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Superlinear elliptic problems under the non-quadraticity condition at infinity

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We present some sufficient conditions to obtain compactness properties for the Euler–Lagrange functional of an elliptic equation. As an application, we extend some existence and multiplicity results for superlinear problems.

Keywords: elliptic equations; superlinear problems; non-quadraticity condition; variational methods

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1. Introduction

In this paper we consider the nonlinear elliptic equation

$$\left.\begin{aligned} -\Delta u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad\ \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 3$, is a bounded smooth domain and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies the standard subcritical growth condition:

(f_0) there exist $a_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,s)| \leqslant a_1(1 + |s|^{p-1}) \quad \text{for any } (x,s) \in \Omega \times \mathbb{R}.$$

Under this condition the weak solutions of the problem are precisely the critical points of the C^1-functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x - \int_\Omega F(x,u) \, \mathrm{d}x, \quad u \in H_0^1(\Omega),$$

where $F(x,s) := \int_0^s f(x,\tau) \, \mathrm{d}\tau$. Hence, we can use all the machinery of the critical point theory to look for weak solutions. It is well known that this theory is based

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on the existence of a linking structure and on deformation lemmas [1, 2, 23, 25]. In general, to be able to derive such deformation results, it is supposed that the functional satisfies some compactness condition. We use here the Cerami condition, which reads as follows: the functional I satisfies the Cerami condition at level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ for short) if any sequence $(u_n) \subset H_0^1(\Omega)$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_{H_0^1(\Omega)'}(1 + \|u_n\|) \to 0$ has a convergent subsequence.

Our main objective is to present sufficient conditions to ensure that the functional satisfies the Cerami condition. More specifically, we shall consider the *non-quadraticity condition at infinity* introduced by Costa and Magalhães [4]:

(NQ) setting $H(x,s) := f(x,s)s - 2F(x,s)$, we have that

$$\lim_{|s|\to\infty} H(x,s) = +\infty \quad \text{uniformly for } x \in \Omega.$$

In the core result of this paper we show that the above condition and (f_0) are suffice to guarantee compactness for the functional I. More specifically, we prove the following result.

Theorem 1.1. *Suppose that f satisfies (f_0) and (NQ). Then the functional I satisfies the Cerami condition at any level $c \in \mathbb{R}$.*

As an application of this theorem we prove some new results for the problem (P) in the case that f is superlinear at infinity and at the origin. Furthermore, we give a unified approach for any superlinear elliptic problem using the non-quadraticity condition. In order to better explain our results we recall that, in their seminal work, Ambrosetti and Rabinowitz [1] introduced the following condition.

(AR) There exist $\theta > 2$ and $s_0 > 0$ such that

$$0 < \theta F(x,s) \leqslant s f(x,s) \quad \text{for any } x \in \Omega,\ |s| > s_0.$$

A straightforward calculation shows that this yields $c_1 > 0$ such that $F(x,s) \geqslant c_1|s|^\theta$ for $|s|$ large. Thus, the problem is called *superlinear* in the sense that the primitive of f lives above any parabola of the type $c_2 s^2$. Unfortunately, there are several nonlinearities which are superlinear but do not satisfy the above inequality. For example, if we take $f(s) = |s|\ln(1 + |s|)$, we can easily check that

$$\lim_{s\to+\infty} F(s)/s^\theta = 0$$

for any $\theta > 2$. So, it is reasonable to ask if we can replace (AR) by a more natural condition.

(SL) The following limit holds:

$$\lim_{|s|\to+\infty} \frac{2F(x,s)}{s^2} = +\infty \quad \text{uniformly for } x \in \Omega.$$

One of the main features of condition (AR) is that it provides the boundedness of Palais–Smale sequences. In the past 40 years many authors have tried to obtain the solution in situations where (AR) is no longer valid. Instead, they consider the

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condition (SL) with extra assumptions. See [4, 10–16, 18, 20, 21] and the references therein; in most of these papers, there is some kind of monotonicity assumption on the functions $F(x,s)$ or $f(x,s)/s$, or some convexity condition on the function $f(x,s)s - 2F(x,s)$.

Our results concerning problem (P) are stated below.

