{k}_\mu.$$

If $\left[\begin{smallmatrix}\mu\\ \lambda\end{smallmatrix}\right]_\varepsilon\neq 0$, then we have $\mu\geqslant\lambda$. This implies that $\mu=\lambda$ since $\lambda,\mu\in\Lambda(n,d)$. Therefore, we have $\mathbf{k}_\lambda\mathbf{k}_\mu=\delta_{\lambda,\mu}\mathbf{k}_\lambda$. Furthermore, since $1=\sum_{\gamma\in\Lambda(n,d)}\mathbf{k}_\gamma$ and $\sigma(\nu)>d$, we have

$$\mathbf{k}_\nu=\sum_{\gamma\in\Lambda(n,d)}\mathbf{k}_\nu\mathbf{k}_\gamma=\sum_{\gamma\in\Lambda(n,d)}\begin{bmatrix}\gamma\\ \nu\end{bmatrix}_\varepsilon\mathbf{k}_\gamma=\sum_{\substack{\gamma\in\Lambda(n,d)\\ \sigma(\gamma)\geqslant\sigma(\nu)>d}}\begin{bmatrix}\gamma\\ \nu\end{bmatrix}_\varepsilon\mathbf{k}_\gamma=0.$$

The proof is completed. ∎

For $a,b\in\mathbb{Z}$, we have

$$\begin{bmatrix}b+a\\ t\end{bmatrix}_\varepsilon=\sum_{0\leqslant j\leqslant t}\varepsilon^{a(t-j)-bj}\begin{bmatrix}a\\ j\end{bmatrix}_\varepsilon\begin{bmatrix}b\\ t-j\end{bmatrix}_\varepsilon. \tag{4.1}$$

LEMMA 4.2. *Let* $\lambda\in\Lambda(n,d)$. *Then we have* $\left[\begin{smallmatrix}\mathbf{k}_i;c\\ t\end{smallmatrix}\right]\mathbf{k}_\lambda=\left[\begin{smallmatrix}\lambda_i+c\\ t\end{smallmatrix}\right]_\varepsilon\mathbf{k}_\lambda$ *for* $1\leqslant i\leqslant n$, $c\in\mathbb{Z}$, $t\in\mathbb{N}$.

Proof. Assume $c\geqslant 0$. By [22, 2.3 (g9), (g10)], we have

$$\begin{bmatrix}\mathbf{k}_i;\pm c\\ t\end{bmatrix}=\sum_{0\leqslant j\leqslant t}\varepsilon^{c(t-j)}\begin{bmatrix}\pm c\\ j\end{bmatrix}_\varepsilon\mathbf{k}_i^{\mp j}\begin{bmatrix}\mathbf{k}_i;0\\ t-j\end{bmatrix}.$$

Hence, by (4.1), we have

$$\begin{bmatrix}\mathbf{k}_i;\pm c\\ t\end{bmatrix}\mathbf{k}_\lambda=\sum_{0\leqslant j\leqslant t}\varepsilon^{c(t-j)\mp j\lambda_i}\begin{bmatrix}\pm c\\ j\end{bmatrix}_\varepsilon\begin{bmatrix}\lambda_i\\ t-j\end{bmatrix}_\varepsilon\mathbf{k}_\lambda=\begin{bmatrix}\lambda_i\pm c\\ t\end{bmatrix}_\varepsilon\mathbf{k}_\lambda.$$

The proof is completed. ∎

By the definition of $\mathcal{T}_{\mathcal{K}}(n,d)_r$, we have the following result.

LEMMA 4.3. *There is an algebra anti-automorphism* τ_d *on* $\mathcal{T}_{\mathcal{K}}(n,d)_r$ *such that*

$$\tau_d(\mathbf{e}_i^{(m)})=\mathbf{f}_i^{(m)},\quad \tau_d(\mathbf{f}_i^{(m)})=\mathbf{e}_i^{(m)},\quad \tau_d(\mathbf{k}_j)=\mathbf{k}_j,\quad \tau_d\left(\begin{bmatrix}\mathbf{k}_j;0\\ t\end{bmatrix}\right)=\begin{bmatrix}\mathbf{k}_j;0\\ t\end{bmatrix},$$

for $1\leqslant i\leqslant n-1$, $1\leqslant j\leqslant n$, $t\in\mathbb{N}$ *and* $0\leqslant m<lp^{r-1}$.

LEMMA 4.4. *Let* $\lambda,\mu\in\Lambda(n,d)$ *and* $a\in\mathbb{N}$.
(1) *If* $\lambda_{i+1}<a$ *for some* $1\leqslant i\leqslant n-1$, *then we have* $\mathbf{e}_i^{(a)}\mathbf{k}_\lambda=\mathbf{k}_\lambda\mathbf{f}_i^{(a)}=0$.
(2) *If* $\mu_j<a$ *for some* $1\leqslant j\leqslant n-1$, *then we have* $\mathbf{k}_\mu\mathbf{e}_j^{(a)}=\mathbf{f}_j^{(a)}\mathbf{k}_\mu=0$.

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Proof. By Lemma 2.1 and 4.2, we have

$$\mathbf{e}_i^{(a)}\mathbf{k}_\lambda=\mathbf{e}_i^{(a)}\prod_{s\neq i,i+1}\begin{bmatrix}\mathbf{k}_s;0\\ \lambda_s\end{bmatrix}\begin{bmatrix}\mathbf{k}_i;a\\ \lambda_i+a\end{bmatrix}\mathbf{k}_\lambda=\mathbf{k}_{\lambda+a\boldsymbol{e}_i-\lambda_{i+1}\boldsymbol{e}_{i+1}}\mathbf{e}_i^{(a)}\mathbf{k}_\lambda,$$

$$\mathbf{k}_\mu\mathbf{e}_j^{(a)}=\mathbf{k}_\mu\prod_{s\neq j,j+1}\begin{bmatrix}\mathbf{k}_s;0\\ \mu_s\end{bmatrix}\begin{bmatrix}\mathbf{k}_{j+1};a\\ \mu_{j+1}+a\end{bmatrix}\mathbf{e}_j^{(a)}=\mathbf{k}_\mu\mathbf{e}_j^{(a)}\mathbf{k}_{\lambda+a\boldsymbol{e}_{j+1}-\mu_j\boldsymbol{e}_j}.$$

Hence, by Lemma 4.1 we have $\mathbf{k}_{\lambda+a\boldsymbol{e}_i-\lambda_{i+1}\boldsymbol{e}_{i+1}}=0$ and $\mathbf{k}_{\lambda+a\boldsymbol{e}_{j+1}-\mu_j\boldsymbol{e}_j}=0$, since $\lambda_{i+1}<a$, $\mu_j<a$ and $\lambda,\mu\in\Lambda(n,d)$. Therefore we have $\mathbf{e}_i^{(a)}\mathbf{k}_\lambda=\mathbf{k}_\mu\mathbf{e}_j^{(a)}=0$. Consequently, we have $\mathbf{k}_\lambda\mathbf{f}_i^{(a)}=\tau_d(\mathbf{e}_i^{(a)}\mathbf{k}_\lambda)=0$ and $\mathbf{f}_j^{(a)}\mathbf{k}_\mu=\tau_d(\mathbf{k}_\mu\mathbf{e}_j^{(a)})=0$. ∎

LEMMA 4.5. *Let $\lambda\in\Lambda(n,d)$ and $a\in\mathbb{N}$. If $\lambda_{i+1}\geqslant a$ for some $1\leqslant i\leqslant n-1$, then we have $\mathbf{e}_i^{(a)}\mathbf{k}_\lambda=\mathbf{k}_{\lambda+a\alpha_i}\mathbf{e}_i^{(a)}$ and $\mathbf{k}_\lambda\mathbf{f}_i^{(a)}=\mathbf{f}_i^{(a)}\mathbf{k}_{\lambda+a\alpha_i}$, where $\alpha_i=\boldsymbol{e}_i-\boldsymbol{e}_{i+1}$.*

Proof. By Lemma 2.1 and 4.2, we have

$$\begin{aligned}\mathbf{e}_i^{(a)}\mathbf{k}_\lambda&=\prod_{j\neq i,i+1}\begin{bmatrix}\mathbf{k}_j;0\\ \lambda_j\end{bmatrix}\begin{bmatrix}\mathbf{k}_i;-a\\ \lambda_i\end{bmatrix}\begin{bmatrix}\mathbf{k}_{i+1};a\\ \lambda_{i+1}\end{bmatrix}\mathbf{e}_i^{(a)}\\ &=\sum_{\mu\in\Lambda(n,d)}\prod_{j\neq i,i+1}\begin{bmatrix}\mathbf{k}_j;0\\ \lambda_j\end{bmatrix}\begin{bmatrix}\mathbf{k}_i;-a\\ \lambda_i\end{bmatrix}\begin{bmatrix}\mathbf{k}_{i+1};a\\ \lambda_{i+1}\end{bmatrix}\mathbf{k}_\mu\mathbf{e}_i^{(a)}.\end{aligned}$$

Hence, by Lemma 4.4, we have

$$\mathbf{e}_i^{(a)}\mathbf{k}_\lambda=\sum_{\substack{\mu\in\Lambda(n,d)\\ \mu_i\geqslant a}}\begin{bmatrix}\mu-a\alpha_i\\ \lambda\end{bmatrix}_\varepsilon\mathbf{k}_\mu\mathbf{e}_i^{(a)}=\sum_{\mu\in\Lambda(n,d),\,\mu-a\alpha_i\geqslant\lambda}\begin{bmatrix}\mu-a\alpha_i\\ \lambda\end{bmatrix}_\varepsilon\mathbf{k}_\mu\mathbf{e}_i^{(a)}=\mathbf{k}_{\lambda+a\alpha_i}\mathbf{e}_i^{(a)}.$$

Therefore, we have $\mathbf{k}_\lambda\mathbf{f}_i^{(a)}=\tau_d(\mathbf{e}_i^{(a)}\mathbf{k}_\lambda)=\tau_d(\mathbf{k}_{\lambda+a\alpha_i}\mathbf{e}_i^{(a)})=\mathbf{f}_i^{(a)}\mathbf{k}_{\lambda+a\alpha_i}$. ∎

For simplicity, we set $\mathbf{k}_\lambda=0$ if $\lambda\notin\mathbb{N}^n$ with $\sigma(\lambda)=d$, where $\sigma(\lambda)=\sum_{1\leqslant i\leqslant n}\lambda_i$. Then, by Lemma 4.4 and 4.5, we have the following result.

