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Mechanical modelling of adhesion and adherence between two solids

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The behaviour of adhesively bonded joints is investigated using a continuum mechanical description for the adhesive. The gradient of the adhesion variable, which describes the volumetric proportion of cavities within the adhesive, is introduced in the free energy, so that the model accounts for the intrinsic cohesion of the adhesive. The adherends are linear elastic materials and the adhesive is first given an elastic behaviour. Using a thermodynamical framework, an adhesion potential function is established, the subdifferential of which is determined in a rigorous way, so that three-dimensional coupled elastic-adhesion evolution equations are derived. Then we consider a generalization to the coupling of adhesion with elastoplasticity. A two-dimensional model of adhesive bonding is derived using a perturbation method. Finally, a finite element discretization of the coupled evolution problem is presented and a resolution scheme based on Newton's method is developed, while the integration of the constitutive law is performed using a three-step operator splitting method.

1 Introduction

In the present paper, we model and simulate the adhesive bonding of two elastic solids brought into contact, assuming that this contact occurs through a thin adhesive solid material. The framework under consideration in the present work is that of continuum mechanics using internal variables in order to describe the effect of local stresses on the intensity of contact between the two surfaces of the adherends. This refers to the general problem of adhesion. The micromechanical process that ultimately leads to fracture through the formation of a macrocrack is usually referred to as material damage. There exists an extensive literature concerning damage mechanics (a pioneering work was that of Kachanov [1] in the late fifties), the establishment of coupled elastoplastic-damaged models in both small and large strain situations (see, for instance, Ju & Simo [2, 3] or Benallal *et al.* [4]), or gradient-dependent damage models, which are generally derived from nonlocal damage theories (see, for example, Bazant & Pijaudier-Cabot [5, 6]. The physical motivation for introducing the gradient of the damage variable is clearly to account for a cohesion effect of the material (see, for example, the quite recent model in Frémond & Nedjar [7]). This is motivated by the further argument that interactions between neighbouring points are effective in polymeric-based adhesives, and the interaction length depends on the length and degree of cross-linking of the macromolecular chains. A model for gradient-dependent damage materials has also been formulated in Ganghoffer *et al.* [8]. The micromechanical

processes leading to initiation and propagation of a crack of polymeric materials, and their interaction with the deformation modes of the material (plasticity, viscous deformation) have been described, for instance, by Kinloch & Young [9] and Perez & Weissman [10]. For a more detailed description of the phenomenology of damage and plasticity in polymers, we refer to Edlund [11] and Edlund & Klarbring [12, 13]. Material softening and its influence on the failure behaviour of adhesive joints have been investigated by Ottosen & Olsson [14] and Gustafson [15]. Much less work has been performed on the adhesion problem itself. It seems that the only available continuum mechanical approach is that of Frémond [16], in which the author defines an internal variable representating the active proportion of bonds between two solid surfaces brought into contact. Dissipation of the elastic energy is here described by a pseudo-potential defined in an appropriate way and the model allows one to recover constitutive laws which were well-established experimentally [17]. Apart from this work, there exists an extensive literature dealing with the mechanics of adherence of solids, using the more classical concept of fracture mechanics (e.g. see Maugis & Barquins [18]). Since however, the modelling of adhesion is very similar to that of damage, we notice that two different approaches can be followed to build up a damage law [19]:

- the direct approach, which takes a phenomenological approach in order to quantify the influence of the damage parameters on the mechanical behaviour, or
- the indirect approach, in which the macroscopic behaviour is derived from considerations of the microscopic mechanisms and the use of homogenization methods.

In this work, we use the direct approach and aim at modelling both brittle and ductile failure, so that we will consider successively elastic and elastoplastic adhesives undergoing continuous damage, under a small strain assumption.

Throughout this work, it is supposed that in the initial state the two surfaces are not in contact. This is simulated by assuming that the set of microcavities completely fills the adhesive. A natural choice for a scalar adhesion variable A in the isotropic case is the ratio of the true volume occupied by the material building the adhesive to the total volume of the adhesive. Adhesion is then promoted by compressive states of stresses, which leads to the closure of the cavities. This process interacts with the development of plastic flow within the adhesive.

2 Elastic adhesives

We first consider the situation of an adhesive undergoing only elastic strains. As a first cornerstone of the thermodynamical formulation of elasticity coupled to adhesion, we specify the free energy density (per unit volume) of the adhesive material, which is supposed to depend on the elastic strain tensor ε^e, on the adhesion variable A and on the gradient of A (through its norm), that is $\psi(\varepsilon^e, A, |\underline{\nabla A}|)$. The adhesion is quantified by a scalar variable A, which represents the true – physical – area of contact at the interface between two solids. The variable A ranges between zero (no contact) and unity (perfect bonding, the physical area is equal to the geometrical area).

The total strain tensor is additively decomposed into the sum of a reversible (elastic) part ε^e and an irreversible one ε^{irr}, that is

$$\varepsilon = \varepsilon^e + \varepsilon^{irr}. \tag{1}$$

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We now require that the free energy ψ of the adhesive material should be equal to the free energy of the pristine material when A is equal to 1 and when $|\underline{\nabla A}|$ is equal to 0, so that the following condition is satisfied:

$$\lim_{\substack{A\to 1\\ |\underline{\nabla A}|\to 0}} \psi(\varepsilon^e, A, |\underline{\nabla A}|) = \psi(\varepsilon^e, A=1, |\underline{\nabla A}|=0) = \psi^0(\varepsilon^e) := \tfrac{1}{2}\varepsilon^e : C : \varepsilon^e. \tag{2}$$

The force variables (σ, Y, G_∇) naturally derive from ψ according to

$$\sigma = \frac{\partial \psi}{\partial \varepsilon^e} = AC:\varepsilon^e; \quad Y = \frac{\partial \psi}{\partial A}; \quad G_\nabla = \frac{\partial \psi}{\partial(|\underline{\nabla A}|)}. \tag{3}$$

Equations (3) define the state laws. The local stress tensor σ is the driving force for the elastic deformation of the material, Y is the adhesion driving force, and G_∇ is the driving force conjugated to the gradient of adhesion.

From the first equation in (3), we deduce that the influence of adhesion on the mechanical behaviour is specified through the concept of strain equivalence, as in Lemaitre & Chaboche [20]. This means that the constitutive law for the adhesive material is given by that of the pristine material replacing the stress tensor σ by the effective stress tensor σ/A. We then consider a free energy potential quadratic in A and set *a priori*

$$\psi(\varepsilon^e, A, |\underline{\nabla A}|) = A\,\tfrac{1}{2}\varepsilon^e : C : \varepsilon^e + \beta(\mathbf{a}A^2 + \mathbf{b}A + \mathbf{c})\,|\underline{\nabla A}|^p,$$

where the Euclidean norm of the gradient of A, $|\underline{\nabla A}|$, accounts for the cohesion of the material. The constants $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the exponent p (which prescribes the intensity of the effect of the gradient of the adhesion variable) must satisfy the following constraints:

- ψ must be convex with respect to each of its argument ε^e, A and $|\underline{\nabla A}|$, which leads to the following sufficient conditions:
  - the inequality $2\mathbf{a}\beta \geqslant 0$ implies that ψ is convex with respect to A;
  - the positivity of the tensor C implies the convexity of ψ with respect to ε^e;
  - the conditions $\beta(\mathbf{a}A^2 + \mathbf{b}A + \mathbf{c}) \geqslant 0$; $p = 0$ or $p \geqslant 1$ imply the convexity of ψ with respect to $|\underline{\nabla A}|$.

The adhesion driving force Y defined by equation (4) becomes

$$Y = \tfrac{1}{2}\varepsilon^e : C : \varepsilon^e + \beta|\underline{\nabla A}|^p\,(2\mathbf{a}A + \mathbf{b})$$

and the second term on the right-hand side is equal to 0 when A is equal to 1 and $|\underline{\nabla A}|$ is equal to 0. Hence, let us choose

$$\psi(\varepsilon^e, A, |\underline{\nabla A}|) = A\,\tfrac{1}{2}\varepsilon^e\, C\,\varepsilon^e + \beta\frac{A^2}{2}|\underline{\nabla A}|^p, \tag{4}$$

which implies

$$Y = \tfrac{1}{2}\varepsilon^e : C : \varepsilon^e + \beta|\underline{\nabla A}|^p\,A; \quad G_\nabla = p\,\beta\frac{A^2}{2}|\underline{\nabla A}|^{p-1}. \tag{5}$$

The adhesion potential $f(\sigma, A, |\underline{\nabla A}|)$ is built from the thermodynamic forces Y and G_∇ as

$$f(\sigma, A, |\underline{\nabla A}|) = -Y - G_\nabla - \gamma A + \eta N\left(\frac{\sigma_m}{A}\right) - K,$$

where K is the initial radius of the adhesion potential surface (when all forces are zero, thus

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it represents an initial force to be overcome), γ is a non-negative constant which determines how the threshold for adhesion (i.e. the actual value of the potential $f(\sigma, A, |\underline{\nabla A}|)$) is affected by adhesion evolution, η is a positive constant, $N(s)$ is the negative part of the real s, that is N is defined by

$$N(s) = -s \quad \text{if } s < 0; \quad N(s) = 0 \quad \text{if } s \geqslant 0 \tag{6}$$

and σ_m is the contraction of the tensor σ and the tensor **1** (the components of which are equal to 1 on the diagonal, and 0 otherwise). In this way, a non-negative pressure has no influence on the propagation of adhesion. Using equation (5), we obtain

$$f(\sigma, A, |\underline{\nabla A}|) = -Y - \mathrm{p}\beta\frac{A^2}{2}|\underline{\nabla A}|^{\mathrm{p}-1} - \gamma A + \eta N\left(\frac{\sigma_m}{A}\right) - K.$$

Notice that in the initial state, the elastic strain energy term is null. The effect of the hydrostatic stress on the growth of adhesion is prescribed by adding the term $\eta N(\sigma_m/A)$. The expression of f will be justified – essentially with regard to convexity properties – in the following section.

