Improved Transversal Polar Navigation Mechanism for Strapdown INS using Ellipsoidal Earth Model

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As the geographical meridians converge rapidly, traditional inertial navigation methods fail in the polar regions. Classic transversal navigation methods can address the problem by transversal rotation of the original north and south poles, but this can introduce errors based on the spherical Earth model. To reduce the principle errors, some fruitful research work using an ellipsoidal Earth model has been done. Under the ellipsoid Earth model, transversal navigation for polar region becomes a complex coupling problem. Considering the coupling of the three-dimensional motion, a more rigorous mechanism for transversal navigation using an ellipsoidal Earth model is proposed. Starting from the relationship between Euclidean coordinates and spherical coordinates, the main equations of transversal polar navigation based on an ellipsoidal Earth model are derived in detail. Complete mechanical arrangements of attitude, position and velocity calculation are presented. The new derivation in this paper completely avoids solving the ellipsoidal radius. Numerical results indicate that the proposed transversal navigation mechanism can outperform the traditional method, especially in the condition of vertical motion.

1. INTRODUCTION.   Polar navigation technology has been a widely researched navigation field with the development of worldwide transportation (Yao et al., 2016; Li et al., 2016; 2015; 2014a; 2014b; Pedersen, 1960). Due to its special geographical characteristics, general navigation methods such as satellite navigation, radio navigation and geomagnetic navigation do not always work efficiently in polar regions (Tang et al., 2009; Naumann, 2011; Zhao, 2017). Inertial navigation is not affected by external conditions such as polar geomagnetic changes and solar storms. With consideration of the self-contained merits, a Strapdown Inertial Navigation System (SINS) is the most preferred system for a vehicle in polar regions (Pedersen, 1960). As the geographical meridians converge rapidly in the polar regions, traditional inertial navigation methods fail in these areas. All mechanisms in SINS using traditional latitude and longitude as the position parameterisations fail to

provide accurate positions and orientations in such cases. If the local geographic navigation frame is chosen as the navigation frame, the turn rate of the navigation frame will be involved in terms of a tangent function. A potential problem will arise when the vehicle travels close to the Earth's poles. This is because when the latitude angle φ approaches $\pm 90°$, $\tan \varphi$ will be infinite (Bekir, 2007). The wander azimuth mechanism is one option to circumvent the polar navigation problem. The corresponding idea is to prevent the navigation frame from rotating about the z-axis. However, for the wander azimuth mechanism, it is impossible to distinguish between the wander azimuth angle and the longitude. This implies that the wander azimuth angle may not be the answer to navigation near the poles (Bekir, 2007). Navigation in an Earth-Centred Earth-Fixed (ECEF) frame is another choice for polar navigation. However, the ECEF mechanism inherits some limitations due to its non-horizontal characteristic. More specifically, the diverging vertical errors will be coupled into an ECEF-based system, affecting the accuracy of the position in other directions. In this respect, it is not suitable for long range and long endurance missions (Titterton and Weston, 2014).

To address the problems of polar navigation, a transversal coordinate system and corresponding navigation mechanism was proposed in Broxmeyer (1964) and Lyon (1984). In this navigation mechanism, the traditional geographic coordinate system is transversely rotated, which makes the original north and south poles change to the equator of the transversal coordinate system. The traditional transversal navigation uses a spherical model for the earth, which brings advantages of brevity and convenience for transverse rotation of the geographic coordinate system. Unfortunately, the Earth is an ellipsoid and the corresponding simplification consequentially introduces principle errors for an Inertial Navigation System (INS). Some interesting research work, such as transformation of navigation parameters and reset and damping of transversal SINS, has been completed based on transversal mechanisation (Li et al., 2015; 2014a; 2014b; Watland and Ariz, 1995).

In Li et al. (2014a; 2014b), performances and error characteristics of transversal navigation based on the sphere model were theoretically analysed. The principle errors caused by the spherical model are mainly in the form of oscillation errors, and damping technology, widely applied in traditional SINS, is used to suppress those oscillation errors (Li et al., 2015; 2014a; 2014b). However, constant errors from velocities remain. Due to these principle errors, transversal navigation using the sphere model is not acceptable for a high-precision INS. In order to further improve the navigation precision, many researchers have addressed these principle errors. In Yao et al. (2016) and Li et al. (2016) the ellipsoidal Earth model is used to address the theoretical error resulting from the inaccurate spherical Earth model. In Yao et al. (2016), implicit equations of radii using a transversal ellipsoidal Earth model are derived in detail. To obtain analytic solutions for these radii, an additional angle parameter is introduced, which unfortunately cannot be obtained analytically in the differential equation. It should be noted that Equation (15) of Yao et al. (2016) has been approximated and simplified. In Li et al. (2016), the radius of the transversal meridian and the radius of the transversal prime vertical are derived. Since in a transversal coordinate system, "latitude" or "longitude" is affected by both "north" velocity and "east" velocity, the individual radius of the transversal meridian or the radius of the transversal prime vertical are not so suitable for coupling of transversal navigation. In addition, the height in transversal navigation is omitted in Li et al. (2016).

Inspired by the ideas of Yao et al. (2016) and Li et al. (2016), a more rigorous transversal navigation mechanism is proposed in this paper. Starting from the relationship between
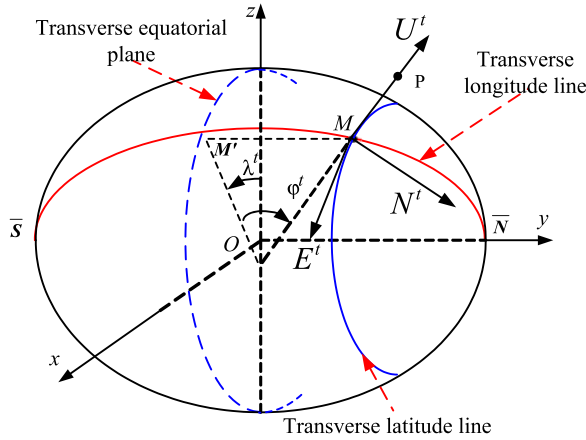
Figure 1. Transversal Navigation Coordinate System.