LEMMA 4.6. *For $\lambda\in\mathbb{Z}^n$ with $\sigma(\lambda)=d$ and $a\in\mathbb{N}$ we have $\mathbf{e}_i^{(a)}\mathbf{k}_\lambda=\mathbf{k}_{\lambda+a\alpha_i}\mathbf{e}_i^{(a)}$ and $\mathbf{k}_\lambda\mathbf{f}_i^{(a)}=\mathbf{f}_i^{(a)}\mathbf{k}_{\lambda+a\alpha_i}$.*

For $A\in\Theta^\pm(n)$ let $\mathbf{e}^{(A^+)}=E^{(A^+)}+I_{d,r}$ and $\mathbf{f}^{(A^-)}=F^{(A^-)}+I_{d,r}$.

LEMMA 4.7. *Let $A\in\Theta^\pm(n)_r$ and $\lambda\in\mathbb{Z}^n$ with $\sigma(\lambda)=d$. Then, we have $\mathbf{e}^{(A^+)}\mathbf{k}_\lambda=\mathbf{k}_{\lambda-co(A^+)+ro(A^+)}\mathbf{e}^{(A^+)}$ and $\mathbf{k}_\lambda\mathbf{f}^{(A^-)}=\mathbf{f}^{(A^-)}\mathbf{k}_{\lambda+co(A^-)-ro(A^-)}$.*

Proof. By Lemma 4.6, we have $\mathbf{e}^{(A^+)}\mathbf{k}_\lambda=\mathbf{k}_\mu\mathbf{e}^{(A^+)}$ and $\mathbf{k}_\lambda\mathbf{f}^{(A^-)}=\mathbf{f}^{(A^-)}\mathbf{k}_\nu$ where

$$\mu=\lambda+\sum_{2\leqslant j\leqslant n}\sum_{1\leqslant k<j}a_{k,j}\sum_{k\leqslant s<j}\alpha_s=\lambda+\sum_{2\leqslant j\leqslant n}\sum_{1\leqslant k<j}a_{k,j}(\boldsymbol{e}_k-\boldsymbol{e}_j)=\lambda-\mathrm{co}(A^+)+\mathrm{ro}(A^+),$$

and

$$\nu=\lambda+\sum_{2\leqslant j\leqslant n}\sum_{1\leqslant k<j}a_{j,k}\sum_{k\leqslant s<j}\alpha_s=\lambda+\sum_{2\leqslant j\leqslant n}\sum_{1\leqslant k<j}a_{j,k}(\boldsymbol{e}_k-\boldsymbol{e}_j)=\lambda+\mathrm{co}(A^-)-\mathrm{ro}(A^-).$$

The proof is completed. ∎

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Recall the sets $\Theta^+(n)_r$ and $\Theta^-(n)_r$ defined in (3.1).

LEMMA 4.8. *Let* $\lambda\in\Lambda(n,d)$.

(1) *If* $A\in\Theta^+(n)_r$ *and* $\lambda_i<\sigma_i(A)$ *for some* i, *then we have* $\mathbf{e}^{(A)}\mathbf{k}_\lambda=0$.
(2) *If* $A\in\Theta^-(n)_r$ *and* $\lambda_i<\sigma_i(A)$ *for some* i, *then we have* $\mathbf{k}_\lambda\mathbf{f}^{(A)}=0$.

Proof. If $A\in\Theta^+(n)_r$ and $\lambda_i<\sigma_i(A)$ for some i, then by Lemma 4.6 we have

$$\mathbf{e}^{(A)}\mathbf{k}_\lambda=\mathbf{m}_n\mathbf{m}_{n-1}\cdots\mathbf{m}_2\mathbf{k}_\lambda=\mathbf{m}_n\mathbf{m}_{n-1}\cdots\mathbf{m}_{i+1}\mathbf{k}_\mu\mathbf{m}_i\mathbf{m}_{i-1}\cdots\mathbf{m}_2,$$

where $\mathbf{m}_j=\mathbf{e}_{j-1}^{(a_{j-1,j})}(\mathbf{e}_{j-2}^{(a_{j-2,j})}\mathbf{e}_{j-1}^{(a_{j-2,j})})\cdots(\mathbf{e}_1^{(a_{1,j})}\mathbf{e}_2^{(a_{1,j})}\cdots\mathbf{e}_{j-1}^{(a_{1,j})})$ and

$$\mu=\lambda+\sum_{2\leqslant j\leqslant i}\sum_{1\leqslant k<j}a_{k,j}\sum_{k\leqslant s<j}\alpha_s=\lambda+\sum_{2\leqslant j\leqslant i}\sum_{1\leqslant k<j}a_{k,j}(\boldsymbol{e}_k-\boldsymbol{e}_j).$$

Since $\mu_i=\lambda_i-\sigma_i(A)<0$ we have $\mathbf{k}_\mu=0$. Hence, we have $\mathbf{e}^{(A)}\mathbf{k}_\lambda=0$. Assume now that $A\in\Theta^-(n)_r$ and $\lambda_i<\sigma_i(A)$ for some i. Then, we have $\mathbf{k}_\lambda\mathbf{f}^{(A)}=\tau_d(\mathbf{e}^{({}^tA)}\mathbf{k}_\lambda)=0$. The proof is completed. ∎

For $A\in\Theta^\pm(n)_r$ and $\lambda\in\Lambda(n,d)$, let

$$\mathbf{m}^{(A,\lambda)}=\mathbf{e}^{(A^+)}\mathbf{k}_\lambda\mathbf{f}^{(A^-)}.$$

PROPOSITION 4.9. *The set* $\mathbf{M}_{d,r}=\{\mathbf{m}^{(A,\lambda)}|A\in\Theta^\pm(n)_r,\lambda\in\Lambda(n,d),\lambda\geqslant\boldsymbol{\sigma}(A)\}$ *is a spanning set for* $\mathcal{T}_{\mathcal{K}}(n,d)_r$.

Proof. By the definition of $I_{d,r}$ we have $\mathbf{k}_i=\sum_{\lambda\in\Lambda(n,d)}\varepsilon^{\lambda_i}\mathbf{k}_\lambda$ and $\mathbf{k}_\lambda=\sum_{\mu\in\Lambda(n,d)}\left[{\mu\atop\lambda}\right]_\varepsilon\mathbf{k}_\mu$ for $\lambda\in\Lambda(n,d)$ and $1\leqslant i\leqslant n$. Hence, by Proposition 3.1, we see that the algebra $\mathcal{T}_{\mathcal{K}}(n,d)_r$ is spanned by the elements $\mathbf{e}^{(A^+)}\mathbf{k}_\lambda\mathbf{f}^{(A^-)}$ for $A\in\Theta^\pm(n)_r$ and $\lambda\in\Lambda(n,d)$. Therefore, to prove the proposition, we have to show that if $\lambda_i<\sigma_i(A)$ for some i, then $\mathbf{m}^{(A,\lambda)}$ lies in the span of $\mathbf{M}_{d,r}$.

