J. Appl. Prob. **55,** 402–415 (2018)
*doi:*10.1017/jpr.2018.26

ON AN AGEING CLASS BASED ON THE MOMENT GENERATING FUNCTION ORDER

SHYAMAL GHOSH * ** AND
MURARI MITRA,* *Indian Institute of Engineering Science and Technology, Shibpur*

Abstract

We develop shock model theory in different scenarios for the $\mathcal{M}$-class of life distributions introduced by Klar and Müller (2003). We also study the cumulative damage model of A-Hameed and Proschan (1975) in the context of $\mathcal{M}$-class and establish analogous results. We obtain moment bounds and explore weak convergence issues within the $\mathcal{M}$-class of life distributions.

Keywords: Moment generating function order; NHPP; nonstationary pure-birth process; cumulative damage model; weak convergence; moment convergence

2010 Mathematics Subject Classification: Primary 62N05
Secondary 60K10

1. Introduction

Stochastic orders and ageing properties play an important role in reliability theory, engineering, social and biological sciences, maintenance, risk analysis, biometrics, and other related fields. A wide variety of stochastic orders have been proposed in the literature; see [24], [25], and [28] for detailed discussions. Based on these orders, various classes of life distributions have been introduced to model different ageing scenarios; for example, increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better than used in expectation (NBUE), and harmonic new better than used in expectation (HNBUE). See [4] and [21] for properties and applications of these ageing concepts.

Among others, two stochastic orders have been proposed based on the Laplace transform (LT) and the moment generating function (MGF). For convenience, we recapitulate the relevant definitions.

Definition 1. Let X and Y be two nonnegative random variables with cumulative distribution functions (CDFs) F and G, respectively. Then X is said to be smaller than Y in

(i) LT order, denoted by $X \leq_{\text{LT}} Y$ or $F \leq_{\text{LT}} G$, if

$$\mathbb{E}[\mathrm{e}^{-tX}] \geq \mathbb{E}[\mathrm{e}^{-tY}] \quad \text{for all } t > 0;$$

(ii) MGF order, denoted by $X \leq_{\text{MGF}} Y$ or $F \leq_{\text{MGF}} G$, if $\mathbb{E}[\mathrm{e}^{t_0 Y}]$ is finite for some $t_0 > 0$ and

$$\mathbb{E}[\mathrm{e}^{tX}] \leq \mathbb{E}[\mathrm{e}^{tY}] \quad \text{for all } t > 0.$$

An important class of life distributions related to the LT order was introduced by Klefsjö [18]. This family, popularly known as the $\mathcal{L}$-class, is defined as follows.

Received 26 July 2017; revision received 9 February 2018.
* Postal address: Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, P.O. Botanic Garden, Howrah, 711103, West Bengal, India.
** Email address: shyamalmath2012@gmail.com

Definition 2. Let F be a life distribution with survival function $\bar{F}$ and finite mean μ. The class of life distributions for which

$$\int_0^\infty \mathrm{e}^{-st}\bar{F}(t)\,\mathrm{d}t \ge \frac{\mu}{1+s\mu} \quad \text{for all } s \ge 0$$

will be denoted by $\mathcal{L}$. If the reverse inequality holds, we say that F belongs to the $\bar{\mathcal{L}}$-class of distributions.

The $\mathcal{L}$-class encompasses all the well-known ageing classes such as IFR, IFRA, NBU, decreasing mean residual life (DMRL), NBUE, HNBUE, and so on. One of the desirable properties of distributions belonging to positive ageing classes is that the residual performance of a unit having already survived up to time t is inferior in some stochastic sense to the performance of a new unit; see [19]. This dictum is obeyed by all the 'usual' positive ageing classes mentioned above but not by many of the distributions (for example, lognormal) belonging to the $\mathcal{L}$-class of life distributions. This, together with the well-known example of Klar [14], raises serious questions about the viability of the $\mathcal{L}$-property as a reasonable notion of positive ageing.

However, in spite of this 'deficiency', the $\mathcal{L}$-class still contains the inverse Gaussian and the Birnbaum–Saunders distributions which are typical examples of fatigue failure models; see [7]. Although very important from a practical standpoint, these distributions do not find a place in any of the usual positive ageing classes.

Motivated by the above thinking, Klar and Müller [16] introduced a new class of life distributions based on the MGF order; the so-called $\mathcal{M}$-class distributions. This family can be viewed as a substantially larger class of *positive ageing* distributions since it contains all HNBUE distributions, does not suffer from the deficiencies of the $\mathcal{L}$-class, and yet manages to include important life distributions such as the inverse Gaussian and the Birnbaum–Saunders distributions.

Definition 3. A nonnegative random variable X with CDF F and mean $\mu = \int_0^\infty \bar{F}(x)\,\mathrm{d}x$ is said to belong to the $\mathcal{M}$-class if $F \le_{\mathrm{MGF}} G$, where G denotes the CDF of an exponential distribution with mean μ, i.e. $X \in \mathcal{M}$-class if and only if

$$\mathbb{E}\mathrm{e}^{sX} = \int_0^\infty \mathrm{e}^{st}\,\mathrm{d}F(t) \le \frac{1}{1-\mu s} \quad \text{for all } 0 \le s < \frac{1}{\mu}$$

or, equivalently,

$$\int_0^\infty \mathrm{e}^{st}\bar{F}(t)\,\mathrm{d}t \le \frac{\mu}{1-\mu s} \quad \text{for all } 0 \le s < \frac{1}{\mu}.$$

If the reverse inequality holds, we say that X belongs to the $\bar{\mathcal{M}}$-class.

See Klar and Müller [16] for more details about the $\mathcal{M}$-class of life distributions.

Li [22] proved that the $\mathcal{M}$-class is closed under both convex linear combination and geometric compounding. Zhang and Li [29] established the closure of this class under a random sum of independent but nonidentically distributed lifetimes thereby generalizing a result of Klar and Müller [16]. Klar [15] developed a procedure to test this exponentiality result against $\mathcal{M}$ alternatives. However, shock model theory, which has been extensively developed for the usual positive ageing classes (see [2], [3], [9], [11], and [17]) is noticeably absent for the $\mathcal{M}$-class. Further, weak convergence issues which have been dealt with in the context of the IFR, DMRL, HNBUE, and $\mathcal{L}$ families (see [5], [6], [10]), and [26]) remain unexplored for the $\mathcal{M}$-class. Consequently, in this paper our objective is twofold: to systematically develop shock model theory for this class and then to investigate weak convergence and related issues.