Euclidean coordinates and spherical coordinates, the main equations of transversal polar navigation based on an ellipsoidal Earth model are derived in detail, including attitude, velocity and position differential equations. The main contribution of the proposed method is a new derivation method to avoid approximation errors in the transformation of transversal navigation. Simulations are carried out to evaluate the feasibility and validity of the proposed navigation mechanism. Compared with Yao et al. (2016) and Li et al. (2016), the proposed navigation mechanism has advantages of high precision and immunity of heaving motions in the vertical direction.

The remainder of this paper is organised as follows. Section 2 presents the definition of the transversal coordinate system. Section 3 is devoted to the derivation of two important formulae, which are the basis for transversal navigation. The transversal navigation mechanism, including differential equations of attitude, velocity and position are proposed in Section 4. The feasibility and effectiveness of the proposed navigation scheme are evaluated through simulation experiments in Section 5. Finally, conclusions are drawn in Section 6.

2. DEFINITION OF TRANSVERSAL COORDINATE SYSTEM. In order to address the problem of traditional inertial navigation methods in the polar regions due to the rapid convergence of meridians, a transversal navigation method was proposed in Broxmeyer (1964), which transforms the traditional geographic coordinate system into a transversal coordinate system. The original geographical north and south poles are transformed to be the equator in the new coordinate system, as shown in Figure 1.

Let the transversal coordinate system be e^t. The origin of the coordinate system is located at the centre of the Earth O, with the x_{et} axis along the Earth rotation axis, the y_{et} axis along the intersection of the meridian (east meridian $0°$) and the z_{et} axis completes the right-handed orthogonal frame. As shown in Figure 1, the relationship between the transversal coordinate system e^t and the Earth coordinate system e is given by:

$$\mathbf{C}_e^{e^t} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{1}$$

Let the transversal navigation coordinate system be t. Its origin is located at the position of the carrier, the E^t axis lies along the tangent of the transversal latitude circle toward the transversal east, the N^t axis lies along the tangent of the transversal meridian circle toward the transversal north and the U^t axis lies along the normal of the sphere toward the zenith. The definition of the transversal latitude and longitude is similar to the definition of geometric latitude and longitude. Let the meridional circle where the prime meridian is located be the transversal equator, and the original geographic coordinate points $[0°, 90°\mathrm{E}]$ and $[0°, 90°\mathrm{W}]$ be the new transversal north and south poles, respectively. The transversal meridian is defined as a contour line by crossing the Earth with the plane through two transversal poles.

M' is the intersecting point of the transversal equator and the transversal meridian passing through point M. The point M is a point on the surface of the Earth. The cross angle between the geometric normal and the transversal equatorial plane is defined as the transversal latitude φ^t at point M. The transversal north hemisphere is located in the geographical eastern hemisphere and the transversal south hemisphere is located in the geographical western hemisphere. We define the initial transversal meridian as the northern hemisphere part of the geographical meridional circle where the geo-longitude is $90°\mathrm{E}$. The cross angle between the transversal meridian surface and the initial transversal meridian is defined as the transversal meridian λ^t at point M. P represents the position of a vehicle with a height h. The position P can be determined by transversal longitude λ^t, transversal latitude φ^t and height h, expressed as (x, y, z) in the Earth coordinate system e and as (φ, λ, h) using the geographic latitude and longitude. For the same point, the transformation formula for (x, y, z) and $(\varphi^t, \lambda^t, h)$ is given by:

$$
\begin{cases}
x = (R_n + h) \cos \varphi^t \sin \lambda^t \\
y = (R_n + h) \sin \varphi^t \\
z = [R_n(1 - e^2) + h] \cos \varphi^t \cos \lambda^t
\end{cases}
\tag{2}
$$

where R_n is the radius of the prime vertical circle and e is the elliptical eccentricity.

The transformation formula between (x, y, z) and (φ, λ, h) is given by:

$$
\begin{cases}
x = (R_n + h) \cos \varphi \cos \lambda \\
y = (R_n + h) \cos \varphi \sin \lambda \\
z = [R_n(1 - e^2) + h] \sin \varphi
\end{cases}
\tag{3}
$$

According to Equations (2) and (3), the relationships of latitude and longitude between the geographic coordinate system and transversal navigation coordinate system are given by:

$$
\begin{cases}
\sin \varphi^t = \cos \varphi \sin \lambda \\
\tan \lambda^t = \cot \varphi \cos \lambda
\end{cases}
\tag{4a}
$$

$$
\begin{cases}
\sin \varphi = \cos \varphi^t \cos \lambda^t \\
\tan \lambda = \tan \varphi^t \lambda^t
\end{cases}
\tag{4b}
$$

As R_n cannot be treated as a constant value in the polar region, the relationship between R_n and φ^t, λ^t becomes:

$$
\begin{aligned}
R_n &= \frac{R_e}{\sqrt{1 - e^2 \sin^2 \varphi}} \\
&= \frac{R_e}{\sqrt{1 - e^2 \cos^2 \varphi^t \cos^2 \lambda^t}}
\end{aligned}
\tag{5}
$$

where R_e is the equatorial radius, and is a constant.

The transversal navigation coordinate system t can be obtained by two rotations of the Earth coordinate system as follows:

$$
x - y - z \frac{y - axis}{\lambda^t} x_1 - y_1 - z_1 \frac{-x_1 - axis}{\varphi^t} E^t - N^t - U^t
$$

We rotate an angle λ^t around the y positive axis to obtain the coordinate system (x_1, y_1, z_1). The rotation matrix is:

$$
\mathbf{C}_e^1 = \begin{bmatrix} \cos \lambda^t & 0 & -\sin \lambda^t \\ 0 & 1 & 0 \\ \sin \lambda^t & 0 & \cos \lambda^t \end{bmatrix}
\tag{6}
$$

We then rotate an angle φ^t around the x_1 axis to obtain the ellipsoidal transversal coordinate system. The rotation matrix is:

$$
\mathbf{C}_1^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi^t & -\sin \varphi^t \\ 0 & \sin \varphi^t & \cos \varphi^t \end{bmatrix}
\tag{7}
$$

Thus, the Direction Cosine Matrix (DCM) $\mathbf{C}_e^t$ is determined:

$$
\mathbf{C}_e^t = \mathbf{C}_1^t \mathbf{C}_e^1 = \begin{bmatrix} \cos \lambda^t & 0 & -\sin \lambda^t \\ -\sin \varphi^t \sin \lambda^t & \cos \varphi^t & -\sin \varphi^t \cos \lambda^t \\ \cos \varphi^t \sin \lambda^t & \sin \varphi^t & \cos \varphi^t \cos \lambda^t \end{bmatrix}
\tag{8}
$$

3. DERIVATION OF TWO IMPORTANT FORMULAE IN TRANSVERSAL NAVIGATION. In order to derive the detailed transversal navigation equations, we firstly determine two important formulae. One is the position differential equation and the other is an equation of the transport rate in transversal navigation. Before the presentation of the explicit form of the following equations, we denote the time differential of one variable as $d(\cdot) = d(\cdot)/dt$.

