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Convergence to Black-Scholes for ergodic volatility models

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We study the effect of stochastic volatility on option prices. In the fast mean-reversion model for stochastic volatility of [5], we show that there is a full asymptotic expansion for the option price, centered at the Black-Scholes price. We show how to callibrate the first two terms in the expansion with the implied volatility surface. We show, however, that this price does not converge in a strong sense to Black-Scholes as the mean-reversion rate increases.

1 Introduction

Much of the current research in continuous-time financial math traces back to the seminal work of Black & Scholes [1]. They assume the risky asset price S_t to follow a log-normal SDE,

$$dS_t = \mu S_t dt + \sigma S_t dB_t^P, \tag{1.1}$$

where B_t^P is Brownian motion for some filtered probability space $(\Omega, \mathscr{F}, P)$. According to this model, the market is complete; thus, the put option pay-off $\max\{K - S(T), 0\}$ can be replicated with a risk-free self-financing strategy. Hence, if Q is the (unique) risk-free measure, then the price of a European put option satisfies

$$P_{BS}(t, S) = e^{-r(T-t)}\mathbf{E}^Q[\max\{K - S(T), 0\} \,|\, S_t = S], \tag{1.2}$$

which by Feyman-Kac, solves a specific parabolic PDE. We denote by $\mathbf{E}^Q$ and $\mathbf{E}^P$ the expectations with respect to Q and P, respectively.

One of the more questionable assumptions in Black & Scholes [1], which much empirical evidence demonstrates to be faulty, is that the asset price has constant volatility. The volatility is roughly the standard deviation of the relative change in asset price over one unit of time. Formally, it is the coefficient σ in (1.1). To cite but a few studies challenging the notion of constant volatility, Canina & Figlewski [2] show that implied volatility is not consistent with historical volatility. Cont & Fonseca [3] study the dynamics of the implied volatility surface of option prices. They show that the surface changes over time with little relation to the underlying.

There has been some success pricing options when the volatility is an Itô process.

Heston [7] was first to use Fourier transforms to derive a semi-explicit formula for the option price. His result was improved by Duffie, Pan & Singleton [4] who considered the additional complication of Poisson jumps in both the stock price and volatility processes.

1.1 The Fouque-Papanicolaou-Sircar model

Fouque, Papanicolaou & Sircar [5] relax the constant volatility assumption by making volatility be a function of an Ornstein–Uhlenbeck process, $\sigma = f(Y_t)$, where

$$M_1 \leqslant f(y) \leqslant M_2 \quad a.e. \tag{1.3}$$

for some positive bounds M_1, M_2. (Since P and Q have the same sets of measure zero, we shall not distinguish $P - a.e.$ from $Q - a.e.$) The model becomes, in the risk-neutral probability measure,

$$dS_t = rS_t dt + f(Y_t)S_t dB_t^Q \tag{1.4}$$

$$dY_t = (\alpha(m - Y_t) - \beta\Lambda(Y_t))\, dt + \beta dW_t^Q. \tag{1.5}$$

Here B_t^Q and W_t^Q are Brownian motions with respect to Q, α, m, β are constant, $\rho = \mathbf{E}^Q[dB_t^Q, dW_t^Q]$ and

$$\Lambda(y) = \frac{\rho(\mu - r)}{f(y)} + \gamma(y)\sqrt{1-\rho^2}$$

is a combined market price of risk. The risk-free interest rate r is assumed to be constant. Because the market is incomplete, the risk-neutral measure is not uniquely specified by the stock price alone. The unspecified functional form of γ reflects this ambiguity.

If we write the dynamics of Y_t in the objective measure P, (1.5) becomes

$$dY_t = \alpha(m - Y_t)dt + \beta dW_t^P. \tag{1.6}$$

For this reason, Y_t (and hence the volatility σ) is said to be *fast mean reverting* if $\alpha \gg 1$. Such behavior has been detected for S&P 500 index data [5]. Fouque *et al.* [5] examine the asymptotic behavior of the option price as they increase α to ∞. More specifically, let $\varepsilon = \alpha^{-1}$ and consider the log-price process $X_t = \ln(S_t)$. They consider the stochastic volatility model $(X_t^\varepsilon, Y_t^\varepsilon)$ given by, in the risk-neutral measure,

$$dX_t^\varepsilon = \left(r - \frac{1}{2}f^2\left(Y_t^\varepsilon\right)\right) dt + f\left(Y_t^\varepsilon\right) dB_t^Q, \tag{1.7}$$

$$dY_t^\varepsilon = \left(\frac{1}{\varepsilon}(m - Y_t) - \frac{v\sqrt{2}}{\sqrt{\varepsilon}}\Lambda\left(Y_t^\varepsilon\right)\right) dt + \frac{v\sqrt{2}}{\sqrt{\varepsilon}} dW_t^Q. \tag{1.8}$$

Note that (1.7) is simply Itô's Lemma applied to (1.4).

Let $P_\varepsilon(t, x, y)$ be the European put option price in a complete market modeled by equations (1.7) and (1.8):

$$P_\varepsilon(t, x, y) = e^{-r(T-t)}\mathbf{E}^Q\left[h\left(X_T^\varepsilon\right) \mid Y_t^\varepsilon = y, X_t^\varepsilon = x\right] \tag{1.9}$$

where $h(x) = \max\{K - e^x, 0\}$ is the pay-off function. In Fouque *et al.* [5] the authors obtain the first two terms in the asymptotic expansion of $P_\varepsilon(t,x,y)$ in powers of $\sqrt{\varepsilon}$,

$$P_\varepsilon(t,x,y) = P_{BS}(t,x) + \sqrt{\varepsilon}P_1(t,x) + O(\varepsilon). \tag{1.10}$$

Note that the two terms on the RHS of (1.10) are independent of the variable y and hence independent of the instantaneous volatility at time t. The correction term $P_1(t,x)$ has been used to derive partial hedging techniques [10] and to price certain barrier options [9].

The Black–Scholes price $P_{BS}(t,x)$ is with respect to a volatility $\bar{\sigma}$ which is the root mean squared average of the stochastic volatility (see (2.5) and (2.12) for a precise definition). The volatility $\bar{\sigma}$ can be calibrated from the implied volatility surface Σ, which is the graph of the volatility (implied by setting equal the Black–Scholes and market prices of the option) as a function of the "moneyness" $K - S_t$ and the time to maturity $T - t$ of the option. The first order correction $P_1(t,x)$ can also be calibrated from the implied volatility surface by fitting the surface to the family of functions

$$a + b\,\frac{\ln(K/S_t)}{T-t}, \tag{1.11}$$

with the two free parameters a, b. Thus if one calibrates implied volatility by a function of the form (1.11), the corresponding Black-Scholes price gives the price of the option correct to order $\sqrt{\varepsilon}$. The parameters a and b therefore depend on ε. The point of course is that a, b can be estimated from data whereas ε may not be. Fouque *et al.* [5] refer to the variable occurring in (1.11) as the log-moneyness-to-maturity ratio, κ defined by

$$\kappa = \frac{\ln(K/S_t)}{T-t}. \tag{1.12}$$

The expansion (1.10) was rigorously established by Fouque *et al.* [5] together with Solna [6]. They prove that for any fixed x, t with $t < T$, there is the limit

$$\lim_{\varepsilon\to 0} |P_\varepsilon(t,x,y) - (P_{BS}(t,x) + \sqrt{\varepsilon}P_1(t,x))|\varepsilon^{p-1} = 0, \tag{1.13}$$

provided $p > 0$.

1.2 Main results

Using Fourier transforms and perturbation techniques, we improve (1.13) to show the following:

Theorem 1 *Consider the model from* § 1.1 *defined by equations* (1.3), (1.7), (1.8) *and* (1.9). *There exist functions* $P_2(t,x,y), P_3(t,x,y), \ldots$ *such that for any positive* $n \in \mathbb{Z}$, *and fixed* x, t *with* $t < T$

$$\lim_{\varepsilon\to 0} \frac{\left| P_\varepsilon(t,x,y) - P_{BS}(t,x) - \sum_{j=1}^{n} \varepsilon^{j/2} P_j(t,x,y) \right|}{\varepsilon^{\frac{n+1}{2}-p}} = 0, \tag{1.14}$$

provided $p > 0$.

The higher order correction terms $P_n(t,x,y)$, $n \geqslant 2$ depend on y.

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In § 3, we compute the asymptotic expansion of the price induced by an expansion of the volatility. Comparing the $O(\varepsilon^{\frac{i}{2}})$ terms to those from the expansion of Theorem 1, for $i = 0, 1, 2$, we derive a functional form for the implied volatility surface, correct to $O(\varepsilon^{\frac{3}{2}})$, which is a generalization of (1.11):

$$\Sigma(\kappa, T-t, y) = \frac{c(y)}{T-t} + \sum_{k=0}^{2}(a_k + b_k(T-t))\kappa^i + \sum_{k=3}^{4} b_k(T-t)\kappa^k, \tag{1.15}$$

where the a_k, $0 \leqslant k \leqslant 2$, and b_k, $0 \leqslant k \leqslant 4$, are constants. The only dependence in y in (1.15) is in the function $c(y)$. In particular, if we fix a strike K_0 and expiration T_0 and let $\kappa = \kappa(x, K, T-t), \kappa_0 = \kappa(x, K_0, T_0 - t)$, we can fit determine a_k, b_k from the observed data of

$$\Sigma(\kappa, T-t, y)(T-t) - \Sigma(\kappa_0, T_0 - t, y)(T_0 - t).$$

Let $P^{K,T}$ denote the price of a European put with strike K and expiration T. To forecast $P^{K,T}$ over some other time period, we need only observe P^{K_0,T_0}, which gives us $\Sigma(\kappa_0, T_0 - t, y)$ and hence $c(y)$, via (1.15). Knowing $c(y), a_k, b_k$, we model $\Sigma(\kappa, T-t, y)$ with (1.15), and hence $P^{K,T}$ from Black–Scholes.

Next we consider the representation of $P_\varepsilon(t, x, y)$ in Fouque *et al.* [5], which generalizes the Hull–White formula. Thus let Ω be the space of volatility paths Y_s^ε, $t \leqslant s \leqslant T$. Then if we define $P_\varepsilon(t, x, Y^\varepsilon)$ by

$$P_\varepsilon(t, x, Y^\varepsilon) = e^{-r(T-t)}\mathbf{E}^Q\left[h\left(X_T^\varepsilon\right) \mid X_t^\varepsilon = x,\ Y^\varepsilon\right], \tag{1.16}$$

it follows from (1.9) that

$$P_\varepsilon(t, x, y) = \mathbf{E}^Q\left[P_\varepsilon(t, x, Y^\varepsilon) \mid Y_t^\varepsilon = y\right]. \tag{1.17}$$

The function $P_\varepsilon(t, x, Y^\varepsilon)$ is explicitly given by equation (2.31) of Fouque *et al.* [5]. It is a Black–Scholes price with volatility determined by a path average over Y^ε, but also with an initial stock price which depends on Y^ε. Only when the correlation $\rho = 0$ is the initial stock price deterministic as in the Hull-White formula. We prove the following:

Theorem 2 *For the § 1.1 model there is mean-squared convergence under the Q measure if and only if* $\rho = 0$*. Thus for* $\rho \neq 0$ *one has*

$$\lim_{\varepsilon \to 0} \|P_\varepsilon(t, x, Y^\varepsilon) - P_{BS}(t, x)\|_{L^2(\Omega, Q)} \neq 0. \tag{1.18}$$

Theorem 2 for $\rho = 0$ was proved by Papanicolaou & Sircar [12]. In § 4 we prove the $\rho \neq 0$ case.