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

In Section 2 we introduce the notion of $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$ families which are discrete versions of the $\mathcal{M}(\bar{\mathcal{M}})$-classes and proceed to develop shock model theory in the context of stationary and nonstationary pure-birth processes. As special cases of our results, we obtain the corresponding theorems for the homogeneous Poisson process (HPP) and the nonhomogeneous Poisson process (NHPP). We also investigate the case when the shocks arrive according to a general counting process. Finally, we prove the corresponding results for the cumulative damage model of A-Hameed and Proschan [3]. In Section 3 we obtain moment bounds and show that the $\mathcal{M}$-class distribution is closed under the formation of weak limits. We also establish the preservation of the MGF order under weak limits. We prove that for the $\mathcal{M}$-class of distributions, weak convergence implies convergence of the moment sequences to appropriate moments of the limit distribution. The converse question is also answered in the affirmative.

2. Shock model theory

Consider a device subjected to a sequence of shocks occurring randomly in time according to a counting process $\{N(t) \colon t \geq 0\}$. Suppose that the device has a probability $\bar{\mathbb{P}}_k$ of surviving the first k shocks, $k = 0, 1, 2, 3, \dots$, and let T denote the time to failure. Throughout this paper, we make the blanket assumption that the survival probabilities satisfy the following natural condition: $1 = \bar{\mathbb{P}}_0 \geq \bar{\mathbb{P}}_1 \geq \bar{\mathbb{P}}_2 \cdots$.

The probability that the device survives beyond time t is given by the survival function

$$\bar{H}(t) := \mathbb{P}(T > t) = \sum_{k=0}^{\infty} \mathbb{P}(N(t) = k)\bar{\mathbb{P}}_k. \tag{1}$$

Shock model theory can be applied to diverse areas such as inventory control, risk analysis, biometry, and dam theory by simply making suitable substitutions. For example, shock, device failure, and survival until time t are substituted respectively by claim, bankruptcy, and all claims met during $[0, t]$ in risk analysis, by demand, shortage, and no shortage during $[0, t]$ in inventory control; see [3] for details. By knowing that the survival function $\bar{H}(t)$ belongs to a certain class of life distributions, it is possible to obtain sharp bounds on the survival function and other parameters of the distribution. Such knowledge leads to greater precision in estimation results, appropriate maintenance strategies in reliability contexts, and effective financial policies in risk analysis.

Reliability shock models are primarily concerned with the following fundamental question: if the sequence $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ possesses a certain discrete ageing property (such as IFR, IFRA, NBU, NBUE, and so on), does $\bar{H}(t)$ inherit the corresponding continuous version of the said ageing property under the transformation (1) in various scenarios (for example, when $N(t)$ is HPP, NHPP, stationary, or nonstationary pure birth process, and so on).

Esary *et al.* [11] answered the above question in the affirmative for the IFR, IFRA, NBU, NBUE, and DMRL classes when $N(t)$ is a HPP. Under some suitable assumptions on the intensity function $\Lambda(t)$, A-Hameed and Proschan [2] proved the same preservation result for a NHPP. They also considered the situation where shocks arrive according to stationary and nonstationary pure birth processes and established the preservation result for the above classes; see [3]. Analogous results for the HNBUE and $\mathcal{L}$ classes can be found in [17] and [18]. Later, Abouammoh *et al.* [1] established such results for the new better than used in failure rate (NBUFR) and new better than average failure rate (NBAFR) classes of life distributions. We now proceed to derive the corresponding results in the context of the $\mathcal{M}$-class of distribution.

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

To this end, we formally introduce the notion of $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$ families which are discrete versions of the $\mathcal{M}(\bar{\mathcal{M}})$-classes. These become highly relevant in our subsequent development of shock model theory. Suppose that ξ is a positive integer-valued random variable and let $\bar{\mathbb{P}}_k := \mathbb{P}(\xi > k)$ denote the corresponding survival probabilities. Let $1 = \bar{\mathbb{Q}}_0 \geq \bar{\mathbb{Q}}_1 \geq \bar{\mathbb{Q}}_2 \geq \cdots$ denote the survival probabilities of the geometric distribution having finite mean $\mu = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k = \sum_{k=0}^{\infty} \bar{\mathbb{Q}}_k$, i.e. $\bar{\mathbb{Q}}_k = (1 - 1/\mu)^k$, $k = 0, 1, 2, \dots$. (ξ can be thought of as the number of shocks required to cause failure). Since the discrete counterpart of the exponential distribution is the geometric distribution, it seems intuitively appealing to say that a discrete distribution with survival probabilities $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ is in the $\mathcal{G}_{\mathcal{M}}$-class if and only if

$$\begin{aligned}\sum_{k=0}^{\infty} \mathrm{e}^{pk} \mathbb{P}(\xi = k) &= \mathrm{e}^{p} \sum_{k=0}^{\infty} \mathrm{e}^{pk} (\bar{\mathbb{P}}_k - \bar{\mathbb{P}}_{k+1}) \\ &\leq \mathrm{e}^{p} \sum_{k=0}^{\infty} \mathrm{e}^{pk} \bigg[\bigg(1 - \frac{1}{\mu}\bigg)^{k} - \bigg(1 - \frac{1}{\mu}\bigg)^{k+1} \bigg] \quad \text{for all } 0 \leq p < \ln \frac{\mu}{\mu - 1},\end{aligned}$$

which is equivalent to

$$\sum_{k=0}^{\infty} \bar{\mathbb{P}}_k \mathrm{e}^{pk} \leq \sum_{k=0}^{\infty} \bigg(1 - \frac{1}{\mu}\bigg)^{k} \mathrm{e}^{pk} \quad \text{for all } 0 \leq p < \ln \frac{\mu}{\mu - 1}.$$

A simple reparametrization now leads to the following formal definition.