3 Construction of the subdifferential of the indicator function

The behaviour law (3_1) of the adhesive can be inverted as $\varepsilon^{\mathrm{e}} = 1/A\, C^{-1}:\sigma$, which implies that the adhesion driving force Y can be equivalently expressed by

$$Y = \frac{1}{2}\frac{1}{A^2}\sigma:C^{-1}:\sigma + \beta A|\underline{\nabla A}|^{\mathrm{p}} \Rightarrow |\underline{\nabla A}| = \left\{\frac{1}{\beta A}\left\{Y - \frac{1}{2}\frac{1}{A^2}\sigma:C^{-1}:\sigma\right\}\right\}^{1/\mathrm{p}},$$

since the quantity $Y - 1/2\, 1/A^2\, \sigma:C^{-1}:\sigma$ must be non-negative. Defining the two non-negative functions of A

$$a = \left\{\mathrm{p}\beta\frac{A^2}{2}\right\}^{\mathrm{p}/(\mathrm{p}-1)}\frac{1}{\beta A}; \quad b = \frac{1}{2}\left\{\mathrm{p}\beta\frac{A^2}{2}\right\}^{\mathrm{p}/(\mathrm{p}-1)}\frac{1}{A^2}\frac{1}{\beta A}, \tag{7}$$

the potential function f for adhesion can be expressed as

$$f: \begin{bmatrix} \mathbf{S}\times\mathbf{R} \to \mathbf{R} \cup \{+\infty\} \\ (\sigma, A) \mapsto -Y + G(\sigma, Y) - \gamma A + \eta N\left(\frac{\sigma_m}{A}\right) - K \end{bmatrix}, \tag{8}$$

where $\mathbf{S}$ denotes the space of symmetric second order tensors and

$$G(\sigma, Y) = \begin{cases} -\{aY - b\sigma:C^{-1}:\sigma\}^{(\mathrm{p}-1)/\mathrm{p}} & \text{if } aY - b\sigma:C^{-1}:\sigma \geqslant 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

For each real K, one defines the set C_K as: $C_K = \{(\sigma, Y) \in \mathbf{S}\times\mathbf{R} \,|\, f(\sigma, Y) \leqslant 0\}$. Note that if (σ, Y) belongs to C_K, the quantity $aY - b\sigma:C^{-1}:\sigma$ is non-negative. We then have the following lemma, the proof of which is a trivial consequence of the concavity and the monotonicity of the function

$$g_{\mathrm{p}}: \begin{pmatrix} \mathbf{R}^+ \to \mathbf{R}^+ \\ x \mapsto x^{(\mathrm{p}-1)/\mathrm{p}} \end{pmatrix}.$$

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Lemma 1

(1) *For each real K, C_K is convex.*
(2) *G defined in equation* (9) *is convex on the space* $\mathbf{S} \times \mathbf{R}$.

In the following subsection, the subdifferential of the indicator function of the convex set C_K is formally established, using the subdifferentials of G and N.

3.1 Formal construction of the subdifferential of the indicator function

We denote by ∂G (respectively, ∂N) the subdifferential of the convex function G (resp. N), given in equation (9) (resp. equation (6)).

Lemma 2 *The subdifferential ∂I_{C_K} of the indicator function I_{C_K} defined by*

$$I_{C_K}(\sigma, Y) = \begin{cases} 0 & \text{if } (\sigma, Y) \in C_K \\ +\infty & \text{otherwise,} \end{cases}$$

is given by

(1) $\partial I_{C_K}(\sigma_0, Y_0) = \varnothing$, *if* (σ_0, Y_0) *does not belong to* C_K, *i.e. if it satisfies the inequality*

$$-Y_0 + G(\sigma_0, Y_0) + \frac{\eta}{A} N(\sigma_{0m}) - \gamma A > K.$$

(2) $\partial I_{C_K}(\sigma_0, Y_0) = (0, 0)$ *if* (σ_0, Y_0) *belongs to the interior of* C_K, *i.e. if it satisfies the strict inequality*

$$-Y_0 + G(\sigma_0, Y_0) + \eta/A\, N(\sigma_{0m}) - \gamma A < K.$$

(3) $\partial I_{C_K}(\sigma_0, Y_0) = \{-\xi(\mathbf{0}, 1) + \xi\, \partial G(\sigma_0, Y_0) + \eta/A\, \xi(\partial N(\sigma_{m0}), 0)/\xi \geqslant 0\}$, *if* (σ_0, Y_0) *lies on the boundary of* C_K, *i.e. if it satisfies the equality* $-Y_0 + G(\sigma_0, Y_0) + \eta/A\, N(\sigma_{0m}) - \gamma A = K$.

Proof Let us recall that an element (α, e) of $\mathbf{S} \times \mathbf{R}$ belongs to the subdifferential $\partial I_{C_K}(\sigma_0, Y_0)$ of I_{C_K} at (σ_0, Y_0) if and only if it satisfies

$$\forall (\sigma, Y) \in \mathbf{S} \times \mathbf{R}: I_{C_K}(\sigma, Y) \geqslant I_{C_K}(\sigma_0\, Y_0) + \alpha : (\sigma - \sigma_0) + e(Y - Y_0). \tag{10}$$

(i) Assume that (σ_0, Y_0) does not belong to C_K. Then equation (10) can never be satisfied for every (σ, Y) in C_K. One then deduces the first point of the lemma.

(ii) In this case, equation (10) is interesting only if (σ, Y) belongs to C_K, and becomes

$$\forall (\sigma, Y) \in \mathbf{S} \times \mathbf{R}: 0 \geqslant 0 + \alpha : (\sigma - \sigma_0) + e(Y - Y_0). \tag{11}$$

Considering the continuity of the functions which appear in the load function f, it is possible to make σ and Y vary independently and in all 'directions', still satisfying the strict inequality $f(\sigma, Y) < 0$ and assuming that (σ, Y) is close to (σ_0, Y_0). From equation (11), one deduces that the only possibility for α and e is both to be equal to 0.

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(iii) In this third part, we suppose $f(\sigma_0, Y_0) = 0$. If (σ, Y) satisfies $f(\sigma, Y) \leqslant 0$, one obtains $0 \geqslant f(\sigma, Y) - f(\sigma_0, Y_0)$. This immediately leads to

$$0 \geqslant f(\sigma, Y) - f(\sigma, Y_0) \geqslant -(Y - Y_0) + \partial G(\sigma_0, Y_0)(\sigma - \sigma_0, Y - Y_0) + \frac{\eta}{A}\partial N(\sigma_{0m})(\sigma_m - \sigma_{0m})$$

and for each real $\xi \geqslant 0$, one has furthermore

$$0 \geqslant -\xi(Y - Y_0) + \xi\partial G(\sigma_0, Y_0)(\sigma - \sigma_0, Y - Y_0) + \frac{\eta}{A}\xi\partial N(\sigma_{0m})(\sigma_m - \sigma_{0m}).$$

Equation (11) is then verified for the following element: $-\xi(\mathbf{0}, 1) + \xi\partial G(\sigma_0, \Theta_0) + \eta/A\,\xi\partial N(\sigma_{0m})(\mathbf{1}, 0)$, which then belongs to the subdifferential of I_{C_K}. Conversely, assume that (σ_0, Y_0) belongs to the boundary of C_K and consider α and e satisfying equation (11). Let us assume in a first step that σ is equal to σ_0, equation (11) becomes

$$0 \geqslant -(Y - Y_0) + G(\sigma_0, Y) - G(\sigma_0, Y_0) \leqslant 0,$$

from which we deduce

$$(Y - Y_0)(-1 + \partial G(\sigma_0, Y_0)(0, 1)) \leqslant 0. \tag{12}$$

If the quantity $(-1 + \partial G(\sigma_0, Y_0)(0, 1)) \leqslant 0$ is negative, equation (12) leads to $Y \geqslant Y_0$, so that $e \leqslant 0$. Suppose now that $\sigma \neq \sigma_0$; hence

$$-(Y - Y_0) + G(\sigma, Y) - G(\sigma_0, Y_0) + \frac{\eta}{A}N(\sigma_m) - \frac{\eta}{A}N(\sigma_{0m}) = -\varepsilon(\varepsilon \geqslant 0)$$

$$\Rightarrow (Y - Y_0)(-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)) + \partial G(\sigma_0, Y_0)(\sigma - \sigma_0, 0) + \frac{\eta}{A}(N(\sigma_m) - N(\sigma_{0m})) = -\varepsilon.$$

This equality implies

$$Y - Y_0 = \frac{-1}{-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)} \times \left\{\partial G(\sigma_0, Y_0)(\sigma - \sigma_0, 0) + \frac{\eta}{A}(N(\sigma_m) - N(\sigma_{0m})) + \varepsilon\right\}.$$

Equation (11) then becomes

$$(-e)\frac{-1}{-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)} \times \left\{\partial G(\sigma_0, Y_0)(\sigma - \sigma_0, 0) + \frac{\eta}{A}(N(\sigma_m) - N(\sigma_{0m})) + \varepsilon\right\} \geqslant \alpha(\sigma - \sigma_0)$$

$$\Leftrightarrow (-e)\frac{-1}{-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)} \times \left\{\frac{\eta}{A}(N(\sigma_m) - N(\sigma_{0m})) + \varepsilon\right\} \geqslant \alpha(\sigma - \sigma_0)$$

$$+ (-e)\frac{1}{-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)}\partial G(\sigma_0, Y_0)(\sigma - \sigma_0, 0).$$

This last inequality means that

$$\alpha + (-e)\frac{-1}{-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)}$$

belongs to the subdifferential of the convex function

$$\sigma \mapsto (-e)\frac{-1}{-1 + \partial G(\sigma_0, Y_0)(\mathbf{0}, 1)} \times \left\{\frac{\eta}{A}(N(\sigma_{0m}) - N(\sigma_{0m}))\right\},$$

since ε can be chosen arbitrarily small. One concludes simply taking $\xi = -e$.