THEOREM 1.2. *Suppose that f satisfies (f_0), (NQ) and (SL). Then problem (1.1) has at least one non-zero weak solution, provided that*

(f_1) *it holds that*

$$\limsup_{s\to 0} \frac{F(x,s)}{s^2} = 0 \quad \textit{uniformly for } x \in \Omega.$$

If $f(x,s)$ is odd in s, then we can drop condition (f_1) and obtain infinitely many weak solutions.

We note that, for the existence result, we can suppose that the limit in (SL) holds only for $x \in \Omega_0$, where $\Omega_0 \subset \Omega$ is a subset with positive measure (see the proof of theorem 1.2). So, we can deal with nonlinearities that are locally superlinear at infinity.

In order to compare our existence result with the literature, we start by again citing [4], where Costa and Magalhães supposed, among other conditions, that

(F_μ) there exist $a_2 > 0$ and $\mu > \frac{1}{2}N(p-2)$ such that

$$\liminf_{|s|\to\infty} \frac{H(x,s)}{|s|^\mu} \geqslant a_2 \quad \text{uniformly for } x \in \Omega,$$

where the number $p \in (2, 2^*)$ comes from (f_0). Since $\mu > 0$, we see that (NQ) is weaker than (F_μ), and therefore our existence result extends theorem 1 of [4]. It also extends the main theorem of [18], where the conditions (f_1) and (NQ) are replaced by

$(\hat{f}_1)$ $f(x,s) = o(s)$ as $s \to 0$, uniformly for $x \in \Omega$,

(M_1) the function $f(x,s)/|s|$ is increasing in $|s|$ for $|s| > s_1$.

In addition to the conditions at the origin in [18] being stronger than ours, the main point is that (M_1) and (SL) together imply (NQ). Indeed, it can be proved that (M_1) implies that $H(x,s)$ is increasing in $|s|$ for $|s| > s_2$. Hence, if $s > s_2$, we have that

$$\begin{aligned} \frac{F(x,s)}{s^2} - \frac{F(x,s_2)}{s_2^2} &= \int_{s_2}^{s} \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{F(x,\tau)}{\tau^2}\right\}\mathrm{d}\tau = \int_{s_2}^{s} \frac{H(x,\tau)}{\tau^3}\,\mathrm{d}\tau \\ &\leqslant H(x,s)\left(-\frac{1}{2s^2} + \frac{1}{2s_2^2}\right), \end{aligned} \tag{1.2}$$

and therefore

$$\frac{F(x,s)}{s^2} \leqslant c_3 + c_4 H(x,s)$$

for some $c_3, c_4 > 0$. It follows from (SL) that $\lim_{s\to+\infty} H(x,s) = +\infty$. An analogous argument shows that the same occurs as $s \to -\infty$. In [7], Fang and Liu obtained

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one non-zero solution by assuming (f_0), (SL), $(\hat{f}_1)$ and the following:

(J) there exists $\theta \geqslant 1$ such that $H(x, ts) \leqslant \theta H(x, s)$ for any $(x, s) \in \Omega \times \mathbb{R}$ and $t \in [0, 1]$.

This quasi-monotonicity condition was introduced by Jeanjean in [11]. The same argument used in (1.2) shows that (J) and (SL) imply (NQ), and therefore theorem 1.2 extends theorem 1.1 of [7].

Our existence result also complements many other works of the updated literature. For example, in [17], Liu and Wang obtained a non-zero solution under $(\hat{f}_1)$, (SL) and the following version of (M_1):

$(\hat{M}_1)$ the function $H(x, s)$ is non-decreasing in $|s|$ and increasing for $|s|$ small.

This hypothesis plays an important role in their proof, since they apply the Nehari method. Finally, Schechter and Zou [22] assumed (f_0), (SL) and $(\hat{f}_1)$ hold. Moreover, they additionally assumed that $H(x, s)$ was convex on s or

(SZ) there exist $\theta > 2$, $a_3 \geqslant 0$ and $s_3 \geqslant 0$ such that

$$\theta F(x, s) - sf(x, s) \leqslant a_3(1 + s^2) \quad \text{for any } x \in \Omega,\ |s| \geqslant s_3.$$

Since we do not require any kind of monotonicity or convexity, our existence result extends or complements the aforementioned works. It also complement other results on superlinear problems (see [17, 20, 21, 24, 26] and the references therein). As a matter of fact, we can consider here the nonlinearity f such that $H(x, s) = a(x)s^2(1+\cos(s))+\ln(1+|s|)$, with $a \in C^\infty(\Omega)$ being positive. Hence, the arguments presented in the cited papers do not work in our setting.

