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Non-uniqueness in G-measures

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Dedicated to the memory of Daniel Rudolph

Abstract. Bramson and Kalikow and Quas showed the phenomenon of non-uniqueness for g-measures in the absence of a C^1 condition on g. We extend this result to show that for a sequence $G = (G_n)$, the class of G-measures can be badly behaved in the sense of containing measures of type III_λ for all λ in a continuous image of an F_σ set.

1. *Introduction*

Let Γ be an infinite sum of finite abelian groups,

$$\Gamma = \coprod_{k=1}^{\infty} G_k.$$

Let $\Gamma_n = \coprod_{k=1}^{n} G_k$, so that $\Gamma_n \nearrow \Gamma$.

Then Γ acts on the direct product $X = \prod_{k=1}^{\infty} G_k$ by termwise multiplication. We will represent $x \in X$ as the infinite sequence $(x_1, x_2, \ldots)$, where $x_i \in G_i$. Notice that we may re-group the coordinates, letting $G'_k = \prod_{j=n_k}^{n_{k+1}-1} G_j$ for some increasing sequence (n_k). Then we may equally well represent Γ as $\coprod_{k=1}^{\infty} G'_k$ and X as $\prod_{k=1}^{\infty} G'_k$.

As each G_k is abelian and finite it is, by Sylow's theorem, a product of cyclic groups. Hence we have an essentially unique representation of Γ and X as

$$\Gamma = \coprod_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}, \quad X = \prod_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}$$

for a suitable sequence $\ell(i)$. (This representation is unique up to the ordering of the $\ell(i)$.)

In [**2**], a study was made of the Γ-non-singular Borel probability measures μ on X. For each such μ, we can define a Γ_n average

$$\mu^{(n)} = \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} \mu \circ \gamma$$

(where we set $\mu^{(0)} = \mu$), and, from these, normalized transition functions

$$g_n(x) = \frac{d\mu^{(n-1)}}{d\mu^{(n)}}(x) = \frac{(1/|\Gamma_{n-1}|) \sum_{\gamma \in \Gamma_{n-1}} ((d\mu \circ \gamma)/d\mu)}{(1/|\Gamma_n|) \sum_{\gamma \in \Gamma_n} ((d\mu \circ \gamma)/d\mu)}.$$

† (1949–2010).

Notice that these functions satisfy

$$\frac{1}{|G_n|}\sum_{\gamma\in G_n} g_n(\gamma x)=1 \tag{1}$$

and

$$g_n(x)=g_n(0,0,\ldots,x_n,x_{n+1},\ldots)=g_n(\gamma,\ldots,x_n,x_{n+1},\ldots) \tag{2}$$

for all $\gamma\in\Gamma_{n-1}$. Furthermore, $((d\mu\circ\gamma)/d\mu)(x)$ is reconstructible from the gs as

$$\frac{d\mu\circ\gamma}{d\mu}=\lim_{n\to\infty}\frac{g_1(\gamma x)}{g_1(x)}\frac{g_2(\gamma x)}{g_2(x)}\cdots\frac{g_n(\gamma x)}{g_n(x)}. \tag{3}$$

If $\gamma\in\Gamma_n$, (2) ensures that this ratio stops changing at n.

Actually, from the point of view of [**2**], the sequence $G_n(x)=g_1(x)\cdots g_n(x)$ was regarded as being of primary interest, and (3) expresses the fact that μ is a G-measure.

Our discussion started from a measure μ and progressed to the construction of a sequence of functions (g_n). In this paper, our interest is to start from a sequence (g_n) and ask how varied the collection of associated μs can be. An alternative statement of this question is, given $\{g_n\}$, to describe the set of possible G-measures.

In the above discussion, the various g_n are defined only μ-almost everywhere. As we no longer posit μ, but rather (g_n), we will assume that $g_n : X\longrightarrow\mathbb{R}^+$ are continuous on X.

It was proved in [**2**, Theorem 2] that any non-singular μ is equivalent to a μ' for which the associated g_ns are continuous. Hence, in global generality, as our interest is in choices of the g_n which lead to a wide variety of associated measures, the assumption of continuity of the g_ns may be restrictive, in the sense that without it, wilder behaviour may be possible. We leave this issue open for further study.

Our results are related to results of Bramson and Kalikow [**7**] and Quas [**14**] on g-measures. These authors studied the case when all the g_n were identical functions, and, under various continuity assumptions, constructed examples where there is not a unique limit. (Keane's celebrated paper showed uniqueness under the hypothesis that g is C^1.) Recently, there has been some new progress on the issue of exactly which g-functions admit uniquely ergodic measures [**1, 12**].