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One proceeds in a similar way if the quantity $-1+\partial G(\sigma_0, Y_0)(\mathbf{0}, 1)$ is non-negative. In that case, one chooses $\xi = e$. We will admit the result when $-1+\partial G(\sigma_0, Y_0)(\mathbf{0}, 1)$ is equal to 0. □

In the next section, we evaluate the subdifferentials of the functions G and N.

3.2 Determination of the subdifferentials of G and N

Set $H(\sigma_m) = \eta N(\sigma_m/A)$ in the sequel. We now have the following results, the proofs of which are trivial.

Lemma 3 *The subdifferential of H is given by*

$$\partial H(\sigma_{0m}) = \begin{cases} -\dfrac{\eta}{A}\dfrac{1}{3}\mathbf{1} & \text{if } \sigma_{0m} < 0, \\ 0 & \text{if } \sigma_{0m} > 0, \\ -\theta\dfrac{\eta}{A}\dfrac{1}{3}\mathbf{1}, \quad \text{with} \quad \theta \in [0,1] & \text{if } \sigma_{0m} = 0, \end{cases}$$

where $\mathbf{1}$ *is the identity tensor.*

Lemma 4 *The subdifferential of G is given by*

(i) $\partial G(\sigma_0, Y_0) = \varnothing$ if $aY_0 - b\sigma_0 : C^{-1} : \sigma_0 < 0$;

(ii) $$\partial G(\sigma_0, Y_0)(\sigma, Y) = \frac{\mathrm{p}-1}{\mathrm{p}} \left\{ \begin{array}{l} -a\{aY_0 - b\sigma_0 : C^{-1} : \sigma_0\}^{-1/\mathrm{p}}\, Y \\ +2b\{aY_0 - b\sigma_0 : C^{-1} : \sigma_0\}^{-1/\mathrm{p}}\, \sigma_0 : C^{-1} : \sigma \end{array} \right\}$$
if $aY_0 - b\sigma_0 : C^{-1} : \sigma_0 > 0$;

(iii) $\partial G(\sigma_0, Y_0)(\sigma, Y) = -\infty$ if $aY_0 - b\sigma_0 : C^{-1} : \sigma_0 = 0$ and $aY - 2b\sigma_0 : C^{-1} : \sigma > 0$;

(iv) $\partial G(\sigma_0, Y_0)(\sigma, Y) = +\infty$ if $aY_0 - b\sigma_0 : C^{-1} : \sigma_0 = 0$ and $aY - 2b\sigma_0 : C^{-1} : \sigma < 0$;

(v) $\partial G(\sigma_0, Y_0)(\sigma, Y) \in \mathbf{R}^+$ if $aY_0 - b\sigma_0 : C^{-1} : \sigma_0 = 0$ and $aY - 2b\sigma_0 : C^{-1} : \sigma = 0$.

We are now in a position to derive the evolution laws for the internal variables.

3.3 Evolution laws for the internal variables

The rate of change of the free energy density is

$$\dot{\psi} = \frac{\partial \psi}{\partial \varepsilon^{\mathrm{e}}} : \dot{\varepsilon}^{\mathrm{e}} + \frac{\partial \psi}{\partial A}\dot{A},$$

so that the intrinsic dissipation becomes $\phi = \sigma : \dot{\varepsilon}^{\mathrm{irr}} - Y\dot{A} \geqslant 0$. The indicator function of the convex set C_K is used as a dissipation potential, so that we write

$$(\dot{\varepsilon}^{\mathrm{irr}}, -\dot{A}) \in \partial I_{C_K}(\sigma, Y), \tag{14}$$

which is a sufficient condition for the dissipation to be non-negative. The previous

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computations written in Lemmas 2, 3 and 4 show that the first evolution law (14) can be expressed as

$$\dot{\varepsilon}^{\mathrm{irr}} = -\mu \frac{\eta}{A} \frac{1}{3} \mathbf{1} + \frac{\mathrm{p}-1}{\mathrm{p}} 2\xi b \{aY - b\sigma : C^{-1} : \sigma\}^{-1/\mathrm{p}} C^{-1} : \sigma,$$

where the multipliers μ and ξ are non-negative and satisfy

$$\mu = 0 \quad \text{if } \sigma_m > 0 \quad \text{or } f(\sigma, Y) < 0; \quad \mu = \xi \geqslant 0 \quad \text{if } f(\sigma, Y) = 0 \quad \text{and} \quad \sigma_m \leqslant 0. \tag{15}$$

The second evolution law in (14) is

$$-\dot{A} = -\xi + \xi \frac{\mathrm{p}-1}{\mathrm{p}} a \{aY - b\sigma : C^{-1} : \sigma\}^{-1/\mathrm{p}},$$

where the second multiplier ξ satisfies

$$\xi = 0 \quad \text{if } f(\sigma, Y) < 0. \tag{16}$$

The potential function for adhesion f is further defined by the following two functions, f_{21}, f_{22}, which intersect in a non-smooth way (there is a discontinuity of slopes at the point $\sigma_m = 0$):

$$f_{21}(\sigma, Y) = -Y + \beta \frac{A^2}{2} |\underline{\nabla A}|^{\mathrm{p}} - \gamma A - K; \quad f_{22}(\sigma, Y) = f_{21}(\sigma, Y) + \frac{\eta}{A^2} (N(\sigma_m))^2.$$

Clearly, the conditions (15) and (16) on the multipliers (μ, ξ) are equivalent to the following statements:

$$\mu \geqslant 0; \quad \mu f_{21} = 0; \quad f_{21} \leqslant 0; \quad \mu = \xi \quad \text{if } f_{22} = 0; \quad \xi \geqslant 0; \quad \xi f_{22} = 0; \quad f_{22} \leqslant 0.$$

The evolution laws for the internal variables can then be expressed in terms of the gradient of adhesion, using equations (6) and (7) as follows:

$$\dot{\varepsilon}^{\mathrm{irr}} = -\mu \frac{\eta}{A} \frac{1}{3} \mathbf{1} + 2\xi b \frac{\mathrm{p}-1}{\mathrm{p}} \left(\mathrm{p}\beta \frac{A^2}{2} \right)^{-1/(\mathrm{p}-1)} |\underline{\nabla A}|^{-1} C^{-1} : \sigma;$$

$$-\dot{A} = -\xi - \xi a \frac{\mathrm{p}-1}{\mathrm{p}} \left(\mathrm{p}\beta \frac{A^2}{2} \right)^{-1/(\mathrm{p}-1)} |\underline{\nabla A}|^{-1};$$

$$\mu \geqslant 0; \quad \mu f_{21} = 0; \quad f_{21} \leqslant 0; \quad \mu = \xi \quad \text{if } f_{22} = 0; \quad \xi \geqslant 0; \quad \xi f_{22} = 0; \quad f_{22} \leqslant 0.$$

Using the thermodynamical framework now established, we further enlarge the modelling to the consideration of plastic flow.

4 Elastoplastic adhesives

The coupling of damage with elastoplasticity is classic in the literature (see, for instance, Ju [2]). In the same spirit, we add to the elastic part of the free energy $\psi^{\mathrm{e}}(\varepsilon^{\mathrm{e}}, A, |\underline{\nabla A}|)$ a plastic

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part $\psi^{\mathrm{p}}(r, A)$, which depends on the accumulated plastic strain r defined by $r = (\frac{2}{3}\dot{\varepsilon}^{\mathrm{p}}:\dot{\varepsilon}^{\mathrm{p}})^{1/2}$, so that the following additive split holds:

$$\psi = \psi^{\mathrm{e}} + \psi^{\mathrm{p}}; \quad \psi^{\mathrm{e}}(\varepsilon^{\mathrm{e}}, A, |\underline{\nabla A}|) = A\,\tfrac{1}{2}\varepsilon^{\mathrm{e}}:C:\varepsilon^{\mathrm{e}} + \beta\frac{A^2}{2}|\underline{\nabla A}|^{\mathrm{p}}; \quad \psi^{\mathrm{p}}; \quad \psi^{\mathrm{p}}(r, A) = \tfrac{1}{2}A\,\theta r^2,$$

where θ represents a hardening parameter: the physical interpretation is that the resistance to plastic deformation increases, due to the increase of the density of dislocations.