The multiplicity statement of theorem 1.2 complements many results on multiplicity of solutions for superlinear problems (see, for instance, [1, 9, 27] and the references therein). The main novelty here is that the non-quadraticity condition is considered in the superlinear setting. We emphasize that, in some of the aforementioned works, the proof of existence is given by showing that the (bounded) Palais–Smale sequence weakly converges to a non-zero critical point of I. Hence, their authors cannot obtain multiple solutions, even if the function f is odd. Since here we prove compactness for I, we are able to use the symmetric mountain pass theorem to obtain infinitely many solutions in this context.

In the next section we prove our main result, namely theorem 1.1. The result is applied in § 3, where we present the proof of theorem 1.2. Note that our ideas can be used in many different linking-type settings, so § 4 gives possible extensions of the study of problem (P).

2. Proof of the main result

Throughout the paper we suppose that the function f satisfies (f_0). For brevity, we write $\int_\Omega g$ instead of $\int_\Omega g(x)\,\mathrm{d}x$. For any $1 \leqslant t < \infty$, $|g|_t$ denotes the norm in $L^t(\Omega)$.

We denote by H the Hilbert space $H_0^1(\Omega)$ endowed with the norm

$$\|u\|^2 = \left(\int |\nabla u|^2\right)^{1/2} \quad \text{for any } u \in H.$$

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As stated in § 1, the weak solutions of (P) are precisely the critical points of the C^1-functional

$$I(u) := \tfrac{1}{2}\|u\|^2 - \int F(x,u) \quad \text{for any } u \in H.$$

By using some careful estimates we can prove our compactness result as follows.

Proof of theorem 1.1. Let $(u_n) \subset H$ be such that

$$I(u_n) \to c, \qquad \|I'(u_n)\|_{H'}(1+\|u_n\|) \to 0,$$

where $c \in \mathbb{R}$. Since f has subcritical growth it suffices to prove that (u_n) is bounded.

Arguing by contradiction we suppose that, along a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. For each $n \in \mathbb{N}$, let $t_n \in [0,1]$ be such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n). \tag{2.1}$$

Setting $v_n := u_n/\|u_n\|$, we obtain $v \in H$ such that, along a subsequence,

$$\left.\begin{aligned} v_n &\rightharpoonup v \text{ weakly in } H,\\ v_n &\to v \text{ strongly in } L^q(\Omega) \quad \text{for any } 1 \leqslant q < 2^*,\\ v_n(x) &\to v(x). \end{aligned}\right\} \tag{2.2}$$

In what follows we prove that $v \neq 0$. Indeed, suppose by contradiction that $v = 0$. Then it follows from (f_0) and the strong convergence in (2.2) that $\int F(x, \sqrt{4m}v_n) \to 0$ as $n \to +\infty$ for any fixed $m > 0$. Since we may suppose that $\sqrt{4m} < \|u_n\|$, it follows from the definition of t_n in (2.1) that

$$I(t_n u_n) \geqslant I\Big(\frac{\sqrt{4m}}{\|u_n\|}u_n\Big) = 2m - \int F(x, \sqrt{4m}v_n) \geqslant m > 0, \tag{2.3}$$

for any $n \geqslant n_0$, where $n_0 \in \mathbb{N}$ depends only on m.

We look for a contradiction by considering two cases.