We are examining the more general setting where the g_ns vary, and proving that the set of possible G-measures is rather arbitrary (Theorem A below). Our approach to measuring this arbitrariness is to seek general behaviour in the Krieger types of the measures that can occur. In the Bramson–Kalikow [**7**] construction of non-uniqueness, although the exhibited measures are distinct, the associated non-singular Γ actions are conjugate and hence of the same Krieger type. What we will show is that for G-measures one can select the set of allowed Krieger types rather generally. The construction focuses on those λ for which there is a non-singular G-measure of type III_λ and show that this set can be made any prechosen F_σ subset of $[0, 1]$. It is easy to see that minor modifications of our construction would lead to more general Borel types for this set. We leave open the two questions of the exact Borel type of the set of λs that can arise for a particular G and how general a set is reachable by the method exhibited here.

The situations where there is a unique G-measure has been discussed in some detail in [**2**], where it was called unique ergodicity. In [**6**], it was shown that the ratio set of a unique G-measure is generated in an appropriate sense, by the set of essential limits of products of the form $\prod_{i=1}^{n}(g_i(x)/g_i(y))$. We show that the same behaviour is manifest in the non-unique case, although the groups generated by these limits can be non-trivially different for different measures μ.

In [**9**], it was shown that every ergodic non-singular dynamical system is orbit equivalent to a Markov odometer on a Bratteli–Vershik system, which is uniquely ergodic in the sense of G-measures. In fact, the Bratteli–Vershik systems obtained are close to infinite product systems, in the sense that they are induced transformations on closed subsets of a full product odometer. The results here throw that observation into a stark light: uniqueness can fail spectacularly in the general case. A consequence of our construction is that we are able to make an explicit realization of the ergodic decomposition of the measures constructed here into uniquely ergodic G-measures on suitable Bratelli–Vershik systems.

2. *Notation and statement of the results*

Suppose that (g_n) is fixed, and let $\mathcal{M}(G)$ be the collection of Γ-non-singular Borel probability measures satisfying (3). Following [**2**], notice that these form a weak*-compact and convex set whose extreme points are the ergodic measures.

One does not necessarily have any recurrent elements in $\mathcal{M}(G)$; a non-recurrent ergodic element is an atomic measure on a single orbit.

Our focus will be on recurrent and ergodic elements and their ratio sets. Notice that μ is recurrent and ergodic if and only if $\sum_{\gamma\in\Gamma}((d\mu\circ\gamma)/d\mu)$ diverges almost surely. In our case, the functions $(d\mu\circ\gamma)/d\mu$ will all be continuous. Hence this sum diverges on an F_σ set in X and we are interested only in ergodic measures supported on this F_σ. Notice that it is a Γ-invariant set.

To begin with, we ignore type *II* measures and consider only measures of type *III*. We associate to (g_n) the set

$$\Lambda(G)=\{\lambda\in[0,1] : \text{there exists } \mu\in\mathcal{M}(G) \text{ ergodic of type } III_\lambda\}.$$

We shall prove the following theorem.

THEOREM A. *If $S\subseteq[0,1]$ is an F_σ, then there is a sequence $g=(g_n)$ of continuous functions satisfying conditions (1) and (2) of §1 with $S=\Lambda(G)$.*

3. *Reduction to the Markov case*

If $g=(g_n)$ is a sequence of continuous functions on X, satisfying conditions (1) and (2), and if $h : X\longrightarrow\mathbb{R}_+$ is a continuous function, then we may define

$$\mu^h(A)=\int_A h\,d\mu,$$

which is an equivalent measure to μ. Let us calculate the associated sequence

$$\begin{aligned}g_n^h(x)&=\frac{d(\mu^h)^{(n-1)}}{d(\mu^h)^{(n)}}(x)=\frac{(1/|\Gamma_{n-1}|)\sum_{\gamma\in\Gamma_{n-1}}((d\mu^h\circ\gamma)/d\mu^h)(x)}{(1/|\Gamma_n|)\sum_{\gamma\in\Gamma_n}((d\mu^h\circ\gamma(x))/d\mu^h)}\\&=\frac{(1/|\Gamma_{n-1}|)\sum_{\gamma\in\Gamma_{n-1}}((d\mu^h\circ\gamma)/d\mu)(x)(h(\gamma x)/h(x))}{(1/|\Gamma_n|)\sum_{\gamma\in\Gamma_n}((d\mu^h\circ\gamma(x))/d\mu)(h(\gamma x)/h(x))}\\&=\frac{(1/|\Gamma_{n-1}|)\sum_{\gamma\in\Gamma_{n-1}}h(\gamma x)g_1(\gamma x)\cdots g_{n-1}(\gamma x)}{(1/|\Gamma_n|)\sum_{\gamma\in\Gamma_n}h(\gamma x)g_1(\gamma x)\cdots g_n(\gamma x)}.\end{aligned}$$