We argue by induction on $\deg(A)$. The result follows from Lemma 4.4 in the cases where $\deg(A)=1$. Assume now that $\deg(A)>1$, and suppose $\lambda_i<\sigma_i(A)$ for some $1\leqslant i\leqslant n$. For $2\leqslant j\leqslant n$ let

$$\mathbf{m}_j=\mathbf{e}_{j-1}^{(a_{j-1,j})}(\mathbf{e}_{j-2}^{(a_{j-2,j})}\mathbf{e}_{j-1}^{(a_{j-2,j})})\cdots(\mathbf{e}_1^{(a_{1,j})}\mathbf{e}_2^{(a_{1,j})}\cdots\mathbf{e}_{j-1}^{(a_{1,j})}),$$
$$\mathbf{m}'_j=(\mathbf{f}_{j-1}^{(a_{j,1})}\cdots\mathbf{f}_2^{(a_{j,1})}\mathbf{f}_1^{(a_{j,1})})\cdots(\mathbf{f}_{j-1}^{(a_{j,j-2})}\mathbf{f}_{j-2}^{(a_{j,j-2})})\mathbf{f}_{j-1}^{(a_{j,j-1})}.$$

Then, we have $\mathbf{e}^{(A^+)}=\mathbf{m}_n\mathbf{m}_{n-1}\cdots\mathbf{m}_2$ and $\mathbf{f}^{(A^-)}=\mathbf{m}'_2\mathbf{m}'_3\cdots\mathbf{m}'_n$. Let A_i be the submatrix of A consisting of the first i rows and columns, and write $\mathbf{e}^{(A^+)}=\mathbf{x}_1\mathbf{e}^{(A_i^+)},\mathbf{f}^{(A^-)}=\mathbf{f}^{(A_i^-)}\mathbf{x}'_1$. Then,

$$\mathbf{m}^{(A,\lambda)}=\mathbf{x}_1\mathbf{e}^{(A_i^+)}\mathbf{k}_\lambda\mathbf{f}^{(A_i^-)}\mathbf{x}'_1,$$

where $\mathbf{x}_1=\mathbf{m}_n\mathbf{m}_{n-1}\cdots\mathbf{m}_{i+1}$ and $\mathbf{x}'_1=\mathbf{m}'_{i+1}\mathbf{m}'_{i+2}\cdots\mathbf{m}'_n$. By Lemma 4.8, we may assume that $\lambda_i\geqslant\sigma_i(A_i^+)=\sigma_i(A^+)$. Furthermore, by Lemma 4.7, we have

$$\mathbf{m}^{(A,\lambda)}=\mathbf{x}_1\mathbf{k}_{\lambda'}\mathbf{e}^{(A_i^+)}\mathbf{f}^{(A_i^-)}\mathbf{x}'_1,$$

where $\lambda'=\lambda-\mathrm{co}(A_i^+)+\mathrm{ro}(A_i^+)$. By Lemma 2.1,

$$\mathbf{e}^{(A_i^+)}\mathbf{f}^{(A_i^-)}=\mathbf{f}^{(A_i^-)}\mathbf{e}^{(A_i^+)}+f,$$

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where f is a $\mathcal{K}$-linear combination of $\mathbf{x}_j^{\mathbf{e}}\mathbf{h}_j\mathbf{x}_j^{\mathbf{f}}$ with $\mathbf{h}_j \in \mathrm{span}_{\mathcal{K}}\{\mathbf{k}_\lambda | \lambda \in \Lambda(n,d)\}$ and $\deg(\mathbf{x}_j^{\mathbf{e}}) + \deg(\mathbf{x}_j^{\mathbf{f}}) < \deg(A_i)$. Here, $\mathbf{x}_j^{\mathbf{e}}$ (resp. $\mathbf{x}_j^{\mathbf{f}}$) denotes a monomial in the $\mathbf{e}_i^{(a)}$ (resp. $\mathbf{f}_i^{(a)}$). Thus, $\deg(\mathbf{x}_1) + \deg(\mathbf{x}_j^{\mathbf{e}}) + \deg(\mathbf{x}_j^{\mathbf{f}}) + \deg(\mathbf{x}_1') < \deg(A)$. Since $\lambda_i < \sigma_i(A)$, we have $\lambda_i' = \lambda_i - \sigma_i(A_i^+) < \sigma_i(A) - \sigma_i(A_i^+) = \sigma_i(A_i^-)$. It follows from Lemma 4.7 that $\mathbf{k}_{\lambda'}\mathbf{f}^{(A_i^-)} = 0$. Hence, we have

$$\mathbf{m}^{(A,\lambda)} = \mathbf{x}_1\mathbf{k}_{\lambda'} f \mathbf{x}_1'.$$

Furthermore, by Proposition 3.1, we see that each $\mathbf{x}_1\mathbf{x}_j^{\mathbf{e}}$ is a $\mathcal{K}$-linear combination of $\mathbf{e}^{(B)}$ with $B \in \Theta^+(n)_r$, $\deg(B) = \deg(\mathbf{x}_1\mathbf{x}_j^{\mathbf{e}})$ and each $\mathbf{x}_j^{\mathbf{f}}\mathbf{x}_1'$ is a $\mathcal{K}$-linear combination of $\mathbf{f}^{(C)}$ with $C \in \Theta^-(n)_r$, $\deg(C) =$ $\deg(\mathbf{x}_j^{\mathbf{f}}\mathbf{x}_1')$. Therefore, by Lemma 4.7 each $\mathbf{x}_1\mathbf{k}_{\lambda'}\mathbf{x}_j^{\mathbf{e}}\mathbf{h}_j\mathbf{x}_j^{\mathbf{f}}\mathbf{x}_1'$ is a $\mathcal{K}$-linear combination of $\mathbf{m}^{(A',\mu)}$ with $\deg(A') < \deg(A)$, since $\deg(\mathbf{x}_1) + \deg(\mathbf{x}_j^{\mathbf{e}}) + \deg(\mathbf{x}_j^{\mathbf{f}}) + \deg(\mathbf{x}_1') < \deg(A)$. Consequently, by induction, we have $\mathbf{m}^{(A,\lambda)} \in \mathrm{span}_{\mathcal{K}}\mathbf{M}_{d,r}$. The proof is completed. ∎

By Lemma 3.5, we have $\zeta_d(U_{\mathcal{K}}(G_rT)) = \mathcal{S}_{\mathcal{K}}(n,d)_r$. Therefore, the map $\zeta_d : U_{\mathcal{K}}(\mathfrak{gl}_n) \to \mathcal{S}_{\mathcal{K}}(n,d)$ given in (3.7) restricts to a surjective algebra homomorphism

$$\zeta_{d,r} : U_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r. \tag{4.2}$$

By (3.8), we have

$$\zeta_{d,r}(K_\lambda) = [\mathrm{diag}(\lambda)]_\varepsilon,$$

for $\lambda \in \Lambda(n,d)$. So, we have $\zeta_{d,r}(I_{d,r}) = 0$. Hence, the map $\zeta_{d,r} : U_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r$ induces a surjective algebra homomorphism

$$\zeta_{d,r}' : \mathcal{T}_{\mathcal{K}}(n,d)_r = U_{\mathcal{K}}(G_rT)/I_{d,r} \to \mathcal{S}_{\mathcal{K}}(n,d)_r.$$

THEOREM 4.10. *The map $\zeta_{d,r}'$ is an algebra isomorphism. In particular, the kernel of the map $\zeta_{d,r} : U_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r$ is generated by the elements $1 - \sum_{\mu\in\Lambda(n,d)} K_\mu$, $K_iK_\lambda - \varepsilon^{\lambda_i}K_\lambda$ and $\left[{K_i;0 \atop t}\right]K_\lambda - \left[{\lambda_i \atop t}\right]_\varepsilon K_\lambda$ for $1 \leqslant i \leqslant n$, $t \in \mathbb{N}$ and $\lambda \in \Lambda(n,d)$.*

Proof. By Proposition 3.7, the set $\zeta_{d,r}'(\mathbf{M}_{d,r})$ forms a $\mathcal{K}$-basis for $\mathcal{S}_{\mathcal{K}}(n,d)_r$. Thus, by Proposition 4.9, we conclude that $\zeta_{d,r}'$ is an algebra isomorphism. ∎

§5. The algebra $\mathfrak{T}_{\mathcal{K}}(n,d)_r$

For $d \in \mathbb{N}$, let

$$\mathfrak{T}_{\mathcal{K}}(n,d)_r = \dot{U}_{\mathcal{K}}(G_rT)/J_{d,r},$$

where $J_{d,r}$ is the two-sided ideal of $\dot{U}_{\mathcal{K}}(G_rT)$ generated by the elements 1_λ for $\lambda \notin \Lambda(n,d)$. For $\lambda \in \mathbb{Z}^n$, let

$$\mathfrak{k}_\lambda = 1_\lambda + J_{d,r}.$$

Then, we have $1 = \sum_{\lambda\in\Lambda(n,d)} \mathfrak{k}_\lambda$. For $1 \leqslant i \leqslant n-1$ and $0 \leqslant m < lp^{r-1}$, let

$$\mathfrak{e}_i^{(m)} = \sum_{\lambda\in\Lambda(n,d)} E_i^{(m)}1_\lambda + J_{d,r},\ \mathfrak{f}_i^{(m)} = \sum_{\lambda\in\Lambda(n,d)} 1_\lambda F_i^{(m)} + J_{d,r}.$$

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For $A\in\Theta^{\pm}(n)_r$, let. Furthermore for $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n-1$, $c\in\mathbb{Z}$ and $t\in\mathbb{N}$ let

$$\mathfrak{k}_i=\sum_{\lambda\in\Lambda(n,d)}K_i1_\lambda+J_{d,r},\ \begin{bmatrix}\mathfrak{k}_i;c\\t\end{bmatrix}=\sum_{\lambda\in\Lambda(n,d)}\begin{bmatrix}K_i;c\\t\end{bmatrix}1_\lambda+J_{d,r},\ \begin{bmatrix}\widetilde{\mathfrak{k}}_j;c\\t\end{bmatrix}=\sum_{\lambda\in\Lambda(n,d)}\begin{bmatrix}\widetilde{K}_j;c\\t\end{bmatrix}1_\lambda+J_{d,r}.$$

$$\mathfrak{e}^{(A^+)}=\sum_{\lambda\in\Lambda(n,d)}E^{(A^+)}1_\lambda+J_{d,r},\ \mathfrak{f}^{(A^-)}=\sum_{\lambda\in\Lambda(n,d)}1_\lambda F^{(A^-)}+J_{d,r}.$$

We will prove in Theorem 5.5 that the algebra $\mathfrak{T}_{\mathcal{K}}(n,d)_r$ is isomorphic to the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$.