Executing differential operations on both sides of Equation (2) and substituting Equation (5) into the resulting equation gives:

$$\mathbf{V}^e = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T = \begin{bmatrix} dx & dy & dz \end{bmatrix}^T$$

$$= \begin{bmatrix} \left(\sin \lambda^t \cos \varphi^t \left(dh - \dfrac{R_e \left(\begin{array}{c} 2d\lambda^t e^2 \sin \lambda^t \cos \lambda^t \cos^2 \varphi^t \\ + 2d\varphi^t e^2 \cos^2 \lambda^t \sin \varphi^t \cos \varphi^t \end{array} \right)}{2\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}} \right) \\[2em] + d\lambda^t \cos \lambda^t \cos \varphi^t \left(\dfrac{R_e}{\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}} + h \right) \\[1em] - d\varphi^t \sin \lambda^t \sin \varphi^t \left(\dfrac{R_e}{\sqrt{1 - e^2 \cos^2(\lambda^t) \cos^2(\varphi^t)}} + h \right) \right) \\[4em] \left(\sin \varphi^t \left(dh - \dfrac{R_e \left(\begin{array}{c} 2d\lambda^t e^2 \sin \lambda^t \cos \lambda^t \cos^2 \varphi^t \\ + 2d\varphi^t e^2 \cos^2 \lambda^t \sin \varphi^t \cos \varphi^t \end{array} \right)}{2\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}} \right) \\[2em] + d\varphi^t \cos \varphi^t \left(\dfrac{R_e}{\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}} + h \right) \right) \\[4em] \left(\cos \lambda^t \cos \varphi^t \left(dh - \dfrac{(1 - e^2)R_e \left(\begin{array}{c} 2d\lambda^t e^2 \sin \lambda^t \cos \lambda^t \cos^2 \varphi^t \\ + 2d\varphi^t e^2 \cos^2 \lambda^t \sin \varphi^t \cos \varphi^t \end{array} \right)}{2\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}} \right) \\[2em] - d\lambda^t \sin \lambda^t \cos \varphi^t \\[1em] \left(\dfrac{(1 - e^2)R_e}{\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}} + h \right) \\[1em] - d\varphi^t \cos \lambda^t \sin \varphi^t \left(\dfrac{(1 - e^2)R_e}{\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}} + h \right) \right) \end{bmatrix} \tag{9}$$

According to the transversal coordinate system definition, the velocity vector in the transversal navigation coordinate t can be expressed as a component form:

$$\mathbf{V}^t = \begin{bmatrix} v_e^t & v_n^t & v_u^t \end{bmatrix} \tag{10}$$

where v_e^t, v_n^t, v_u^t represent the east velocity, north velocity and vertical velocity in transversal coordinate t, respectively.

According to Equations (8)–(10), the velocity in the transversal navigation coordinate t can be obtained:

$$\mathbf{V}^t = \begin{bmatrix} v_e^t & v_n^t & v_u^t \end{bmatrix}^T = (\mathbf{C}_e^t)\mathbf{V}^e \tag{11a}$$

$$v_e^t = -\frac{\begin{pmatrix} d\lambda^t e^2 \cos^4 \lambda^t \cos^3 \varphi^t \left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + R_e\right) + d\lambda^t \cos^2 \lambda^t \cos \varphi^t \\ \left(e^2 \sin^2 \lambda^t \cos^2 \varphi^t - 1\right) \left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + R_e\right) + d\lambda^t \sin^2 \lambda^t \cos \varphi^t \\ \left((e^2 - 1)R_e - h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\right) + d\varphi^t e^2 R_e \sin \lambda^t \cos \lambda^t \sin \varphi^t \end{pmatrix}}{\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}}$$

$$v_n^t = -\frac{\begin{pmatrix} e^2 \cos(2\lambda^t - 2\varphi^t) \left(d\lambda^t R_e + d\varphi^t h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\right) \\ - d\lambda^t e^2 R_e \cos(2(\lambda^t + \varphi^t)) \\ + 2d\varphi^t e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + d\varphi^t e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + 2d\varphi^t e^2 h \sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} - 8d\varphi^t h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + 2d\varphi^t e^2 \cos(2\lambda^t) \left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 2R_e\right) + 4d\varphi^t e^2 Re - 8d\varphi^t R_e \end{pmatrix}}{8\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}}$$

$$v_u^t = dh \tag{11b}$$

If $d\lambda^t, d\varphi^t, dh$ are treated as unknown variables in Equation (11), the linear equations determined by Equation (11) are solved, and the position differential equation in the transversal coordinate system can be obtained:

$$d\lambda^t = \frac{\begin{pmatrix} \left(v_{et}^{\text{sec}} \varphi^t(e^2 \cos(2(\lambda^t + \varphi^t)) + e^2 \cos(2(\lambda^t - \varphi^t)) \right. \\ \left. + 2e^2 \cos(2\varphi^t) + 2e^2 - 8\right)^2 \\ \left(2e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \right. \\ + e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + e^2 h \cos(2(\lambda^t - \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + 2e^2 h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ - 8h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 2e^2 \cos(2\lambda^t) \\ \left. \left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 2R_e\right) + 4e^2 R_e - 8R_e\right) \end{pmatrix}}{\begin{pmatrix} \left(64\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2} \left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + R_e\right) \cdot \right. \\ \begin{pmatrix} 2e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + e^2 h \cos(2(\lambda^t - \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + 2e^2 h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ + 2e^2 h \cos(2\lambda^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ - 8h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 8e^2 R_e - 8R_e \end{pmatrix} \end{pmatrix}}$$