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The last ratio is defined and continuous everywhere. Hence replacing g_n with g_n^h will replace $\mathcal{M}(G)$ with a family of measures all simultaneously equivalent to those in $\mathcal{M}(G)$, with h the common Radon derivative.

LEMMA 3.1. *Let μ be a G-measure. There exists a measure $\nu \cong \mu$ with ν being a G^h measure for a sequence (g_i^h), each of which depends on finitely many coordinates. (Here, $h = d\mu/d\nu$), that is there exists a sequence m_n, with $m_n \geq n$ so that $g_n^h(x) = g_n^h(x_n, x_{n+1}, \ldots, x_{m_n})$.*

Proof. This is a minor modification of the argument of [**2**, Theorem 2], where it is shown that every G-measure is equivalent to an H-measure, where the sequence H_n is continuous. In that proof, Lusin's theorem is used to approximate each function g_n by a continuous function h_n on a set whose complement has small measure (depending on n). Since every continuous function on X may be uniformly approximated arbitrarily closely by a function which depends on finitely many coordinates, we may suppose in that proof that h_n has this property. Now the proof proceeds as in [**2**]. □

In the Introduction, we pointed out that the space $X = \prod_{i=1}^{\infty} G_i$ is isomorphic to the space $X = \prod_{i=1}^{\infty} G_i'$ obtained by 're-grouping' the coordinates, that is where we take some strictly increasing sequence n_i and set

$$G_i' = G_{n_i} \times \cdots \times G.$$

After this re-grouping, for a measure μ on X, we find (with the obvious notation)

$$\mu'^{(i)} = \mu^{(n_i)}$$

and

$$g_j'(x) = \frac{d\mu^{(n_{j-1})}}{d\mu^{(n_j)}} = g_{n_j}(x) \cdots g_{n_{j+1}-1}(x).$$

The next lemma asserts that if we have a G-measure as in the conclusion of Lemma 3.1, after suitable re-grouping of coordinates, it becomes a Markov measure.

LEMMA 3.2. *Suppose μ is a G-measure and each g_n depends only on a finite number of coordinates $(x_n, x_{n+1}, \ldots, x_{m_n})$, where $m_n \geq n$. Then, after a suitable re-grouping of coordinates, μ is a Markov measure, that is we have*

$$g_n'(x') = g_n'(x_n', x_{n+1}').$$

Proof. Choose $n_1 = 1$. Now $g_1(x) = g_1(x_1, \ldots, x_{m_1})$, and we choose $n_2 = m_1$.

Now suppose that n_{k-1} and n_k have been chosen.

Note that $g_{n_{k-1}}$ depends on $(x_{n_{k-1}}, \ldots, x_{m_{n_{k-1}}})$ $g_{n_{k-1}+1}$ depends on $(x_{n_{k-1}+1}, \ldots, x_{m_{n_{k-1}+1}})$ and so forth, finishing with the observation that g_{n_k-1} depends on $(x_{n_k-1}, \ldots, x_{m_{n_k-1}})$. Let $n_{k+1} = \max\{m_j : j = n_{k-1}, n_{k-1}+1, \ldots, n_k - 1\}$. Then

$$g_k' = \prod_{j=n_{k-1}}^{n_k - 1} g_j(x)$$

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depends on the coordinates $(x_{n_{k-1}}, \ldots, x_{n_{k+1}})$. Thus, if we set $x'_k = (x_{n_{k-1}}, \ldots, x_{n_k-1})$, then g'_k depends on x'_k and x'_{k+1}. So, re-grouping $X = \prod_{n=1}^{\infty} G'_k$, where

$$G'_k = G_{n_{k-1}} \times \cdots \times G_{n_k-1},$$

we see that μ is a Markov measure with respect to these coordinates. □

This result should be compared with [**3**, Theorem 2].

The conclusion of this section is that, without loss of generality, we may concentrate on the case of Markov measures.