By [3, Lem. 3.10 and Prop. 4.2], we have the following result.

Lemma 5.1. *There is an unique algebra antiautomorphism $\dot\tau_d$ on $\mathfrak{T}_{\mathcal{K}}(n,d)_r$ such that $\dot\tau_d(\mathfrak{e}_i^{(m)})=\mathfrak{f}_i^{(m)}$, $\dot\tau_d(\mathfrak{f}_i^{(m)})=\mathfrak{e}_i^{(m)}$ and $\dot\tau_d(\mathfrak{k}_\lambda)=\mathfrak{k}_\lambda$ for $1\leqslant i\leqslant n-1$, $0\leqslant m<lp^{r-1}$ and $\lambda\in\Lambda(n,d)$.*

Clearly we have the following result.

Lemma 5.2. *Let $1\leqslant i\leqslant n, 1\leqslant j\leqslant n-1$, $c\in\mathbb{Z}$, $t\in\mathbb{N}$ and $\lambda\in\mathbb{Z}^n$. The following formulas hold in $\mathfrak{T}_{\mathcal{K}}(n,d)_r$:*

$$\mathfrak{k}_i\mathfrak{k}_\lambda=\varepsilon^{\lambda_i}\mathfrak{k}_\lambda,\ \begin{bmatrix}\mathfrak{k}_i;c\\t\end{bmatrix}\mathfrak{k}_\lambda=\begin{bmatrix}\lambda_i+c\\t\end{bmatrix}_\varepsilon\mathfrak{k}_\lambda,\ \begin{bmatrix}\widetilde{\mathfrak{k}}_j;c\\t\end{bmatrix}\mathfrak{k}_\lambda=\begin{bmatrix}\lambda_j-\lambda_{j+1}+c\\t\end{bmatrix}_\varepsilon\mathfrak{k}_\lambda.$$

Recall from (3.1) that $\Theta^{\pm}(n)_r=\{A\in\Theta^{\pm}(n)\mid 0\leqslant a_{ij}<lp^{r-1},\forall i\neq j\}$, $\Theta^{+}(n)_r=\{A\in\Theta^{+}(n)\mid 0\leqslant a_{ij}<lp^{r-1},\forall i<j\}$ and $\Theta^{-}(n)_r=\{A\in\Theta^{-}(n)\mid 0\leqslant a_{ij}<lp^{r-1},\forall i>j\}$.

Lemma 5.3. *Let $\lambda\in\Lambda(n,d)$. The following results hold in $\mathfrak{T}_{\mathcal{K}}(n,d)_r$. (1) If $A\in\Theta^{+}(n)_r$ and $\lambda_i<\sigma_i(A)$ for some i, then $\mathfrak{e}^{(A)}\mathfrak{k}_\lambda=0$. (2) If $A\in\Theta^{-}(n)_r$ and $\lambda_i<\sigma_i(A)$ for some i, then $\mathfrak{k}_\lambda\mathfrak{f}^{(A)}=0$.*

Proof. Assume $A\in\Theta^{+}(n)_r$ and $\lambda_i<\sigma_i(A)$ for some i. For $\mu\in\mathbb{Z}^n, 1\leqslant j\leqslant n-1$, we have $\mathfrak{e}_j\mathfrak{k}_\mu=\mathfrak{k}_{\mu+\alpha_j}\mathfrak{e}_j$. This implies that

$$\mathfrak{m}_i(A)\mathfrak{m}_{i-1}(A)\cdots\mathfrak{m}_2(A)\mathfrak{k}_\lambda=\mathfrak{k}_{\lambda+\nu}\mathfrak{m}_i(A)\mathfrak{m}_{i-1}(A)\cdots\mathfrak{m}_2(A),$$

where $\mathfrak{m}_j=\mathfrak{e}_{j-1}^{(a_{j-1,j})}(\mathfrak{e}_{j-2}^{(a_{j-2,j})}\mathfrak{e}_{j-1}^{(a_{j-2,j})})\cdots(\mathfrak{e}_1^{(a_{1,j})}\mathfrak{e}_2^{(a_{1,j})}\cdots\mathfrak{e}_{j-1}^{(a_{1,j})})$ and

$$\nu=\sum_{2\leqslant j\leqslant i}\bigg(\sum_{1\leqslant s\leqslant j-1}a_{s,j}\boldsymbol{e}_s-(\sum_{1\leqslant s\leqslant j-1}a_{s,j})\boldsymbol{e}_j\bigg).$$

Thus, we have $\mathfrak{e}^{(A)}\mathfrak{k}_\lambda=\mathfrak{m}_n\mathfrak{m}_{n-1}\cdots\mathfrak{m}_{i+1}\mathfrak{k}_{\lambda+\nu}\mathfrak{m}_i\mathfrak{m}_{i-1}\cdots\mathfrak{m}_2$. Since $\lambda_i<\sigma_i(A)$, we have $\lambda+\nu\notin\Lambda(n,d)$. It follows that $\mathfrak{k}_{\lambda+\nu}=0$. Therefore, we have $\mathfrak{e}^{(A)}\mathfrak{k}_\lambda=0$. Applying $\dot\tau_d$ to the identity in (1) gives that in (2). □

Proposition 5.4. *Let $\mathfrak{M}_{d,r}=\{\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda\mathfrak{f}^{(A^-)}\mid A\in\Theta^{\pm}(n)_r, \lambda\in\Lambda(n,d), \lambda\geqslant\boldsymbol{\sigma}(A)\}$. Then, the algebra $\mathfrak{T}_{\mathcal{K}}(n,d)_r$ is spanned as a $\mathcal{K}$-module by the elements in $\mathfrak{M}_{d,r}$.*

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Proof. By Proposition 3.3, we have

$$\mathfrak{T}_{\mathcal{K}}(n,d)_r = \mathrm{span}_{\mathcal{K}}\{\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda \mathfrak{f}^{(A^-)} \mid A \in \Theta^{\pm}(n)_r, \lambda \in \Lambda(n,d)\}.$$

Thus, it is enough to prove that if $\lambda \in \Lambda(n,d)$ and $\lambda_i < \sigma_i(A)$ for some i, $\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda \mathfrak{f}^{(A^-)} \in \mathrm{span}_{\mathcal{K}}\mathfrak{M}_{d,r}$. We apply induction on $\deg(A)$. If $\deg(A)=1$, then by Lemma 5.3 we have $\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda \mathfrak{f}^{(A^-)} = 0$. Now suppose $\deg(A)>1$. For $2 \leqslant j \leqslant n$ let

$$\begin{aligned}\mathfrak{m}_j &= \mathfrak{e}_{j-1}^{(a_{j-1,j})}(\mathfrak{e}_{j-2}^{(a_{j-2,j})}\mathfrak{e}_{j-1}^{(a_{j-2,j})})\cdots(\mathfrak{e}_{1}^{(a_{1,j})}\mathfrak{e}_{2}^{(a_{1,j})}\cdots\mathfrak{e}_{j-1}^{(a_{1,j})})\\ \mathfrak{m}'_j &= (\mathfrak{f}_{j-1}^{(a_{j,1})}\cdots\mathfrak{f}_{2}^{(a_{j,1})}\mathfrak{f}_{1}^{(a_{j,1})})\cdots(\mathfrak{f}_{j-1}^{(a_{j,j-2})}\mathfrak{f}_{j-2}^{(a_{j,j-2})})\mathfrak{f}_{j-1}^{(a_{j,j-1})}.\end{aligned}$$

Then, we have

$$\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda \mathfrak{f}^{(A^-)} = X_1(X_2\mathfrak{k}_\lambda)Y_1Y_2 = X_1(\mathfrak{k}_{\lambda'}X_2)Y_1Y_2,$$

where $X_1 = \mathfrak{m}_n\mathfrak{m}_{n-1}\cdots\mathfrak{m}_{i+1}$, $X_2 = \mathfrak{m}_i\mathfrak{m}_{i-1}\cdots\mathfrak{m}_2$, $Y_1 = \mathfrak{m}'_2\mathfrak{m}'_3\cdots\mathfrak{m}'_i$, $Y_2 = \mathfrak{m}'_{i+1}\mathfrak{m}'_{i+2}\cdots\mathfrak{m}'_n$ and

$$\lambda' = \lambda + \sum_{2\leqslant j\leqslant i}\left(\sum_{1\leqslant s\leqslant j-1} a_{s,j}\boldsymbol{e}_s - \left(\sum_{1\leqslant s\leqslant j-1} a_{s,j}\right)\boldsymbol{e}_j\right).$$