Definition 4. A discrete life distribution with survival probabilities $\bar{\mathbb{P}}_k$, $k = 0, 1, \dots$, and finite mean $\mu = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k$ is said to belong to the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$ class if and only if

$$\sum_{k=0}^{\infty} \bar{\mathbb{P}}_k p^k \leq [\geq] \frac{\mu}{p + (1 - p)\mu} \quad \text{for all } 1 \leq p < \frac{\mu}{\mu - 1}.$$

We are now in a position to develop a shock model theory for pure birth processes.

2.1. Pure birth shock models

Consider a pure birth shock model in which a device is subjected to shocks governed by a birth process with intensities λ_k, $k = 0, 1, 2, \dots$. Further, let V_{k+1} denote the interarrival time between the kth and $(k + 1)$th shocks. Then V_{k+1}, $k = 0, 1, 2, \dots$, are independent and exponentially distributed with means $1/\lambda_k$, i.e.

$$\bar{F}_{k+1}(t) := \mathbb{P}(V_{k+1} > t) = \mathrm{e}^{-\lambda_k t} \quad \text{for all } t \geq 0.$$

The survival function $\bar{H}(t)$ of the device can then be written as

$$\bar{H}(t) = \sum_{k=0}^{\infty} S_k(t) \bar{\mathbb{P}}_k \tag{2}$$

with

$$S_k(t) := \mathbb{P}(N(t) = k), \tag{3}$$

where $N(t)$ is the pure birth process defined above. We assume that the intensities $(\lambda_k)_{k=0}^{\infty}$ are such that the probability of infinitely many shocks in $(0, t]$ is 0 (i.e. $\sum_{k=0}^{\infty} S_k(t) = 1$), which is equivalent to the condition $\sum_{k=0}^{\infty} \lambda_k^{-1} = \infty$; see [12, p. 452].

We first need to establish the following theorem which will be essential in proving the main preservation result of this section.

Theorem 1. *Let* $\alpha_0 = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k/\lambda_k$. *Then the survival function* $\bar{H}(t)$ *in (2) belongs to the* $\mathcal{M}(\bar{\mathcal{M}})$*-class if*

$$\sum_{k=0}^{\infty} \bar{P}_k m_k(s) \le [\ge] \frac{\alpha_0}{1-\alpha_0 s} \quad \textit{for all } 0 \le s < \frac{1}{\alpha_0}, \tag{4}$$

where the $m_k(s) = \int_0^{\infty} \mathrm{e}^{st} S_k(t)\,\mathrm{d}t$ *are given by*

$$m_0(s) = \frac{1}{\lambda_0 - s}, \qquad m_k(s) = \Bigg(\prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_j - s}\Bigg) \frac{1}{\lambda_k - s} \quad \textit{for } k = 1, 2, 3, \dots. \tag{5}$$

Proof. For $k = 0$,

$$\int_0^{\infty} \mathrm{e}^{st} S_0(t)\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{st-\lambda_0 t}\,\mathrm{d}t = \frac{1}{\lambda_0 - s} = m_0(s).$$

Defining $W_k = V_1 + V_2 + \cdots + V_k$ and $G_k = F_1 \star F_2 \star F_3 \star \cdots \star F_k$ for $k \ge 1$, where '$\star$' denotes convolution, we have

$$\begin{aligned}
\int_0^{\infty} \mathrm{e}^{st} S_k(t)\,\mathrm{d}t &= \int_0^{\infty} \mathrm{e}^{st} \mathbb{P}(N(t) = k)\,\mathrm{d}t \\
&= \int_0^{\infty} \mathrm{e}^{st} \{\mathbb{P}(W_k \le t) - \mathbb{P}(W_{k+1} \le t)\}\,\mathrm{d}t \\
&= \int_0^{\infty} \mathrm{e}^{st} \bar{G}_{k+1}(t)\,\mathrm{d}t - \int_0^{\infty} \mathrm{e}^{st} \bar{G}_k(t)\,\mathrm{d}t \\
&= \frac{1}{s}\bigg[\int_0^{\infty} \mathrm{e}^{st}\,\mathrm{d}G_{k+1}(t) - \int_0^{\infty} \mathrm{e}^{st}\,\mathrm{d}G_k(t)\bigg] \\
&= \frac{1}{s}\bigg[\prod_{j=0}^{k} \frac{\lambda_j}{\lambda_j - s} - \prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_j - s}\bigg] \\
&= \Bigg(\prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_j - s}\Bigg) \frac{1}{\lambda_k - s} \\
&= m_k(s).
\end{aligned}$$

Note that

$$\int_0^{\infty} \mathrm{e}^{st} \bar{H}(t)\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{st} \Bigg(\sum_{k=0}^{\infty} S_k(t) \bar{\mathbb{P}}_k\Bigg)\,\mathrm{d}t = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k \bigg(\int_0^{\infty} \mathrm{e}^{st} S_k(t)\,\mathrm{d}t\bigg) = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k m_k(s).$$

Therefore, noting that $\alpha_0 = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k/\lambda_k = \int_0^{\infty} \bar{H}(t)\,\mathrm{d}t$, $\bar{H}(t)$ will be in the $\mathcal{M}(\bar{\mathcal{M}})$-class if

$$\sum_{k=0}^{\infty} \bar{\mathbb{P}}_k m_k(s) \le [\ge] \frac{\alpha_0}{1-\alpha_0 s} \quad \text{for all } 0 \le s < \frac{1}{\alpha_0}. \qquad \square$$

In the next theorem we show how the discrete $\mathcal{M}$ property is inherited by the survival function $\bar{H}(t)$ defined in (2).