$$
- \frac{
\begin{aligned}
&(v_{nt}e^2 R_e \sin\lambda^t \cos\lambda^t \sin\varphi^t (e^2 \cos(2(\lambda^t + \varphi^t)) \\
&+ e^2 \cos(2(\lambda^t - \varphi^t)) + 2e^2 \cos(2\lambda^t) + 2e^2 \cos(2\varphi^t) + 2e^2 - 8)^2)
\end{aligned}
}{
\left(8\left(1 - e^2 \cos^2\lambda^t \cos^2\varphi^t\right)^{3/2} \left(h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} + R_e\right) \cdot \right)
\begin{pmatrix}
2e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} \\
+ e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} \\
+ e^2 h \cos(2(\lambda^t - \varphi^t))\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} \\
+ 2e^2 h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} \\
+ 2e^2 h \cos 2\lambda^t \sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} \\
- 8h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t} + 8e^2 R_e - 8R_e
\end{pmatrix}
}
\tag{12}
$$

$$
d\varphi^t = \frac{
\left(16e^2 R_e v_{et} \left(1 - e^2 \cos^2\lambda^t \cos^2\varphi^t\right)^{3/2} \lambda^t \sec\lambda^t \sin^2(2\lambda^t) \sin\varphi^t \right)
}{
\begin{pmatrix}
\left(\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\, h + R_e\right) \\
(2\cos(2\lambda^t)e^2 + 2\cos(2\varphi^t)e^2 + \cos(2(\lambda^t + \varphi^t))e^2 \\
+ \cos(2(\lambda^t - \varphi^t))e^2 + 2e^2 - 8)\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2\varphi^t)e^2 \\
(8R_e e^2 + 2h + h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t + \varphi^t))e^2 \\
+ h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t - \varphi^t))e^2 \\
+ 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 + 2h\cos(2\lambda^t)\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 \\
- 8R_e - 8h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t})
\end{pmatrix}
}
$$

$$
- \frac{
\begin{pmatrix}
4v_{nt}\left(1 - e^2 \cos^2\lambda^t \cos^2\varphi^t\right)^{3/2}\left(6R_e e^2 - 2\cos(2\lambda^t)\right. \\
\left(R_e - h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\right)e^2 + 2R_e \cos(2\varphi^t)e^2 \\
+ 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2\varphi^t)e^2 + R_e \cos(2(\lambda^t + \varphi^t))e^2 \\
+ h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t + \varphi^t))e^2 \\
+ \left(\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\, h + R_e\right)\cos(2(\lambda^t - \varphi^t))e^2 \\
+ 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 \\
\left. - 8R_e - 8h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\right)\lambda^t \sec\lambda^t \sin(2\lambda^t)
\end{pmatrix}
}{
\begin{pmatrix}
\left(\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\, h + R_e\right)(2\cos(2\lambda^t)e^2 + 2\cos(2\varphi^t)e^2 \\
+ \cos(2(\lambda^t + \varphi^t))e^2 + \cos(2(\lambda^t - \varphi^t))e^2 + 2e^2 - 8) \\
(8R_e e^2 + 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2\varphi^t)e^2 \\
+ h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t + \varphi^t))e^2 \\
+ h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t - \varphi^t))e^2 \\
+ 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 + 2h\cos(2\lambda^t)\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 \\
- 8R_e - 8h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t})
\end{pmatrix}
}
\tag{13}
$$

$$
dh = v_{ut} \tag{14}
$$

According to Equations (12)–(14), we obtain the position differential relationship equation in the transversal navigation system, which relates velocity $\mathbf{V}^t$ with λ^t, φ^t and h.

According to the rule of the DCM differential equation, we know:

$$\dot{\mathbf{C}}_e^t = \mathbf{C}_e^t \left[\boldsymbol{\omega}_{te}^e \times \right] \tag{15}$$

where $\left[\boldsymbol{\omega}_{te}^e \times \right]$ is the skew symmetric DCM corresponding to the angular velocity vector $\boldsymbol{\omega}_{te}^e = \left[\omega_{te(x)}^e \quad \omega_{te(y)}^e \quad \omega_{te(z)}^e \right]^T$, expressed as:

$$\left[\boldsymbol{\omega}_{te}^e \times \right] = \begin{bmatrix} 0 & -\omega_{te(z)}^e & \omega_{te(y)}^e \\ \omega_{te(z)}^e & 0 & -\omega_{te(x)}^e \\ -\omega_{te(y)}^e & \omega_{te(x)}^e & 0 \end{bmatrix} \tag{16}$$

According to the orthogonality of the DCM:

$$(\mathbf{C}_e^t)^{-1} = (\mathbf{C}_e^t)^T \tag{17}$$

Substituting Equation (8) into Equation (15) yields:

$$\left[\boldsymbol{\omega}_{te}^e \times \right] = (\mathbf{C}_e^t)^{-1} \dot{\mathbf{C}}_e^t = \begin{bmatrix} 0 & d\varphi^t \sin\lambda^t & -d\lambda^t \\ -d\varphi^t \sin\lambda^t & 0 & -d\varphi^t \cos\lambda^t \\ -d\lambda^t & d\varphi^t \cos\lambda^t & 0 \end{bmatrix} \tag{18}$$

Then $\boldsymbol{\omega}_{et}^t$ can be determined by Equation (18).

Equations (12)–(14) determine the relationship between the velocity and the position of the vehicle in the transversal coordinate system. Equation (18) determines the relationship between the linear velocity and the relative rotation called transport rate in the transversal coordinate system. The two formulae are the basis for the following transversal navigation mechanism.