By Lemma 2.1 and 5.2, we have $\mathfrak{k}_{\lambda'}X_2Y_1 = \mathfrak{k}_{\lambda'}Y_1X_2 + \mathfrak{k}_{\lambda'}f_1f_2$ where f_1 is a $\mathcal{K}$-linear combination of monomials $f_{1,k}$ in the $\mathfrak{e}_s^{(a)}$, f_2 is a $\mathcal{K}$-linear combination of monomials $f_{2,k}$ in the $\mathfrak{f}_s^{(a)}$, and $\deg(f_{1,k})+\deg(f_{2,k}) < \deg(X_2)+\deg(Y_1)$. Since $\lambda_i < \sigma_i(A)$, we have $\lambda'_i < \sigma_i(A^-)$. Hence by Lemma 5.3, we have $\mathfrak{k}_{\lambda'}Y_1 = 0$. This implies that

$$\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda \mathfrak{f}^{(A^-)} = X_1\mathfrak{k}_{\lambda'}f_1f_2Y_2 = \mathfrak{k}_{\lambda''}X_1f_1f_2Y_2,$$

where

$$\lambda'' = \lambda + \sum_{2\leqslant j\leqslant n}\left(\sum_{1\leqslant s\leqslant j-1} a_{s,j}\boldsymbol{e}_s - \left(\sum_{1\leqslant s\leqslant j-1} a_{s,j}\right)\boldsymbol{e}_j\right).$$

By Proposition 3.1, we have $X_1f_{1,k} \in \mathrm{span}_{\mathcal{K}}\{\mathfrak{e}^{(B)} \mid B \in \Theta^+(n)_r, \deg(B) = \deg(X_1f_{1,k})\}$ and $f_{2,k}Y_2 \in \mathrm{span}_{\mathcal{K}}\{\mathfrak{f}^{(C)} \mid C \in \Theta^-(n)_r, \deg(C) = \deg(f_{2,k}Y_2)\}$. Thus,

$$X_1f_1f_2Y_2 \in \mathrm{span}_{\mathcal{K}}\{\mathfrak{e}^{(B^+)}\mathfrak{f}^{(B^-)} \mid B \in \Theta^{\pm}(n)_r, \deg(B) < \deg(A)\}.$$

By induction, we have $\mathfrak{k}_{\lambda''}\mathfrak{e}^{(B^+)}\mathfrak{f}^{(B^-)} \in \mathrm{span}_{\mathcal{K}}\mathfrak{M}_{d,r}$ for $B \in \Theta^{\pm}(n)_r$ with $\deg(B) < \deg(A)$. Therefore, $\mathfrak{e}^{(A^+)}\mathfrak{k}_\lambda \mathfrak{f}^{(A^-)} = \mathfrak{k}_{\lambda''}X_1f_1f_2Y_2 \in \mathrm{span}_{\mathcal{K}}\mathfrak{M}_{d,r}$. ∎

By Lemma 3.6, we have $\dot{\zeta}_d(\dot{U}_{\mathcal{K}}(G_rT)) = \mathcal{S}_{\mathcal{K}}(n,d)_r$. Therefore, the map $\dot{\zeta}_d : \dot{U}_{\mathcal{K}}(\mathfrak{gl}_n) \to \mathcal{S}_{\mathcal{K}}(n,d)$ given in (3.10) restricts to a surjective algebra homomorphism

$$\dot{\zeta}_{d,r} : \dot{U}_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r. \tag{5.1}$$

Since $\dot{\zeta}_{d,r}(J_{d,r}) = 0$, the map $\dot{\zeta}_{d,r} : \dot{U}_{\mathcal{K}}(G_rT) \to \mathcal{S}_{\mathcal{K}}(n,d)_r$ induces a surjective algebra homomorphism

$$\dot{\zeta}'_{d,r} : \mathfrak{T}_{\mathcal{K}}(n,d)_r = \dot{U}_{\mathcal{K}}(G_rT)/J_{d,r} \to \mathcal{S}_{\mathcal{K}}(n,d)_r.$$

A presentation of $\mathcal{S}_{\mathcal{K}}(n,d)_r$ was given in [16, Th. 3.9] in the case where $r=1$, $\mathcal{K}$ is a field and l' is odd. We now generalize this result to the general case.

THEOREM 5.5. *The map $\dot{\zeta}'_{d,r}:\mathfrak{T}_{\mathcal{K}}(n,d)_r\to\mathcal{S}_{\mathcal{K}}(n,d)_r$ is an algebra isomorphism. In particular, the kernel of the map $\dot{\zeta}_{d,r}:\dot{U}_{\mathcal{K}}(G_rT)\to\mathcal{S}_{\mathcal{K}}(n,d)_r$ is generated by the elements 1_λ for $\lambda\notin\Lambda(n,d)$.*

Proof. By Proposition 3.7, the set $\dot{\zeta}'_{d,r}(\mathfrak{M}_{d,r})$ forms a $\mathcal{K}$-basis for $\mathcal{S}_{\mathcal{K}}(n,d)_r$. Therefore, by Proposition 5.4, we conclude that $\dot{\zeta}'_{d,r}$ is an algebra isomorphism. □

§6. The classical case

Let $\mathcal{U}(\mathfrak{gl}_n)$ be the $\mathbb{Q}$-algebra defined by the generators

$$\bar{E}_i,\ \bar{F}_i\quad(1\leqslant i\leqslant n-1),\ H_j\quad(1\leqslant j\leqslant n),$$

and the relations

(a) $H_iH_j=H_jH_i$;
(b) $H_i\bar{E}_j-\bar{E}_jH_i=(\delta_{i,j}-\delta_{i,j+1})\bar{E}_j$;
(c) $H_i\bar{F}_j-\bar{F}_jH_i=(-\delta_{i,j}+\delta_{i,j+1})\bar{F}_j$;
(d) $\bar{E}_i\bar{E}_j=\bar{E}_j\bar{E}_i,\ \bar{F}_i\bar{F}_j=\bar{F}_j\bar{F}_i$ *when* $|i-j|>1$;
(e) $\bar{E}_i\bar{F}_j-\bar{F}_j\bar{E}_i=\delta_{i,j}H_i$;
(f) $\bar{E}_i^2\bar{E}_j-2\bar{E}_i\bar{E}_j\bar{E}_i+\bar{E}_j\bar{E}_i^2=0$ *when* $|i-j|=1$;
(g) $\bar{F}_i^2\bar{F}_j-2\bar{F}_i\bar{F}_j\bar{F}_i+\bar{F}_j\bar{F}_i^2=0$ *when* $|i-j|=1$.

Then, $\mathcal{U}(\mathfrak{gl}_n)$ is the universal enveloping algebra of $\mathfrak{gl}_n$. Let $\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n)$ be the $\mathbb{Z}$-subalgebra of $\mathcal{U}(\mathfrak{gl}_n)$ generated by $\bar{E}_i^{(m)}$, $\bar{F}_i^{(m)}$, and $\binom{H_j}{t}$ for $1\leqslant i\leqslant n-1$, $1\leqslant j\leqslant n$ and $m,t\in\mathbb{N}$, where

$$\bar{E}_i^{(m)}=\frac{\bar{E}_i^m}{m!},\ \bar{F}_i^{(m)}=\frac{\bar{F}_i^m}{m!},\ \binom{H_j}{t}=\frac{H_j(H_j-1)\cdots(H_j-t+1)}{t!}.$$

Let $U_{\mathbb{Z}}(\mathfrak{gl}_n)=U_{\mathcal{Z}}(\mathfrak{gl}_n)\otimes_{\mathcal{Z}}\mathbb{Z}$, where $\mathbb{Z}$ is viewed as $\mathcal{Z}$-modules by specializing v to 1. Let $\bar{U}_{\mathbb{Z}}(\mathfrak{gl}_n)=U_{\mathbb{Z}}(\mathfrak{gl}_n)/\langle K_i-1\mid 1\leqslant i\leqslant n\rangle$. We shall denote the images of $E_i^{(m)}$, $F_i^{(m)}$, etc. in $\bar{U}_{\mathbb{Z}}(\mathfrak{gl}_n)$ by the same letters. By [22, 6.7(c)], there is an algebra isomorphism

$$\theta:\bar{U}_{\mathbb{Z}}(\mathfrak{gl}_n)\to\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n),\tag{6.1}$$

such that $\theta(E_i^{(m)})=\bar{E}_i^{(m)}$, $\theta(F_i^{(m)})=\bar{F}_i^{(m)}$, $\theta(\left[\begin{smallmatrix}K_j;0\\ t\end{smallmatrix}\right])=\binom{H_j}{t}$ for $1\leqslant i\leqslant n-1$, $1\leqslant j\leqslant n$, $m,t\in\mathbb{N}$. We will identify $\bar{U}_{\mathbb{Z}}(\mathfrak{gl}_n)$ with $\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n)$.