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

Theorem 2. *If $\lambda_j \geq [\leq]\, \lambda_0$ for $j = 1, 2, \dots$ and $(\bar{\mathbb{P}}_k/\lambda_k)_{k=0}^{\infty}$ is decreasing and in the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$-class, then $\bar{H}(t)$ will be in the $\mathcal{M}(\bar{\mathcal{M}})$-class.*

Proof. According to Theorem 3.3 of [16], the $\mathcal{M}(\bar{\mathcal{M}})$ property is scale invariant. So, without loss of generality, we can choose $\lambda_0 = 1$. Then $\lambda_j \geq [\leq]\, \lambda_0$ implies that

$$m_k(s) \leq [\geq] \frac{1}{(1-s)^{k+1}\lambda_k} \quad \text{for } k = 0, 1, 2, \dots.$$

Now, using the definition of the discrete $\mathcal{M}(\bar{\mathcal{M}})$ property,

$$\begin{aligned}
\sum_{k=0}^{\infty} \bar{\mathbb{P}}_k m_k(s) &\leq [\geq] \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k \left(\frac{1}{1-s}\right)^{k+1} \frac{1}{\lambda_k} \\
&= \frac{1}{1-s} \sum_{k=0}^{\infty} \left(\frac{1}{1-s}\right)^{k} \frac{\bar{\mathbb{P}}_k}{\lambda_k} \\
&\leq [\geq] \frac{1}{1-s} \cdot \frac{\alpha_0}{(1 - 1/(1-s))\alpha_0 + 1/(1-s)} \\
&= \frac{\alpha_0}{1 - \alpha_0 s}.
\end{aligned}$$

An application of Theorem 1 completes the proof. □

If all the birth intensities λ_k are equal to λ then the pure-birth process reduces to a HPP and thus Theorem 2 leads to the following corollary.

Corollary 1. *When shocks arrive according to a HPP, the survival function $\bar{H}(t)$ possesses the $\mathcal{M}(\bar{\mathcal{M}})$ property if the sequence $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ belongs to the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$-class of distribution.*

We now consider the nonstationary pure birth shock model of [3] with shocks arrive according to a Markov process; given that k shocks have occurred in $(0, t]$, the probability of a shock occurring in $(t, t + \Delta t]$ is $\lambda_k \lambda(t)\Delta t + o(\Delta t)$, while the probability of more than one shock occurring in $(t, t + \Delta t)$ is $o(\Delta t)$.

Using Remark 1.1 of [3], the survival function $\bar{H}^*(t)$ for this shock model is given by

$$\bar{H}^*(t) = \sum_{k=0}^{\infty} S_k(\Lambda(t)) \bar{\mathbb{P}}_k, \tag{6}$$

where $S_k(t)$ is as defined in (3) and $\Lambda(t) = \int_0^t \lambda(x)\, \mathrm{d}x$.

To prove the preservation result in this more general setup, we first require the following definition.

Definition 5. A nonnegative function g is defined on $[0, \infty)$ with $g(0) = 0$ is said to be star-shaped (anti-star-shaped) if $x^{-1}g(x)$ is increasing (decreasing) on $(0, \infty)$.

Lemma 1. *We say that $\bar{H}^*(t) = \bar{H}(\Lambda(t))$ belongs to the $\mathcal{M}(\bar{\mathcal{M}})$-class if $\bar{H} \in \mathcal{M}(\bar{\mathcal{M}})$ and $\Lambda(t)$ is star-shaped (anti-star-shaped) for $t \geq 0$ with $\Lambda(0) = 0$.*

Proof. The proof follows similar arguments to those of Lemma 4.2 of [17]. □

Theorem 3. *Let $\alpha_0 = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k/\lambda_k$ and $m_k(s)$ be defined as in (5). Then $\bar{H}^*(t)$ defined by (6) belongs to the $\mathcal{M}(\bar{\mathcal{M}})$ class if $\Lambda(t)$ is star-shaped (anti-star-shaped) and (4) holds.*

Proof. Let $\bar{H}(t) = \sum_{k=0}^{\infty} S_k(t)\bar{\mathbb{P}}_k$ with $S_k(t)$ defined as in (3). Since (4) holds, Theorem 1 implies that $\bar{H}(t) \in \mathcal{M}$-class. Now the result follows from Lemma 1 by simply observing that $\bar{H}^*(t) = \bar{H}(\Lambda(t))$. The proof for the dual class is analogous. □

We are now in a position to prove the main theorem.

Theorem 4. *Let $\alpha_0 = \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k/\lambda_k$ and $m_k(s)$ be defined as in (5). Further, assume that $\lambda_j \geq [\leq]\, \lambda_0$ for $j = 1, 2, \ldots$ and $(\bar{\mathbb{P}}_k/\lambda_k)_{k=0}^{\infty}$ is decreasing and in the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$-class. Then $\bar{H}(t)$ defined by (6) belongs to the $\mathcal{M}(\bar{\mathcal{M}})$-class if $\Lambda(t)$ is star-shaped (anti-star-shaped).*

Proof. From the proof of Theorem 2, it follows that (4) holds. An application of Theorem 3 now completes the proof. □

Here, if all the birth intensities λ_k are equal to λ then the counting process reduces to an NHPP and in this case we have the following result as a consequence of Theorem 4.

Corollary 2. *When shocks arrive according to a NHPP, the survival function $\bar{H}(t)$ possesses the $\mathcal{M}(\bar{\mathcal{M}})$ property if $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ belongs to the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$-class and $\Lambda(t)$ is star-shaped (anti-star-shaped).*

For a further illustration of the above result, consider the delayed Poisson process model of [17]. Here every shock is delayed by a random amount of time before reaching the device. Let $K(x)$ be the delay distribution function. This delay can, for instance, be thought of as a period of incubation in biometry. Now, the probability of k shocks in $(0, t]$ is given by

$$\mathbb{P}_k(t) = \exp\biggl(-\lambda \int_0^t K(x)\,\mathrm{d}x\biggr) \frac{\{-\lambda \int_0^t K(x)\,\mathrm{d}x\}^k}{k!};$$

see [17]. Thus, the intensity function of this nonhomogeneous process is $\Lambda(t) = \lambda \int_0^t K(x)\,\mathrm{d}x$. As K is increasing, by using a result of Hardy *et al.* [13, Section 6.3] and arguing as in [17], we conclude that Λ is star-shaped. Therefore, from Corollary 2, it follows that the survival function $\bar{H}(t)$ with this $\Lambda(t)$ has the $\mathcal{M}$ property if $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ belongs to the $\mathcal{G}_{\mathcal{M}}$-class.