4. *Construction of the examples*

In this section, we will show how to choose a sequence $\ell(i)$, which determines the space $X = \prod_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}$, and a sequence $\{g_i\}$ of functions on X, with $g_i(x)$ depending on the values $(x_i, x_{i+1}) x_i \in \mathbb{Z}_{\ell(i)}, x_{i+1} \in \mathbb{Z}_{\ell(i+1)}$, in such a way that we can describe the subset Λ of $[0, 1]$ so that $\lambda \in \Lambda$ belongs to the ratio set of a G-measure.

In order to do this, we first construct a tree $\widehat{S}$ in $\mathbb{N} \times \mathbb{N}$. We then identify the infinite branches of $\widehat{S}$ with a Cantor set C, and Λ as a continuous image of $C \setminus \{0\}$.

4.1. *Construction of $\widehat{S}$.*

First, choose a sequence $\{t(i) : i \in \mathbb{N}\}$ of positive integers ≥ 2. The nodes of the tree $\widehat{S}$ are the elements of $\mathbb{N} \times \mathbb{N}$ of the form $(i, 0), (i, 1), \ldots, (i, t(i) - 1)$ arranged in levels indexed by i.

The nodes at level i are connected to certain nodes at level $i + 1$; that is we partition $(i + 1, 0), \ldots, (i + 1, t(i + 1))$ into subsets $\{F(i, j) : j = 0, \ldots, t(i) - 1\}$ and say that the node (i, j) connects to the node $(i + 1, k)$ if and only if $(i + 1, k) \in F(i, j)$.

The sets $F(i, j)$ are chosen so that $(i + 1, 0) \in F(i, 0)$. (In this way, the nodes $(i, 0)$ will play a special role as a reservoir of new material in the construction.)

We let $\widehat{S}$ be the tree defined by the sequence $t(i)$ and the partitions $F(i, j)$.

Notice that the infinite branches of $\widehat{S}$ may be identified with a Cantor set (a totally disconnected compact set without isolated points) $C \subseteq [0, 1]$ as follows.

If $B = \{(i, j(i))\}$ is such a branch (i.e. $(i + 1, j(i + 1)) \in F(i, j(i))$ for all i), then set

$$\alpha(B) = \sum_{i=1}^{\infty} \frac{j(i)}{t(i) \cdot 2^i}.$$

Then α maps branches to distinct points, and these points form the Cantor set $C \subseteq [0, 1]$. If we delete 0 from this set, we are looking at the image of all branches save the **0** branch. This set is a countable disjoint union of Cantor sets. To see this, consider the subtrees $\widehat{S}_t$, each of which consists of those branches which pass through nodes $(i, 0)$ for $i \leq t - 1$, do *not* pass through node $(t, 0)$ and proceed in any way for larger i. Notice $\widehat{S} \setminus \{\mathbf{0}\} = \bigcup_i \widehat{S}_i$ and the α image of each $\widehat{S}_i$ is another Cantor set.

4.2. *Construction of Λ from $\widehat{S}$.*

Given the tree $\widehat{S}$ above, we will construct a set $\Lambda \subseteq [0, 1]$ related to a certain function $\lambda(i, j)$ on the nodes of $\widehat{S}$. Specifically, suppose λ is a real-valued function on the nodes of $\widehat{S}$ such that:

(1) $\lambda(i, 0) = 0$; and

(2) if $j \neq 0$ $|\lambda(i + 1, k) - \lambda(i, j)| < 2^{-(i+1)}$,

whenever $(i + 1, k) \in F(i, j)$.

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Given such a function λ and a branch B, it is clear that $\lambda(i, j(i))$ converges as i tends to ∞. Denote the limiting value by $\lambda(B)$.

Since the function α is one-to-one from the set of infinite branches of $\widehat{S}$ to C, we may thus use λ to define a mapping from C to $\mathbb{R}$. By abuse of notation, denote this mapping also as $\lambda : C \longrightarrow \mathbb{R}$. It is clear from the construction that λ is continuous except perhaps at

$$\mathbf{0} = \alpha(\{(i, 0) : i \in \mathbb{N}\}).$$

Finally, let $\Lambda = \lambda(C \setminus \{\mathbf{0}\})$.