Let $\mathcal{S}_{\mathbb{Z}}(n,d)=\mathcal{S}_{\mathcal{Z}}(n,d)\otimes_{\mathcal{Z}}\mathbb{Z}$ where $\mathbb{Z}$ is viewed as $\mathcal{Z}$-modules by specializing v to 1. The map ζ_d given in (3.6) induces, upon tensoring with $\mathbb{Z}$, a surjective algebra homomorphism

$$\xi_d:U_{\mathbb{Z}}(\mathfrak{gl}_n)\to\mathcal{S}_{\mathbb{Z}}(n,d).$$

Since $\xi_d(K_i)=1$, the map ξ_d induces a surjective algebra homomorphism

$$\xi_d:\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n)=\bar{U}_{\mathbb{Z}}(\mathfrak{gl}_n)\to\mathcal{S}_{\mathbb{Z}}(n,d).$$

In the remainder of this section, we assume that $l'=l=1$. Let

$$\mathcal{U}_{\mathcal{K}}(\mathfrak{gl}_n)=\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n)\otimes_{\mathbb{Z}}\mathcal{K},\quad \bar{\mathcal{S}}_{\mathcal{K}}(n,d)=\mathcal{S}_{\mathbb{Z}}(n,d)\otimes_{\mathbb{Z}}\mathcal{K}.$$

We shall denote the images of $\bar{E}_i^{(m)}$, $\bar{F}_i^{(m)}$, etc. in $\mathcal{U}_{\mathcal{K}}(\mathfrak{gl}_n)$ by the same letters. For $A\in\Theta(n,d)$, let $[A]_1$ be the image of $[A]$ in $\bar{\mathcal{S}}_{\mathcal{K}}(n,d)$. The map ξ_d induces, upon tensoring with

$\mathcal{K}$, a surjective algebra homomorphism

$$\xi_d : \mathcal{U}_{\mathcal{K}}(\mathfrak{gl}_n) \to \bar{\mathcal{S}}_{\mathcal{K}}(n,d).$$

Let $\bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$ be the infinitesimal Schur algebra introduced in [11]. By [11, (5.3.4)], the set $\{[A]_1 \mid A \in \Theta(n,d), a_{i,j} < p^r, \forall i,j\}$ forms a $\mathcal{K}$-basis for $\bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$. Hence, we have

$$\bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r \cong \mathcal{S}_{\mathcal{K}}(n,d)_{r+1}.$$

Let $\mathcal{U}_{\mathcal{K}}(G_rT)$ be the $\mathcal{K}$-subalgebra of $\mathcal{U}_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $\bar{E}_i^{(m)}$, $\bar{F}_i^{(m)}$, $\binom{H_j}{t}$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, $t \in \mathbb{N}$ and $0 \leqslant m < p^r$. Then, by (6.1), we have

$$\mathcal{U}_{\mathcal{K}}(G_rT) \cong U_{\mathcal{K}}(G_{r+1}T)/\langle K_i - 1 \mid 1 \leqslant i \leqslant n \rangle.$$

Therefore, by Lemma 3.5, we have $\xi_d(\mathcal{U}_{\mathcal{K}}(G_rT)) = \bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$. Hence, by restricting the map ξ_d to $\mathcal{U}_{\mathcal{K}}(G_rT)$, we obtain a surjective algebra homomorphism

$$\xi_{d,r} : \mathcal{U}_{\mathcal{K}}(G_rT) \to \bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r.$$

By Theorem 4.10, we obtain the following result.

THEOREM 6.1. *The kernel of the map $\xi_{d,r} : \mathcal{U}_{\mathcal{K}}(G_rT) \to \bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$ is generated by the elements $1 - \sum_{\mu \in \Lambda(n,d)} H_\mu$ and $\binom{H_i}{t} H_\lambda - \binom{\lambda_i}{t} H_\lambda$ for $1 \leqslant i \leqslant n$, $t \in \mathbb{N}$ and $\lambda \in \Lambda(n,d)$, where $H_\lambda = \prod_{1 \leqslant i \leqslant n} \binom{H_i}{\lambda_i}$.*

Let

$$\dot{\mathcal{U}}(\mathfrak{gl}_n) := \bigoplus_{\lambda,\mu \in \mathbb{Z}^n} {}_\lambda\mathcal{U}(\mathfrak{gl}_n)_\mu,$$

where

$${}_\lambda\mathcal{U}(\mathfrak{gl}_n)_\mu = \mathcal{U}(\mathfrak{gl}_n) \Big/ \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} (H^{\mathbf{j}} - \lambda^{\mathbf{j}}) \mathcal{U}(\mathfrak{gl}_n) + \sum_{\mathbf{j} \in \mathbb{Z}^n} \mathcal{U}(\mathfrak{gl}_n)(H^{\mathbf{j}} - \mu^{\mathbf{j}}) \right),$$

$H^{\mathbf{j}} = \prod_{1 \leqslant i \leqslant n} H_i^{j_i}$ and $\lambda^{\mathbf{j}} = \prod_{1 \leqslant i \leqslant n} \lambda_i^{j_i}$. Let $\bar{\pi}_{\lambda,\mu} : \mathcal{U}(\mathfrak{gl}_n) \to {}_\lambda\mathcal{U}(\mathfrak{gl}_n)_\mu$ be the canonical projection. Let $\bar{1}_\lambda = \bar{\pi}_{\lambda,\lambda}(1)$. As in the case of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$, there is a natural associative $\mathbb{Q}$-algebra structure on $\dot{\mathcal{U}}(\mathfrak{gl}_n)$ inherited from that of $\mathcal{U}(\mathfrak{gl}_n)$, and $\dot{\mathcal{U}}(\mathfrak{gl}_n)$ is naturally a $\mathcal{U}(\mathfrak{gl}_n)$-bimodule. Let $\dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{gl}_n)$ be the $\mathbb{Z}$-subalgebra of $\dot{\mathcal{U}}(\mathfrak{gl}_n)$ generated by the elements $\bar{E}_i^{(m)} \bar{1}_\lambda$ and $\bar{1}_\lambda \bar{F}_i^{(m)}$ for $1 \leqslant i \leqslant n-1$, $\lambda \in \mathbb{Z}^n$ and $m \in \mathbb{N}$.

Let $\dot{U}_{\mathbb{Z}}(\mathfrak{gl}_n) = \dot{U}_{\mathcal{Z}}(\mathfrak{gl}_n) \otimes_{\mathcal{Z}} \mathbb{Z}$, where $\mathbb{Z}$ is viewed as a $\mathcal{Z}$-module by specializing v to 1. By (6.1), we have

$$\dot{U}_{\mathbb{Z}}(\mathfrak{gl}_n) \cong \dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{gl}_n). \tag{6.2}$$

We will identify $\dot{U}_{\mathbb{Z}}(\mathfrak{gl}_n)$ with $\dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{gl}_n)$. The map $\dot{\zeta}_d$ given in (3.9) induces, upon tensoring with $\mathbb{Z}$, a surjective algebra homomorphism

$$\dot{\xi}_d : \dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{gl}_n) \to \mathcal{S}_{\mathbb{Z}}(n,d).$$

Let $\dot{\mathcal{U}}_{\mathcal{K}}(\mathfrak{gl}_n) = \dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{gl}_n) \otimes_{\mathbb{Z}} \mathcal{K}$. We shall denote the images of $\bar{E}_i^{(m)} \bar{1}_\lambda$, $\bar{1}_\lambda \bar{F}_i^{(m)}$ in $\dot{\mathcal{U}}_{\mathcal{K}}(\mathfrak{gl}_n)$ by the same letters. The map $\dot{\xi}_d$ induces, upon tensoring with $\mathcal{K}$, a surjective algebra

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homomorphism

$$\dot{\xi}_d : \dot{\mathcal{U}}_{\mathcal{K}}(\mathfrak{gl}_n) \to \bar{\mathcal{S}}_{\mathcal{K}}(n,d).$$

Let $\dot{\mathcal{U}}_{\mathcal{K}}(G_rT)$ be the $\mathcal{K}$-subalgebra of $\dot{\mathcal{U}}_{\mathcal{K}}(\mathfrak{gl}_n)$ generated by the elements $\bar{E}_i^{(m)}\bar{1}_\lambda$, $\bar{1}_\lambda\bar{F}_i^{(m)}$ for $1 \leqslant i \leqslant n-1$, $\lambda \in \mathbb{Z}^n$ and $0 \leqslant m < p^r$. Then, by (6.2), we have

$$\dot{\mathcal{U}}_{\mathcal{K}}(G_rT) \cong \dot{U}_{\mathcal{K}}(G_{r+1}T).$$

By Lemma 3.6, we have $\dot{\xi}_d(\dot{\mathcal{U}}_{\mathcal{K}}(G_rT)) = \bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$. Hence, by restricting the map $\dot{\xi}_d$ to $\dot{\mathcal{U}}_{\mathcal{K}}(G_rT)$, we obtain a surjective algebra homomorphism

$$\dot{\xi}_{d,r} : \dot{\mathcal{U}}_{\mathcal{K}}(G_rT) \to \bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r.$$

By Theorem 5.5, we obtain the following result.

THEOREM 6.2. *The kernel of the map* $\dot{\xi}_{d,r} : \dot{\mathcal{U}}_{\mathcal{K}}(G_rT) \to \bar{\mathcal{S}}_{\mathcal{K}}(n,d)_r$ *is generated by the elements* $\bar{1}_\lambda$ *for* $\lambda \notin \Lambda(n,d)$.

§7. Borel subalgebras of the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$

In this section, we investigate Borel subalgebras of the infinitesimal q-Schur algebra $\mathcal{S}_{\mathcal{K}}(n,d)_r$. In what follows, we focus entirely on the quantum case as the corresponding results for the classical cases are essentially the same.

Let $U_{\mathcal{K}}(B_r^+T) = U_{\mathcal{K}}^+(G_rT)U_{\mathcal{K}}^0(\mathfrak{gl}_n)$ and $U_{\mathcal{K}}(B_rT) = U_{\mathcal{K}}^0(\mathfrak{gl}_n)U_{\mathcal{K}}^-(G_rT)$. These algebras are called Borel subalgebras of $U_{\mathcal{K}}(G_rT)$. Furthermore, let $\dot{U}_{\mathcal{K}}(B_r^+T)$ (resp. $\dot{U}_{\mathcal{K}}(B_rT)$) be the $\mathcal{K}$-subalgebra of $\dot{U}_{\mathcal{K}}(G_rT)$ generated by the elements $E_i^{(m)}1_\lambda$ (resp. $1_\lambda F_i^{(m)}$) for $1 \leqslant i \leqslant n-1$, $\lambda \in \mathbb{Z}^n$ and $0 \leqslant m < lp^{r-1}$.