$$\boldsymbol{\omega}_{et}^t = \left[\omega_{et(x)}^t \quad \omega_{et(y)}^t \quad \omega_{et(z)}^t \right] = \mathbf{C}_e^t(-\boldsymbol{\omega}_{te}^e) \tag{19a}$$

$$\omega_{et(x)}^t = \frac{\left(\begin{array}{c} 4v_{nt}\left(1 - e^2 \cos^2\lambda^t \cos^2\varphi^t\right)^{3/2}\left(6R_e e^2 - 2\cos(2\lambda^t)\right) \\ \left(R_e - h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\right)e^2 + 2R_e \cos(2\varphi^t)e^2 \\ +2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2\varphi^t)e^2 + R_e \cos(2(\lambda^t + \varphi^t))e^2 \\ +h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t + \varphi^t))e^2 \\ +\left(\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}h + R_e\right)\cos(2\lambda^t - 2\varphi^t)e^2 \\ +2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 \\ -8R_e - 8h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\right)\lambda^t\sec\lambda^t\sin(2\lambda^t) \end{array}\right)}{\left(\begin{array}{c} \left(\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}h + R_e\right)\left(2\cos(2\lambda^t)e^2 \right. \\ \left. +2\cos(2\varphi^t)e^2 + \cos(2(\lambda^t + \varphi^t))e^2 + \cos(2\lambda^t - 2\varphi^t)e^2 + 2e^2 - 8\right) \\ \left(8R_e e^2 + 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2\varphi^t)e^2 \right. \\ +h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2(\lambda^t + \varphi^t))e^2 \\ +h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\cos(2\lambda^t - 2\varphi^t)e^2 + 2h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 \\ +2h\cos(2\lambda^t)\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}e^2 - 8R_e \\ \left. - 8h\sqrt{1 - e^2 \cos^2\lambda^t \cos^2\varphi^t}\right) \end{array}\right)}$$

$$- \frac{16e^2 R_e v_{et} \left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2} \lambda^t \sec\lambda^t \sin^2(2\lambda^t) \sin\varphi^t}{\begin{pmatrix} \left(\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\, h + R_e\right)\left(2\cos(2\lambda^t)e^2 + 2\cos(2\varphi^t)e^2\right. \\ \left. + \cos(2(\lambda^t + \varphi^t))e^2 + \cos(2\lambda^t - 2\varphi^t)e^2 + 2e^2 - 8\right) \\ \left(8R_e e^2 + 2h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\cos(2\varphi^t)e^2\right. \\ +h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\cos(2(\lambda^t + \varphi^t))e^2 \\ +h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\cos(2\lambda^t - 2\varphi^t)e^2 \\ +2h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}e^2 + 2h\cos(2\lambda^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}e^2 \\ \left. - 8R_e - 8h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t}\right) \end{pmatrix}} \quad (19b)$$

$$\omega_{et(y)}^t = \frac{\begin{pmatrix} v_{et}(e^2 \cos(2(\lambda^t + \varphi^t)) + e^2 \cos(2\lambda^t - 2\varphi^t) + 2e^2 \cos(2\varphi^t) + 2e^2 - 8)^2) \\ \begin{pmatrix} 2e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +e^2 h \cos(2\lambda^t - 2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +2e^2 h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} - 8h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \end{pmatrix} \\ +2e^2 \cos(2\lambda^t)\left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 2R_e\right) + 4e^2 R_e - 8R_e \end{pmatrix}}{\begin{pmatrix} 64\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}\left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + R_e\right) \cdot \\ \begin{pmatrix} 2e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +e^2 h \cos(2\lambda^t - 2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +2e^2 h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +2e^2 h \cos(2\lambda^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ -8h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 8e^2 R_e - 8R_e \end{pmatrix} \end{pmatrix}}$$

$$- \frac{\begin{pmatrix} (e^2 R_e v_{nt}\sin\lambda^t \cos\lambda^t \sin\varphi^t(e^2 \cos(2(\lambda^t + \varphi^t)) + e^2 \cos(2\lambda^t - 2\varphi^t) \\ +2e^2 \cos(2\lambda^t) + 2e^2 \cos(2\varphi^t) + 2e^2 - 8)^2) \end{pmatrix}}{\begin{pmatrix} 8\left(1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t\right)^{3/2}\left(h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + R_e\right) \\ \begin{pmatrix} 2e^2 h \cos(2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +e^2 h \cos(2(\lambda^t + \varphi^t))\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +e^2 h \cos(2\lambda^t - 2\varphi^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +2e^2 h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ +2e^2 h \cos(2\lambda^t)\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} \\ - 8h\sqrt{1 - e^2 \cos^2 \lambda^t \cos^2 \varphi^t} + 8e^2 R_e - 8R_e \end{pmatrix} \end{pmatrix}} \quad (19c)$$

$$\omega_{et(z)}^t = \frac{\begin{pmatrix} v_{et}\tan\varphi^t(e^2\cos(2(\lambda^t+\varphi^t)) + e^2\cos(2\lambda^t - 2\varphi^t) \\ +2e^2\cos(2\lambda^t) + 2e^2\cos(2\varphi^t) + 2e^2 - 8)^2 \\ \begin{pmatrix} 2e^2h\cos(2\varphi^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +e^2h\cos(2(\lambda^t+\varphi^t))\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +e^2h\cos(2\lambda^t-2\varphi^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +2e^2h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ -8h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} + 2e^2\cos(2\lambda^t) \\ \left(h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}+2R_e\right)+4e^2R_e-8R_e\end{pmatrix}\end{pmatrix}}{\begin{pmatrix} 64\left(1-e^2\cos^2\lambda^t\cos^2\varphi^t\right)^{3/2}\left(h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}+Re\right)\cdot \\ \begin{pmatrix} 2e^2h\cos(2\varphi^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +e^2h\cos(2(\lambda^t+\varphi^t))\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +e^2h\cos(2\lambda^t-2\varphi^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +2e^2h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ +2e^2h\cos(2\lambda^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t} \\ -8h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}+8e^2R_e-8R_e\end{pmatrix}\end{pmatrix}}$$

$$-\frac{\begin{pmatrix}(e^2R_ev_{nt}\sin\lambda^t\cos\lambda^t\sin\varphi^t\tan\varphi^t(e^2\cos(2(\lambda^t+\varphi^t))+e^2\cos(2\lambda^t-2\varphi^t)\\+2e^2\cos(2\lambda^t)+2e^2\cos(2\varphi^t)+2e^2-8)^2)\end{pmatrix}}{\begin{pmatrix}8\left(1-e^2\cos^2\lambda^t\cos^2\varphi^t\right)^{3/2}\\\left(h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}+R_e\right.\\\begin{pmatrix}2e^2h\cos(2\varphi^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}\\+e^2h\cos(2(\lambda^t+\varphi^t))\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}\\+e^2h\cos(2\lambda^t-2\varphi^t)\sqrt{1-e^2\cos^2(\lambda^t)\cos^2\varphi^t}\\+2e^2h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}\\+2e^2h\cos(2\lambda^t)\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}\\-8h\sqrt{1-e^2\cos^2\lambda^t\cos^2\varphi^t}+8e^2R_e-8R_e\end{pmatrix}\end{pmatrix}} \tag{19d}$$

4. DIFFERENTIAL EQUATIONS OF ATTITUDE, VELOCITY AND POSITION FOR TRANSVERSAL NAVIGATION.

4.1. *Differential equation of attitude.* The differential equation of attitude matrix $\mathbf{C}_b^t$ is given by:

$$\dot{\mathbf{C}}_b^t = \mathbf{C}_b^t(\omega_{tb}^t\times) \tag{20}$$

where b is the body coordinate system.