The volumetric dissipation of the elastoplastic energy is then $\phi = \sigma:\dot{\varepsilon}^{\mathrm{irr}} - Y\dot{d} - R\dot{r} \geqslant 0$. The force associated with r is the radius of the flow surface R, defined by $R = \partial\psi^{\mathrm{p}}/\partial^r = A\,\theta r$. We then deduce that

$$Y = \frac{\partial\psi}{\partial A} = \tfrac{1}{2}\varepsilon^{\mathrm{e}}:C:\varepsilon^{\mathrm{e}} + \beta|\underline{\nabla A}|^{\mathrm{p}}A + \tfrac{1}{2}\theta r^2, \tag{20}$$

while G_{∇} keeps the same expression as that given In equation (3). The plasticity criterion is defined from a plastic potential surface given by

$$f_1\left(\frac{\sigma^D}{A}, \frac{R}{A}, \varepsilon^{\mathrm{e}}\right) = J_2\left(\frac{\sigma^D}{A}\right) + \xi\tfrac{1}{3}\mathrm{Tr}(E:\varepsilon^{\mathrm{e}}) - \frac{R}{A} - k, \tag{21}$$

where $J_2(s) = (\frac{3}{2}s:s)^{1/2}$ is the second stress invariant, k is the yield strength and the parameter ξ indicates the influence of the hydrostatic stress on the flow criterion. The surface $f_1 = 0$ can be interpreted as a flow surface, and if we set $A = 1$ and $\xi = 0$, we recover the flow surface for the von Mises yield criterion with an isotropic hardening written in Lemaitre & Chaboche [20]. Note that, since we consider in the general case a pressure-dependent plasticity criterion, and at the same time impose the plastic flow to be isochoric (volume-preserving), this would mean that the plasticity is of a non-associated character (another potential is then needed, which gives the direction of plastic flow). See Hill [21]. This difficulty is circumvented when expressing the pressure as a function of the elastic strain, resulting in the term $\xi\frac{1}{3}\mathrm{Tr}(E:\varepsilon^{\mathrm{e}})$ in equation (20). The gradient of the adhesion variable is still eliminated from equation (19), so that we get

$$|\underline{\nabla A}| = \left\{\frac{1}{\beta A}\left[Y - \frac{1}{2}\frac{1}{A^2}\alpha:C^{-1}:\sigma - \frac{1}{2}\theta r^2\right]\right\}^{1/\mathrm{p}}. \tag{22}$$

The potential function for adhesion then becomes

$$f(\sigma, Y, |\underline{\nabla A}|) = -Y - \left\{aY - b\,\sigma:C^{-1}:\sigma - \frac{a}{2}\theta r^2\right\}^{(\mathrm{p}-1)/\mathrm{p}} + \eta N\left(\frac{\sigma_m}{A}\right) + \alpha r - \gamma A - K,$$

where a, b are defined in equation (7) and α is a non-negative constant. The adhesion surface $f = 0$ is defined by the two surfaces $f_{21} = 0$ and $f_{22} = 0$ which intersect in a non-differentiable way, with

$$f^{21}(\sigma, Y, |\underline{\nabla A}|) = -Y - \left\{aY - b\,\sigma:C^{-1}:\sigma - \frac{a}{2}\theta r^2\right\}^{(\mathrm{p}-1)/\mathrm{p}} + \alpha r - \gamma A - K \leqslant 0;$$

$$f_{22}(\sigma, Y, |\underline{\nabla A}|) = f_{21} + \eta N\left(\frac{\sigma_m}{A}\right) \leqslant 0.$$

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The evolution laws for the internal variables are defined by the following subdifferential identity:

$$(\dot{\varepsilon}^{\mathrm{irr}}, -\dot{r}, -A) \in \partial I_{C_K}(\sigma, R, Y; \varepsilon^{\mathrm{e}}, r, A);\ C_K = \{(\sigma, R, Y) | f_1 \leqslant 0; f_{21} \leqslant 0; f_{22} \leqslant 0\}.$$

Because the potential function for plasticity is differentiable with respect to the stress tensor and the variable R, we deduce as in section 3.5, that the previous statement is equivalent to

$$\dot{\varepsilon}^{\mathrm{irr}} = \lambda \frac{\partial f_1}{\partial \sigma^D} : \frac{\partial \Sigma_D}{\partial \sigma} - \mu \frac{\eta}{A} \frac{1}{3} \mathbf{1} + 2b\xi \frac{\mathrm{p}-1}{\mathrm{p}} |\underline{\nabla A}|^{-1} \left\{ \mathrm{p}\beta \frac{A^2}{2} \right\}^{-1/(\mathrm{p}-1)} C^{-1} : \sigma,$$

so that we get the following evolution laws:

$$\dot{\varepsilon}^{\mathrm{irr}} = \frac{\lambda}{A} \frac{3}{2} \frac{\sigma^D}{J_2(\sigma^D)} - \mu \frac{\eta}{A} \frac{1}{3} \mathbf{1} + 2b\xi \frac{\mathrm{p}-1}{\mathrm{p}} \left\{ \mathrm{p}\beta \frac{A^2}{2} \right\}^{-1/(\mathrm{p}-1)} C^{-1} : \sigma |\underline{\nabla A}|^{-1}$$

$$-\dot{A} = -\xi - a\xi \frac{\mathrm{p}-1}{\mathrm{p}} \left\{ \mathrm{p}\beta \frac{A^2}{2} \right\}^{-1/(\mathrm{p}-1)} |\underline{\nabla A}|^{-1};\ -\dot{r} = \lambda \frac{\partial f_1}{\partial R} = -\frac{\lambda}{A}, \quad (23)$$

where the multipliers λ, μ and ξ satisfy the following conditions:

$$\lambda \geqslant 0; \quad A f_1 = 0; \quad f_1 \leqslant 0; \quad \mu \geqslant 0; \quad \mu f_{21} = 0; \quad f_{21} \leqslant 0; \quad \mu = \xi \quad \text{if } f_{22} = 0;$$
$$\xi \geqslant 0; \quad \xi f_{22} = 0; \quad f_{22} \leqslant 0, \quad (24)$$

and the plastic strain velocity is identified as

$$\dot{\varepsilon}^{\mathrm{p}} = \frac{\lambda}{A} \frac{3}{2} \frac{\sigma^D}{J_2(\sigma^D)}.$$

Due to the additive split of strains, the elastic strain is then deduced from the knowledge of the irreversible strain according to

$$\dot{\varepsilon}^{\mathrm{e}} = \dot{\varepsilon} - \dot{\varepsilon}^{\mathrm{irr}}. \quad (25)$$

Equations (23)–(25), associated to the equilibrium equations for the three bodies, the conditions of continuity of displacement and traction at each interfaces adherends/adhesive, represent the complete constitutive law for the three-dimensional problem.

5 A two-dimensional model of the adhesive

We are now interested in the behaviour of the three-body system when the adhesive is a thin layer of an isotropic elastoplastic material, and aim at deriving a simplified constitutive behaviour taking the limit when the thickness vanishes. The constitutive law for a thin isotropic adhesive film undergoing elastic deformations can be established from the asymptotic analysis of the behaviour of a thin elastic adhesive sandwiched between two elastic adherends, using an asymptotic analysis as described by Klarbring [22] and the

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principle of equivalence (3_1), so that the effective stress σ/A is substituted for the stress. We now specialize the general constitutive law of the adhesive (3_1) to the isotropic case, as

$$\sigma_{ij} = A\frac{E}{1+\nu}\left\{\varepsilon^{e}_{ij}+\frac{\nu}{1-2\nu}\varepsilon^{2}_{kk}\,\delta_{ij}\right\} \tag{26}$$

involving Young's modulus E and Poisson's ratio ν. Introducing the expressions

$$\sigma = \sigma^0+\varepsilon\sigma^1+\ldots; \quad u = u^0+\varepsilon u^1+\ldots$$

into equations (22) leads to the following equalities for the first order in-plane stresses:

$$\sigma^0_{12} = 0; \quad \sigma^0_{11} = \frac{\nu}{1-\nu}\sigma^0_{33}; \quad \sigma^0_{22} = \frac{\nu}{1-\nu}\sigma^0_{22} \tag{27}$$

and a normal traction satisfying

$$\left(\frac{\sigma}{A}\right)^0_{i3,3} = 0; \quad \frac{\sigma^0_{33}}{A} = u^0_{3,3}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}. \tag{28}$$

Equation (28) and the expression of the in-plane stress components (27) show that the effective stress tensor σ^0/A is constant through the adhesive thickness. Assuming that the adhesion variable $A(x_1, x_2, x_3)$ asymptotically converges to $A(x_1, x_2, 0)$, and since only the thin layer is an adhesive material, there is no continuity condition for A at each adhesive/adherend interface S_i, $i = 1, 2$. One can then suppose that A is constant through the adhesive thickness, so that (27) and (28) imply that the stress tensor σ^0 itself is constant through the adhesive thickness.

Moreover, one proves that the displacement $\underline{u}^0$ varies linear through the adhesive thickness, and we can therefore write

$$\underline{u}^0 = \frac{x_3}{2}(\gamma_1\underline{u}^0-\gamma_2\underline{u}^0)+\tfrac{1}{2}(\gamma_1 u^0+\gamma_2 u^0),$$

where $\gamma_i\underline{u}^0$ denotes the trace of $\underline{u}^0$ on S_i, $i = 1, 2$.