Case 1 (along a subsequence, $t_n < (2/\|u_n\|)$). In this case we first use the condition (f_0) and the Sobolev embeddings to obtain $c_1, c_2 > 0$ such that

$$\left|\int H(x, t_n u_n)\right| \leqslant c_1 t_n \|u_n\| + c_2 t_n^p \|u_n\|^p \leqslant 2c_1 + c_2 2^p = c_3.$$

If $t_n > 0$, it follows from $I'(t_n u_n)(t_n u_n) = 0$ that

$$0 = t_n^2\|u_n\|^2 - \int f(x, t_n u_n)(t_n u_n) = 2I(t_n u_n) - \int H(x, t_n u_n),$$

and therefore

$$I(t_n u_n) = \frac{1}{2}\int H(x, t_n u_n) \leqslant \tfrac{1}{2}c_3.$$

The above inequality also holds if $t_n = 0$, and therefore we obtain a contradiction with (2.3), since the number $m > 0$ in that expression is arbitrary. Hence, case 1 cannot occur.

It remains to disprove the following.

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CASE 2 (along a subsequence, $t_n \geqslant (2/\|u_n\|)$). In this setting we fix $\gamma > 0$ such that

$$3\gamma|\Omega| > 4, \tag{2.4}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. In view of (NQ) we can obtain $s_0 > 0$ such that $H(x,s) \geqslant \gamma$ for any $x \in \Omega$, $|s| \geqslant s_0$. On the other hand, since H has a subcritical growth, we have that $H(x,s) \geqslant -C|s|$ for any $x \in \Omega$, $|s| \leqslant s_1$, where $s_1 > 0$ is small.

We consider the non-negative cut-off function $\psi_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi_\varepsilon(s) = \begin{cases} \mathrm{e}^{-\varepsilon/s^2} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0, \end{cases}$$

with $\varepsilon > 0$ free for now. Note that ψ_ε is smooth and

$$\lim_{s\to 0} \psi_\varepsilon(s) = \lim_{s\to 0} \psi'_\varepsilon(s) = 0.$$

These limits, (f_0) and the continuity of H provide $C_{\gamma,\varepsilon} > 0$ such that

$$H(x,s) \geqslant \gamma\psi_\varepsilon(s) - C_{\gamma,\varepsilon}|s| \quad \text{for any } (x,s) \in \Omega \times \mathbb{R}.$$

Given $0 < s < t$, we can use the above inequality and the definition of H to get

$$\begin{aligned} \frac{I(tu_n)}{t^2\|u_n\|^2} - \frac{I(su_n)}{s^2\|u_n\|^2} &= -\int_\Omega \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{F(x,\tau u_n)}{\tau^2\|u_n\|^2}\right) \mathrm{d}\tau\,\mathrm{d}x \\ &= -\int_\Omega \int_s^t \frac{H(x,\tau u_n)}{\tau^3\|u_n\|^2}\,\mathrm{d}\tau\,\mathrm{d}x \\ &\leqslant \int_\Omega \int_s^t \left(\frac{C_{\gamma,\varepsilon}}{\|u_n\|}\frac{|u_n|}{\|u_n\|}\tau^{-2} - \frac{\gamma\psi_\varepsilon(\tau u_n)}{\|u_n\|^2}\tau^{-3}\right) \mathrm{d}\tau\,\mathrm{d}x, \end{aligned}$$

from which it follows that

$$\frac{I(tu_n)}{t^2\|u_n\|^2} \leqslant \frac{I(su_n)}{s^2\|u_n\|^2} + C_{\gamma,\varepsilon}\frac{|v_n|_1}{s\|u_n\|} - \gamma\int_\Omega\int_s^t \frac{\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\,\mathrm{d}\tau\,\mathrm{d}x.$$

We now set

$$s = s_n = \frac{1}{\|u_n\|} < \frac{2}{\|u_n\|} \leqslant t_n.$$

Since

$$\int_{s_n}^{t_n} \tau^{-3}\,\mathrm{d}\tau = \tfrac{1}{2}(\|u_n\|^2 - t_n^2),$$

we have that

$$\begin{aligned} \frac{I(t_nu_n)}{t_n^2\|u_n\|^2} &\leqslant I(v_n) + C_{\gamma,\varepsilon}|v_n|_1 - \frac{\gamma|\Omega|}{2}\left(1 - \frac{1}{t_n^2\|u_n\|^2}\right) + \gamma A_n \\ &\leqslant B_\gamma + C_{\gamma,\varepsilon}|v_n|_1 - \int F(x,v_n) + \gamma A_n \end{aligned} \tag{2.5}$$

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with

$$A_n = \int_{s_n}^{t_n} \int_{\Omega} \frac{1 - \psi_{\varepsilon}(|\tau u_n|)}{\|u_n\|^2} \tau^{-3} \, dx \, d\tau \geqslant 0$$

and

$$B_{\gamma} = \tfrac{1}{2}(1 - \tfrac{3}{4}\gamma|\Omega|) < 0,$$

where we have used (2.4) in the last inequality.