In order to solve this differential equation, we must know the rotation rate ω_{tb}' which is calculated by ω_{ib}^b, ω_{ie}^t and ω_{et}^t as follows:

$$\omega_{tb}^b = \omega_{ib}^b - \mathbf{C}_t^b(\omega_{ie}^t + \omega_{et}^t) \tag{21}$$

where ω_{ib}^b is the actual measurement of the gyroscope. ω_{ie}^t is the rotation rate of the Earth and can be determined by:

$$\boldsymbol{\omega}_{ie}^t = \mathbf{C}_e^t \boldsymbol{\omega}_{ie} = \left[-\sin \lambda^t \omega_{ie} \ -\sin \varphi^t \cos \lambda^t \omega_{ie} \ \cos \varphi^t \cos \lambda^t \omega_{ie} \right]^T \tag{22}$$

$\boldsymbol{\omega}_{et}^t$ is the key to the attitude solution in transversal navigation, which is already known from Equation (19).

4.2. *Differential equation of velocity.*   In the ellipsoidal transversal coordinate system, the transversal velocity equation can be obtained as follows:

$$\dot{\mathbf{V}}^t = \mathbf{C}_b^t \mathbf{f}^b - (2\boldsymbol{\omega}_{ie}^t + \boldsymbol{\omega}_{et}^t) \times \mathbf{V}^t + \mathbf{g}^t \tag{23}$$

where $\mathbf{V}^t$ is the transversal velocity and $\mathbf{g}^t$ is the gravitational acceleration vector expressed in the t coordinate. $\boldsymbol{\omega}_{ie}^t$ and $\boldsymbol{\omega}_{et}^t$ can be determined by Equations (22) and (19).

According the definition of transversal navigation coordinate t in Section 2, it is also a horizontal coordinate system, therefore:

$$\mathbf{g}^t = \mathbf{g}^n \tag{24}$$

4.3. *Differential equation of position.*   Considering the coupling of the motion in the transversal coordinate system, the differential equations of position are determined by Equations (12) and (13). Since an INS cannot independently determine the vertical position h without external reference information sources (such as an altimeter), the differential equation of vertical position can be removed.

It is well known that in a traditional INS, latitude is only determined by the change rate of north velocity and longitude is only determined by the change rate of east velocity. From Equations (12) and (13), it can be seen that in the transversal coordinate system t, latitude φ^t and longitude λ^t are affected by both north velocity v_n^t and east velocity v_e^t. It can be concluded that there are coupling motions in transversal navigation, which are totally different from a traditional INS.

5. SIMULATION TEST ANALYSIS.   In order to verify the feasibility and effectiveness of the proposed navigation scheme, a simulation study was carried out. The trajectory generator was designed as follows: the initial geographical position is (70°N, 0°E) and geographic east velocity and north velocity are both 6 m/s. The heaving motion of the vehicle was set as $H \cdot \sin(\omega \cdot t)$, where the heave cycle $\omega = 2\pi/3600(rad/s)$ and the heave amplitude H was set as 1 m, 5 m, 10 m, 15 m and 20 m, respectively. The corresponding values are $H_1 = 1$, $H_2 = 5$, $H_3 = 10$, $H_4 = 15$, $H_5 = 20$. The geographical heading angle was 45°, roll angle was set to $5° \sin(\pi t/4)$ and the pitch angle was set to $3° \cos(\pi t/5)$. The simulation period was 24 hours. The proposed transversal navigation mechanism in this paper is called "mechanism 1" and the transversal navigation mechanism proposed in Li et al. (2016) is called "mechanism 2". Both were tested in the simulation experiments. It was found that with the increase of the heaving motions, the principle errors gradually increase, among which the errors of velocity and horizontal attitude are more obviously amplified, as shown in Figure 2 (for brevity, only the curves of east velocity and pitch errors are presented). In Figure 2(a), $\delta V_{et}01$, $\delta V_{et}05$, $\delta V_{et}10$, $\delta V_{et}15$, $\delta V_{et}20$ are east velocity errors and $\Phi_{nt}01$, $\Phi_{nt}05$, $\Phi_{nt}10$, $\Phi_{nt}15$, $\Phi_{nt}20$ are pitch errors in the t coordinate system, under the conditions of heaving motion with, $H_1 = 1$, $H_2 = 5$, $H_3 = 10$, $H_4 = 15$, $H_5 = 20$ respectively.
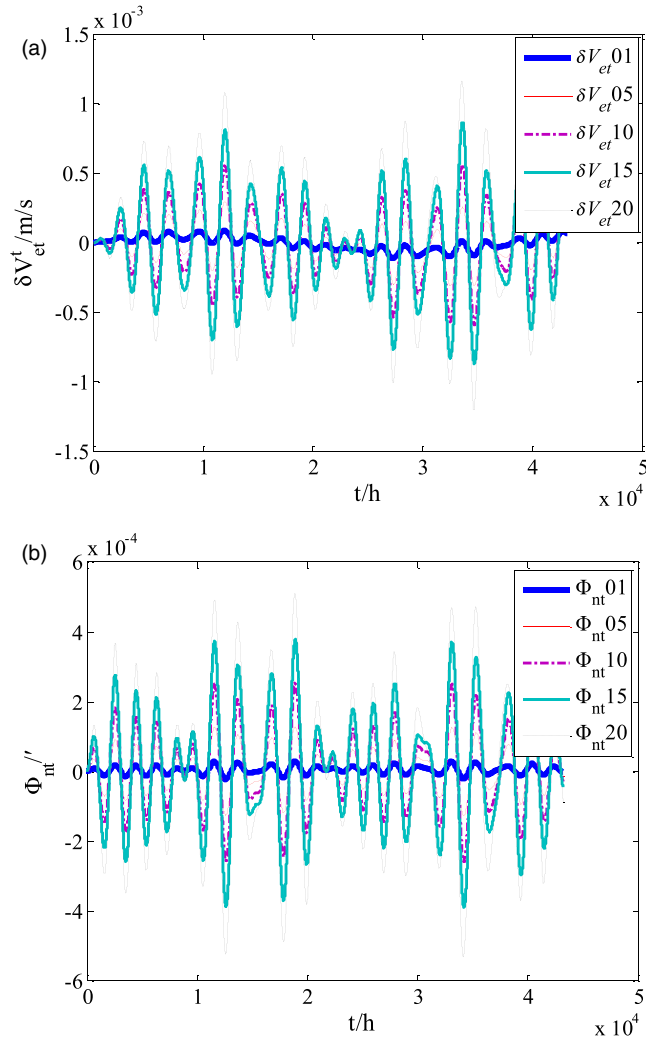
Figure 2.   (a) Curves of east velocity error ("mechanism 2"). (b) Curves of pitch error ("mechanism 2").