LEMMA 4.1. *For $\Lambda \subset [0, 1]$ an F_σ, there is a choice for tree $\widehat{S}$ and λ defined on the nodes of $\widehat{S}$ satisfying (1) and (2) so that the range of $\lambda : C \to \mathbb{R}$ is precisely Λ.*

Proof. We have seen that C will always be a countable union of Cantor sets (totally disconnected compact sets without isolated points). As is well known, all Cantor sets are homeomorphic. Moreover, it is easy to see that any compact subset of $\mathbb{R}$ is the continuous image of a Cantor set. If we did not want the precise estimates on the rate of convergence given by (1) and (2), we would be done. Obtain these as follows. For $\Lambda = \bigcup F_i$, take each set F_i to be the image of those branches descending from $(i, 0)$ but not through $(i + 1, 0)$. We describe this subtree of $\widehat{S}$ as a tree of compact subsets of F_i. The sets at each level of this tree will cover F_i, and those sets linking upward to a single node of the tree will be contained in and cover the set associated with that node. The root of the tree will be node $(i, 0)$ and will be associated with the entire set F_i. The levels thus are labelled by $i, i + 1, \ldots$ and we require that the sets associated with level $i + k$ should all have diameter $<2^{-(i+k+1)}$. Compactness ensures that such a tree of subsets can be constructed. We define λ of a node to be some point in the set assigned to it. We paste all these subtrees together, giving all nodes at a particular level i a distinct index (i, j) to construct $\widehat{S}$ and the associated λ, completing the result. □

4.3. *Construction of the space X and the functions (g_i).* Given a tree $\widehat{S}$ and a map λ satisfying conditions (1) and (2) above, we now show how to build the space $X = \prod_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}$ and functions $g_i(x_i, x_{i+1})$ so that the set of G-measures is exactly of type III_λ, $\lambda \in \Lambda$.

The sequence $\ell(i)$ and functions g_i will be defined inductively.

Let $\ell(1) = 30(t(1) + 1)$ and partition $\{0, \ldots, \ell(1) - 1\}$ into $t(1) + 1$ sets each of cardinality 30; call these sets $S(1, 0), \ldots, S(1, t(1))$. Now choose $\ell(2)$ large enough so the set $\{0, 1, \ldots, \ell(2) - 1\}$ can be broken into $t(2) + 1$ sets $S(2, 0), \ldots, S(2, t(2))$ (one for each node at level 2), each of the same cardinality equal to 30.

Now partition each of the sets $S(1, j)$ into three equal subsets (each of 10 elements), denoted by $S_{-1}(1, j)$, $S_0(1, j)$, $S_1(1, j)$. We define $g_1(i, j) > 0$ to satisfy the following three conditions:

- $$\frac{g_1(x_1, x_2)}{g_1(x_1', x_2)} = \lambda(1, j)^{t_1 - t_2} \qquad \begin{cases} \text{if} & x_1 \in S_{t_1}(1, j), \\ & x_1' \in S_{t_2}(1, j) \\ \text{and} & x_2 \in S(2, k), \end{cases}$$

  where $k \in F(i, j)$;

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- $g_1(x_1, x_2) < 10^{-6}$ if $x_1 \in S(1, j)$ and $x_2 \in S(2, k)$ $k \notin F(1, j)$ $\sum_{x_1 \notin S(1,j)} g_1(x_1, x_2) < 10^{-6}$ for all x_2; and
- g_1 is normalized, i.e.

$$\frac{1}{\ell(1)} \sum_{x_1 \in \{0,\ldots,\ell(1)-1\}} g_1(x_1, x_2) = 1.$$

(Such a choice is clearly possible.)

Now suppose inductively that $\{0, 1, \ldots, \ell(k) - 1\}$ has been broken into $t(k) + 1$ sets $S(k, 0), \ldots, S(k, t(k))$, one for each node of $\widehat{S}$ at level k, and that each of these sets has cardinality $(2k + 1) \cdot 5 \times 2^k$. (Then we must have $\ell(k) = (2k + 1) \cdot 5 \times 2^k(t(k) + 1)$ for $k = 1, 2, 3, \ldots.$)

Next, we define the functions $g_k = g_k(x_k, x_{k+1})$. Partition $S(k, j)$ into $(2k + 1)$ sets, each of cardinality $5 \cdot 2^k$:

$$S_{-k}(k, j), S_{-k+1}(k, j), \ldots, S_0(k, j), \ldots, S_k(k, j).$$

We define $g_k(x_k, x_{k+1})$ so that:

- $$\frac{g_k(x_k, x_{k+1})}{g_k(x'_k, x_{k+1})} = \lambda(k, j)^{t_1 - t_2}$$

  if $x_k \in S_{t_1}(k, j)$ and $x'_k \in S_{t_2}(k, j)$ and $x_{k+1} \in S(k + 1, \ell)$ with $\ell \in F(k, j)$;
- $g_k(x_k, x_{k+1})$ is a positive constant on the complement of

$$E = \{(x_k, x_{k+1}) : x_k \in S(k, j) x_{k+1} \in S(k + 1, \ell) \text{ with } \ell \in F(k, j)\}$$

  with

$$\sum_{\{x_k:(x_k,x_{k+1})\in E^c\}} g_k(x_k, x_{k+1}) < \frac{1}{2^k \cdot 10^6}; \text{ and}$$