As shown by Klarbring [22] for an elastic undamaged adhesive, the solution of the first order problem of the asymptotic expansion does not involve any dependence of the field variables on the thickness coordinate. Therefore, the adhesive can be treated as a material surface, letting the mechanical fields within the adhesive depend only on their boundary values on it. We therefore determine from (26) and (27) the in-plane elastic strains

$$\varepsilon^{e}_{11} = \frac{-\nu}{E}\frac{1}{A}\mathrm{Tr}(\sigma)\mathbf{1}+\frac{1+\nu}{E}\frac{1}{A}\sigma_{11} = \frac{1}{A}\left\{-\frac{\nu}{E}\left(\frac{2\nu}{1-\nu}+1\right)\sigma_{33}+\frac{1+\nu}{E}\frac{\nu}{1-\nu}\sigma_{33}\right\} = 0;$$

$$\varepsilon^{e}_{22} = \varepsilon^{e}_{11} = 0;$$

$$\varepsilon^{e}_{12} = 0,$$

so it is shown that the in-plane strain tensor can be neglected: $\varepsilon^{e}_{\alpha\beta} = 0$.

We now make assumptions concerning the irreversible part of the strain tensor compatible with that obtained for the elastic part, so that a simplified constitutive law will

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be derived. Indeed, we assume that the plastic part of the strain tensor can be neglected: $\varepsilon^{\mathrm{pl}}_{\alpha\beta} \cong 0$. We then obtain the following simplified strain measures for both the elastic and irreversible parts:

$$\varepsilon_{3\beta} = \tfrac{1}{2} W_\beta; \quad \varepsilon_{33} = W_3; \quad W_i = \frac{1}{h}(u_i^1 - u_i^2),$$

$$W^{\mathrm{e}}_\alpha = 2\varepsilon^{\mathrm{e}}_{3\alpha}; \quad W^{\mathrm{irr}}_\alpha = 2\varepsilon^{\mathrm{irr}}_{3\alpha}; \quad W^{\mathrm{e}}_3 = \varepsilon^{\mathrm{e}}_{33}; \quad W^{\mathrm{irr}}_3 = \varepsilon^{\mathrm{pl}}_{33} + \varepsilon^{d}_{33},$$

where the following partition holds: $W = W^{\mathrm{e}} + W^{\mathrm{irr}}$. The integration of the traction vector through the adhesive thickness defines the following simplified stress measures:

$$P_i = \int_{-h/2}^{h/2} \sigma_{3i}\, dx_3,$$

using similar notations as those in Edlung & Klarbring [1–13]. The two-dimensional constitutive law of the adhesive is then in matrix form:

$$P_i = A C_{ij} W^{\mathrm{e}}_j; \quad (C_{ij})_{ij} = \frac{Eh}{1+\nu} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \dfrac{1-\nu}{1-2\nu} \end{bmatrix}.$$

From the behaviour law (26) of the adhesive, we deduce

$$\sigma_{\alpha\beta} = \frac{\nu}{h(1-\nu)} P_3\, \delta_{\alpha\beta}; \quad \sigma_{3i} = \frac{P_i}{h} \Rightarrow P_m := \sigma_m(P) = \frac{1+\nu}{3h(1-\nu)} P_3.$$

We next express the potential functions for adhesion and plasticity in terms of the simplified stress and strain measures previously introduced. The functions f, f_1 now become new functions $\hat{f}, \hat{f}_1$, of the variables $\hat{Y} = hY$, $\hat{R} = hR$, W^{e}, r, A according to

$$\hat{f}\left(\frac{P}{A}, \frac{\hat{R}}{A}; \quad W^{\mathrm{e}}, r, A\right) = f\left(\frac{\sigma^D(P)}{A}, \frac{R(\hat{R})}{A}; \quad W^{\mathrm{e}}, r, A\right);$$

$$\hat{f}_1\left(\frac{P_m}{A}, \hat{Y}, r, A\right) = f_1\left(\frac{\sigma_m(P)}{A}, Y(\hat{Y}), r, A\right),$$

so that we explicitly obtain the following expressions:

$$\hat{f}_1 = \frac{1}{hA}\left\{3P_1^2 + 3P_2^2 + \frac{(1-2\nu)^2}{(1-\nu)^2} P_3^2\right\}^{1/2} + \xi \frac{1}{3} \frac{E}{1-2\nu} W^{\mathrm{e}}_3 - \frac{\hat{R}}{hA} - k;$$

$$\hat{f} = -\frac{\hat{Y}}{h} - \mathrm{p}\beta \frac{A^2}{2}\left(\frac{\hat{Y} - \hat{W}^{\mathrm{e}} - \frac{1}{2}h\theta r^2}{\beta h A}\right)^{(\mathrm{p}-1)/\mathrm{p}} + \eta N\left(\frac{P_m}{A}\right) + \alpha r - \gamma A - K;$$

$$P_m := \sigma_m(P) = \frac{1+\nu}{3h(1-\nu)} P_3;$$

$$\hat{Y} = \hat{W}^{\mathrm{e}} + h\left(\beta |\underline{\nabla A}|^{\mathrm{p}} A + \frac{1}{2}\theta r^2\right); \quad \hat{W}^{\mathrm{e}} = \frac{1}{2A^2} \boldsymbol{P} : \hat{\boldsymbol{C}}^{-1} : \boldsymbol{P}; \quad \hat{C}_{ij} = \frac{1}{h} C_{ij}.$$

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The subdifferential equality $(\dot{\varepsilon}^{\mathrm{irr}}, -\dot{A}, -\dot{r}) \in \partial I_{C_K}(\sigma, Y, R)$ is thus equivalent to the following variational inequality:

$$A = (\sigma, Y, R) \in C_K : \int_\omega ((\sigma^* - \sigma) : \dot{w}^{ir} - (Y^* - Y)\dot{A} - (R^* - R)\dot{r})\, d\sigma \leqslant 0,$$

$$\forall A^* = (\sigma^*, Y^*, R^*) \in C_K = \{(\sigma, Y, R) \mid f_1 \leqslant 0; \quad f_{21} \leqslant 0; \quad f_{22} \leqslant 0\},$$

which is equivalent to

$$\hat{A} = (P, \hat{Y}, \hat{R}) : \int_\omega ((P^* - P) : \dot{w}^{ir} - (\hat{Y}^* - \hat{Y})\dot{A} - (\hat{R}^* - \hat{R})\dot{r})\, d\sigma \leqslant 0,$$

$$\forall \hat{A}^* = (P^*, \hat{Y}^*, \hat{R}^*) \in \hat{C}_0 = \{(P, \hat{Y}, \hat{R}) \mid \hat{f}_1 \leqslant 0; \quad \hat{f}_{21} \leqslant 1; \quad \hat{f}_{22} \leqslant 0\},$$

with $\hat{R} = hR$; $\hat{Y} = hY$. This last statement implies the following subdifferential equality:

$$(\dot{w}^{ir}, -\dot{r}, -\dot{A}) \in \partial I_{\hat{C}_K}(P, \hat{R}, \hat{Y}).$$

The derivation of the evolution laws for the internal variables in a two-dimensional description then follows the same steps as those needed for the establishment of the evolution laws in the three-dimensional case. The expression for the function $\hat{f}$ can be recast in a form similar to that given by equation (8): $\hat{f}(\sigma, A) = -\hat{Y}/h + \hat{G} + \eta N(P_m/A) + \alpha r - \gamma A - K$, with

$$\hat{G}(\sigma, A) = \{-\hat{a}\hat{Y} + \hat{b}P : \hat{C}^{-1} : P - \hat{a}\frac{1}{2}h\theta r^2\}^{(\mathrm{p}-1)/\mathrm{p}}; \quad \hat{a} = \frac{a}{h}; \quad \hat{b} = \frac{b}{h}.$$

The subdifferential equality $(\dot{w}^{ir}, -\dot{r}, -\dot{A}) \in \partial I_{\hat{C}_K}(P, \hat{R}, \hat{Y})$ then leads to the following evolution laws for the internal variables in the two-dimensional description of the adhesive:

$$\dot{w}_\alpha^{\mathrm{irr}} = \lambda \frac{3P_\alpha}{hA} \frac{1}{\left\{3P_1^2 + 3P_2^2 + \dfrac{(1-2\nu)^2}{(1-\nu)^2} P_3^2\right\}^{1/2}} + 2\hat{b}\xi \frac{\mathrm{p}-1}{\mathrm{p}} \left\{\mathrm{p}\beta \frac{A^2}{2}\right\}^{-1/(\mathrm{p}-1)} (\hat{C}^{-1} : P)_\alpha |\underline{\nabla A}|^{-1},$$

$$\dot{w}_3^{\mathrm{irr}} = \lambda \frac{P_3}{hA} \left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{1}{\left\{3P_1^2 + 3P_2^2 + \dfrac{(1-2\nu)^2}{(1-\nu)^2} P_3^2\right\}^{1/2}} - \mu \frac{1}{3} \frac{\eta}{hA} \frac{1+\nu}{1-\nu}$$

$$+ \xi(\mathrm{p}-1)\hat{b}(\hat{C}^{-1} : P)_3 \left\{\mathrm{p}\beta \frac{A^2}{2}\right\}^{-1/(\mathrm{p}-1)} |\underline{\nabla A}|^{-1}.$$

Moreover, one has

$$\dot{A} = \frac{\xi}{h} + \xi \frac{\mathrm{p}-1}{\mathrm{p}} \hat{a} \left\{\mathrm{p}\beta \frac{A^2}{2}\right\}^{-1/(\mathrm{p}-1)} |\underline{\nabla A}|^{-1} = \frac{\xi}{h} + \frac{\xi}{h}(\mathrm{p}-1)\frac{A}{2} |\underline{\nabla A}|^{-1},$$

$$\dot{r} = -\lambda \frac{\partial \hat{f}_1}{\partial \hat{Y}} = \lambda \frac{1}{hA}; \quad \hat{f}_1 \leqslant 0; \quad \lambda \geqslant 0; \quad \hat{f}_1 \lambda = 0;$$

$$\mu \geqslant 0; \quad \mu \hat{f}_{21} \leqslant 0; \quad \hat{f}_{21} \leqslant 0; \quad \xi \geqslant 0; \quad \xi \hat{f}_{22} = 0; \quad \hat{f}_{22} \leqslant 0.$$

We next describe the numerical procedure associated with this problem.