Compared with "mechanism 2", the precision advantages of "mechanism 1" in position, velocity and attitude error are obviously shown in Figures 3–5 (for brevity, only three cases of curves are given).

Next, some gyroscope and accelerometer errors were introduced, and simulations were repeated. It was found that the proposed improved navigation mechanism still has a precision advantage compared with "mechanism 2" under exactly the same experimental conditions (gyro drifts $\varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \cdot 0001°/h$, accelerometer biases $\nabla_x = \nabla_y = \nabla_z = 10^{-7}g$), as shown in Figures 6–8. However, it can be deduced that when the inertial sensor errors become the dominating error sources, the performance improvement using the improved mechanism will be less obvious.
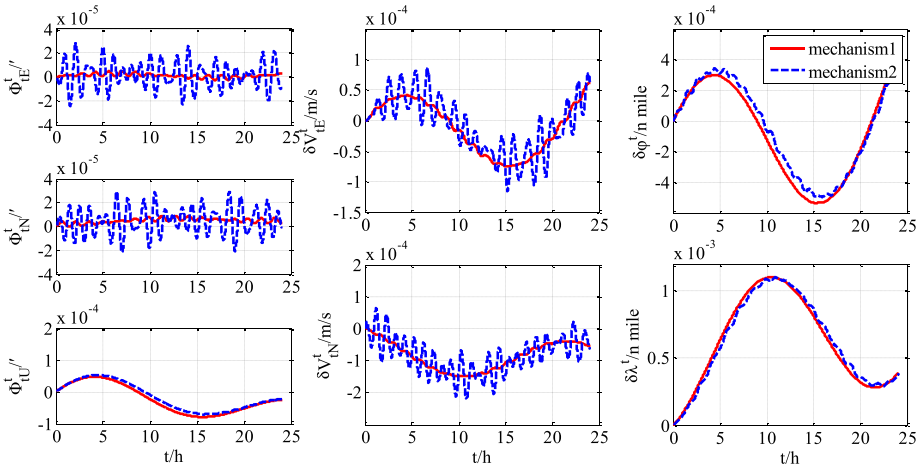
Figure 3.   Transversal navigation errors ($H_1 = 1$).
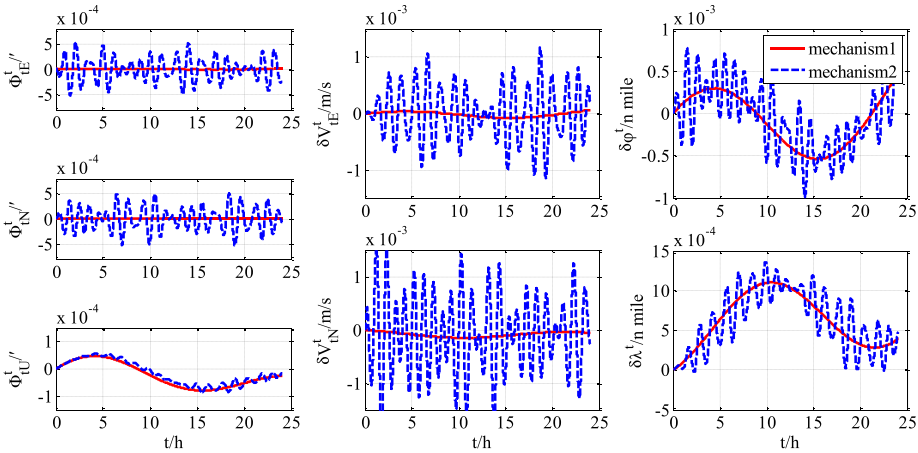
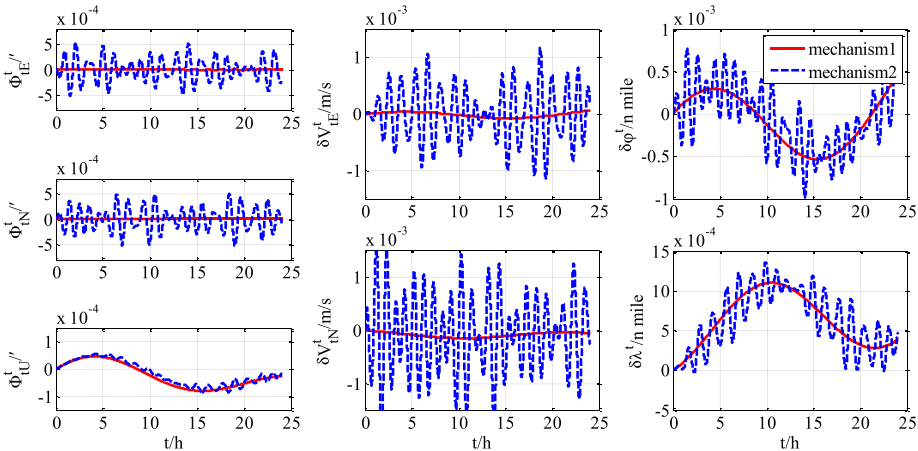Figure 4.   Transversal navigation errors ($H_3 = 10$).