2.2. A more general shock model

Here we consider a more general shock model where a counting process $\{N(t) \colon t \geq 0\}$ governs the arrival of the shocks. Let U_{k+1} be the interarrival time between the kth and $(k+1)$th shocks, and $A_k := \mathbb{E}(U_{k+1})$ for $k = 0, 1, 2, \ldots$. The survival function is given by

$$\bar{S}(t) = \sum_{k=0}^{\infty} a_k(t)\bar{\mathbb{P}}_k, \tag{7}$$

where $a_k(t) := \mathbb{P}(N(t) = k)$. Such a model has received much attention; see [3] for the NBU class, [9] for the NBUE class, [17] for the HNBUE class, and [27] for the classes NBUFR, NBAFR, new better than used in second order, and new better than used in convex ordering. First, we introduce the following lemma.

Lemma 2. *Suppose that the interarrival times U_k, $k = 1, 2, 3, \ldots,$ are independent and belong to the $\mathcal{M}$ class. If $\bar{S}(t)$ is the survival function defined in (7) and $\bar{H}(t)$ is the survival function defined in (2) with $\lambda_k = 1/A_k$, then*

$$\int_0^{\infty} \mathrm{e}^{st}\,\bar{S}(t)\,\mathrm{d}t \leq \int_0^{\infty} \mathrm{e}^{st}\,\bar{H}(t)\,\mathrm{d}t.$$

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

Proof. Our approach here is similar to that of Klefsjö [17]. The first part of the inequality can be written as

$$\int_0^\infty e^{st}\bar{S}(t)\,dt = \int_0^\infty e^{st}\bigg(\sum_{j=0}^\infty\bigg(\sum_{k=0}^j a_k(t)\bigg)(\bar{\mathbb{P}}_j - \bar{\mathbb{P}}_{j+1})\bigg)\,dt$$
$$= \sum_{j=0}^\infty\bigg\{\int_0^\infty e^{st}\mathbb{P}\bigg(\sum_{k=1}^{j+1} U_k > t\bigg)\,dt\bigg\}(\bar{\mathbb{P}}_j - \bar{\mathbb{P}}_{j+1}). \tag{8}$$

Since U_k, $k = 1, 2, 3, \ldots$, belong to the $\mathcal{M}$-class and $\mathbb{P}(V_k > t) = \exp(-t/A_{k-1})$, $k = 1, 2, 3, \ldots$, we have

$$\int_0^\infty e^{st}\mathbb{P}(U_k > t)\,dt \le \int_0^\infty e^{st}\mathbb{P}(V_k > t)\,dt \quad \text{for } k = 1, 2, 3, \ldots.$$

Now, using Theorem 2.1 of [29], we obtain

$$\int_0^\infty e^{st}\mathbb{P}\bigg(\sum_{k=1}^{j+1} U_k > t\bigg)\,dt \le \int_0^\infty e^{st}\mathbb{P}\bigg(\sum_{k=1}^{j+1} V_k > t\bigg)\,dt. \tag{9}$$

Now using (9) in (8), we have

$$\int_0^\infty e^{st}\bar{S}(t)\,dt \le \sum_{j=0}^\infty\bigg\{\int_0^\infty e^{st}\mathbb{P}\bigg(\sum_{k=1}^{j+1} V_k > t\bigg)\,dt\bigg\}(\bar{\mathbb{P}}_j - \bar{\mathbb{P}}_{j+1}) = \int_0^\infty e^{st}\bar{H}(t)\,dt. \qquad \square$$

The corresponding result for the dual class follows by reversing the direction of the inequalities in the above proof. The next theorem is a consequence of the above lemma and Theorem 2.

Theorem 5. *Suppose that the interarrival times U_k, $k = 1, 2, 3, \ldots$, are independent with distributions in $\mathcal{M}(\bar{\mathcal{M}})$ and $\mathbb{E}(U_{k+1}) = A_k$, $k = 0, 1, 2, \ldots$. If $A_k \le [\ge]\, A_0$, $k = 1, 2, 3, \ldots$, and $(\bar{\mathbb{P}}_k A_k)_{k=0}^\infty$ is decreasing and belongs to the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_{\mathcal{M}})$-class, then the survival function $\bar{S}(t)$ belongs to $\mathcal{M}(\bar{\mathcal{M}})$-class.*

2.3. A cumulative damage shock model

Suppose that a device is exposed to shocks causing random damages which accumulate additively. When the accumulated damage exceeds a critical threshold, the device fails. If there is significant individual variation in the ability to withstand shocks, this threshold must, of necessity, be regarded as a random variable having a CDF F with $F(0) = 0$. If the damages $\xi_1, \xi_2, \xi_3, \ldots$ from successive shocks are independent with CDFs $K_1, K_2, K_3, \ldots$ and are further independent of the threshold, the survival probabilities are given by

$$\bar{\mathbb{P}}_0 = 1, \qquad \bar{\mathbb{P}}_k = \int_0^\infty K_1 \star K_2 \star \cdots \star K_k(x)\,dF(x) \quad \text{for } k = 1, 2, 3, \ldots.$$

See [3], [11], [17], and [18] for earlier work on the cumulative damage model. Esary *et al.* [11] proved that the survival probabilities have the discrete NBU property for all choices of life distributions $(K_j)_{j=1}^\infty$ with $K_j(x)$ decreasing in j for all x if and only if F is NBU. A-Hameed

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

and Proschan [3] and Klefsjö [17] considered a more special choice of the distributions $(K_j)_{j=1}^{\infty}$; namely,

$$K_j(x) = \int_0^x \frac{b^{c_j} t^{c_j-1}}{\Gamma(c_j)} \mathrm{e}^{-bt}\,\mathrm{d}t \quad \text{for } b, c_j > 0.$$

Then the survival probabilities are given by

$$\bar{\mathbb{P}}_k = b\int_0^{\infty} \mathrm{e}^{-bx} \frac{(bx)^{C_k-1}}{\Gamma(C_k)} \bar{F}(x)\,\mathrm{d}x \quad \text{for } k = 1, 2, 3 \ldots$$

with $C_k = \sum_{j=1}^{k} c_j$. In particular, they proved that if $C_k = k$ and $\bar{F}$ is NBUE (HNBUE), then the survival probabilities $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ have the discrete NBUE (HNBUE) properties. Later, Klefsjö [18] considered the special case where the random damages are exponentially distributed with mean $1/s$ and showed that the survival probabilities $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ enjoy the discrete $\mathcal{L}$ property if F belongs to the $\mathcal{L}$-class.