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6 Numerical implementation

Let $\hat{X} := \{\hat{A} \in \hat{C}_K \text{ on } \omega\}$ be the set of thermodynamic forces defined on the interface $\omega x\{0\}$, where the set $\hat{C}_K := \{\hat{A} \,|\, \hat{f}_1 \leqslant 0, \hat{f}_{21} \leqslant 0, \hat{f}_{22} \leqslant 0\}$ determines the domain of the forces through the potential functions for plasticity $\hat{f}_1$ and adhesion $\hat{f}_{21}, \hat{f}_{22}$.

The problem $(\hat{P})$, which involves a surface description of the adhesive layer, including the equilibrium of the three-body system and the complementary laws expressed in weak form, can be stated as

$$\text{Find } u(t) \in V, w^{\text{irr}}(t), r(t) \quad \text{and} \quad \hat{A}(t) \in \hat{X} \quad \text{such that for every} \quad t \in [0, T]$$

$$a(u, v) + \frac{1}{h} \int_\omega P \cdot (v^1 - v^2)\, dS - g(v) = 0, \quad \forall v \in V, \tag{$\hat{\text{P}}$}$$

$$\int_\omega [\dot{w}^{\text{irr}} \cdot (P^* - P) - \dot{r}(\hat{R}^* - \hat{R}) - \dot{A}(\hat{Y}^* - \hat{Y})]\, dS \leqslant 0, \quad \forall \hat{A}^* \in \hat{X},$$

where v^1 (resp. v^2) denotes the restriction of v to Ω_1 (resp. Ω_2). Here, a and g are given by

$$a(u, v) = \int_{\Omega_1 \cup \Omega_2} E_{ijkl} u_{i,j} v_{k,l}\, dx; \quad g(v) = \int_{S_t^1 \cup S_t^2} g_i v_i\, dS,$$

and are the standard bilinear and linear functions representing the internal and external virtual work, respectively, associated with the traction forces, here denoted by g, a being defined on the space $V \times V$, where $V = \{v \in (H^1(\Omega_1 \cup \Omega_2))^3 \,|\, v = 0 \text{ on } S_u\}$, S_u being a non-void part of the smooth boundary $\partial\Omega$. The equations of $(\hat{\text{P}})$ respectively, hold together with the 'change of variables' $\hat{R} = hR$; $\hat{Y} = hY$ and the state law starts with the initial conditions

$$u(0) = 0, \quad r(0) = 0 \quad \text{and} \quad \hat{A}(0) = 0.$$

The numerical solution procedure is very similar to the scheme proposed in Edlung & Klarbring [13, 16] for the modelling of damage in adhesive joints, and we essentially use the same notation. We consider the following approximate resolution procedure for $(\hat{\text{P}})$.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the time interval $[0, T]$. The integration in time is performed by a backward Euler (implicit) scheme, which means that the rate of change of a variable f at time t_N is approximated as $\dot{f}(t_N) \equiv \dot{f}_N \approx (f_N - f_{N-1})/\Delta t$, where $\Delta t = t_N - t_{N-1}$. For the spatial discretization we replace V and $\hat{X}$ by finite-dimensional approximations

$$V^h = \left\{ v^{h(r)} \,\middle|\, v^{h(r)}(x) = \sum_{n=1}^{N^{(r)}} \Psi_n^{(r)}(x)\, v^{n(r)}, v^{h(r)}(x_n) = 0 \quad \text{for} \quad x_n \in S_u^{(r)}, r = 1, 2 \right\},$$

where $\Psi_n^{(r)}$ are the global finite element basis (polynomial) functions, $N^{(r)}$ is the total number of nodes in the rth body and $v^{n(r)} := v^{h(r)}(x_n)$, x_n being the coordinate of the nth node,

$$\hat{X}^h = \{\hat{A}^h(x_g^e) \in \hat{C}_K, e = 1, \ldots, E, g = 1, \ldots, G\},$$

where it is required that $\hat{A}^h$ belongs to $\hat{C}_K$ at the integration points.

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The integrals over ω appearing in $(\hat{\mathrm{P}})$ are evaluated using a gaussian integration method, that is

$$\int_{\omega} f dS = \sum_{e=1}^{E} \left(\int_{\omega_e} f dS \right) \approx \sum_{e=1}^{E} I_e[f] = \sum_{e=1}^{E} \sum_{g=1}^{G} W_g^{\mathrm{e}} f(x_g^{\mathrm{e}}), \tag{29}$$

where E is the number of integration cells, G is the number of integration points x_g^{e} in each **cell, and W_g^{e} are the integration weights.** After discretization in time and space of $(\hat{\mathrm{P}})$, we **obtain the following problem** (assuming from now on that the coefficients C_{ij} are constant on ω):

For $N = 1, 2, \ldots, M$; find $u_N^h \in V^h$, $w_N^{h\mathrm{irr}}, r_N^h$ and $\hat{A}_N^h \in \hat{X}^h$ such that

$$a(u_N^h, v^h) + \int_{\omega_e} \left[P_{iN}^h \frac{1}{h} (v_i^{h1} - v_i^{h2}) \right] dS = g_N(v^h) = 0, \quad \forall v^h \in V^h, \tag{30}$$

$$\int_{\omega_e} [(w_N^{h\mathrm{irr}} - w_{N-1}^{h\mathrm{irr}})(P^{*h} - P_N^h) - (r_N^h - r h_{N-1}^h)(\hat{R}^{*h} - \hat{R}_N^h) - (A_N^h - A_{N-1}^h)(\hat{Y}^{*h} - \hat{Y}_N^h)]$$

$$dS \leqslant 0, \quad \forall \hat{A}^{*h} \in \hat{X}^h,$$

and at each integration point x_g^{e} and for every component i

$$w_{iN}^h = w_{iN}^{h\mathrm{e}} + w_{iN}^{h\mathrm{irr}}, \tag{$\hat{M}_\Delta^h$}$$

$$P_{iN}^h = A_N^h C_{ij} w_{jN}^{h\mathrm{e}}; \quad \hat{R}_N^h = h A_N^h \theta r_N^h, \tag{31}$$

$$\hat{Y}_N^h = h(\tfrac{1}{2} w_{iN}^{h\mathrm{e}} C_{ij} w_{jN}^{h\mathrm{e}} + \beta |\nabla A_N|^{\mathrm{p}} A_N + \tfrac{1}{2} \theta r_N^{h^2}), \tag{32}$$

with the initial conditions $u_0^h = 0$, $r_0^h = 0$ and $\hat{A}_0^h = 0$.

With the integration rule (29), the virtual work equation (30) can be written as follows:

$$B(u_N, v) \equiv \sum_{r=1}^{2} \sum_{n=1}^{N^{(r)}} \sum_{m=1}^{N^{(r)}} K_{ik}^{nm(r)} u_{kN}^{m(r)} v_i^{n(r)} + \sum_{e=1}^{E} \sum_{g=1}^{G} W_g^{\mathrm{e}} P_{iN}^h(x_g^{\mathrm{e}}) \frac{1}{h} \sum_{r=1}^{2} \sum_{n=1}^{N^{(r)}} B_g^{\mathrm{e}n(r)} v_i^{n(r)}$$

$$- \sum_{r=1}^{2} \sum_{n=1}^{N^{(r)}} G_{iN}^{n(r)} v_i^{n(r)} = 0 \quad \forall v_i^{n(r)},$$

where

$$K_{ik}^{nm(r)} := \int_{\Omega^{(r)}} E_{ijkl} \psi_{n,j}^{(r)} \psi_{m,l}^{(r)} dV; \quad G_{iN}^{n(r)} := \int_{Sg^{(r)}} g_{iN} \Psi_n^{(r)} dS, \tag{33}$$

$$B_g^{\mathrm{e}n(1)} := \Psi_n^{(1)}(x_g^{\mathrm{e}}), \quad B_g^{\mathrm{e}n(2)} := -\Psi_n^{(2)}(x_g^{\mathrm{e}}).$$

For each time-step we have to solve the following problem: Given the state at t_{N-1}, find $u_N, P_{iN}^h, R_N^h, Y_N^h, w_{iN}^{h\mathrm{irr}}, r_N^h$ and A_N^h at each integration point, such that

$$B(u_N, v) = 0, \quad \forall v \in R^q, \quad q = 3N^{(1)} + 3N^{(2)}, \tag{$\hat{\mathrm{P}}_\Delta^h$}$$

and the constitutive law $(\hat{M}_\Delta^h)$ are satisfied.