Let $\mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ (resp. $\mathcal{S}_{\mathcal{K}}^{\leqslant 0}(n,d)_r$) be the $\mathcal{K}$-subalgebra of $\mathcal{S}_{\mathcal{K}}(n,d)_r$ generated by $\mathtt{e}_i^{(m)}$ (resp. $\mathtt{f}_i^{(m)}$) and $\mathtt{k}_\lambda$ for $1 \leqslant i \leqslant n-1$, $0 \leqslant m < lp^{r-1}$ and $\lambda \in \Lambda(n,d)$. These algebras are called Borel subalgebras of $\mathcal{S}_{\mathcal{K}}(n,d)_r$.

Let $\mathcal{S}_{\mathcal{K}}^+(n,d)_r$ (resp. $\mathcal{S}_{\mathcal{K}}^-(n,d)_r$) be the $\mathcal{K}$-subalgebra of $\mathcal{S}_{\mathcal{K}}(n,d)_r$ generated by $\mathtt{e}_i^{(m)}$ (resp. $\mathtt{f}_i^{(m)}$) for $1 \leqslant i \leqslant n-1$ and $0 \leqslant m < lp^{r-1}$.

LEMMA 7.1. *Each of the following set forms a* $\mathcal{K}$*-basis of* $\mathcal{S}_{\mathcal{K}}^+(n,d)_r$*:*

(1) $\mathscr{M}_{d,r}^+ := \{\mathtt{e}^{(A)} \mid A \in \Theta^+(n)_r, \sigma(A) \leqslant d\}$;
(2) $\mathscr{P}_{d,r}^+ := \{A(\mathbf{0},d) \mid A \in \Theta^+(n)_r, \sigma(A) \leqslant d\}$.

A similar result holds for $\mathcal{S}_{\mathcal{K}}^-(n,d)_r$.

Proof. By [8, Prop. 8.2], we have $1 = \sum_{\lambda \in \Lambda(n,d)} \mathtt{k}_\lambda$. Hence, by [15, Lem. 4.10], we have

$$\mathtt{e}^{(A)} = \sum_{\lambda \in \Lambda(n,d)} \mathtt{e}^{(A)}\mathtt{k}_\lambda = \sum_{\substack{\lambda \in \Lambda(n,d) \\ \lambda \geqslant \sigma(A)}} \mathtt{e}^{(A)}\mathtt{k}_\lambda, \tag{7.1}$$

for $A \in \Theta^+(n)_r$. Furthermore, we have

$$A(\mathbf{0},d) = \sum_{\lambda \in \Lambda(n,d)} A(\mathbf{0},d)[\mathrm{diag}(\lambda)]_\varepsilon = \sum_{\substack{\lambda \in \Lambda(n,d) \\ \lambda \geqslant \sigma(A)}} A(\mathbf{0},d)[\mathrm{diag}(\lambda)]_\varepsilon, \tag{7.2}$$

for $A \in \Theta^+(n)_r$, since $1 = \sum_{\lambda \in \Lambda(n,d)} [\mathrm{diag}(\lambda)]_\varepsilon$. Therefore, we have

$$\mathsf{e}^{(A)} = A(\mathbf{0}, d) = 0,$$

for $A \in \Theta^+(n)_r$ with $\sigma(A) > d$. Hence, by Proposition 3.1 and 3.2, we conclude that $\mathcal{S}_{\mathcal{K}}^+(n,d)_r = \mathrm{span}_{\mathcal{K}} \mathscr{M}_{d,r}^+ = \mathrm{span}_{\mathcal{K}} \mathscr{P}_{d,r}^+$. Furthermore, by Proposition 3.7, the sets $\mathscr{M}_{d,r}^+$ and $\mathscr{P}_{d,r}^+$ are both linearly independent. Our assertion follows. ∎

LEMMA 7.2. *Each of the following set forms a $\mathcal{K}$-basis of $\mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$:*

(1) $\mathscr{M}_{d,r}^{\geqslant 0} := \{\mathsf{e}^{(A)}\mathtt{k}_\lambda \mid A \in \Theta^+(n)_r, \lambda \in \Lambda(n,d), \lambda \geqslant \boldsymbol{\sigma}(A)\}$;
(2) $\mathscr{L}_{d,r}^{\geqslant 0} := \{[A + \mathrm{diag}(\lambda)]_\varepsilon \mid A \in \Theta^+(n)_r, \lambda \in \Lambda(n, d - \sigma(A))\}$.

A similar result holds for $\mathcal{S}_{\mathcal{K}}^{\leqslant 0}(n,d)_r$.

Proof. From Lemma 7.1, (7.1) and (7.2), it follows that $\mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r = \mathrm{span}_{\mathcal{K}} \mathscr{M}_{d,r}^{\geqslant 0} = \mathrm{span}_{\mathcal{K}} \mathscr{L}_{d,r}^{\geqslant 0}$. Therefore, the result follows from Proposition 3.7. ∎

Let $\mathcal{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ be the quotient of $U_{\mathcal{K}}(B_r^+T)$ by $I_{d,r}^{\geqslant 0}$, where $I_{d,r}^{\geqslant 0}$ is the two-sided ideal of $U_{\mathcal{K}}(B_r^+T)$ generated by the elements $1 - \sum_{\mu \in \Lambda(n,d)} K_\mu, K_i K_\lambda - \varepsilon^{\lambda_i} K_\lambda$ and $\left[{K_i;0 \atop t} \right] K_\lambda - \left[{\lambda_i \atop t} \right]_\varepsilon K_\lambda$ for $1 \leqslant i \leqslant n$, $t \in \mathbb{N}$ and $\lambda \in \Lambda(n,d)$.

By restricting the map $\zeta_{d,r}$ given in (4.2) to $U_{\mathcal{K}}(B_r^+T)$, we obtain a surjective algebra homomorphism $\zeta_{d,r} : U_{\mathcal{K}}(B_r^+T) \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$. Since $\zeta_{d,r}(I_{d,r}^{\geqslant 0}) = 0$, the map $\zeta_{d,r}$ induces an epimorphism

$$\zeta'_{d,r} : \mathcal{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r = U_{\mathcal{K}}(B_r^+T)/I_{d,r}^{\geqslant 0} \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r.$$

THEOREM 7.3. *The map $\zeta'_{d,r} : \mathcal{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ is an algebra isomorphism. In particular, the kernel of the map $\zeta_{d,r} : U_{\mathcal{K}}(B_r^+T) \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ is $I_{d,r}^{\geqslant 0}$. A similar result holds for $\mathcal{S}_{\mathcal{K}}^{\leqslant 0}(n,d)_r$.*

Proof. Using an argument similar to the proof of Proposition 4.9, we can show that the algebra $\mathcal{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ is spanned as a $\mathcal{K}$-module by the elements $E^{(A)}K_\lambda + I_{d,r}^{\geqslant 0}$ for $A \in \Theta^+(n)_r$, $\lambda \in \Lambda(n,d)$ and $\lambda \geqslant \boldsymbol{\sigma}(A)$. Furthermore, by Lemma 7.2, the set

$$\{\zeta_{d,r}(E^{(A)}K_\lambda + I_{d,r}^{\geqslant 0}) \mid A \in \Theta^+(n)_r, \lambda \in \Lambda(n,d), \lambda \geqslant \boldsymbol{\sigma}(A)\},$$

forms a $\mathcal{K}$-basis for $\mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$. Hence, $\zeta'_{d,r}$ is an algebra isomorphism. ∎

Let $\mathfrak{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ be the quotient of $\dot{U}_{\mathcal{K}}(B_r^+T)$ by $J_{d,r}^{\geqslant 0}$, where $J_{d,r}^{\geqslant 0}$ is the two-sided ideal of $\dot{U}_{\mathcal{K}}(B_r^+T)$ generated by the elements 1_λ for $\lambda \notin \Lambda(n,d)$.