We end with the obvious remark that because of put-call parity, all our convergence results hold for European calls as well.

2 Convergence to an asymptotic expansion

We now prove Theorem 1. From Fouque *et al.* [5] the value $P_\varepsilon(t,x,y)$ of the option satisfies the initial value problem

$$\begin{aligned}&\frac{\partial P_\varepsilon}{\partial t}+\frac{1}{2}f(y)^2\frac{\partial^2 P_\varepsilon}{\partial x^2}+\left[r-\frac{1}{2}f(y)^2\right]\frac{\partial P_\varepsilon}{\partial x}-rP_\varepsilon\\&\quad+\frac{1}{\sqrt{\varepsilon}}\left\{\sqrt{2}\rho v f(y)\frac{\partial P_\varepsilon}{\partial x\partial y}-\sqrt{2}v\Lambda(y)\frac{\partial P_\varepsilon}{\partial y}\right\}\\&\quad+\frac{1}{\varepsilon}\left\{v^2\frac{\partial^2 P_\varepsilon}{\partial y^2}+(m-y)\frac{\partial P_\varepsilon}{\partial y}\right\}=0,\quad t<T,\\&P_\varepsilon(T,x,y)=h(x).\end{aligned}\tag{2.1}$$

To begin the asymptotic analysis of (2.1) we Fourier transform the equation in the x variable. Thus putting

$$\hat{P}_\varepsilon(t,\xi,y)=\int_{-\infty}^{\infty}P_\varepsilon(t,x,y)e^{ix\xi}\,dx,\tag{2.2}$$

we see that $\hat{P}_\varepsilon(t,\xi,y)$ satisfies the initial value problem,

$$\begin{aligned}&\frac{\partial \hat{P}_\varepsilon}{\partial t}-\frac{1}{2}f(y)^2\xi^2\hat{P}_\varepsilon-i\xi\left[r-\frac{1}{2}f(y)^2\right]\hat{P}_\varepsilon-r\hat{P}_\varepsilon\\&\quad-\frac{1}{\sqrt{\varepsilon}}\left\{\sqrt{2}\rho v f(y)i\xi\frac{\partial \hat{P}_\varepsilon}{\partial y}+\sqrt{2}v\Lambda(y)\frac{\partial \hat{P}_\varepsilon}{\partial y}\right\}\\&\quad+\frac{1}{\varepsilon}\left\{v^2\frac{\partial^2 \hat{P}_\varepsilon}{\partial y^2}+(m-y)\frac{\partial \hat{P}_\varepsilon}{\partial y}\right\}=0,\quad t<T,\\&\hat{P}_\varepsilon(T,\xi,y)=\hat{h}(\xi).\end{aligned}\tag{2.3}$$

If we denote the solution of (2.3) when $\hat{h}(\xi)=1$ by $\hat{G}_\varepsilon(t,\xi,y)$ then it is clear that

$$\hat{P}_\varepsilon(t,\xi,y)=\hat{G}_\varepsilon(t,\xi,y)\hat{h}(\xi),\quad t<T.$$

We define a function $u_\varepsilon(s,\xi,z), s\geqslant 0,\xi,z\in\mathbb{R}$ by

$$\hat{G}_\varepsilon(t,\xi,y)=u_\varepsilon\left(\frac{T-t}{\varepsilon},\ \xi,\ \frac{y-m}{\sqrt{2}v}\right),$$

whence the variable s corresponds to $(T-t)/\varepsilon$ and z corresponds to $(y-m)/\sqrt{2}v$. Note however that, although $T-t$ has a fixed positive value in the statement of Theorem 1, we shall in the following be defining variables $s_0,s_1,\dots$ analagous to s which vary in the entire interval $0<s<(T-t)/\varepsilon$. This is because we generate the function u_ε by means of a perturbation series expansion. From (2.3) we see that u_ε satisfies the initial value problem

$$
\begin{aligned}
\frac{\partial u_\varepsilon}{\partial s} &= \frac{1}{2}\frac{\partial^2 u_\varepsilon}{\partial z^2} - z\,\frac{\partial u_\varepsilon}{\partial z} - \sqrt{\varepsilon}\{\Gamma(z) + i\xi\rho g(z)\}\frac{\partial u_\varepsilon}{\partial z} \\
&\quad -\varepsilon\left\{\frac{1}{2}\,g(z)^2\xi^2 + i\xi\left[r - \frac{1}{2}\,g(z)^2\right] + r\right\}u_\varepsilon, \quad s > 0, \\
u_\varepsilon(0,\xi,z) &= 1,
\end{aligned} \tag{2.4}
$$

where the functions g, Γ are defined by

$$
g\left(\frac{y-m}{\sqrt{2}v}\right) = f(y), \qquad \Gamma\left(\frac{y-m}{\sqrt{2}v}\right) = \Lambda(y).
$$

Next define a function $v_\varepsilon(s,\xi,z)$ by

$$
v_\varepsilon(s,\xi,z) = u_\varepsilon(s,\xi,z)\exp\left[\varepsilon s\left\{\frac{1}{2}\,\langle g^2\rangle\xi^2 + i\xi\left[r - \frac{1}{2}\,\langle g^2\rangle\right] + r\right\}\right] - 1, \tag{2.5}
$$

where $\bar{\sigma}^2 = \langle g^2\rangle$ is the average of g^2 with respect to a probability measure to be determined later. Then it follows from (2.4) that v_ε satisfies the initial value problem

$$
\begin{aligned}
\frac{\partial v_\varepsilon}{\partial s} &= \frac{1}{2}\frac{\partial^2 v_\varepsilon}{\partial z^2} - z\,\frac{\partial v_\varepsilon}{\partial z} - \sqrt{\varepsilon}\{\Gamma(z) + i\xi\rho g(z)\}\frac{\partial v_\varepsilon}{\partial z} \\
&\quad - \frac{\varepsilon}{2}[g(z)^2 - \langle g^2\rangle](\xi^2 - i\xi)[v_\varepsilon + 1], \quad s > 0, \\
v_\varepsilon(0,\xi,z) &= 0.
\end{aligned} \tag{2.6}
$$

We wish to obtain the solution of (2.6) as a convergent perturbation series expansion from which we can obtain the terms in the expansion of Theorem 1. To do this we introduce a parameter $\lambda \in \mathbb{R}$ and let $v_{\varepsilon,\lambda}(s,\xi,z)$ be the solution to the initial value problem

$$
\begin{aligned}
\frac{\partial v_{\varepsilon,\lambda}}{\partial s} &= \frac{1}{2}\frac{\partial^2 v_{\varepsilon,\lambda}}{\partial z^2} - z\,\frac{\partial v_{\varepsilon,\lambda}}{\partial z} - \lambda\sqrt{\varepsilon}\{\Gamma(z) + i\xi\rho g(z)\}\frac{\partial v_{\varepsilon,\lambda}}{\partial z} \\
&\quad - \frac{\varepsilon}{2}[g(z)^2 - \langle g^2\rangle](\xi^2 - i\xi)[\lambda v_{\varepsilon,\lambda} + 1], \quad s > 0, \\
v_{\varepsilon,\lambda}(0,\xi,z) &= 0.
\end{aligned} \tag{2.7}
$$

The solutions of (2.6) and (2.7) are related by the identity $v_\varepsilon(s,\xi,z) = v_{\varepsilon,1}(s,\xi,z)$. We write the solution of (2.7) in a perturbation series expansion in λ,

$$
v_{\varepsilon,\lambda} = \sum_{n=0}^{\infty} v_{\varepsilon,n}\,\lambda^n. \tag{2.8}
$$

Substituting (2.8) into (2.7) and equating the coefficients of powers of λ on both sides of

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(2.7), we see that the $v_{\varepsilon,n}$ can be solved for inductively by the equations

$$\begin{aligned}
\frac{\partial v_{\varepsilon,0}}{\partial s} &= \frac{1}{2}\,\frac{\partial^2 v_{\varepsilon,0}}{\partial z^2} - z\,\frac{\partial v_{\varepsilon,0}}{\partial z} - \frac{\varepsilon}{2}[g(z)^2 - \langle g^2\rangle](\xi^2 - i\xi), \quad s > 0,\\
v_{\varepsilon,0}(0,\xi,z) &= 0,\\
\frac{\partial v_{\varepsilon,n+1}}{\partial s} &= \frac{1}{2}\,\frac{\partial^2 v_{\varepsilon,n+1}}{\partial z^2} - z\frac{\partial v_{\varepsilon,n+1}}{\partial z} - \sqrt{\varepsilon}\{\Gamma(z) + i\xi\rho g(z)\}\frac{\partial v_{\varepsilon,n}}{\partial z}\\
&\quad - \frac{\varepsilon}{2}[g(z)^2 - \langle g^2\rangle](\xi^2 - i\xi)v_{\varepsilon,n}, \quad s > 0, n \geqslant 0,\\
v_{\varepsilon,n+1}(0,\xi,z) &= 0.
\end{aligned}$$

Observe now that the solution of the initial value problem

$$\begin{aligned}
\frac{\partial v}{\partial s} &= \frac{1}{2}\,\frac{\partial^2 v}{\partial z^2} - z\,\frac{\partial v}{\partial z} + f(s,z), \quad s > 0\\
v(0,z) &= 0,
\end{aligned}$$

has the representation

$$v(s,z) = \int_0^s \int_{-\infty}^{\infty} G(s-s',z,z')f(s',z')\,dz'\,ds', \tag{2.9}$$

where $G(s,z,z')$ is given by Mehler's formula

$$G(s,z,z') = \frac{1}{a(s)\sqrt{2\pi}}\,\exp[-(z' - ze^{-s})^2/2a(s)^2], \tag{2.10}$$

and $a(s)$ is defined by

$$a(s)^2 = \frac{1}{2}[1 - e^{-2s}], \quad s > 0.$$

There are many derivations of Mehler's formula. See Simon [13] for a derivation using stochastic processes. Note that

$$\lim_{s\to\infty} G(s,z,z') = p(z'), \tag{2.11}$$

where the function $p(z')$, $z' \in \mathbb{R}$, on the RHS of the identity (2.11) is the probability density function for the normal variable with mean 0 and variance 1/2. We define now $\langle g^2\rangle$ by

$$\langle g^2\rangle = \int_{-\infty}^{\infty} g^2(z)p(z)\,dz. \tag{2.12}$$

Lemma 1 *For $z \in \mathbb{R}$, $n = 0,1,2,\ldots$ there exists a polynomial $P_{n,z}$ in ξ and $\sqrt{\varepsilon}$ with the property that*

$$\begin{aligned}
&\left| v_{\varepsilon,n}\left(\frac{T-t}{\varepsilon},\xi,z\right) - (\xi^2 - i\xi)P_{n,z}(\xi,\sqrt{\varepsilon})\right|\\
&\quad\leqslant C_n[1 + |z|^{n+1}](|\xi|^2 + |\xi|)\|g\|_\infty^2[\|g\|_\infty^2(\xi^2 + |\xi|) + \|g\|_\infty|\xi| + \|\Gamma\|_\infty]^n\\
&\qquad \exp[-(T-t)/2\varepsilon], \quad \varepsilon < \min[T-t,1].
\end{aligned} \tag{2.13}$$