To update the discretized constitutive law $(\hat{M}_\Delta^h)$, the constitutive response is required when the residual force vector is computed and when the tangent stiffness matrix is computed from the consistent tangent modulus. The constitutive response is also needed in each integration point and in each iteration for a current state of deformation defined by Newton's procedure. Since the output from Newton's procedure is a displacement increment, it is natural to consider the problem of updating the material state as a strain-driven problem which has to be solved locally at each integration point. Thus, we

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eliminate the thermodynamic forces p_i, $\hat{R}$, $\hat{Y}$ in order to deal with a law only depending on the internal variables and rewrite $(\hat{M}^h_\Delta)$ using equations (31)–(32). We then obtain the following form of the law which will be referred to as $(\bar{M}^h_\Delta)$:

$$w^{\mathrm{e}}_{iN} = \bar{w}_{iN} - w^{\mathrm{irr}}_{iN}; \quad w^{\mathrm{irr}}_{\alpha N} = w^{\mathrm{irr}}_{\alpha N-1} + \Delta\lambda \frac{1}{hA_N} A_{\alpha N} + \frac{\Delta\mu_1}{2}(\mathrm{p}-1)\, w_{\alpha N-1} |\nabla A_N|^{-1}; \tag{$\bar{\mathrm{M}}^h_\Delta$}$$

$$w^{\mathrm{irr}}_{\alpha N} = w^{\mathrm{irr}}_{\alpha N-1} + \Delta\lambda \frac{1}{hA_N} A_{\alpha N} + \frac{\Delta\mu_1}{2}(\mathrm{p}-1)\, w_{\alpha N-1} |\nabla A_N|^{-1};$$

$$w^{\mathrm{irr}}_{3N} = w^{\mathrm{irr}}_{3N-1} - \frac{2\mu}{3}\eta E \frac{w^{\mathrm{e}}_{3N-1}}{1-2\nu}\Delta\mu_2 + \frac{\Delta\mu_1}{2}(\mathrm{p}-1)\, w^{\mathrm{e}}_{3N-1} |\nabla A_N|^{-1}; \tag{34}$$

$$r_N = r_{N-1} + \frac{1}{hA_N}\Delta\lambda; \quad A_N = A_{N-1} + \frac{\Delta\mu_1}{h} + \frac{\Delta\mu_1}{h}(\mathrm{p}-1)|\nabla A_N|^{-1}\frac{A_N}{2};$$

$$\bar{f}_1 \leqslant 0; \quad \lambda \geqslant 0; \quad \bar{f}_1 \lambda = 0; \quad \mu_1 \geqslant 0; \quad \mu_1 \bar{f}_{21} = 0; \quad \bar{f}_{21} \leqslant 0; \quad \mu_2 \geqslant 0; \quad \mu_2 \bar{f}_{22} = 0; \quad \bar{f}_{22} \leqslant 0.$$

Here

$$A_i(w^{\mathrm{e}}) = \frac{\sqrt{3} w^{\mathrm{e}}_i}{[(w^{\mathrm{e}}_1)^2 + (w^{\mathrm{e}}_2)^2 + \frac{4}{3}(w^{\mathrm{e}}_3)^2]^{1/2}}$$

and the potential functions are now expressed in the generalized strain variables, according to the constitutive law (29), as

$$\bar{f}_1 = \frac{\sqrt{3}}{2}\frac{E}{1+\nu}\left[(w^{\mathrm{e}}_1)^2 + (w^{\mathrm{e}}_2)^2 + \frac{4}{3}(w^{\mathrm{e}}_3)^2\right]^{1/2} - k + \xi\frac{E}{3(1-2\nu)} w^{\mathrm{e}}_3 - \theta r,$$

$$\bar{f}_{21} = \frac{Eh}{4(1+\nu)}\left[(w^{\mathrm{e}}_1)^2 + (w^{\mathrm{e}}_2)^2 + 2\left(\frac{1-\nu}{1-2\nu}\right)(w^{\mathrm{e}}_3)^2\right] - \beta A |\underline{\nabla A}|^{\mathrm{p}}$$

$$- \frac{1}{2}\theta r^2 - \mathrm{p}\beta\frac{A^2}{2}|\underline{\nabla A}|^{\mathrm{p}-1} + \alpha r - \gamma A - K,$$

$$\bar{f}_{22} = \bar{f}_{21} + \eta\frac{E}{3(1-2\nu)} w^{\mathrm{e}}_3.$$

The problem (U) of updating the material state can be stated as follows:
Find $\Delta\lambda$, $\Delta\mu_1$ and $\Delta\mu_2$ such that $(\bar{M}^h_\Delta)$ is satisfied for

$$w_N = w_{N-1} + \Delta w, \tag{U}$$

where Δw is a given strain increment. As a consequence of the solution of (U), the new state (w^{irr}_N, w^{e}_N, r_N and A_N) is obtained. As previously described, $(\hat{\mathrm{P}}^H_\Delta)$ is solved iteratively using Newton's method. From the result of an increment (say K), the increment given in the generalised strain in (U) is obtained as

$$\Delta w = w({}^K u_N - u_{N-1}),$$

that is computed with respect to the displacement u_{N-1} corresponding to the last equilibrium (convergent) state.

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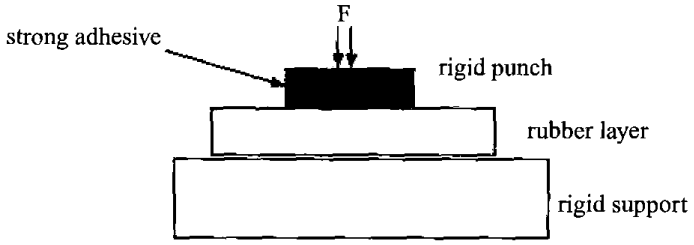

FIGURE 1. Adhesion between a rubber layer and a rigid support.

Table 1. *The material coefficients and parameters for the adhesive*

E (MPa)	ν	εh	lc (m)	p	η	γ (MPa)	β_1 (MPa)	K (MPa)
1	0.48	10^{-4}	4.10^{-4}	2	0.2	1	2	0.01

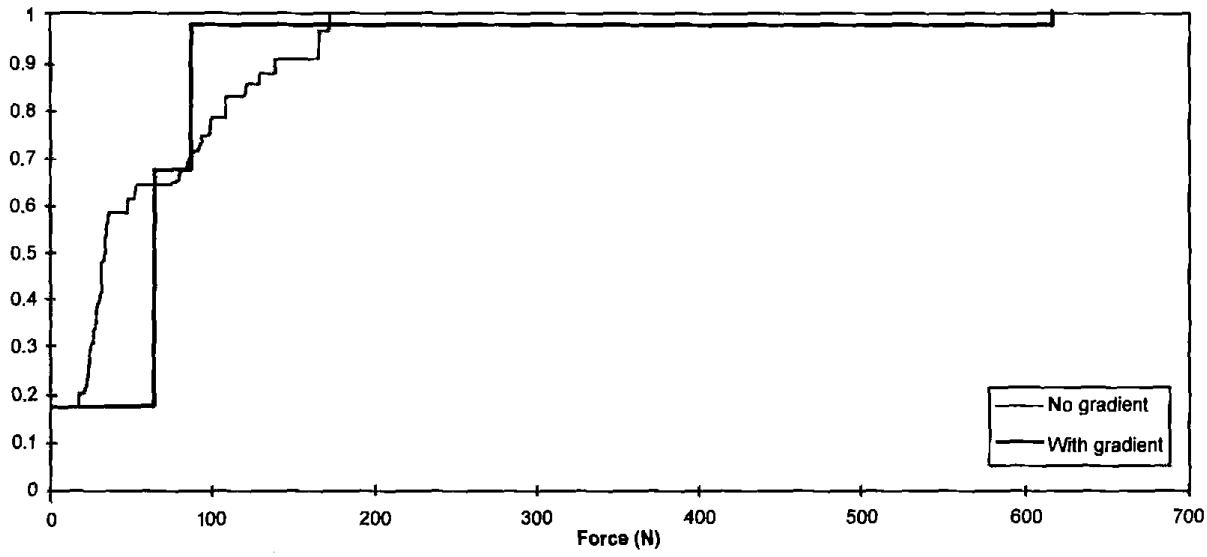

FIGURE 2. Evolution of the true contact area vs. the compressive effort.

For the resolution of (U) the operator splitting methodology is used. It has been used in plasticity problems [23] and in coupled plasticity-damage problems – [24–26]. The three-steps procedure proposed in Ju [2] has been used in the present computations.

7 Numerical examples

Two numerical examples involving two elastic adherends brought into contact through a thin adhesive material will be considered. In the first situation, it is assumed that adhesion is instantaneous and promoted by a gradual increase of the global compression effort. In the second case, the adhesion process is considered as time-dependent, and therefore the regularised form of the evolution equations will be used. In both examples, the adhesive itself is considered as a linear elastic material, so that only the adhesion process itself might be time-dependent.