We shall verify below that, uniformly in $n \in \mathbb{N}$, the following limit holds:

$$\lim_{\varepsilon \to 0} \int_{s_n}^{t_n} \int_{\Omega} \frac{1 - \psi_{\varepsilon}(|\tau u_n|)}{\|u_n\|^2} \tau^{-3} \, dx \, d\tau = 0. \tag{2.6}$$

If this is true, we can choose $\varepsilon > 0$ such that $\gamma A_n < -\frac{1}{2} B_{\gamma}$ for all $n \in \mathbb{N}$. Since we are supposing that $v = 0$, it follows from (2.2) and (f_0) that $|v_n|_1 = o_n(1)$ and $\int F(x, v_n) = o_n(1)$ as $n \to +\infty$. Hence, we can take the limit in (2.5) to obtain

$$\limsup_{n \to +\infty} \frac{I(t_n u_n)}{t_n^2 \|u_n\|^2} \leqslant B_{\gamma} - \frac{B_{\gamma}}{2} = \frac{B_{\gamma}}{2} < 0,$$

and therefore $I(t_n u_n) < 0$, for n large, again contradicting (2.3).

We proceed now with the proof that the limit in (2.6) is uniform. We start by considering $\delta > 0$ and splitting the term A_n into two integrals

$$\int_{s_n}^{t_n} \int_{\Omega} \frac{1 - \psi_{\varepsilon}(|\tau u_n|)}{\|u_n\|^2} \tau^{-3} \, dx \, d\tau = \int_{s_n}^{t_n} \int_{|\tau u_n| \geqslant \delta} (\cdots) + \int_{s_n}^{t_n} \int_{|\tau u_n| < \delta} (\cdots).$$

For ease of notation we call the first integral on the right-hand side above $A_{n,\delta}^{+}$, and the second one $A_{n,\delta}^{-}$. It suffices to show that these quantities go to 0, uniformly in n, as $\varepsilon \to 0$.

Since ψ_{ε} is non-decreasing we have that

$$\begin{aligned} A_{n,\delta}^{+} &\leqslant \frac{1 - e^{-\varepsilon/\delta^2}}{\delta \|u_n\|^2} \int_{s_n}^{t_n} \int_{|\tau u_n| \geqslant \delta} |\tau u_n| \tau^{-3} \, dx \, d\tau \\ &\leqslant \frac{1 - e^{-\varepsilon/\delta^2}}{\delta \|u_n\|} \left(\frac{1}{s_n} - \frac{1}{t_n} \right) \int_{\Omega} \frac{|u_n|}{\|u_n\|} \\ &\leqslant \left(\frac{1 - e^{-\varepsilon/\delta^2}}{\delta} \right) |v_n|_1, \end{aligned}$$

since $s_n \|u_n\| = 1$. Recalling that $(|v_n|_1)$ is uniformly bounded, we conclude that the limit $\lim_{\varepsilon \to 0} A_{n,\delta}^{+} = 0$ is uniform.

The calculations for $A_{n,\delta}^{-}$ are more involved. We first note that, for each $|s| \leqslant \delta$ fixed, the function $\varepsilon \mapsto \psi_{\varepsilon}(s)$ is smooth. Hence, it follows from Taylor's theorem that, for $h(s) = s^{-2} e^{-(\varepsilon, s^2)}$,

$$1 - \psi_{\varepsilon}(s) = \varepsilon s^{-2} e^{-\varepsilon/s^2} + r(\varepsilon, s) = \varepsilon \left(h(s) + \frac{r(\varepsilon, s)}{\varepsilon} \right) \leqslant \varepsilon (h(s) + 1)$$

holds, since the continuous remainder term r is such that $\lim_{\varepsilon \to 0} r(\varepsilon, s)/\varepsilon = 0$ uniformly in the compact set $|s| \leqslant \delta$. By applying Taylor's theorem again, we get,