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The polynomial $P_{n,z}$ has the properties:

(a) *$P_{n,z}$ has degree at most $2n$ in ξ and $2n+2$ in $\sqrt{\varepsilon}$.*

(b) *If n is odd the lowest power of $\sqrt{\varepsilon}$ which occurs in $P_{n,z}$ is $(n+1)/2$. If $n=0$ the lowest power of $\sqrt{\varepsilon}$ which occurs is 2. If $n \geqslant 2$ is even the lowest power of $\sqrt{\varepsilon}$ which occurs is $n/2+1$.*

(c) *There exists a constant C_n depending only on n such that all the coefficients of $P_{n,z}$ are bounded by*

$$C_n[1+|z|^{n+1}]\|g\|_\infty^2[\|g\|_\infty^2+\|g\|_\infty+\|\Gamma\|_\infty]^n. \tag{2.14}$$

Proof We first consider the case $n=0$. In view of (2.11) we have that

$$G(s,z,z')-p(z') = -\int_s^\infty \frac{\partial G}{\partial s'}(s',z,z')\,ds'. \tag{2.15}$$

One can see from (2.10) that

$$\frac{\partial G}{\partial s}(s,z,z') = e^{-s}H(s,z,z')+ze^{-s}K(s,z,z'), \tag{2.16}$$

where for $s \geqslant 1$ there is a universal constant C such that

$$|H(s,z,z')|+|K(s,z,z')| \leqslant C\exp[-(z'-ze^{-s})^2/2]. \tag{2.17}$$

From (2.9), (2.12) we see that

$$v_{\varepsilon,0}(s,\xi,z) = \frac{-\varepsilon}{2}(\xi^2-i\xi)\int_0^s\int_{-\infty}^\infty \{G(s',z,z')-p(z')\}\,[g(z')^2-\langle g^2\rangle]\,dz'\,ds'.$$

We define the polynomial $P_{0,z}$ by

$$\begin{aligned} P_{0,z}(\xi,\sqrt{\varepsilon}) &= \frac{-\varepsilon}{2}\int_0^\infty\int_{-\infty}^\infty \{G(s',z,z')-p(z')\}\,[g(z')^2-\langle g^2\rangle]\,dz'\,ds' \\ &= \varepsilon\, a(z). \end{aligned} \tag{2.18}$$

It follows from (2.15), (2.16), (2.17) that there is a universal constant C such that $|a(z)| \leqslant C[1+|z|]\|g\|_\infty^2$. We can also see that

$$\left|v_{\varepsilon,0}\left(\frac{T-t}{\varepsilon},\xi,z\right)-(\xi^2-i\xi)P_{0,z}(\xi,\sqrt{\varepsilon})\right| \leqslant C\varepsilon[1+|z|](|\xi|^2+|\xi|)\|g\|_\infty^2\exp[-(T-t)/\varepsilon],$$

for some universal constant C. We have proved the lemma when $n=0$.

To deal with general $n \geqslant 1$ we introduce various integral operators. For $n \geqslant 1$ we define the integral operator $A(s)$ on functions $f:\mathbb{R}\to\mathbb{C}$ by

$$A(s)f(z) = \int_{-\infty}^\infty \left[G(s,z,z')-p(z')\right]f(z')\,dz'. \tag{2.19}$$

We define the operator B by

$$Bf(z) = \int_{-\infty}^\infty p(z')f(z')\,dz',$$

whence B projects to the constant function. The operators $T_1(s), T_2(s)$ are given in terms of $A(s), B$ by the formulas

$$T_1(s)f(z) = \frac{1}{2}[g(z)^2 - \langle g^2\rangle]A(s)f(z),$$
$$T_2(s)f(z) = \frac{1}{2}[g(z)^2 - \langle g^2\rangle]Bf(z).$$

Note that $T_2(s) = T_2$ is independent of s and $T_2^2 = 0$. We define $T_3(s)$ by

$$T_3(s)f(z) = \Gamma(z)\int_{-\infty}^{\infty}\frac{\partial G}{\partial z}(s,z,z')f(z')\,dz'.$$

It is evident from (2.10) that

$$\frac{\partial G}{\partial z}(s,z,z') = \frac{e^{-s}}{\sigma(s)}K(s,z,z'), \tag{2.20}$$

where

$$|K(s,z,z')| \leqslant C\ G(s,\ z/2,\ z'/2), \tag{2.21}$$

for some universal constant C. We similarly define an operator $T_4(s)$ by

$$T_4(s)f(z) = \rho g(z)\int_{-\infty}^{\infty}\frac{\partial G}{\partial z}(s,z,z')f(z')\,dz'.$$

The function $v_{\varepsilon,n},\ n \geqslant 1$, can be expressed in terms of the operators $T_j(s),\ 1 \leqslant j \leqslant 4$. In fact $v_{\varepsilon,n}(s,\xi,z)$ is a sum of terms,

$$(-1)^{n+1}(\sqrt{\varepsilon})^p(i\xi)^q[\varepsilon(\xi^2 - i\xi)]^r\int_{\{s_0+\ldots+s_n\langle s,s_k\rangle 0,k=0,\ldots,n\}} ds_0\ldots ds_n$$
$$T(s_0)T_{j_1}(s_1)\ldots T_{j_n}(s_n)T_2[1](z), \tag{2.22}$$

where $1 \leqslant j_k \leqslant 4,\ k = 1,\ldots,n$ and $T(s_0)$ can be either the operator $A(s_0)$ or B. The integers p,q,r are given by the formulas

$$p = \sum_{k=1}^{n}[\delta(j_k - 3) + \delta(j_k - 4)],$$
$$q = \sum_{k=1}^{n}\ \delta(j_k - 4),$$
$$r = 1 + \sum_{k=1}^{n}[\delta(j_k - 1) + \delta(j_k - 2)],$$

where $\delta(x),\ x \in \mathbb{Z}$, is the Kronecker $\delta,\ \delta(0) = 1,\ \delta(x) = 0,\ x \neq 0$.

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We consider first the situation in (2.22) where $j_k \neq 2,\ 1 \leqslant k \leqslant n$, and $T(s_0) = A(s_0)$. In that case one has, on putting

$$K_n(z) = \int_0^\infty \cdots \int_0^\infty ds_0 \cdots ds_n T(s_0) T_{j_1}(s_1) \cdots T_{j_n}(s_n) T_2[1](z),$$

that there is a constant C_n depending only on n such that

$$|K_n(z)| \leqslant C_n[1 + |z|^r] \|\Gamma\|_\infty^{p-q}\ \|g\|_\infty^{2r+q}. \tag{2.23}$$

The contribution to the polynomial $P_{n,z}(\xi, \sqrt{\varepsilon})$ from (2.22) is given by

$$(-1)^{n+1}(\sqrt{\varepsilon})^p (i\xi)^q\ \varepsilon^r [\xi^2 - i\xi]^{r-1} K_n(z). \tag{2.24}$$

Note that the estimate (2.23) is consistent with the estimate (2.14) on the coefficients of $P_{n,z}$. Using the fact that

$$\int_{\{s_0+\ldots+s_n>s,\ s_k>0,\ k=0,\ldots,n\}} ds_0 \ldots ds_n \exp[-(s_0 + \ldots + s_n)] = \sum_{k=0}^n \frac{s^k}{k!}\, e^{-s},$$

we see that the difference between $K_n(z)$ and the integral in (2.22) is bounded by

$$C_n[1 + |z|^r]\ \|\Gamma\|_\infty^{p-q}\ \|g\|_\infty^{2r+q}\ \exp[-(T-t)/2\varepsilon],$$

for a constant C_n depending only on n, when $s = (T-t)/\varepsilon$. We have therefore proved the estimate (2.13) corresponding to the term (2.22).

We consider now the general case of (2.22). We may write

$$T_{j_1}(s_1) \cdots T_{j_n}(s_n) T_2 = \prod_{i=1}^{n_1} T_{j_i}(s_i) T_2$$
$$\prod_{i=n_1+2}^{n_2} T_{j_i}(s_i) T_2 \cdots \prod_{i=n_k+2}^{n} T_{j_i}(s_i) T_2,$$

where $j_i = 2$ if and only if $i = n_1+1, \ldots, n_k+1$, with $0 \leqslant n_1,\ n_k + 2 \leqslant n$. Assuming now that $T(s_0) = A(s_0)$ we have that the integral in (2.22) is the same as

$$\int_{\{s_0+s_1+\cdots+s_{n_1}+s_{n_1+2}+\ldots+s_{n_2}+\ldots+s_{n_k+2}+\ldots+s_n<s\}} ds_0\, ds_1..ds_{n_1}\, ds_{n_1+2} \ldots ds_n$$
$$\frac{1}{k!}\left[s - (s_0 + s_1 + \cdots + s_{n_1} + s_{n_1+2} + \ldots + s_{n_2} + \ldots + s_{n_k+2} + \ldots + s_n)\right]^k$$
$$T(s_0) \prod_{i=1}^{n_1} T_{j_i}(s_i) T_2 \prod_{i=n_1+2}^{n_2} T_{j_i}(s_i) T_2\ \cdots \prod_{i=n_k+2}^{n} T_{j_i}(s_i) T_2[1](z). \tag{2.25}$$

If we expand out the monomial of degree k in the last expression we can write it as

$$\sum_{m=0}^k s^m\ K_{n,m}(z, s).$$

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Arguing as in the previous paragraph we see that the infinite integral $K_{n,m}(z,\infty)$ exists and

$$|K_{n,m}(z,\ (T-t)/\varepsilon) - K_{n,m}(z,\infty)| \leqslant C_n[1+|z|^r]\ \|\Gamma\|_\infty^{p-q}\ \|g\|_\infty^{2r+q}\ \exp[-(T-t)/2\varepsilon], \quad (2.26)$$

for some constant C_n depending only on n. The contribution to the polynomial $P_{n,z}(\xi,\sqrt{\varepsilon})$ from (2.22) is given by

$$(-1)^{n+1}(\sqrt{\varepsilon})^p(i\xi)^q\ \varepsilon^r[\xi^2 - i\xi]^{r-1}\sum_{m=0}^{k}\left(\frac{T-t}{\varepsilon}\right)^m K_{n,m}(z,\infty). \quad (2.27)$$

Evidently (2.26) implies the bound (2.13) corresponding to the term (2.22). Similarly one also obtains the bound on the coefficients corresponding to (2.14).