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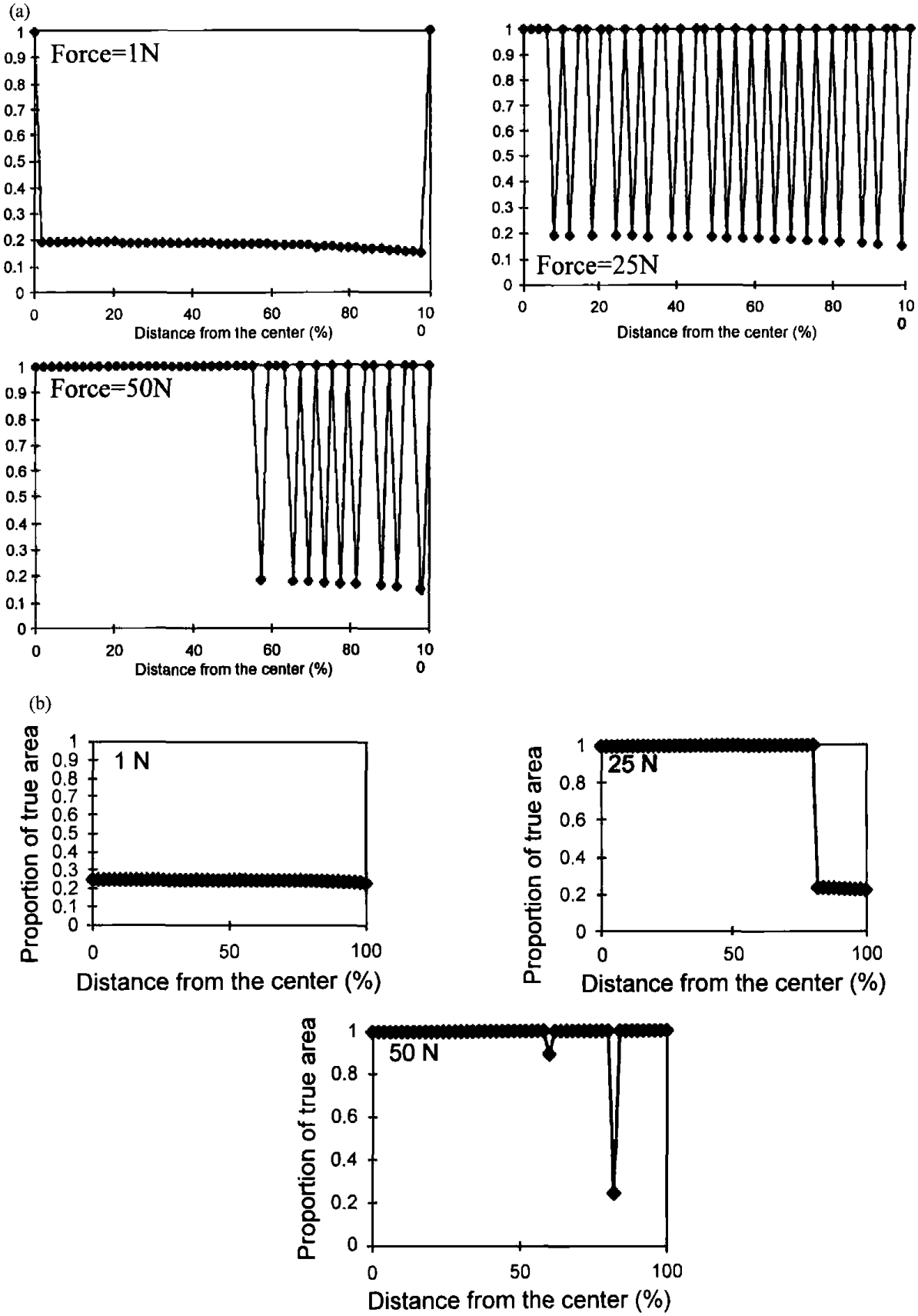

FIGURE 3. Profiles of adhesion along the interface (a) without gradient, (b) with gradient.

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The experimental system is the following (Fig. 1). A cylindrical elastomer block of thickness 2 mm, radius 5 mm and modulus 1 mpa is brought into contact with a plane rigid support. A rigid flat cylindrical punch (radius 5 mm) is supposed to be strongly bonded on the rubber layer. Note that the punch is chosen to be a rigid material in order to distribute the applied traction uniformly on the upper interface of the rubber layer. Considering the terminology described in the previous sections, the rubber layer and the rigid support are the two adherends.

The elastic modulus of the adhesive is taken as equal to that of the elastomeric layer (1 MPa), and its thickness $2\varepsilon h \approx 10^{-4}$ m. Poisson's coefficient of both adhesive and rubber are equal to 0.48, thus representing a quasi-incompressible behaviour. The punch is then pressed at a constant low speed (1 mm/min) on the rubber layer (so that the process can be considered as quasi-static), leading to an increase of adhesion at the interface rubber layer/support.

The parameter β is written as the product $\beta = \beta_1 l_c^p$, where l_c is an internal length scale which accounts for interactions between neighbouring points.

In Ganghoffer & Gent [27], the fracture of an elastomeric layer adhered to an aluminium plane baseplate has been measured by peeling a rubber strip at a speed equal to the true velocity of the crack front (about 100 mm/min), and found equal to 200 J/m^2, which corresponds to the parameter $2\varepsilon\beta h$, implying that $\beta = 10^6$ J/m^3.

We first assume that the adhesive has a non-dissipative behaviour (over the range of time length of the experiments), and the adhesive is given an elastic behaviour. Both the rubber layer and the adhesive are discretized using plane linear elements, according to the axial symmetry of the model. This implies that stresses are constant in each element, but they can vary between neighbouring elements. Since the rubber layer is initially in contact with its support, the displacements of the corresponding interface nodes are equal to 0 and the corresponding degrees of freedom are eliminated. The state of adhesion is assumed to be identical at the interfaces rubber/glue or glue/support, so that the model has been further simplified, assuming that the adhesive between rubber and the support can be described by only one line of nodes. A typical mesh of five layers of elements and 20 nodes on a radius has been used. In the initial state, the two adherends are brought into contact, and it is assumed that the adhesion variable A is equal to 0. Furthermore, the adhesion is supposed to be perfect at a material point at the interface when A reaches a critical value $A_{cr} = 0.99$.

The material data of the adhesive (the plastic properties are not considered here, since the adhesive is supposed to be given an elastic behaviour) are presented in Table 1.

From the computed contact area at each integration point, an average true contact area over the whole interfacial zone is further defined, according to the expression

$$A_m = \sum_{i=2}^{G} \pi \left(\left(i \frac{R}{G} \right)^2 - \left(\left(i-1 \right) \frac{R}{G} \right)^2 \right) / (\pi R^2).$$

Figure 2 presents the effect of the gradient of the adhesion variable. Clearly, there is a drastic evolution of adhesion when the gradient is not considered (the evolution laws are derived from the potential functions in which l_c is taken equal to 0), but this is mollified by the regularizing effect of the gradient.

We further represent the propagation of adhesion for different force levels (Fig. 3) first when the gradient of adhesion is not considered. In that case, it appears that the profile of

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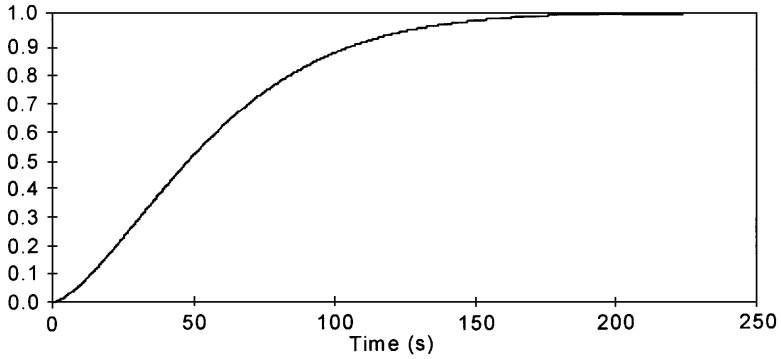

FIGURE 4. Time evolution of the contact area for a constant load.

adhesion is very chaotic (when the load is 20N, there is an alternance of nodes completely adhered followed by nodes which partially adhere). When the load increases (80N), the adhesion front is clearly observed (about half of the nodes from the centre are fully adhered), but the same chaotic behaviour is observed for the remaining nodes. Notice that all nodes are partially adhered for a low load level (1N), which physically corresponds to an avalanche effect (adhesion occurs on the whole geometrical contact surface).

Again, the effect of the gradient of the adhesion variable is clearly a regularizing effect (see Fig. 3(b)).

In the second example, a compressive effort of 50N is applied to the punch, and the adhesion process is assumed to be governed by the diffusion of polymeric chains (from the elastomer to the support), which is a long time process. The parameter β is supposed to be given by the following function of time [4, 13]: $\beta = \beta_{\max}\sqrt{t/\tau}$, with $\beta_{\max} = 6.10^6$ MPa and the relaxation time $\tau = 200$ s. The values of the other parameters are kept unchanged (Table 1). However, in that case, the gradient of the adhesion variable is not considered. The resulting contact area is seen to increase with time in a sigmoïdal way (Fig. 4).

8. Conclusion

A mechanical model of the adhesion between two elastic solids has been established, considering that the contact occurs through a thin third elastoplastic body. The strength of adhesion between both solids has been quantified via an internal variable, representing the proportion of microvoids at the interface between the solids, the evolution of which is coupled to the local state of stress within the framework of continuum thermodynamics. Numerical simulations performed on elastic layers being adhered to a rigid baseplate have shown a quasi-linear variation of the true contact area versus the compressive effort. When a time-dependent adhesion process is considered, the evolution vs. time of the true contact area has been obtained. The mechanical model proposed in this work has been further validated by experiments. Very few papers in the literature in fact deal with the measurements of the contact area in plane conditions. The work [28] on the wringing of gauge blocks on glass indicates an area of contact that linearly increases with the wringing torque. These results are in good agreement with the predictions of the present simulation.

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