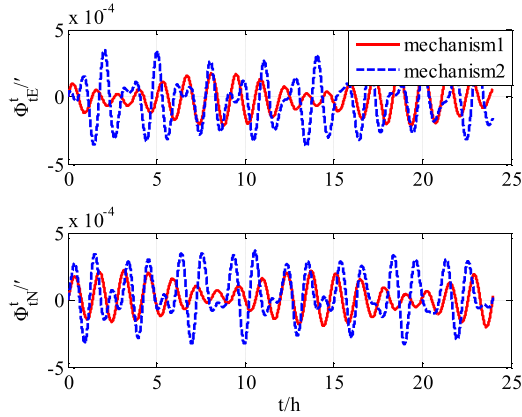Figure 5.   Transversal navigation errors ($H_5 = 20$).

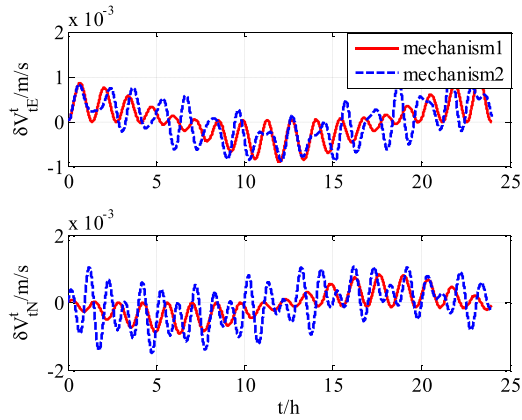Figure 6. Transversal attitude errors (with gyroscope and accelerometer errors).

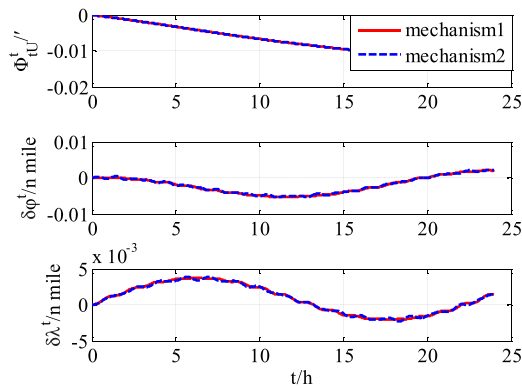Figure 7. Transversal velocity errors (with gyroscope and accelerometer errors).

Figure 8. Transversal heading and position errors (with gyroscope and accelerometer errors).

From the experimental results, it can be concluded that the transversal navigation mechanism ("mechanism 2") presented in Li et al. (2016) has errors in the presence of obvious heaving motion, and the performance will deteriorate with motion intensity increases. Errors of horizontal velocity and horizontal attitude are obviously affected by the heaving motion, and the effect on the errors of the position and heading are relatively small. The more rigorous improved mechanism of this paper ("mechanism 1") has an obvious precision advantage over "mechanism 2". As the intensity of heaving motion increases, the advantages will increase further.

6. CONCLUSIONS. The traditional transversal navigation algorithm incorporates errors by adopting a simplified Earth model. Establishing an accurate ellipsoid model with elliptic curvature radius can significantly reduce the transversal navigation principle errors. After theoretical analysis by rigorous transversal navigation equations, it is found that under the ellipsoid Earth model, the transversal navigation of the polar region is a complex coupling problem. In this paper, the mechanism of transversal polar region navigation based on an ellipsoidal Earth model is theoretically re-derived, and the complete mechanical arrangement of attitude, position and velocity calculation is presented. The new derivation in this paper completely avoids solving the ellipsoidal radius, and the coupling of the three-dimensional motion is fully considered. The simulation experiment validates the precision advantage of the transversal navigation mechanism proposed in this paper, especially in the condition of vertical motion.

REFERENCES

Bekir, E. (2007). *Introduction to Modern Navigation Systems*. World Scientific Publishing Co Pte Ltd.
Broxmeyer, C. (1964). *Inertial Navigation System*. New York, McGraw-Hill.
Li, Q., Ben, Y.Y. and Yu, F. (2015). System reset of transversal strapdown INS for ship in polar region. *Measurement*, **60**(1), 247–257.
Li, Q., Ben, Y.Y., Sun, F. and Huo, L. (2014a). Transversal Strapdown INS and Damping Technology for Marine in Polar Region. *Proceedings of IEEE/ION Position, Location and Navigation Symposium, Monterey, California*, 1365–1370.
Li, Q., Ben, Y.Y., Yu, F. and Tan, J. (2016). Transversal Strapdown INS based on Reference Ellipsoid for Vehicle in Polar Region. *IEEE Transactions on Vehicular Technology*, **65**(9), 7791–7795.
Li, Q., Sun, F., Ben, Y.Y. and Yu, F. (2014b). Transversal strapdown INS and damping design in polar region. *Systems Engineering and Electronics*, **36**(12), 2496–2503.
Lyon, W.K. (1984). The Navigation of Arctic Polar Submarines. *Journal of Navigation*, **37**(2), 155–179.
Naumann, J. (2011). Grid Navigation with Polar Stereographic Charts. *European Journal of Navigation*, **9**(1), 4–8.
Pedersen, E.S. (1960). Self-contained Polar Navigation. *Journal of Navigation*, **13**(1), 76–78.
Tang, Y.G., Wu, Y.X. and Wu, M.P. (2009). INS/GPS Integration: Global Observability Analysis. *IEEE Transactions on Vehicular Technology*, **58**(3), 1129–1142.
Titterton, D.H. and Weston, J.L. (2014). *Strapdown Inertial Navigation Technology*. AIAA.

Watland, D.R. and Ariz, P. (1995). *Orthogonal Polar Coordinate System to Accommodate Polar Navigation*. United States Patent: 5448486.

Yao, Y.Q., Xu, X.S., Li, Y, Liu,Y.T., Sun, J. and Tong, J.W. (2016). Transverse Navigation under the Ellipsoidal Earth Model and its Performance in both Polar and Non-polar areas. *Journal of Navigation*, **69**, 335–352.

Zhao, Y.W. (2017). Applying Time-Differenced Carrier Phase in Non-Differential GPS/IMU Tightly-Coupled Navigation Systems to Improve the Positioning Performance. *IEEE Transactions on Vehicular Technology*, **66**(2), 992–1003.