We are left then to establish the bounds (a) and (b) on the degree of the polynomial $P_{n,z}$. The bound (a) follows from the fact that $p+r=n+1$, $q+r \leqslant n+1$. To prove (b) we need an upper bound on k in (2.27). Since $T_2^2 = 0$ one sees that among $T(s_0)$, $T_{j_1}(s_1)\ldots T_{j_n}(s_n)$ there can be at most $(n+1)/2$ equal to T_2 or B if n is odd and $n/2$ if n is even. Now the lowest degree of $\sqrt{\varepsilon}$ in the polynomial (2.27) is $p+2(r-k) = n+1-r+2(r-k)$ $= n+1-k+(r-k)$. Since $k \leqslant r$ we obtain the estimate $(n+1)/2$ for the lowest degree of $\sqrt{\varepsilon}$. Similarly for n even the lowest degree of $\sqrt{\varepsilon}$ is $n/2+1$ since $k \leqslant n/2$ and $r-k \geqslant 0$. □

We consider next the Green's function corresponding to the evolution equation (2.7). Thus let $G_{\varepsilon,\lambda,\xi}(s,z,z')$ have the property that the function

$$v(s,z) = \int_{-\infty}^{\infty} G_{\varepsilon,\lambda,\xi}(s,z,z')f(z')\,dz'$$

is the solution to the initial value problem,

$$\begin{aligned}\frac{\partial v}{\partial s} &= \frac{1}{2}\frac{\partial^2 v}{\partial z^2} - z\,\frac{\partial v}{\partial z} - \lambda\sqrt{\varepsilon}\{\Gamma(z) + i\xi\rho g(z)\}\frac{\partial v}{\partial z}\\ &\quad - \frac{\varepsilon}{2}\left[g(z)^2 - \langle g^2\rangle\right](\xi^2 - i\xi)\lambda v, \quad s>0,\\ v(0,z) &= f(z).\end{aligned} \quad (2.28)$$

Evidently, if $\varepsilon = 0$ or $\lambda = 0$ then $G_{\varepsilon,\lambda,\xi}$ coincides with the Green's function G of (2.10).

Lemma 2 *Suppose the function g of* (2.28) *satisfies the inequality* $g(z)^2 \geqslant m^2 > 0$, $z \in \mathbb{R}$. *Then there is the inequality*

$$\int_{-\infty}^{\infty} |G_{\varepsilon,\lambda,\xi}(s,z,z')|\,dz' \leqslant \exp\left[\frac{-\varepsilon s}{2}[(1-\lambda\rho^2)m^2 - \langle g^2\rangle]\xi^2\lambda\right].$$

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Proof We make the transform $v = \exp[w]$. Then if v satisfies (2.28) w satisfies

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{1}{2}\left[\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial w}{\partial z}\right)^2\right] - z\,\frac{\partial w}{\partial z} - \lambda\sqrt{\varepsilon}\{\Gamma(z) + i\xi\rho g(z)\}\frac{\partial w}{\partial z}\\ &\quad - \frac{\varepsilon}{2}[g(z)^2 - \langle g^2\rangle](\xi^2 - i\xi)\lambda, \quad s > 0\\ w(0,z) &= \log f(z).\end{aligned}$$

Writing $w = w_1 + iw_2$ with w_1, w_2 real we have that

$$\begin{aligned}\frac{\partial w_1}{\partial s} &= \frac{1}{2}\left[\frac{\partial^2 w_1}{\partial z^2} + \left(\frac{\partial w_1}{\partial z}\right)^2 - \left(\frac{\partial w_2}{\partial z}\right)^2\right] - z\,\frac{\partial w_1}{\partial z}\\ &\quad - \lambda\sqrt{\varepsilon}\Gamma(z)\frac{\partial w_1}{\partial z} + \lambda\sqrt{\varepsilon}\xi\rho g(z)\frac{\partial w_2}{\partial z} - \frac{\varepsilon}{2}[g(z)^2 - \langle g^2\rangle]\xi^2\lambda.\end{aligned}$$

Using the Schwarz inequality in the last expression we obtain a differential inequality,

$$\begin{aligned}\frac{\partial w_1}{\partial s} &\leqslant \frac{1}{2}\left[\frac{\partial^2 w_1}{\partial z^2} + \left(\frac{\partial w_1}{\partial z}\right)^2\right] - z\,\frac{\partial w_1}{\partial z}\\ &\quad - \lambda\sqrt{\varepsilon}\Gamma(z)\frac{\partial w_1}{\partial z} - \frac{\varepsilon}{2}[(1-\lambda\rho^2)g(z)^2 - \langle g^2\rangle]\xi^2\lambda,\\ w_1(0,z) &= Re\ \log f(z).\end{aligned}$$

Making the inverse transformation $v_1 = \exp[w_1]$ therefore yields the differential inequality

$$\begin{aligned}\frac{\partial v_1}{\partial s} &\leqslant \frac{1}{2}\,\frac{\partial^2 v_1}{\partial z^2} - [z + \lambda\sqrt{\varepsilon}\Gamma(z)]\frac{\partial v_1}{\partial z} - \frac{\varepsilon}{2}[(1-\lambda\rho^2)m^2 - \langle g^2\rangle]\xi^2\lambda v_1\\ 0 &\leqslant v_1(0,z) \leqslant |f(z)|.\end{aligned} \tag{2.29}$$

Let $H(s,z,z')$ be the Green's function for the equation

$$\frac{\partial v}{\partial s} = \frac{1}{2}\,\frac{\partial^2 v}{\partial z^2} - [z + \lambda\sqrt{\varepsilon}\Gamma(z)]\frac{\partial v}{\partial z}.$$

Then by the maximum principle $H(s,z,z') \geqslant 0$ and the solution v_1 of (2.29) satisfies the inequality

$$0 \leqslant v_1(s,z) \leqslant \exp\left[\frac{-\varepsilon s}{2}[(1-\lambda\rho^2)m^2 - \langle g^2\rangle]\xi^2\lambda\right]\int_{-\infty}^{\infty} H(s,z,z')|f(z')|\,dz'.$$

The result follows now by choosing a suitable f with $|f(z')| = 1,\ z' \in \mathbb{R}$ and using the identity

$$\int_{-\infty}^{\infty} H(s,z,z')\,dz' = 1. \qquad \square$$

Lemma 3 *Let $u_\varepsilon(s,\xi,z)$ be the solution of* (2.4). *Then there is the inequality*

$$\left|u_\varepsilon(s,\xi,z) - \exp\left[-\varepsilon s\left\{\frac{1}{2}\langle g^2\rangle\xi^2 + i\xi\left[r - \frac{1}{2}\langle g^2\rangle\right] + r\right\}\right]\right| \leqslant \varepsilon s|\xi^2 - i\xi|\ \|g\|_\infty^2 \exp\left[-\varepsilon s\left\{\frac{1}{4}(1-\rho^2)m^2\xi^2 + r\right\}\right]. \tag{2.30}$$

Proof We have from (2.5), (2.6) that the LHS of (2.30) is bounded by

$$\varepsilon|\xi^2 - i\xi|\ \|g\|_\infty^2 \exp\left[-\varepsilon s\left\{\frac{1}{2}\ \langle g^2\rangle\xi^2 + r\right\}\right] \int_0^s \int_{-\infty}^{\infty} |G_{\varepsilon,1,\xi}(s',z,z')|\ dz'\ ds'. \tag{2.31}$$

If we now use the estimate from Lemma 2,

$$\int_{-\infty}^{\infty} |G_{\varepsilon,1,\xi}(s',z,z')|\, dz' \leqslant \exp\left[\frac{\varepsilon s}{4}\langle g^2\rangle\xi^2\right],\ 0 < s' < s/2,$$

$$\int_{-\infty}^{\infty} |G_{\varepsilon,1,\xi}(s',z,z')|\, dz' \leqslant \exp\left[\frac{-\varepsilon s}{4}(1-\rho^2)m^2\xi^2 + \frac{\varepsilon s}{2}\langle g^2\rangle\xi^2\right],\ s/2 < s' < s,$$

the result follows from (2.31). □

Lemma 4 *There is a universal constant $C > 0$ such that*

$$\left|v_{\varepsilon,n}\left(\frac{T-t}{\varepsilon},\xi,z\right)\right| \leqslant C^n \exp[e|z|/(e-1)]\varepsilon^{n/4} \max\left[1,(T-t)^{n/2}\right](|\xi|^2 + |\xi|)\|g\|_\infty^2 \times [\|g\|_\infty^2(\xi^2 + |\xi|) + \|g\|_\infty|\xi| + \|\Gamma\|_\infty]^n. \tag{2.32}$$

Proof Let G be the Green's function (2.10). Then it is easy to see that for any $\gamma > 0$ there is the inequality,

$$\int_{-\infty}^{\infty} dz_1 \ldots \int_{-\infty}^{\infty} dz_k\ G(s_0,z_0,z_1)G(s_1,z_1,z_2)\cdots G(s_{k-1},z_{k-1},z_k)e^{\gamma|z_k|} \leqslant C(\gamma)^k e^{\gamma|z_0|}, \tag{2.33}$$

where $C(\gamma)$ is a constant depending only on γ. We also have that if $s_0,\ldots,s_{k-1} \geqslant 1$,

$$\int_{-\infty}^{\infty} dz_1 \ldots \int_{-\infty}^{\infty} dz_k\ G(s_0,z_0,z_1)G(s_1,z_1,z_2)\cdots G(s_{k-1},z_{k-1},z_k) \exp\left[\sum_{i=0}^{k-1}|z_i| + \gamma|z_k|\right] \leqslant C(\gamma)^k \exp\left[\left(\sum_{j=0}^{k-1} e^{-j} + e^{-k}\gamma\right)|z_0|\right], \tag{2.34}$$

where $C(\gamma)$ depends only on γ. We have already observed that $v_{\varepsilon,n}(s,\xi,z)$ is a sum of terms of the form (2.22). The number of such terms is at most 4^{n+1}. Hence if we bound (2.22) by the RHS of (2.32) we shall be done. The inequalities (2.33), (2.34) enable us to do this.