- g_k is normalized, i.e.

$$\frac{1}{\ell_k} \sum_{x_k \in \{0,\ldots,\ell_k-1\}} g_k(x_k, x_{k+1}) = 1.$$

(To see that this is possible, first define $g_k(x_k, x_{k+1}) = \lambda(k, j)^{t_j}$ if $x_k \in S_{t_i}(k, j)$ and $x_{k+1} \in S(k + 1, \ell)$, $\ell \in F(k, j)$, and then normalize.)

This completes the description of the functions g_k.

The group $\Gamma = \coprod_{k=1}^{\infty} \mathbb{Z}_{\ell(k)}$ acts on X by finite coordinate changes, and, as mentioned above, any G-measure is non-singular for this action.

Our next two steps are to show that:

(1) for each $\lambda \in \Lambda$, there is a recurrent and ergodic G-measure μ of type III_λ; and
(2) if $\lambda \notin \Lambda$, there is no such μ.

5. *Proof of the theorem*

LEMMA 5.1. *Let $E_k = \{x = (x_k) : ((k + 1), j(k + 1)) \notin F(k, j(k))\}$.*

Then, for any G-measure μ,

$$\mu(E_k) < \frac{1}{10^6 \cdot 2^k}.$$

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Proof. For $x_{k+1} \in S(k+1, t)$ with $(k+1, t) \in F(k, j)$, we have, by the definition of g_k,

$$\sum_{x_k \notin S(k,j)} g_k(x_k, x_{k+1}) < 10^{-6} \cdot 2^{-k}. \tag{4}$$

The left-hand side dominates the measure of E_k. □

Definition 5.1. Let

$$X_0 = \{x \in X : \exists N_0 \text{ so that } k \geq N_0 \Longrightarrow (k+1, j(k+1)) \in F(k, j(k))\}.$$

Notice that X_0 is Γ-invariant, as it is defined by a 'tail' property—i.e. is invariant under perturbations of the initial coordinates.

COROLLARY 5.1. *For all G-measures μ,*

$$\mu(X_0) = 1.$$

Proof. Notice that

$$X_0 = X \Bigg\backslash \bigcap_{N=1}^{\infty} \left(\bigcup_{k=N}^{\infty} E_k \right)$$

and

$$\mu\left(\bigcup_{k=N}^{\infty} E_k \right) < \sum_{k=N}^{\infty} \frac{1}{10^6 \cdot 2^k} = \frac{1}{10^6 \cdot 2^{N-1}},$$

so

$$\mu\left(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k \right) = 0.$$ □

Notation. To $x \in X_0$, we associate the set $B(x)$ of branches $\{(i, t(i))\}$ in $\widehat{S}$ such that

$$x_k \in S(k, t(k))$$

for k sufficiently large.

Conversely, to a branch B in $\widehat{S}$ we may associate the subset $X(B)$ of X_0 given by

$$X(B) = \{x \in X_0 : B(x) = B\}.$$

Notice that each $X(B)$ is Γ-invariant and Borel. Indeed, X_0 is measurably partitioned as $X_0 = \bigcup_B X(B)$.

From this observation, the following theorem follows.

THEOREM 5.1. *Suppose that μ is an ergodic G-measure. Then, for some unique B,*

$$\mu(X(B)) = 1.$$

Proof. Recall that $\mu(X_0) = 1$ and that the $X(B)$s give a measurable partition of X_0. Now the theorem follows from ergodicity of μ. □

THEOREM 5.2. *Suppose that B is not the zero branch $\{(i, 0) : i \in \mathbb{N}\}$. Then, if $\mu(X(B)) = 1$, μ is of type $III_{\lambda(B)}$.*

Proof. Let $B = \{(j, t(j))\}$ and $\lambda = \lambda(B) \in (0, 1)$. Let $A \subseteq X(B)$ have positive measure and $\varepsilon > 0$. Choose a rectangle

$$R = \{x_0\} \times \{x_1\} \times \cdots \times \{x_{n_1}\} \times \mathbb{Z}_{\ell(n_1+1)} \times \cdots$$