We now establish the appropriate theorem in the context of the $\mathcal{M}$-class of distributions.

Theorem 6. *If $C_k = k$ then the survival probabilities $(\bar{\mathbb{P}}_k)_{k=0}^{\infty}$ belong to the $\mathcal{G}_{\mathcal{M}}(\bar{\mathcal{G}}_M)$-class if and only if $\bar{F}$ belongs to the $\mathcal{M}(\bar{\mathcal{M}})$-class.*

Proof. Suppose that $\bar{F} \in \mathcal{M}$-class. In order to show that $(\bar{\mathbb{P}}_k)_{k=0}^{\infty} \in \mathcal{G}_{\mathcal{M}}$, we need to verify that

$$\sum_{k=0}^{\infty} \bar{\mathbb{P}}_k p^k \leq \frac{\mu}{p + \mu(1-p)} \quad \text{for all } 1 \leq p < \frac{\mu}{\mu - 1} \tag{10}$$

holds, where

$$\begin{aligned}
\mu &= \sum_{k=0}^{\infty} \bar{\mathbb{P}}_k \\
&= 1 + \sum_{k=1}^{\infty} \bar{\mathbb{P}}_k \\
&= 1 + \sum_{k=1}^{\infty} \bigg\{ b^k \int_0^{\infty} \mathrm{e}^{-bx} \frac{x^{k-1}}{(k-1)!} \bar{F}(x)\,\mathrm{d}x \bigg\} \\
&= 1 + b\int_0^{\infty} \bigg\{ \bar{F}(x)\mathrm{e}^{-bx} \sum_{k=1}^{\infty} \frac{(bx)^{k-1}}{(k-1)!} \bigg\}\,\mathrm{d}x \\
&= 1 + b\int_0^{\infty} \bar{F}(x)\,\mathrm{d}x \\
&= 1 + b\mu_F.
\end{aligned} \tag{11}$$

Now,

$$\begin{aligned}
\sum_{k=0}^{\infty} \bar{\mathbb{P}}_k p^k &= 1 + \sum_{k=1}^{\infty} p^k \bigg\{ b^k \int_0^{\infty} \mathrm{e}^{-bx} \frac{x^{k-1}}{(k-1)!} \bar{F}(x)\,\mathrm{d}x \bigg\} \\
&= 1 + \sum_{k=0}^{\infty} bp^{k+1} \bigg\{ \int_0^{\infty} \mathrm{e}^{-bx} \frac{(bx)^k}{k!} \bar{F}(x)\,\mathrm{d}x \bigg\}
\end{aligned}$$

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

$$= 1 + \int_0^\infty bp\mathrm{e}^{-bx}\bar{F}(x)\bigg\{\sum_{k=0}^\infty \frac{(pbx)^k}{k!}\bigg\}\,\mathrm{d}x$$

$$= 1 + bp\int_0^\infty \mathrm{e}^{b(p-1)x}\bar{F}(x)\,\mathrm{d}x. \tag{12}$$

From the $\mathcal{M}$ property of $\bar{F}$, we have

$$\int_0^\infty \mathrm{e}^{b(p-1)x}\bar{F}(x)\,\mathrm{d}x \le \frac{\mu_F}{1-b(p-1)\mu_F} \quad \text{for } 1 \le p < 1 + \frac{1}{b\mu_F}.$$

This together with (12) and the fact that $\mu = 1 + b\mu_F$ implies that (10) holds.

The converse follows easily from (11) and (12) and the definitions of the $\mathcal{M}$ and $\mathcal{G}_{\mathcal{M}}$ classes via the simple reparametrization $s = b(p-1)$. The proof for the dual class is analogous. □

3. Weak convergence within the $\mathcal{M}$ class

Let $\mathcal{C}$ be a particular ageing class. In the case of several monotonic ageing classes, it is known that if $F_n \in \mathcal{C}$ and $F_n \to F$ in law, where F is continuous, then $F \in \mathcal{C}$. Basu and Simons [6] first established this result in the context of the IFR class. Later, Basu and Bhattacharjee [5] and Chaudhuri [10] proved the same result for the HNBUE and $\mathcal{L}$-classes, respectively. The same exercise was carried out for the DMRL class by Nanda and Paul [26]. We complete the investigation by settling the question for the $\mathcal{M}$-class.

We first obtain bounds for the moments of an $\mathcal{M}$ distribution in terms of the mean. This will be useful in establishing the main result of this section. Theorem 5.1 of [16] yields the following explicit reliability bound. If $F \in \mathcal{M}$ with mean μ then

$$\bar{F}(t) \le \frac{t}{\mu}\exp\bigg(1 - \frac{t}{\mu}\bigg) \quad \text{for all } t > \mu. \tag{13}$$

We use this result to obtain an appropriate moment bound.

Theorem 7. *Let $F \in \mathcal{M}$-class with mean μ. Then, for $r > 0$,*

$$\mu_r := \mathbb{E}(X^r) \le \mu^r(1 + er\Gamma(r+1)),$$

where e is Euler's number.

Proof. For $r > 0$, by (13),

$$\mu_r \le r\bigg[\int_0^\mu x^{r-1}\,\mathrm{d}x + \int_\mu^\infty x^{r-1}\bar{F}(x)\,\mathrm{d}x\bigg]$$
$$\le \bigg[\mu^r + \frac{r}{\mu}\int_\mu^\infty x^r\mathrm{e}^{(1-x/\mu)}\,\mathrm{d}x\bigg]$$
$$= \mu^r(1 + er\Gamma(r+1)).$$ □

The main result of this section is contained in the following theorem.

Theorem 8. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of life distributions belonging to the $\mathcal{M}$-class with mean μ_n, and also be the probability law obeyed by the sequence of random variables $\{X_n\}_{n=1}^\infty$. Suppose that*

(i) *$F_n \to F$ in law, where F is a continuous distribution function;*

(ii) *the sequence $\{\mu_n\}_{n=1}^\infty$ is bounded.*

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

Then $F \in \mathcal{M}$-class. Further, for every $r > 0$,

$$\lim_{n\to\infty} \int_0^\infty x^r \, \mathrm{d}F_n(x) = \int_0^\infty x^r \, \mathrm{d}F(x). \tag{14}$$

To prove the above theorem, we use Theorem 3 of [20] which we state below for convenience.