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We consider first the situation in (2.22) where $j_k \neq 2$, $1 \leqslant k \leqslant n$, and $T(s_0) = A(s_0)$. We can write the integral in (2.22) as a sum of 2^{n+1} integrals,

$$\int_{E_0 \cap E_s} ds_0 \int_{E_1 \cap E_s} ds_1 \cdots \int_{E_n \cap E_s} ds_n, \tag{2.35}$$

where

$$E_s = \{s_0 + .. + s_n < s,\ s_k > 0,\ k = 0, \dots, n\},$$

and the sets E_j, $0 \leqslant j \leqslant n$, are either the interval $(0,1)$ or $(1,\infty)$. It follows now from (2.16), (2.17), (2.20), (2.21) and the inequalities (2.33), (2.34) that the integral (2.35) is bounded by

$$C^n \exp\left[\frac{e|z|}{e-1}\right] \|\Gamma\|_\infty^{p-q}\ \|g\|_\infty^{2r+q} \int_{E_0} a(s_0)\, ds_0 \int_{E_1} a(s_1)\, ds_1 \cdots \int_{E_n} a(s_n)\, ds_n, \tag{2.36}$$

where C is a universal constant and the function $a : (0,\infty) \to \mathbb{R}$ is defined by

$$a(s) = 1/\sqrt{s},\ 0 < s < 1,\ a(s) = e^{-s},\ s > 1.$$

It is clear now that the bound (2.36) implies that (2.22) is bounded by the RHS of (2.32).

One can generalize the previous argument to deal with the situation where $j_k = 2$ for some k. In that case we use the fact that the integral in (2.22) is the same as (2.25). □

Proof of Theorem 1 For the put option $h(x) = \max\{K - e^x, 0\}$ one has

$$\hat{h}(\xi) = K^{1+i\xi}/i\xi(1+i\xi). \tag{2.37}$$

Observe that the Fourier integral of the function h exists only if $\Im(\xi) < 0$ but one can justify setting $\Im(\xi) = 0$ in the following. The function $\hat{h}(\xi)$, $\xi \in \mathbb{R}$, is bounded as $|\xi| \to \infty$ and $\xi\hat{h}(\xi)$ is bounded as $\xi \to 0$. We have by the Fourier inversion theorem that

$$\begin{aligned} P_\varepsilon(t,x,y) - P_{BS}(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x}\hat{h}(\xi) \Bigg\{ u_\varepsilon\left(\frac{T-t}{\varepsilon}, \xi, z\right) \\ - \exp\left[-(T-t)\left\{\frac{1}{2}\langle g^2\rangle\xi^2 + i\xi\left[r - \frac{1}{2}\langle g^2\rangle\right] + r\right\}\right]\Bigg\}\, d\xi, \end{aligned} \tag{2.38}$$

where $z = (y-m)/\sqrt{2}v$. Let $\alpha > 0$ be arbitrary. Then it follows from Lemma 3 that for the integral on the RHS of (2.38) we have the estimate,

$$\left|\int_{|\xi|>1/\varepsilon^\alpha} d\xi\right| \leqslant C(\rho, m, T-t)K\|g\|_\infty^2 \exp[-(T-t)(1-\rho^2)m^2/8\varepsilon^{2\alpha}],$$

where $C(\rho, m, T-t)$ is a constant depending only on $\rho, m, T-t$. From (2.5) we may write

$$\begin{aligned} \int_{|\xi|<1/\varepsilon^\alpha} d\xi = \frac{1}{2\pi} \int_{-1/\varepsilon^\alpha}^{1/\varepsilon^\alpha} e^{-i\xi x}\hat{h}(\xi) \\ \times \exp\left[-(T-t)\left\{\frac{1}{2}\langle g^2\rangle\xi^2 + i\xi\left[r - \frac{1}{2}\langle g^2\rangle\right] + r\right\}\right] v_\varepsilon\left(\frac{T-t}{\varepsilon},\ \xi, z\right) d\xi. \end{aligned}$$

From (2.8) we have that

$$v_\varepsilon\left(\frac{T-t}{\varepsilon},\ \xi, z\right) = \sum_{n=0}^{\infty} v_{\varepsilon,n}\left(\frac{T-t}{\varepsilon},\ \xi, z\right). \tag{2.39}$$

Lemma 4 implies that the series in (2.39) converges for $|\xi| < 1/\varepsilon^\alpha$ provided $\alpha < 1/8$ and ε is sufficiently small. In particular we see that

$$\begin{aligned}&\Bigg|\frac{1}{2\pi}\int_{-1/\varepsilon^\alpha}^{1/\varepsilon^\alpha} e^{-i\xi x}\hat{h}(\xi)\exp\left[-(T-t)\left\{\frac{1}{2}\langle g^2\rangle\xi^2 + i\xi\left[r-\frac{1}{2}\langle g^2\rangle\right]+r\right\}\right]\\ &\quad\times\left\{v_\varepsilon\left(\frac{T-t}{\varepsilon},\ \xi, z\right) - \sum_{n=0}^{N} v_{\varepsilon,n}\left(\frac{T-t}{\varepsilon},\ \xi, z\right)\right\}d\xi\Bigg| \leqslant C\exp\left[\frac{e|z|}{e-1}\right]\varepsilon^{N(.25-2\alpha)},\end{aligned}$$

where C is independent of z and ε. The result now follows from Lemma 1. □

3 Evaluation of coefficients

In this section we evaluate the coefficients in the asymptotic expansion correct to $O(\varepsilon)$. This has also been carried out in Howison [8] by a different methodology, simultaneously and independently of the present work. We shall further compute the corresponding functional form of the implied volatility. Our starting point is the identity

$$\begin{aligned}P_\varepsilon(t,x,y) - P_{BS}(t,x) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{h}(\xi)\exp\left[-(T-t)\left\{\frac{1}{2}\langle g^2\rangle\xi^2 + i\xi[r-\langle g^2\rangle]+r\right\}\right]\\ &\times\sum_{n=0}^{\infty} v_{\varepsilon,n}\left(\frac{T-t}{\varepsilon}, \xi, z\right)d\xi,\end{aligned} \tag{3.1}$$

where $z = (y-m)/\sqrt{2}v$. The identity (3.1) follows from (2.5), (2.38), (2.39). The coefficients in the asymptotic expansion can be obtained then by computing the polynomials $P_{n,z}(\xi, \sqrt{\varepsilon})$ defined by Lemma 1. It follows in particular that all the terms in the expansion depend on the observables, the stock price $S = e^x$ and the strike price K, only through the derivatives of $P_{BS}(t,x)$ with respect to x. This is consistent with equation (5.43) of Fouque *et al.* [5].

We can derive (5.43) of Fouque *et al.* [5] by identifying the terms in $\sqrt{\varepsilon}$ in (3.1). The only polynomial which contributes a term in $\sqrt{\varepsilon}$ is $P_{1,z}$. From (2.22) the contribution is obtained from the expression,

$$\begin{aligned}&\sqrt{\varepsilon}[\varepsilon(\xi^2 - i\xi)]\int_{s_0+s_1<s} ds_0\, ds_1 B\ \ T_3(s_1)T_2[1](z)\\ &+\sqrt{\varepsilon}i\xi[\varepsilon(\xi^2-i\xi)]\int_{s_0+s_1<s} ds_0\, ds_1 B\ \ T_4(s_1)T_2[1](z),\end{aligned}$$

where $s = (T-t)/\varepsilon$. Doing the integration with respect to s_0 and then letting $\varepsilon \to 0$ we see that the coefficient of $\sqrt{\varepsilon}$ term in $P_{1,z}$ is given by

$$(T-t)\sqrt{\varepsilon}(\xi^2-i\xi)\int_0^\infty ds_1 B\, T_3(s_1)T_2[1](z)+(T-t)\sqrt{\varepsilon}i\xi(\xi^2-i\xi)\int_0^\infty ds_1 B\, T_4(s_1)T_2[1](z)$$
$$=(T-t)(\xi^2-i\xi)[V_2-2V_3-i\xi V_3], \tag{3.2}$$

where $V_2/\sqrt{\varepsilon}$, $V_3/\sqrt{\varepsilon}$ are constants depending only on ρ, g, Γ. Observing now that we have the correspondence of operators given by

$$\xi^2-i\xi \leftrightarrow -\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}=-S^2\frac{\partial^2}{\partial S^2},$$
$$(\xi^2-i\xi)i\xi \leftrightarrow \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x}\right]=2S^2\frac{\partial^2}{\partial S^2}+S^3\frac{\partial^3}{\partial S^3},$$

it follows from (3.1), (3.2) that

$$P_\varepsilon(t,x,y)=P_{BS}(t,x)-(T-t)\left[V_2S^2\frac{\partial^2 P_{BS}}{\partial S^2}+V_3S^3\frac{\partial^3 P_{BS}}{\partial S^3}\right]+O(\varepsilon), \tag{3.3}$$

which is (5.43) of Fouque *et al.* [5].

The asymptotic expansion (3.3) yields an implied volatility curve (5.55) of Fouque *et al.* [5] which is not flat. We give a derivation of this which we will then generalize to the expansion correct to $O(\varepsilon^{3/2})$. Let $P_{BS}(t,S,\Sigma)$ be the Black–Scholes price of the put option with strike price K, S the stock price at time t, and Σ the volatility. Then the Black–Scholes formula yields,

$$P_{BS}(t,S,\Sigma)=-S\,\frac{1}{\sqrt{2\pi}}\int_{d_1}^\infty e^{-z^2/2}dz+Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{d_2}^\infty e^{-z^2/2}dz, \tag{3.4}$$

where

$$d_{1,2}=\frac{\ln(S/K)+(r\pm\frac{1}{2}\Sigma^2)(T-t)}{\Sigma\sqrt{T-t}}.$$

We now write Σ as an expansion in powers of $\sqrt{\varepsilon}$,

$$\Sigma=\overline{\sigma}+a_1\sqrt{\varepsilon}+O(\varepsilon), \tag{3.5}$$

which in turn yields an expansion

$$P_{BS}(t,S,\Sigma)=P_{BS}(t,S,\overline{\sigma})+b_1\sqrt{\varepsilon}+O(\varepsilon). \tag{3.6}$$

We need to obtain b_1 as a function of a_1. To do this we write d_1, d_2 as a function of Σ and note that

$$d_1(\overline{\sigma}+a_1\sqrt{\varepsilon}+O(\varepsilon))-d_1(\overline{\sigma})=a_1\sqrt{\varepsilon}\left[\sqrt{T-t}-\frac{d_1(\bar{\sigma})}{\bar{\sigma}}\right]+O(\varepsilon),$$
$$d_2(\overline{\sigma}+a_1\sqrt{\varepsilon}+O(\varepsilon))-d_2(\overline{\sigma})=-a_1\sqrt{\varepsilon}\left[\sqrt{T-t}+\frac{d_2(\bar{\sigma})}{\bar{\sigma}}\right]+O(\varepsilon). \tag{3.7}$$

Now from (3.4), (3.7) one has

$$\begin{aligned}
&P_{BS}(t,S,\bar\sigma+a_1\sqrt{\varepsilon}+O(\varepsilon))-P_{BS}(t,S,\bar\sigma)\\
&\quad=\frac{S}{\sqrt{2\pi}}\exp\left[-d_1^2(\bar\sigma)/2\right]a_1\sqrt{\varepsilon}\left[\sqrt{T-t}-\frac{d_1(\bar\sigma)}{\bar\sigma}\right]\\
&\qquad+\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\exp\left[-d_2^2(\bar\sigma)/2\right]a_1\sqrt{\varepsilon}\left[\sqrt{T-t}+\frac{d_2(\bar\sigma)}{\bar\sigma}\right]+O(\varepsilon)\\
&\quad=\frac{S}{\sqrt{2\pi}}\exp\left[-d_1^2(\bar\sigma)/2\right]a_1\sqrt{\varepsilon}\sqrt{T-t}+O(\varepsilon).
\end{aligned}$$