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such that

$$\mu(A \cap R) > (1-\varepsilon)\mu(R). \tag{5}$$

Choose $n_2 > \max(\log_2(1/\lambda), \log_2(1/\varepsilon), n_1)$ so large that

$$j > n_2 \Longrightarrow |\lambda - \lambda(j, t(j))| < \frac{1}{2^j}.$$

Then, if γ is any element of $\mathbb{Z}_{\ell(j)} \subseteq \Gamma$ which takes $S_{t_1}(j, t(j))$ to $S_{t_1-1}(j, t(j))$, we have $(d\mu \circ \gamma(x))/d\mu = \lambda(j, t(j))$ for all x such that $x_j \in S_{t_1}(j, t(j))$. If $j > n_1$, then such a γ maps R to itself.

Choose $j > n_2$. We may partition $S(j, t(j))$ as follows:

$$S(j, t(j)) = \bigcup_{t=-t(j)}^{t(j)} S_t(j, t(j)).$$

Thus, if we let $R_t = R \cap \{x : x_j \in S_t(j, t(j))\}$, we may partition $R = \bigcup_{t=-t(j)}^{t(j)} R_t$, up to a set of measure less than $1/10^6 \cdot 2^j$.

Now, by the construction of the sets $S_t(j, t(j))$, each has cardinality $5 \cdot 2^j$. Thus we can choose a finite coordinate change γ in $\mathbb{Z}_{\ell(j)}$ which maps each $S_t(j, t(j))$ to $S_{t-1}(j, t(j))$ for all $t = -t(j)+1, \ldots, t(j)$. This maps $R' = \bigcup_{t=-t(j)+1}^{t(j)} R_t$ to $R'' = \bigcup_{t=-t(j)}^{t(j)-1} R_t$ with $|((d\mu \circ \gamma)/d\mu)(x) - \lambda| < \varepsilon$ for all $x \in R'$.

Notice that

$$\mu\bigg(\bigcup_{t=-t(j)}^{t(j)} R_t\bigg) = \sum_{t=-t(j)}^{t(j)} \mu(R_t).$$

It follows that $\mu(R_{-t(j)})/\mu(R_t)$ lies between $\sum_t (\lambda+\varepsilon)^t$ and $\sum_t (\lambda-\varepsilon)^t$. The sums are equal to $\cosh(t(j) + (1/2)\log(\lambda \pm \varepsilon))/\cosh((1/2)\log(\lambda \pm \varepsilon))$, which are both bounded away from 1, provided $t(j) > 1$. A similar calculation holds for $\mu(R_{t(j)})/\mu(R_t)$.

It is now easy to see, using (5), that both $R' \cap A$ and $R'' \cap A$ have positive measure.

Since $R' \cap A$ approximates arbitrarily closely an arbitrary subset of A of positive measure, it follows that λ belongs to the ratio set $r(X, \mu, \Gamma)$.

We now show that μ is of type III_λ. Suppose that $r \in r(X, \mu, \Gamma)$. Let $\varepsilon > 0$ and choose A to be the set (of positive measure) defined by

$$A = \{x : x_0 = x_1 \cdots = x_{k-1} = 0 \text{ and } x_k \in S_1(k, t(k))\},$$

where k is so large that for $j > k$, $|\lambda - \lambda(j, t(j))| < \varepsilon$. By the definition of the ratio set, we may find a set $B \subseteq A$ of positive measure, and $\gamma \in \Gamma$ such that $\gamma B \subseteq A$ and

$$\left|\frac{d\mu \circ \gamma}{d\mu}(x) - r\right| < \frac{\varepsilon}{2} \quad \text{for all } x \in B.$$

Since B has positive measure, there exists a rectangle

$$R'' = \{x_0\} \times \{x_1\} \times \cdots \times \{x_{k-1}\} \times \{x_k\} \times \cdots \times \{x_\ell\} \times \mathbb{Z}_{\ell(k+1)}$$

such that $\mu(B \cap R'') \geq (1-\varepsilon)\mu(R'')$. For $k \leq j \leq \ell$, let $\ell(j)$ be such that $x_j \in S_{\ell(j)}(j, t(j))$, so that $\ell(k) = 1$, and let $\ell'(k)$ be such that $\gamma x_j \in S_{\ell'(j)}(j, t(j))$. For $x \in R''$, we have

$$\frac{d\mu \circ \gamma}{d\mu}(x) = \prod_{j=k}^{\ell} \lambda(j, t(j))^{\ell(j)-\ell'(j)}.$$