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for $|s| \leqslant \delta$,

$$h(s) = h(0) + h'(0)s + r_1(\varepsilon, s) = r_1(\varepsilon, s),$$

with $r_1(\varepsilon, s) = o(|s|)$ as $s \to 0$. Thus, we conclude that if $\delta > 0$ is small,

$$1 - \psi_\varepsilon(s) \leqslant \varepsilon(1 + |s|) \quad \text{for any } |s| \leqslant \delta.$$

The above inequality and the definition of $A^-_{n,\delta}$ give

$$\begin{aligned} A^-_{n,\delta} &= \int_{s_n}^{t_n} \int_{|\tau u_n| < \delta} \frac{1 - \psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2} \tau^{-3} \, \mathrm{d}x \, \mathrm{d}\tau \\ &\leqslant \varepsilon \int_{s_n}^{t_n} \int_\Omega \frac{\tau^{-3}}{\|u_n\|^2} \, \mathrm{d}x \, \mathrm{d}\tau + \varepsilon \int_{s_n}^{t_n} \int_\Omega \frac{|u_n|}{\|u_n\|^2} \tau^{-2} \, \mathrm{d}x \, \mathrm{d}\tau \\ &= \varepsilon \frac{|\Omega|}{2} \left(1 - \frac{1}{t_n^2 \|u_n\|^2}\right) + \frac{\varepsilon}{\|u_n\|} \left(1 - \frac{1}{t_n \|u_n\|}\right) \int_\Omega |v_n| \, \mathrm{d}x \\ &\leqslant \varepsilon \left(\frac{|\Omega|}{2} + |v_n|_1\right), \end{aligned}$$

since we may assume that $\|u_n\| > 1$. This implies that, uniformly in n,

$$\lim_{\varepsilon \to 0} A^-_{n,\delta} = 0$$

holds. This completes the proof that the weak limit v is non-zero.

After proving that $v \neq 0$ we can prove the theorem in the following way: the set $\tilde{\Omega} := \{x \in \Omega \colon v(x) \neq 0\}$ has positive measure. Moreover, since $\|u_n\| \to +\infty$, we have that $|u_n(x)| \to +\infty$ almost everywhere in $\tilde{\Omega}$. Thus, the continuity of H, Fatou's lemma and (NQ) yield

$$\begin{aligned} 2c &= \lim_{n \to +\infty} (2I(u_n) - I'(u_n)u_n) \\ &\geqslant \operatorname{meas}(\Omega \setminus \tilde{\Omega}) \min_{\bar{\Omega} \times \mathbb{R}} H + \int_{\tilde{\Omega}} \liminf_{n \to +\infty} H(x, u_n) = +\infty, \end{aligned}$$

which is a contradiction. Hence, we have that (u_n) is bounded and the theorem is proved. □

3. Proof of theorem 1.2

In this section we prove our results concerning problem (P). For the multiplicity part we need the following version of the symmetric mountain pass theorem [19, theorem 9.12] (see [2, theorem 1.3] for the proof that the deformation lemma used in [19] also holds with the Cerami condition).

Theorem 3.1. *Let X be an infinite-dimensional Banach space, let $\mathcal{I} \in C^1(X, \mathbb{R})$ be even and satisfy $(\mathrm{Ce})_c$ for any $c \in \mathbb{R}$ and satisfy $\mathcal{I}(0) = 0$. If $X = V \oplus W$, where V is finite dimensional, and $\mathcal{I}$ satisfies the following conditions, then $\mathcal{I}$ possesses an unbounded sequence of critical values.*

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($\mathcal{I}_1$) *There exist* $\alpha, \rho > 0$ *such that*

$$\mathcal{I}(u) \geqslant \alpha \quad \textit{for any } u \in \partial B_\rho(0) \cap W.$$

($\mathcal{I}_2$) *For any finite-dimensional subspace* $\hat{X} \subset X$ *there exists* $R = R(\hat{X})$ *such that*

$$\mathcal{I}(u) \leqslant 0 \quad \textit{for any } u \in \hat{X} \setminus B_R(0).$$

We are now ready to obtain the solutions for (P).