We conclude that a_1, b_1 are related by

$$b_1=\frac{S}{\sqrt{2\pi}}\exp\left[-d_1^2(\bar\sigma)/2\right]\sqrt{T-t}\,a_1. \tag{3.8}$$

The implied volatility curve correct to order $\sqrt{\varepsilon}$ is now obtained by setting $b_1\sqrt{\varepsilon}$ equal to the $\sqrt{\varepsilon}$ term on the RHS of (3.3) and then using (3.8) to solve for a_1. We have now that

$$\frac{\partial^2 P_{BS}}{\partial S^2}=\frac{\exp[-d_1^2(\bar\sigma)/2]}{\sqrt{2\pi}S\bar\sigma\sqrt{T-t}},\qquad \frac{\partial^3 P_{BS}}{\partial S^3}=-\frac{\exp[-d_1^2(\bar\sigma)/2]}{\sqrt{2\pi}S^2\bar\sigma\sqrt{T-t}}\left\{1+\frac{d_1(\bar\sigma)}{\bar\sigma\sqrt{T-t}}\right\}. \tag{3.9}$$

Hence from (3.3), (3.8) we conclude that

$$a_1\sqrt{\varepsilon}=-\frac{V_2}{\bar\sigma}+\frac{V_3}{\bar\sigma}\left\{1+\frac{d_1(\bar\sigma)}{\bar\sigma\sqrt{T-t}}\right\}=a\frac{\ln(K/S_t)}{T-t}+b=a\kappa+b, \tag{3.10}$$

where a, b are constants of order $\sqrt{\varepsilon}$. We have derived equation (5.55) of [5].

Next we compute the terms in ε in (3.1). By Lemma 1 the polynomials $P_{n,z}$ which contribute terms in ε are $n=0,1,2,3$. The $n=0$ contribution is given by (2.18), and depends on z. The other contributions are independent of z. From (2.22) the contribution from $P_{1,z}$ is the term

$$[\varepsilon(\xi^2-i\xi)]^2\int_{s_0+s_1<s} ds_0\,ds_1 BT_1(s_1)T_2[1](z).$$

Arguing as before this yields an ε term given by

$$(T-t)\varepsilon(\xi^2-i\xi)^2\int_0^\infty ds_1 BT_1(s_1)T_2[1](z). \tag{3.11}$$

The contribution from $P_{2,z}$ is the sum of the terms,

$$(-1)\varepsilon(i\xi)^q[\varepsilon(\xi^2-i\xi)]\int_{s_0+s_1+s_2<s} ds_0\,ds_1\,ds_2 BT_j(s_1)T_{j'}(s_2)T_2[1](z),$$

where j, j' can be either 3 or 4 and q is the number of them that are 4. The ε term corresponding to these is given by

$$-(T-t)\varepsilon(i\xi)^q(\xi^2-i\xi)\int_0^\infty\int_0^\infty ds_1\,ds_2 BT_j(s_1)T_{j'}(s_2)T_2[1](z). \tag{3.12}$$

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The contribution from $P_{3,z}$ is the sum of the terms,

$$\varepsilon(i\xi)^q[\varepsilon(\xi^2 - i\xi)]^2 \int_{s_0+s_1+s_2+s_3<s} ds_0\, ds_1\, ds_2\, ds_3 B\, T_j(s_1) T_2\ T_{j'}(s_3) T_2[1](z),$$

where again j, j' can be either 3 or 4 and q is the number of them that are 4. The ε term corresponding to these is given by

$$\frac{(T-t)^2}{2}\varepsilon(i\xi)^q(\xi^2 - i\xi)^2 \int_0^\infty \int_0^\infty ds_1\, ds_3 B\, T_j(s_1) T_2 T_{j'}(s_3) T_2[1](z). \tag{3.13}$$

It follows then that the $O(\varepsilon)$ contribution to the RHS of (3.3) is given from (2.18), (3.11), (3.12), (3.13) as,

$$\begin{aligned}\varepsilon\Bigg[&\alpha_1(z)S^2\frac{\partial^2 P_{BS}}{\partial S^2} + (T-t)\left\{\alpha_2 S^2\frac{\partial^2 P_{BS}}{\partial S^2} + \alpha_3 S^3\frac{\partial^3 P_{BS}}{\partial S^3} + \alpha_4 S^4\frac{\partial^4 P_{BS}}{\partial S^4}\right\}\\ &+(T-t)^2\left\{\alpha_5\left(S^2\frac{\partial^2}{\partial S^2}\right)^2 P_{BS} + \alpha_6\left(S\frac{\partial}{\partial S}\right)\left(S^2\frac{\partial^2}{\partial S^2}\right)^2 P_{BS}\right.\\ &\left.+\alpha_7\left(S\frac{\partial}{\partial S}\right)^2\left(S^2\frac{\partial^2}{\partial S^2}\right)^2 P_{BS}\right\}\Bigg],\end{aligned} \tag{3.14}$$

where only the coefficient $\alpha_1(z)$ depends on $z = (y-m)/\sqrt{2}v$. Note that $\alpha_1(z)$ does not depend option-specific data, like K and $T-t$.

We wish now to determine the relation between the $O(\varepsilon)$ contributions in (3.5), (3.6) analogous to (3.8). We write therefore,

$$\begin{aligned}\Sigma &= \bar\sigma + a_1\sqrt{\varepsilon} + a_2\varepsilon + O\left(\varepsilon^{3/2}\right),\\ P_{BS}(t,S,\Sigma) &= P_{BS}(t,S,\bar\sigma) + b_1\sqrt{\varepsilon} + b_2\varepsilon + O\left(\varepsilon^{3/2}\right).\end{aligned} \tag{3.15}$$

In analogy to (3.7) we have that

$$\begin{aligned}d_1\left(\bar\sigma + a_1\sqrt{\varepsilon} + a_2\varepsilon + O\left(\varepsilon^{3/2}\right)\right) - d_1(\bar\sigma) &= [a_1\sqrt{\varepsilon} + a_2\varepsilon]\left[\sqrt{T-t} - \frac{d_1(\bar\sigma)}{\bar\sigma}\right]\\ &\quad + \frac{a_1^2\varepsilon}{\bar\sigma}\left[\frac{d_1(\bar\sigma)}{\bar\sigma} - \frac{1}{2}\sqrt{T-t}\right] + O\left(\varepsilon^{3/2}\right),\\ d_2\left(\bar\sigma + a_1\sqrt{\varepsilon} + a_2\varepsilon + O\left(\varepsilon^{3/2}\right)\right) - d_2(\bar\sigma) &= -[a_1\sqrt{\varepsilon} + a_2\varepsilon]\left[\sqrt{T-t} + \frac{d_2(\bar\sigma)}{\bar\sigma}\right]\\ &\quad + \frac{a_1^2\varepsilon}{\bar\sigma}\left[\frac{d_2(\bar\sigma)}{\bar\sigma} + \frac{1}{2}\sqrt{T-t}\right] + O\left(\varepsilon^{3/2}\right).\end{aligned} \tag{3.16}$$

If we write the LHS of (3.16) as $d_1' - d_1, d_2' - d_2$ with $d_1 = d_1(\bar\sigma), d_2 = d_2(\bar\sigma)$, then we have from (3.4) that

$$\begin{aligned}&P_{BS}\left(t,S,\bar\sigma + a_1\sqrt{\varepsilon} + a_2\varepsilon + O\left(\varepsilon^{3/2}\right)\right) - P_{BS}(t,S,\bar\sigma)\\ &\quad = \frac{S}{\sqrt{2\pi}}\exp\left[-d_1^2(\bar\sigma)/2\right]\left\{(d_1' - d_1) - (d_2' - d_2) - \frac{d_1(d_1'-d_1)^2}{2} + \frac{d_2(d_2'-d_2)^2}{2} + O(\varepsilon^{3/2})\right\}.\end{aligned} \tag{3.17}$$

It follows from (3.16), (3.17) that b_2 is given by the formula

$$b_2 = \frac{S}{\sqrt{2\pi}} \exp\left[-d_1^2(\bar\sigma)/2\right]\sqrt{T-t}\left\{a_2 + a_1^2 d_1(\bar\sigma)d_2(\bar\sigma)/2\bar\sigma\right\}. \tag{3.18}$$

We can derive now from (3.10), (3.14), (3.18) the functional form of a_2 analogous to (3.10). In order to do this we need to obtain derivatives of P_{BS} higher than those given in (3.9). We first note that

$$\begin{aligned}
\left(S\frac{\partial}{\partial S}\right) S\frac{\partial^2 P_{BS}}{\partial S^2} &= -\frac{\exp[-d_1^2(\bar\sigma)/2]d_1(\bar\sigma)}{\sqrt{2\pi}\bar\sigma^2(T-t)},\\
\left(S\frac{\partial}{\partial S}\right)^2 S\frac{\partial^2 P_{BS}}{\partial S^2} &= \frac{\exp[-d_1^2(\bar\sigma)/2]}{\sqrt{2\pi}[\bar\sigma^2(T-t)]^{3/2}}\left[-1+d_1(\bar\sigma)^2\right],\\
\left(S\frac{\partial}{\partial S}\right)^3 S\frac{\partial^2 P_{BS}}{\partial S^2} &= \frac{\exp[-d_1^2(\bar\sigma)/2]}{\sqrt{2\pi}[\bar\sigma^2(T-t)]^{2}}\{3d_1(\bar\sigma)-d_1(\bar\sigma)^3\},\\
\left(S\frac{\partial}{\partial S}\right)^4 S\frac{\partial^2 P_{BS}}{\partial S^2} &= \frac{\exp[-d_1^2(\bar\sigma)/2]}{\sqrt{2\pi}[\bar\sigma^2(T-t)]^{5/2}}\{3-6d_1(\bar\sigma)^2+d_1(\bar\sigma)^4\}.
\end{aligned}$$