Now

$$\left|\prod_{j=k}^{\ell} \lambda(j, t(j))^{\ell(j)-\ell'(j)} - \lambda^{\sum_{j=k}^{\infty} \ell(j)-\ell'(j)}\right|$$
$$= \lambda^{\sum_{j=k}^{\ell} \ell(j)-\ell'(j)} \left|1 - \prod_{j=k}^{\ell} \left(\frac{\lambda(j, t(j))}{\lambda}\right)^{\ell(j)-\ell'(j)}\right|$$
$$= \lambda^{\sum_{j=k}^{\infty} \ell(j)-\ell'(j)} (1 - e^{\sum_{j=k}^{\ell} (\ell(j)-\ell'(j))\{\log \lambda(j,t(j)) - \log \lambda\}}).$$

Now, since $\lambda < 1$, we have

$$|\log \lambda(j, t(j)) - \log \lambda| \leq 2\, |\lambda(j, t(j)) - \lambda| \leq \frac{2}{2^{j+1}} = \frac{1}{2^j}$$

and

$$|\ell(j) - \ell'(j)| \leq 2j + 1.$$

Thus

$$\left|\frac{\prod_{j=k}^{\ell} \lambda(j, t(j))^{\ell(j)-\ell'(j)}}{\lambda^{\sum_{j=k}^{\ell} \ell(j)-\ell'(j)}} - 1\right| \leq |1 - e^{\sum_{j=k}^{\infty} ((2j+1)/2j)}|.$$

For ℓ sufficiently large, this can be made smaller than $\varepsilon/2$.

Thus r may be arbitrarily closely approximated by a power of λ. Hence r belongs to the closure of the subgroup generated by λ.

We have shown that (X, Γ, μ) is of type III_λ. □

PROPOSITION 5.1. *Let B be the zero branch. If $\mu(X(B)) = 1$, then (X, Γ, μ) is of type III_0.*

Proof. The proof is a simple modification of the preceding proof and is left to the reader. □

The preceding propositions give that

$$\Lambda(G) \subseteq \Lambda.$$

To prove equality, it will suffice to show that for each branch B, there is a recurrent ergodic G-measure μ with $\mu(X(B)) = 1$. In fact, we prove more.

PROPOSITION 5.2. *Let B be a branch of $\widehat{S}$. Then there is a G-measure μ concentrated on $X(B)$ which is uniquely ergodic in the sense of* [**3**].

Proof. For $b \in B$, consider the sequence of probability measures μ_k^b given by

$$\mu_k^b(f) = \frac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} f(\gamma b) G_k(\gamma b)$$

for $f \in C(X)$.

This is a sequence of measures in the unit ball of $C(X)^*$. By the Banach–Alaoglu theorem, weak*-limits exist, and an easy calculation given in [**4**] shows that any weak*-limit is a G-measure.

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Of course, for a fixed b, we might get a number of different weak*-limits—at least in principle—and as b varies, the set of weak*-limits might also vary. We would like to show that there is a unique weak*-limit.

To do this, consider the set $P \subseteq X(B)$ given by

$$P = \{x : x_k \in S(k, t(k)) \text{ for all } k\}.$$

The definition of μ_k^b above yields readily that P has full measure in $X(B)$ for any weak*-limit of μ_k^b.

Now notice that from the definition of X, we have that $S(k, t(k))$ is a set of cardinality $2t(k) - 1$, with

$$g_k(S_t(k, t(k))) = \frac{\lambda(k, t(k))^t}{C(k)} \quad \text{for } t = -t(k), \ldots, t(k).$$

Thus P is the infinite product space $\prod_{k=1}^{\infty} \mathbb{Z}_{2t(k)-1}$. Furthermore, P is equipped with an infinite product measure $\mu = \bigotimes_{k=1}^{\infty} \mu_k$, where $\mu_k(x_i) = \lambda(k, t(k))^{x_i}/C(k)$ for $i = -t(k), \ldots, t(k)$ and, in fact, the definition of g_k-functions is such that any weak*-limit of $\{\mu_k^b\}$ actually coincides with μ on P. By the (standard) argument that there is a unique infinite product measure, we see that for each b, there is a unique weak*-limit μ^b on P and that this limit is independent of b. Since $\mu^b(P) = 1$, this completes the proof. □

The above results shows the following proposition.

PROPOSITION 5.3. *The measurable partition of X into branches $X(B)$ implements the ergodic decomposition of (X, Γ, μ) into uniquely ergodic Markov G-measures on the Bratteli–Vershik systems which are the $X(B)$s.*

See [**9**] for the definitions of uniquely ergodic Markov G-measures.

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