Proof of theorem 1.2. Conditions (f_0), (f_1) and standard arguments imply that $\int F(x,u) = o(\|u\|^2)$ as $\|u\| \to 0$. Hence, there exist $\alpha, \rho > 0$ such that $I(u) \geqslant \alpha$ whenever $u \in \partial B_\rho(0) \subset H$. Suppose that the limit in (SL) holds for $x \in \Omega_0 \subset \Omega$ of positive measure. If we take a positive function $\phi \in H_0^1(\Omega_0)$, we can use (SL) to conclude that $I(t\phi) \to -\infty$ as $t \to +\infty$. Since I satisfies the Cerami condition, it follows from the mountain pass theorem that I has a non-zero critical point.

To prove the multiplicity part we shall apply theorem 3.1 with $X = H$ and $\mathcal{I} = I$. Since f is odd in the second variable, I is even. Recalling that $I(0) = 0$ and I satisfies the Cerami condition, it remains to check the geometric conditions ($\mathcal{I}_1$) and ($\mathcal{I}_2$).

Let $\hat{X} \subset H$ be a finite-dimensional subspace. Since all the norms in $\hat{X}$ are equivalent, there exists $c_1 > 0$ such that $\|u\|^2 \leqslant c_1 \int u^2$ for any $u \in \hat{X}$. Given $M > (2/c_1)$, it follows from (SL) that $F(x,s) \geqslant Ms^2 - c_2$ for any $x \in \Omega$ and $s \in \mathbb{R}$. Hence,

$$I(u) \leqslant \frac{1}{2}\left(1 - \frac{2M}{c_2}\right)\|u\|^2 + c_1|\Omega|,$$

and we conclude that $I(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in \hat{X}$. This establishes ($\mathcal{I}_2$).

In order to verify ($\mathcal{I}_1$) we set, for each $k \in \mathbb{N}$,

$$V_k := \operatorname{span}\{\varphi_1, \dots, \varphi_k\}, \qquad W_k = V_k^\perp,$$

where $(\varphi_k)_{k\in\mathbb{N}}$ are the eigenfunctions of $(-\Delta, H_0^1(\Omega))$. Integrating the inequality in (f_0), we get

$$I(u) \geqslant \tfrac{1}{2}\|u\|^2 - c_3|u|_p^p - c_4,$$

for some $c_3, c_4 > 0$. Since $2 < p < 2^*$, the interpolation inequality $|u|_p \leqslant |u|_2^\theta |u|_{2^*}^{1-\theta}$, for some $\theta \in (0,1)$, yields

$$I(u) \geqslant \tfrac{1}{2}\|u\|^2 - c_3|u|_2^{p\theta}|u|_{2^*}^{p(1-\theta)} - c_4 \geqslant \tfrac{1}{2}\|u\|^2 - c_5|u|_2^{p\theta}\|u\|^{p(1-\theta)} - c_4,$$

where $c_5 > 0$ and we have used the embedding $H \hookrightarrow L^{2^*}(\Omega)$.

The above inequality holds for any $u \in H$. If we take $u \in W_k$, we can use the variational inequality $\|u\|^2 \geqslant \lambda_{k+1}|u|_2^2$ to obtain

$$I(u) \geqslant \tfrac{1}{2}\|u\|^2 - \frac{c_5}{\lambda_{k+1}^{p\theta/2}}\|u\|^{p\theta}\|u\|^{p(1-\theta)} - c_4 = \left(\frac{1}{2} - \frac{c_5}{\lambda_{k+1}^{p\theta/2}}\|u\|^{p-2}\right)\|u\|^2 - c_4.$$

We now set $\rho = 2\sqrt{c_4 + 1}$ and choose $k \in \mathbb{N}$ such that

$$\frac{c_5}{\lambda_{k+1}^{p\theta/2}}\rho^{p-2} \leqslant \frac{1}{4}. \tag{3.1}$$