It is clear now that a_2 is a polynomial of degree 4 in $\kappa = \ln(K/S)/(T-t)$ with coefficients which depend on $T-t$. More precisely we have from (3.14) the identity,

$$\begin{aligned}
a_2 + a_1^2 d_1(\bar\sigma)d_2(\bar\sigma)/2\bar\sigma = {}& \frac{\alpha_1(z)}{\bar\sigma(T-t)} + \frac{\alpha_2}{\bar\sigma} - \frac{\alpha_3}{\bar\sigma}\left[1+\frac{d_1(\bar\sigma)}{\bar\sigma\sqrt{T-t}}\right]\\
&+\frac{\alpha_4}{\bar\sigma}\left[2+\frac{3d_1(\bar\sigma)}{\bar\sigma\sqrt{T-t}}+\frac{d_1(\bar\sigma)^2-1}{\bar\sigma^2(T-t)}\right]\\
&+\frac{\alpha_5}{\bar\sigma^3}\left[-\bar\sigma\sqrt{T-t}\,d_1(\bar\sigma)-1+d_1(\bar\sigma)^2\right]\\
&+\frac{\alpha_6}{\bar\sigma^3}\left[-1+d_1(\bar\sigma)^2+\frac{3d_1(\bar\sigma)-d_1(\bar\sigma)^3}{\bar\sigma\sqrt{T-t}}\right]\\
&+\frac{\alpha_7}{\bar\sigma^3}\left[\frac{3d_1(\bar\sigma)-d_1(\bar\sigma)^3}{\bar\sigma\sqrt{T-t}}+\frac{3-6d_1(\bar\sigma)^2+d_1(\bar\sigma)^4}{\bar\sigma^2(T-t)}\right].
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
a_2 + a_1^2 d_1(\bar\sigma)d_2(\bar\sigma)/2\bar\sigma &= Q_{0,z}(T-t) + \sum_{i=1}^4 Q_i(T-t)\kappa^i \quad \text{where}\\
Q_{0,z}(T-t) &= \frac{1}{T-t}\left[p_0^0(z)+p_1^0(T-t)+p_2^0(T-t)^2\right],\\
Q_1(T-t) &= p_0^1 + p_1^1(T-t),\\
Q_2(T-t) &= p_0^2 + p_1^2(T-t),\\
Q_3(T-t) &= p_1^3(T-t),\\
Q_4(T-t) &= p_1^4(T-t),
\end{aligned}$$

and the p_j^i are constants except for $p_0^0(z)$ which depends explicitly on z. The above equations, (3.10) and (3.15) give (1.15).

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4 Proof of Theorem 2

We now prove Theorem 2 in the case $\rho \neq 0$. First we obtain a formula for the Fourier transform of the function $P_\varepsilon(t,x,\omega)$ of (1.16). One can easily derive from this by Fourier inversion the formula for $P_\varepsilon(t,x,\omega)$ given in Fouque *et al.* [5].

To derive the formula we first note that at least formally the function

$$P_\varepsilon(t,x,Y^\varepsilon) = e^{-r(T-t)} E\left[h\left(X_T^\varepsilon\right) \middle| X_t^\varepsilon = x\right],$$

defined for a given volatility path $Y_s^\varepsilon,\ t \leqslant s \leqslant T$, solves the initial value problem

$$\frac{\partial P_\varepsilon}{\partial t} + \left[r - \frac{1}{2} f\left(Y_t^\varepsilon\right)^2 + f\left(Y_t^\varepsilon\right)\rho\, \dot{W}_t^Q\right] \frac{\partial P_\varepsilon}{\partial x} - r\, P_\varepsilon + \frac{1}{2}(1-\rho^2) f\left(Y_t^\varepsilon\right)^2 \frac{\partial^2 P_\varepsilon}{\partial x^2} = 0,\ t < T,$$
$$P_\varepsilon(T,x,Y^\varepsilon) = h(x). \tag{4.1}$$

In (4.1) the process $\dot{W}_t^Q$ is the white noise process which is the derivative of the Brownian motion W_t^Q of (1.8). On taking the Fourier transform of (4.1) we obtain an initial value problem for the transform $\hat{P}_\varepsilon(t,\xi,Y^\varepsilon)$ of $P_\varepsilon(t,x,Y^\varepsilon)$,

$$\frac{\partial \hat{P}_\varepsilon}{\partial t} - i\xi \left[r - \frac{1}{2} f\left(Y_t^\varepsilon\right)^2 + f\left(Y_t^\varepsilon\right)\rho\, \dot{W}_t^Q\right] \hat{P}_\varepsilon - r\, \hat{P}_\varepsilon - \frac{1}{2}\xi^2(1-\rho^2) f\left(Y_t^\varepsilon\right)^2 \hat{P}_\varepsilon = 0,\ t < T,$$
$$\hat{P}_\varepsilon(T,\xi,Y^\varepsilon) = \hat{h}(\xi). \tag{4.2}$$

We may solve (4.2) to obtain an explicit formula for $\hat{P}_\varepsilon$,

$$\hat{P}_\varepsilon(t,\xi,Y^\varepsilon) = \hat{h}(\xi)\exp\left[-\int_t^T ds \left\{ i\xi\left[r - \frac{1}{2} f\left(Y_s^\varepsilon\right)^2 + f\left(Y_s^\varepsilon\right)\rho \dot{W}_s^Q\right] + r + \frac{1}{2}\xi^2(1-\rho^2) f\left(Y_s^\varepsilon\right)^2 \right\}\right]. \tag{4.3}$$

To see why one might expect Theorem 2 to hold, consider

$$\begin{aligned}|\hat{P}_\varepsilon(t,\xi,Y^\varepsilon)|^2 &= |\hat{h}(\xi)|^2 \exp\left[-2\int_t^T ds\left\{r + \frac{1}{2}\xi^2(1-\rho^2) f\left(Y_s^\varepsilon\right)^2\right\}\right] \\ &= |\hat{P}_{BS}(t,\xi)|^2 \exp\left[\xi^2 \int_t^T ds\left\{\langle g^2\rangle - (1-\rho^2) f\left(Y_s^\varepsilon\right)^2\right\}\right],\end{aligned}$$

where g is as in (2.5) and $\langle g^2\rangle$ is given by (2.12). If we now take the expectation in the last equation conditioned on $Y_t^\varepsilon = y$ we have, by Jensen's inequality,

$$E[|\hat{P}_\varepsilon(t,\xi,Y^\varepsilon)|^2 | Y_t^\varepsilon = y] \geqslant |\hat{P}_{BS}(t,\xi)|^2 \exp\left[\xi^2 \int_t^T ds\{\langle g^2\rangle - (1-\rho^2) E\left[f\left(Y_s^\varepsilon\right)^2 | Y_t^\varepsilon = y\right]\}\right].$$

It is clear that

$$\lim_{\varepsilon\to 0} \int_t^T ds\{\langle g^2\rangle - (1-\rho^2)E\left[f\left(Y_s^\varepsilon\right)^2 | Y_t^\varepsilon = y\right]\} = \rho^2 (T-t)\langle g^2\rangle.$$

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We conclude therefore that

$$\lim_{\varepsilon\to 0} E\left[|\hat{P}_\varepsilon(t,\xi,Y^\varepsilon)|^2 | Y_t^\varepsilon = y\right] > |\hat{P}_{BS}(t,\xi)|^2,$$

if $\rho \neq 0$ and $\xi \neq 0$.

We proceed now to the proof of Theorem 2. First note that the function

$$\hat{P}_\varepsilon(t,\xi,y) = E\left[\hat{P}_\varepsilon(t,\xi,Y^\varepsilon) | Y_t^\varepsilon = y\right]$$

is the solution of (2.3). For $\xi,\xi' \in \mathbb{R}$ define the function $\hat{P}_\varepsilon(t,\xi,\xi',y)$ by

$$\hat{P}_\varepsilon(t,\xi,\xi',y) = E\left[\hat{P}_\varepsilon(t,\xi,Y^\varepsilon)\overline{\hat{P}_\varepsilon(t,\xi',Y^\varepsilon)} | Y_t^\varepsilon = y\right]. \tag{4.4}$$

It is easy to see that $\hat{P}_\varepsilon(t,\xi,\xi',y)$ is the solution to an initial value problem similar to (2.3). In fact if we note that

$$\begin{aligned}\hat{P}_\varepsilon(t,\xi,Y^\varepsilon)\overline{\hat{P}_\varepsilon(t,\xi',Y^\varepsilon)} = \hat{h}(\xi)\overline{\hat{h}(\xi')}\exp\Bigg[-\int_t^T ds\Bigg\{ & i(\xi-\xi')\left[r-\frac{1}{2}f\left(Y_s^\varepsilon\right)^2 + f\left(Y_s^\varepsilon\right)\rho\,\dot{W}_s^Q\right] \\ & + 2r + \frac{1}{2}(\xi^2+\xi'^2)(1-\rho^2)f\left(Y_s^\varepsilon\right)^2\Bigg\}\Bigg],\end{aligned} \tag{4.5}$$

and compare (4.5) with (4.3) we see from (2.3) that $\hat{P}_\varepsilon(t,\xi,\xi',y)$ satisfies

$$\begin{aligned}&\frac{\partial \hat{P}_\varepsilon}{\partial t} - \frac{1}{2}f(y)^2[(\xi-\xi')^2\rho^2 + (\xi^2+\xi'^2)(1-\rho^2)]\hat{P}_\varepsilon \\ &\quad - i(\xi-\xi')\left[r - \frac{1}{2}f(y)^2\right]\hat{P}_\varepsilon - 2r\hat{P}_\varepsilon \\ &\quad - \frac{1}{\sqrt{\varepsilon}}\left\{\sqrt{2}\rho v f(y) i(\xi-\xi')\frac{\partial \hat{P}_\varepsilon}{\partial y} + \sqrt{2}\,v\Lambda(y)\frac{\partial \hat{P}_\varepsilon}{\partial y}\right\} \\ &\quad + \frac{1}{\varepsilon}\left\{v^2\frac{\partial^2 \hat{P}_\varepsilon}{\partial y^2} + (m-y)\frac{\partial \hat{P}_\varepsilon}{\partial y}\right\} = 0,\ t<T, \\ &\quad \hat{P}_\varepsilon(T,\xi,\xi',y) = \hat{h}(\xi)\overline{\hat{h}(\xi')}.\end{aligned} \tag{4.6}$$

Let us define $\hat{P}_0(t,\xi,\xi')$ by

$$\begin{aligned}\hat{P}_0(t,\xi,\xi') = \hat{h}(\xi)\hat{h}(-\xi')\exp\Bigg[-(T-t)\Bigg\{&\frac{1}{2}\langle g^2\rangle[\xi^2+\xi'^2-2\xi\xi'\rho^2] \\ &+ i(\xi-\xi')\left[r-\frac{1}{2}\langle g^2\rangle\right] + 2r\Bigg\}\Bigg],\quad t<T.\end{aligned}$$

We shall show that $\lim_{\varepsilon\to 0}\ \hat{P}_\varepsilon(t,\xi,\xi',y) = \hat{P}_0(t,\xi,\xi')$. To do this we proceed in a similar way to how we obtained the asymptotic expansion for the solution to (2.3). Thus we define $u_\varepsilon(s,\xi,\xi',z)$ by

$$\hat{P}_\varepsilon(t,\xi,\xi',y) = u_\varepsilon\left(\frac{T-t}{\varepsilon},\xi,\xi',\frac{y-m}{\sqrt{2}v}\right)\hat{h}(\xi)\overline{\hat{h}(\xi')}.$$