By restricting the map $\dot{\zeta}_{d,r}$ given in (5.1) to $\dot{U}_{\mathcal{K}}(B_r^+T)$, we obtain a surjective algebra homomorphism $\dot{\zeta}_{d,r} : \dot{U}_{\mathcal{K}}(B_r^+T) \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$. Since $\dot{\zeta}_{d,r}(J_{d,r}^{\geqslant 0}) = 0$, the map $\dot{\zeta}_{d,r}$ induces an epimorphism

$$\dot{\zeta}'_{d,r} : \mathfrak{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r = \dot{U}_{\mathcal{K}}(B_r^+T)/J_{d,r}^{\geqslant 0} \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r.$$

THEOREM 7.4. *The map $\dot{\zeta}'_{d,r} : \mathfrak{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ is an algebra isomorphism. In particular, the kernel of the map $\dot{\zeta}_{d,r} : \dot{U}_{\mathcal{K}}(B_r^+T) \to \mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ is $J_{d,r}^{\geqslant 0}$. A similar result holds for $\mathcal{S}_{\mathcal{K}}^{\leqslant 0}(n,d)_r$.*

Proof. Using an argument similar to the proof of Proposition 5.4, we can show that the algebra $\mathfrak{T}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$ is spanned as a $\mathcal{K}$-module by the elements $E^{(A)}1_\lambda + J_{d,r}^{\geqslant 0}$ for $A \in \Theta^+(n)_r$, $\lambda \in \Lambda(n,d)$ and $\lambda \geqslant \boldsymbol{\sigma}(A)$. Furthermore, by Lemma 7.2, the set

$$\{\dot{\zeta}'_{d,r}(E^{(A)}1_\lambda + J_{d,r}^{\geqslant 0}) \mid A \in \Theta^+(n)_r, \lambda \in \Lambda(n,d), \lambda \geqslant \boldsymbol{\sigma}(A)\},$$

forms a $\mathcal{K}$-basis of $\mathcal{S}_{\mathcal{K}}^{\geqslant 0}(n,d)_r$. Therefore, $\dot{\zeta}'_{d,r}$ is an algebra isomorphism. □

§8. Irreducible $\mathcal{S}_{\mathcal{K}}(n,d)_r$-modules

In this section, we assume that $\mathcal{K}$ is a field, $p > 0$ and $l' = l$ is odd. Let $X = \mathbb{Z}^n$ and $X^+ = \{\lambda \in X \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\}$. For $\lambda \in X^+$, let $L(\lambda)$ be the simple integrable $U_{\mathcal{K}}(\mathfrak{gl}_n)$-module of highest weight λ. Let $\mathrm{Ind}_{U_1}^{U_2}(-) = H^0(U_2/U_1, -)$ be the induction functor for quantized enveloping algebras defined in [1], [2].

For $\lambda \in X$, let

$$\widehat{Z}_r(\lambda) = \mathrm{Ind}_{U_{\mathcal{K}}(B_rT)}^{U_{\mathcal{K}}(G_rT)}\lambda, \quad \widehat{L}_r(\lambda) = \mathrm{soc}_{U_{\mathcal{K}}(G_rT)}\widehat{Z}_r(\lambda).$$

Let $P_r = \{\lambda \in \mathbb{N}^n \mid 0 \leqslant \lambda_i - \lambda_{i+1} < lp^{r-1}$ for $1 \leqslant i \leqslant n\}$, where $\lambda_{n+1} = 0$. The following result was given in [12, Ths. 3.4.1 & 3.4.3].

THEOREM 8.1. (1) *The set* $\{\widehat{L}_r(\lambda) \mid \lambda \in X\}$ *form a complete set of pairwise nonisomorphic irreducible integrable* $U_{\mathcal{K}}(G_rT)$*-modules.*

(2) *For* $\lambda, \mu \in X$*, we have* $\widehat{L}_r(\lambda + lp^{r-1}\mu) \cong \widehat{L}_r(\lambda) \otimes lp^{r-1}\mu$.

(3) *For* $\lambda \in P_r$*, we have* $L(\lambda)|_{U_{\mathcal{K}}(G_rT)} \cong \widehat{L}_r(\lambda)$.

If M is a $U_{\mathcal{K}}(G_rT)$-module and $\lambda \in X$ let

$$M_\lambda = \{w \in M \mid K_i w = \varepsilon^{\lambda_i} w, \begin{bmatrix} K_i;0 \\ t \end{bmatrix} w = \begin{bmatrix} \lambda_i \\ t \end{bmatrix}_\varepsilon w \text{ for } 1 \leqslant i \leqslant n, t \in \mathbb{N}\}.$$

Let $\Gamma_r = P_r + lp^{r-1}\mathbb{N}^n$ and $\Gamma_r^d = \left\{\lambda \in \Gamma_r \middle| \sum_{i=1}^n \lambda_i = d\right\}$. For $\lambda, \mu \in \mathbb{Z}^n$ with $\sum_{1\leqslant i\leqslant n} \lambda_i = \sum_{1\leqslant i\leqslant n} \mu_i$ we write $\lambda \trianglelefteq \mu$ if $\sum_{1\leqslant s\leqslant i} \lambda_s \leqslant \sum_{1\leqslant s\leqslant i} \mu_s$ for $1 \leqslant i \leqslant n$.

LEMMA 8.2. *For* $\lambda \in \Gamma_r^d$ *we have* $\widehat{L}_r(\lambda) = \oplus_{\mu\in\Lambda(n,d)} \widehat{L}_r(\lambda)_\mu$.

Proof. We write $\lambda = \alpha + lp^{r-1}\beta$ with $\alpha \in P_r$ and $\beta \in \mathbb{N}^n$. By Theorem 8.1, we have $\widehat{L}_r(\lambda) \cong L(\alpha) \otimes lp^{r-1}\beta$. Hence, it suffices to show that $L(\alpha) = \oplus_{\mu\in\Lambda(n,d')} L(\alpha)_\mu$, where $d' = \sum_{1\leqslant i\leqslant n} \alpha_i$. If $L(\alpha)_\mu \neq 0$ for some $\mu \in \mathbb{Z}^n$ with $\sum_{1\leqslant i\leqslant n} \mu_i = d'$. We claim that $\mu \in \mathbb{N}^n$. Otherwise, there exists some element w in the symmetric group $\mathfrak{S}_n$ such that $\gamma = (\mu_{w(1)}, \ldots, \mu_{w(n)})$ and $\gamma_n < 0$. Since $L(\alpha)_\mu \neq 0$, we have $L(\alpha)_\gamma \neq 0$ and hence $\gamma \trianglelefteq \alpha$. This implies that $\sum_{1\leqslant i\leqslant n-1} \gamma_i \leqslant \sum_{1\leqslant i\leqslant n-1} \alpha_i \leqslant d'$. Hence, since $\gamma_n < 0$, we have $\sum_{1\leqslant i\leqslant n} \gamma_i < d'$. This is a contradiction. The assertion follows. □

The irreducible modules for infinitesimal q-Schur algebras were classified in [4, Sec. 5.1]. We now use Theorem 4.10 to give a classification of irreducible $\mathcal{S}_{\mathcal{K}}(n,d)_r$-modules.

THEOREM 8.3. *The set* $\{\widehat{L}_r(\lambda) \mid \lambda \in \Gamma_r^d\}$ *forms a completed set of pairwise nonisomorphic irreducible* $\mathcal{S}_{\mathcal{K}}(n,d)_r$*-modules.*

Proof. Let $\lambda \in \Gamma_r^d$. By Lemma 8.2, we have $\widehat{L}_r(\lambda) = \oplus_{\mu\in\Lambda(n,d)} \widehat{L}_r(\lambda)_\mu$. Let w_μ be a nonzero vector in $\widehat{L}_r(\lambda)_\mu$ for some $\mu \in \Lambda(n,d)$. Since $\mu \in \Lambda(n,d)$, we have $K_\alpha w_\mu = \left[\begin{smallmatrix} \mu \\ \alpha \end{smallmatrix}\right]_\varepsilon w_\mu =$

$\delta_{\alpha,\mu}w_\mu$ for $\alpha\in\Lambda(n,d)$. It follows that

$$\sum_{\beta\in\Lambda(n,d)} K_\beta w_\mu = \sum_{\beta\in\Lambda(n,d)} \delta_{\beta,\mu}w_\mu = w_\mu,$$
$$(K_i-\varepsilon^{\alpha_i})K_\alpha w_\mu = \delta_{\alpha,\mu}K_i w_\mu - \varepsilon^{\alpha_i}\delta_{\alpha,\mu}w_\mu = 0,$$
$$\left(\begin{bmatrix} K_i;0 \\ t\end{bmatrix} - \begin{bmatrix}\alpha_i \\ t\end{bmatrix}\right)K_\alpha w_\mu = \delta_{\alpha,\mu}\begin{bmatrix} K_i;0 \\ t\end{bmatrix} w_\mu - \begin{bmatrix}\alpha_i \\ t\end{bmatrix}\delta_{\alpha,\mu}w_\mu = 0,$$

for $\alpha\in\Lambda(n,d)$, $1\leqslant i\leqslant n$ and $t\in\mathbb{N}$. Thus, by Theorem 4.10, we conclude that $\widehat{L}_r(\lambda)$ can be regarded as a $\mathcal{S}_{\mathcal{K}}(n,d)_r$-module.

On the other hand, let L be an irreducible $\mathcal{S}_{\mathcal{K}}(n,d)_r$-module. By Theorem 8.1, we conclude that $L\cong\widehat{L}_r(\nu+lp^{r-1}\delta)\cong L(\nu)\otimes lp^{r-1}\delta$ for some $\nu\in P_r$ and $\delta\in\mathbb{Z}^n$. Hence, since L is a $\mathcal{S}_{\mathcal{K}}(n,d)_r$-module, we have $(\nu_{w(1)},\nu_{w(2)},\ldots,\nu_{w(n)})+lp^{r-1}\delta\in\Lambda(n,d)$ for any w in the symmetric group $\mathfrak{S}_n$. It follows that $\nu_n+lp^{r-1}\delta_j\geqslant 0$ for $1\leqslant j\leqslant n$. Furthermore, since $\nu\in P_r$, we have $0\leqslant\nu_n<lp^{r-1}$. Therefore, we have $\delta_j\geqslant 0$ for $1\leqslant j\leqslant n$. Consequently, we have $\nu+lp^{r-1}\delta\in\Gamma_r^d$. The proof is completed. ∎

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