Theorem 9. (Kozakiewicz [20, Theorem 3].) *Let $\{F_n(x)\}$ be a sequence of distribution functions and let $\{\phi_n(t)\}$ be the corresponding sequence of MGFs which are all assumed to exist for $|t| < \alpha$, where $\alpha > 0$. For any fixed nonnegative x, let $M(x)$ be the least upper bound of the sequence $\{F_n(-x) + 1 - F_n(x)\}$. The necessary and sufficient conditions for the convergence of $\{\phi_n(t)\}$ in the interval $|t| < \alpha$ are*

$$\lim_{x\to\infty} M(x)\mathrm{e}^{|t|x} = 0, \qquad |t| < \alpha, \tag{15}$$

and the sequence $\{F_n(x)\}$ converges to a distribution function $F(x)$ at each point of continuity of $F(x)$.

Further, the MGF of $F(x)$ exists for $|t| < \alpha$ and is equal in this interval to the limit of the sequence $\{\phi_n(t)\}$.

In order to use Theorem 9, we essentially need to verify that condition (15) holds for a sequence of $\mathcal{M}$ distributions with bounded means. To this end, we require the following two lemmas.

Lemma 3. *The following conditions are equivalent:*

(i) $\lim_{x\to\infty} M(x)\mathrm{e}^{|t|x} = 0$, $|t| < \alpha$;

(ii) $\limsup_{x\to\infty} \ln M(x)/x \leq -\alpha$.

Proof. (i) This statement implies that for each fixed $t \ni |t| < \alpha$, there exists $x_0 \ni M(x)\mathrm{e}^{|t|x} < 1$ for all $x > x_0$. This, in turn, implies that $\ln M(x)/x \leq -|t|$ for all $x > x_0$ further implying that $\ln M(x)/x \leq -\alpha$ for all $x > x_0$, letting $|t| \to \alpha$. Again, this implies $\limsup_{x\to\infty} \ln M(x)/x \leq -\alpha$.

(ii) This statement implies that for all sufficiently large x, we have $M(x) \leq \mathrm{e}^{-\alpha x}$, so that for $|t| < \alpha$, $0 \leq M(x)\mathrm{e}^{|t|x} \leq \mathrm{e}^{|t|x}\mathrm{e}^{-\alpha x} = \mathrm{e}^{-(\alpha-|t|)x}$. An application of the squeeze lemma now completes the proof. □

Lemma 4. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of $\mathcal{M}$ distributions with mean μ_n. Assume that the sequence $\{\mu_n\}_{n=1}^\infty$ is bounded. Then*

$$\limsup_{x\to\infty} \frac{\ln M(x)}{x} \leq -\alpha.$$

Proof. For the sequence of life distributions $\{F_n\}_{n=1}^\infty$ belonging to the $\mathcal{M}$ class,

$$M(x) = \sup_{n\geq 1} \bar{F}_n(x).$$

By Theorem 3.1 of [16], we have, for all $n \geq 1$,

$$-\limsup_{x\to\infty} \frac{\ln \bar{F}_n(x)}{x} \geq \frac{1}{\mu_n}.$$

https://doi.org/10.1017/jpr.2018.26 Published online by Cambridge University Press

Supposing that $\mu_n \le B$ for all $n \ge 1$, we obtain

$$\limsup_{x\to\infty} \frac{\ln \bar{F}_n(x)}{x} \le -\frac{1}{\mu_n} \le -\alpha_1, \tag{16}$$

where $\alpha_1 = 1/B \le 1/\mu_n$ for all $n \ge 1$. Note that

$$\sup_{n\ge1} \limsup_{x\to\infty} \frac{\ln \bar{F}_n(x)}{x} = \limsup_{x\to\infty} \frac{\ln M(x)}{x}. \tag{17}$$

Now a simple use of (16) in (17) establishes the lemma. □

Proof of Theorem 8. Assumption (i) in conjunction with Lemmas 3 and 4 imply that the conditions of Theorem 9 (Theorem 3 of [20]) hold. Thus,

$$\int_0^\infty \mathrm{e}^{tx}\,\mathrm{d}F_n(x) \to \int_0^\infty \mathrm{e}^{tx}\,\mathrm{d}F(x) \quad \text{as } n\to\infty. \tag{18}$$

Now, by Fatou's lemma,

$$\int_0^\infty \bar{F}(x)\,\mathrm{d}x \le B < \infty.$$

By Theorem 7, $\mathbb{E}X_n^2 \le \mu_n^2(1+4e)$ for all $n \ge 1$ and so, for any $c > 0$,

$$\mathbb{E}X_n^2 = \int X_n^2\,\mathrm{d}P \ge c\int_{\{X_n\ge c\}} X_n\,\mathrm{d}P,$$

$$\text{which implies that } \int_{\{X_n\ge c\}} X_n\,\mathrm{d}P \le \frac{\mathbb{E}X_n^2}{c} \le \frac{\mu_n^2(1+4e)}{c} \le \frac{B^2(1+4e)}{c},$$

which, in turn, implies that $\{X_n\}_{n=1}^\infty$ is uniformly integrable. From a theorem of [8, p. 348], we have

$$\lim_{n\to\infty}\int_0^\infty x\,\mathrm{d}F_n(x) = \int_0^\infty x\,\mathrm{d}F(x). \tag{19}$$

Thus, as $F_n \in \mathcal{M}$,

$$\int_0^\infty \mathrm{e}^{tx}\,\mathrm{d}F_n(x) \le \frac{1}{1-\mu_n t}, \qquad 0 \le t < \frac{1}{\mu_n}.$$

Taking limits as $n \to \infty$, from (18) and (19), we have,

$$\int_0^\infty \mathrm{e}^{tx}\,\mathrm{d}F(x) \le \frac{1}{1-\mu t}, \qquad 0 \le t < \frac{1}{\mu}.$$

This completes the proof of the first part of the theorem.