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This is always possible, since $\lambda_k \to +\infty$. It follows that, for any $u \in \partial B_\rho(0) \cap W_k$,

$$I(u) \geqslant (\tfrac{1}{2} - \tfrac{1}{4})\rho^2 - c_4 = \tfrac{1}{4}(2\sqrt{c_4+1})^2 - c_4 = 1$$

holds. Therefore, $(\mathcal{I}_1)$ is satisfied with $\alpha = 1$, $\rho = 2\sqrt{c_4+1}$ and the decomposition of H being $H = V_k \oplus W_k$. The multiplicity result follows from theorem 3.1. □

4. Further remarks

In this final section we present many variants for consideration. For example, concerning the condition at the origin, we could suppose that

$$\lim_{s\to 0} \frac{2F(x,s)}{s^2} = K_0(x) \quad \text{uniformly for } x \in \Omega,$$

where $K_0 \in L^t(\Omega)$ for some $t > \frac{1}{2}N$ and the positive part of K_0 is non-trivial. In this case the linear problem

$$-\Delta u = \lambda K_0(x)u, \quad u \in H_0^1(\Omega),$$

has a sequence of eigenvalues $(\lambda_j(K_0))_{j\in\mathbb{N}}$ with $\lambda_1(K_0) > 0$. A simple inspection of the proof of theorem 1.2 shows that it remains true if we suppose that $\lambda_1(K_0) > 1$ instead of applying condition (f_1). Indeed, we can deal with *non-resonance at the origin* in the following sense: suppose that $\lambda_m(K_0) < 1 < \lambda_{m+1}(K_0)$ for some $m \geqslant 1$. In this case we can apply the local linking theorem given by Li and Willem [15], together with our compactness result, to obtain a non-zero solution. So, it is possible to generalize the main theorems contained in [6, 12, 13].

We could also treat the asymptotically linear case, by replacing (SL) by the following condition:

$$\lim_{|s|\to +\infty} \frac{2F(x,s)}{s^2} = K_\infty(x) \quad \text{uniformly for } x \in \Omega,$$

where $K_\infty \in L^t(\Omega)$ for some $t > \frac{1}{2}N$ and the positive part of K_∞ is non-trivial. If $\lambda_m(K_\infty) = 1$ for some $m \geqslant 1$, we could use the saddle-point theorem to extend the existence result of [4, theorem 2] (see also [8] for related results). This means that, under the non-quadraticity condition, we give here a unified approach for nonlinear elliptic problems that are superlinear or asymptotically linear at infinity. Actually, our theorem 1.1 presents another proof of [4, lemma 1.2] but with weaker conditions. Hence, we could also consider the *double-resonant* case:

(DR) there exists $j \geqslant 1$ such that

$$\lambda_j \leqslant \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \lambda_{j+1},$$

where λ_j is the sequence of eigenvalues on $(-\Delta, H_0^1(\Omega))$. In this case the resonant phenomena are allowed in two consecutive eigenvalues. The main point here is to obtain compactness for the associated functional, but now this is a consequence of condition (NQ). Thus, we can obtain a non-trivial solution under the assumption

$$\lambda_m < \lim_{s\to 0} \frac{f(x,s)}{s} < \lambda_{m+1}$$

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for some $m \geqslant 1$ as a consequence of the local linking theorem (see [3, 5, 8] for more details on double-resonant problems).

Finally, under the hypothesis of theorem 1.2, it is possible to argue as in [1] to obtain two solutions, one positive and other negative. Indeed, to obtain the first one we define

$$f^+(x,s) := \begin{cases} f(x,s) & \text{if } s \geqslant 0, \\ 0 & \text{if } s < 0, \end{cases}$$

and consider the functional

$$I^+(u) := \tfrac{1}{2}\|u\|^2 - \int F^+(x,u), \quad u \in H_0^1(\Omega),$$

where

$$F^+(x,s) := \int_0^s f^+(x,\tau)\,\mathrm{d}\tau.$$

We have that F^+ is superlinear at infinity and non-quadratic at infinity in one direction. More precisely,

$$\lim_{s\to\infty}(sf^+(x,s) - 2F^+(x,s)) = +\infty \quad \text{uniformly in } x \in \Omega,$$

and we can argue as in the proof of theorem 1.2 to obtain a positive solution. The negative solution can be obtained with the analogous truncation, f^-.

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