If we define $v_\varepsilon(s,\xi,\xi',z)$ by

$$v_\varepsilon(s,\xi,\xi',z) = u_\varepsilon(s,\xi,\xi',z)\exp\left[\varepsilon s\left\{\frac{1}{2}\langle g^2\rangle[\xi^2+\xi'^2-2\xi\xi'\rho^2] + i(\xi-\xi')\left[r-\frac{1}{2}\langle g^2\rangle\right]+2r\right\}\right]-1,$$

then it follows from (4.6) that v_ε is the solution to the initial value problem

$$\begin{aligned}\frac{\partial v_\varepsilon}{\partial s} &= \frac{1}{2}\frac{\partial^2 v_\varepsilon}{\partial z^2} - z\frac{\partial^2 v_\varepsilon}{\partial z} - \sqrt{\varepsilon}\{\Gamma(z)+i(\xi-\xi')\rho g(z)\}\frac{\partial v_\varepsilon}{\partial z} \\ &\frac{-\varepsilon}{2}[g(z)^2-\langle g^2\rangle]\{\xi^2+\xi'^2-2\xi\xi'\rho^2-i(\xi-\xi')\}[v_\varepsilon+1],\ s>0, \\ &v_\varepsilon(0,\xi,\xi',z)=0. \end{aligned} \tag{4.7}$$

Let $G_{\varepsilon,\xi,\xi'}(s,z,z')$ be the Green's function for the initial value problem

$$\begin{aligned}\frac{\partial v}{\partial s} &= \frac{1}{2}\frac{\partial^2 v}{\partial z^2} - z\frac{\partial v}{\partial z} - \sqrt{\varepsilon}\{\Gamma(z)+i(\xi-\xi')\rho g(z)\}\frac{\partial v}{\partial z} \\ &\quad -\frac{\varepsilon}{2}[g(z)^2-\langle g^2\rangle]\{\xi^2+\xi'^2-2\xi\xi'\rho^2-i(\xi-\xi')\}v, \quad s>0, \\ v(0,z) &= f(z). \end{aligned} \tag{4.8}$$

Thus the solution of (4.8) is given by

$$v(s,z) = \int_{-\infty}^{\infty} G_{\varepsilon,\xi,\xi'}(s,z,z')f(z')\,dz'.$$

Note that $G_{\varepsilon,\xi,\xi'}$ is an analytic function of $\xi,\xi'\in\mathbb{C}$. We have in analogy to Lemma 2 the following:

Lemma 5 *Suppose the function g of* (2.28) *satisfies the inequality,* $g(z)^2 \geqslant m^2 > 0,\ z\in\mathbb{R}$. *Then there is the inequality*

$$\int_{-\infty}^{\infty}|G_{\varepsilon,\xi,\xi'}(s,z,z')|\,dz' \leqslant \exp\left[\frac{\varepsilon s}{2}\{10\|g\|_\infty^2 + \langle g^2\rangle[(Re\,\xi)^2+(Re\,\xi')^2-2(Re\,\xi)(Re\,\xi')\rho^2] - m^2[(Re\,\xi)^2+(Re\,\xi')^2](1-\rho^2)\right\}\right],$$

provided $|Im\,\xi|,\ |Im\,\xi'|\leqslant 1$

Proof Same as for Lemma 2. □

Next we define $v_{\varepsilon,0}(s,\xi,\xi',z)$ by

$$v_{\varepsilon,0}(s,\xi,\xi',z) = \frac{-\varepsilon}{2}\{\xi^2+\xi'^2-2\xi\xi'\rho^2-i(\xi-\xi')\}\int_0^s ds' A(s')[g^2-\langle g^2\rangle](z),$$

where $A(s)$ is the operator (2.19). It follows now from (4.7), (4.8) that $v_\varepsilon(s,\xi,\xi',z)$ has the representation

$$v_\varepsilon(s,\xi,\xi',z) = v_{\varepsilon,0}(s,\xi,\xi',z) - \sqrt{\varepsilon}\int_0^s\int_{-\infty}^{\infty} G_{\varepsilon,\xi,\xi'}(s-s',z,z')\{\Gamma(z') + i(\xi-\xi')\rho g(z')\}\frac{\partial v_{\varepsilon,0}}{\partial z'}(s',\xi,\xi',z')\,dz'\,ds' - \frac{\varepsilon}{2}\{\xi^2+\xi'^2-2\xi\xi'\rho^2 - i(\xi-\xi')\} \times \int_0^s\int_{-\infty}^{\infty} G_{\varepsilon,\xi,\xi'}(s-s',z,z')[g(z')^2-\langle g^2\rangle]v_{\varepsilon,0}(s',\xi,\xi',z')\,dz'\,ds'. \quad (4.9)$$

It is clear that $v_\varepsilon(s,\xi,\xi',z)$ is analytic for $\xi,\xi'\in\mathbb{C}$.

Lemma 6 *There is a universal constant C such that if $|Im\,\xi|$, $|Im\,\xi'|\leqslant 1$, then*

$$|v_\varepsilon(s,\xi,\xi',z)| \leqslant \varepsilon\, C\|g\|_\infty^2|\xi^2+\xi'^2-2\xi\xi'\rho^2-i(\xi-\xi')| + \exp\left[\frac{\varepsilon s}{2}\{10\|g\|_\infty^2 + \langle g^2\rangle[(Re\,\xi)^2+(Re\,\xi')^2-2(Re\,\xi)(Re\,\xi')\rho^2] - m^2[(Re\,\xi)^2+(Re\,\xi')^2](1-\rho^2)\}\right]\{C\varepsilon^{3/2}s[\|\Gamma\|_\infty + |\xi-\xi'|\,\|g\|_\infty]\|g\|_\infty^2|\xi^2+\xi'^2-2\xi\xi'\rho^2-i(\xi-\xi')| + C\,\varepsilon^2 s|\xi^2+\xi'^2-2\xi\xi'\rho^2-i(\xi-\xi')|^2\,\|g\|_\infty^4\}.$$

Proof Use Lemma 5 and the fact that there is a universal constant C such that

$$\left|\int_0^s ds' A(s')[g^2-\langle g^2\rangle](z)\right| + \left|\frac{\partial}{\partial z}\int_0^s ds' A(s')[g^2-\langle g^2\rangle](z)\right| \leqslant C\|g\|_\infty^2. \qquad \square$$

Let us put

$$P_\varepsilon(t,x,x',y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\,d\xi' e^{-ix\xi}e^{ix'\xi'}\hat{P}_\varepsilon(t,\xi,\xi',y),$$
$$P_0(t,x,x') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\,d\xi' e^{-ix\xi}e^{ix'\xi'}\hat{P}_0(t,\xi,\xi').$$

It is clear from (4.4) that

$$P_\varepsilon(t,x,x,y) = E\left[|P_\varepsilon(t,x,Y^\varepsilon)|^2|Y_t^\varepsilon = y\right].$$

Lemma 7 *For any $x,x',y\in\mathbb{R}, t<T$, there is the limit*

$$\lim_{\varepsilon\to 0} P_\varepsilon(t,x,x',y) = P_0(t,x,x').$$

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Proof We have that

$$\hat{P}_\varepsilon(t,\xi,\xi',y) - \hat{P}_0(t,\xi,\xi') = \hat{h}(\xi)\hat{h}(-\xi')$$
$$\exp\left[-(T-t)\left\{\frac{1}{2}\langle g^2\rangle[\xi^2+\xi'^2-2\xi\xi'\rho^2] + i(\xi-\xi')\left[r-\frac{1}{2}\langle g^2\rangle\right]+2r\right\}\right]$$
$$v_\varepsilon\left(\frac{T-t}{\varepsilon},\xi,\xi',\frac{y-m}{\sqrt{2}v}\right), \quad \xi,\xi'\in\mathbb{R}. \tag{4.10}$$

We can represent the function $P_\varepsilon(t,x,x',y)$ and $P_0(t,x,x')$ as integrals along the lines $Im(\xi) = -1$, $Im(\xi') = 1$. In that way we avoid the singularity of $\hat{h}(\xi)$, $\hat{h}(-\xi')$ at $\xi = 0$, $\xi' = 0$. The result follows from (4.10) and Lemma 6. In fact we have

$$|P_\varepsilon(t,x,x',y) - P_0(t,x,x')| \leqslant C\sqrt{\varepsilon},$$

for some constant C. □

Proof of Theorem 2 In view of Lemma 7 it will be sufficient to show that

$$P_0(t,x,x) \neq P_{BS}(t,x)^2, \quad \rho\neq 0, \quad t<T. \tag{4.11}$$

To see this we observe that

$$\hat{P}_0(t,\xi,\xi') - \hat{P}_{BS}(t,\xi)\hat{P}_{BS}(t,-\xi') = \hat{P}_{BS}(t,\xi)\hat{P}_{BS}(t,-\xi')(T-t)\langle g^2\rangle\xi\xi'\rho^2$$
$$\times\int_0^1 d\alpha\exp[\alpha(T-t)\langle g^2\rangle\xi\xi'\rho^2].$$

Let μ be the probability measure for the standard normal variable. Then

$$\exp[a\xi\xi'] = \exp\left[\frac{-a\xi^2}{2}-\frac{a\xi'^2}{2}\right]\int_{-\infty}^{\infty} d\mu(\beta)\exp[\sqrt{a}\beta(\xi+\xi')],$$

for any $a>0$. Hence we have that

$$P_0(t,x,x) - P_{BS}(t,x)^2 = \int_0^1 d\alpha\int_{-\infty}^{\infty} d\mu(\beta)|F(\alpha,\beta)|^2, \tag{4.12}$$

where $F(\alpha,\beta)$ is given by the formula

$$F(\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi e^{-ix\xi}\hat{P}_{BS}(t,\xi)\xi\rho\sqrt{T-t}$$
$$\langle g^2\rangle^{1/2}\exp\left[\frac{-\alpha(T-t)\langle g^2\rangle\rho^2}{2}\xi^2+\sqrt{\alpha(T-t)\langle g^2\rangle}\rho\beta\xi\right].$$

Observe that $F(0,\beta) = i\rho\sqrt{T-t}\langle g^2\rangle^{1/2}\partial P_{BS}(t,x)/\partial x \neq 0$. Hence (4.11) follows from (4.12). □

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5 Conclusion

In this paper we have studied a model of Fouque, Papanicolaou and Sircar for the pricing of options on a stock which has stochastic volatility. These authors showed that under the assumption of fast mean reverting volatility, measured by a small parameter ε, the price of the option can be written in an expansion in powers of $\sqrt{\varepsilon}$. They gave a formula for the terms in the expansion correct to order $\sqrt{\varepsilon}$ and also a formula for the corresponding functional form of the implied volatility surface. This paper continues from their work by obtaining an expansion to all orders in $\sqrt{\varepsilon}$, and proving rigorously that it is asymptotic. The paper also contains a formula for the functional form of the implied volatility correct to order ε. Finally, the paper proves a result showing that a homogenization theorem of Fouque *et al*., which holds in the case of zero correlation between stock price and volatility, does not extend to the case of non-zero correlation.

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