To prove (14) note that $\mathbb{E}[X_n^{r+1}] \le \mu_n^{r+1}[1 + e(r+1)\Gamma(r+2)]$ and $\{\mu_n\}$ is bounded. So, $\sup_n \mathbb{E}[X_n^{r+1}] < \infty$. Therefore, by another result of [8, p. 348], $\mathbb{E}[X^r] < \infty$ and $\mathbb{E}[X_n^r] \to \mathbb{E}[X^r]$ as $n \to \infty$. □

Lemma 5. *A distribution function which belongs to the $\mathcal{M}$-class is uniquely determined by its moment sequence.*

Proof. For any distribution belonging to the $\mathcal{M}$-class, by Theorem 7, it follows easily that the radius of convergence of the power series $\sum_{r=0}^\infty (u^r/r!)\mathbb{E}[X^r]$ is $1/\mu$ and, hence, nonnull. The lemma now follows from a result of Loève [23, p. 217]. □

In the previous theorem we showed that under certain conditions, weak convergence of a sequence of $\mathcal{M}$ distributions implies convergence of moments of the sequence of the distribution functions to the corresponding moments of the limiting distribution. The following provides an affirmative answer to the converse question.

Theorem 10. *Suppose $\{F_n\}$ is a sequence of distribution functions belonging to the $\mathcal{M}$-class and let F be an $\mathcal{M}$ distribution such that, for all integers $r \geq 1$,*

$$\lim_{n\to\infty} \int_0^\infty x^r \, \mathrm{d}F_n(x) = \int_0^\infty x^r \, \mathrm{d}F(x). \tag{20}$$

Then $F_n \to F$ in law.

Proof. Lemma 5 together with (20) imply that the limiting distribution of every weakly convergent subsequence of $\{F_n\}$ happens to be necessarily F, thereby completing the proof. □

Remark 1. It is interesting to note that MGF order is preserved under the formation of weak limits. Suppose that $\{F_n\}$ and $\{G_n\}$ are two sequences of life distribution functions such that $F_n \to F$ and $G_n \to G$ in law. Further, assume that $F_n \leq_{\mathrm{MGF}} G_n$ for each $n \geq 1$ and both $\{F_n\}$ and $\{G_n\}$ satisfy any one of the conditions of Lemma 3. Then $F \leq_{\mathrm{MGF}} G$.

Acknowledgements

The authors are grateful to the anonymous reviewers for their useful comments and suggestions which have led to a substantial improvement on an earlier version of this manuscript.

References

[1] Abouammoh, A. M., Hindi, M. I. and Ahmed, A. N. (1988). Shock models with NBUFR and NBAFR survivals. *Trabajos Estadistica* **3,** 97–113.
[2] A-Hameed, M. S. and Proschan, F. (1973). Nonstationary shock models. *Stoch. Process. Appl.* **1,** 383–404.
[3] A-Hameed, M. S. and Proschan, F. (1975). Shock models with underlying birth process. *J. Appl. Prob.* **12,** 18–28.
[4] Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston, New York.
[5] Basu, S. K. and Bhattacharjee, M. C. (1984). On weak convergence within the HNBUE family of life distributions. *J. Appl. Prob.* **21,** 654–660.
[6] Basu, S. K. and Simons, G. (1983). Moment spaces for IFR distributions, applications and related materials. In *Contributions to Statistics*, North Holland, Amsterdam, pp. 27–46.
[7] Bhattacharyya, G. and Fries, A. (1982). Fatigue failure models: Birnbaum-Saunders vs. inverse Gaussian. *IEEE Trans. Reliab.* **31,** 439–441.
[8] Billingsley, P. (1979). *Probability and Measure*. John Wiley, New York.
[9] Block, H. W. and Savits, T. H. (1978). Shock models with NBUE survival. *J. Appl. Prob.* **15,** 621–628.
[10] Chaudhuri, G. (1995). A note on the L-class of life distributions. *Sankhyā A* **57,** 158–160.
[11] Esary, J. D., Marshall, A. W. and Proschan, F. (1973). Shock models and wear processes. *Ann. Prob.* **1,** 627–649.
[12] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd edn. John Wiley, New York.
[13] Hardy, G. H., Littlewood, J. E. and Pólya, G. (1952). *Inequalities*, 2nd edn. Cambridge University Press.
[14] Klar, B. (2002). A note on the $\mathcal{L}$-class of life distributions. *J. Appl. Prob.* **39,** 11–19.
[15] Klar, B. (2005). Tests for exponentiality against the $\mathcal{M}$ and $\mathcal{LM}$-classes of life distributions. *Test* **14,** 543–565.
[16] Klar, B. and Müller, A. (2003). Characterizations of classes of lifetime distributions generalizing the NBUE class. *J. Appl. Prob.* **40,** 20–32.
[17] Klefsjö, B. (1981). HNBUE survival under some shock models. *Scand. J. Statist.* **8,** 39–47.
[18] Klefsjö, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Prob.* **20,** 615–626.

[19] Kochar, S. C. and Deshpande, J. V. (1985). On exponential scores statistic for testing against positive ageing. *Statist. Prob. Lett.* **3,** 71–73.
[20] Kozakiewicz, W. (1947). On the convergence of sequences of moment generating functions. *Ann. Math. Statist.* **18,** 61–69.
[21] Lai, C.-D. and Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability.* Springer, New York.
[22] Li, X. (2004). Some properties of ageing notions based on the moment-generating-function order. *J. Appl. Prob.* **41,** 927–934.
[23] Loève, M. (1963). *Probability Theory*, 3rd edn. Van Nostrand, Princeton, NJ.
[24] Marshall, A. W. and Olkin, I. (2007). *Life Distributions*. Springer, New York.
[25] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. John Wiley, Chichester.
[26] Nanda, A. K. and Paul, P. (2004). An observation from DMRL life distribution. *Statist. Meth.* **6,** 57–65, 235.
[27] Pellerey, F. (1994). Shock models with underlying counting process. *J. Appl. Prob.* **31,** 156–166.
[28] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.
[29] Zhang, S. and Li, X. (2010). Some new results on the moment generating function order and related life distributions. *J. Appl. Prob.* **47,** 923–933.