SOME PROPERTIES OF THE EIGENFUNCTIONS OF THE LAPLACE-OPERATOR ON RIEMANNIAN MANIFOLDS

S. MINAKSHISUNDARAM AND Å. PLEIJEL

Introduction. Let V be a connected, compact, differentiable Riemannian manifold. If V is not closed we denote its boundary by S. In terms of local coordinates (x^i), $i = 1, 2, \ldots N$, the line-element dr is given by[1]

$$dr^2 = g_{ik}(x^1, x^2, \ldots x^N)\, dx^i dx^k$$

where $g_{ik}(x^1, x^2, \ldots x^N)$ are the components of the metric tensor on V We denote by Δ the Beltrami-Laplace-Operator

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ik} \frac{\partial u}{\partial x^k}\right)$$

and we consider on V the differential equation

$$\Delta u + \lambda u = 0. \tag{1}$$

If V is closed this equation will in general have an infinite number of eigenvalues $\lambda = \lambda_m$, $m = 1, 2, \ldots$, and corresponding eigenfunctions $\phi_m(P)$ where P is a point in V. When V has a boundary we have to consider in addition to (1), certain boundary-conditions in order to define eigenvalue-problems. We consider the following conditions, either

$$u = 0 \text{ on } S, \tag{2}$$

or

$$\frac{\partial u}{\partial n} = 0 \text{ on } S, \tag{3}$$

where $\frac{\partial}{\partial n}$ denotes a differentiation in the direction of the normal of S. Eigenvalues and eigenfunctions shall be denoted in the same way as was indicated in the case of a closed manifold. We assume in all cases the eigenvalues to have been arranged in non-decreasing order of magnitude and the eigenfunctions to form a complete *orthonormal* set

$$\int_V \phi_i(P)\phi_k(P)dV = \delta_{ik}$$

$(dV = \sqrt{g}dx^1dx^2 \ldots dx^N)$. In the problem with a closed manifold and in (1), (3) the value $\lambda_0 = 0$ is a simple eigenvalue $(\lambda_1 > 0)$ with the corresponding eigenfunction equal to the constant $\frac{1}{\sqrt{V}}$ where V denotes the volume of the manifold. In the problem (1), (2) we have $\lambda_0 > 0$.

Received July 4, 1948.

[1]We use the usual notations of tensor-calculus (g is the determinant of the covariant metric tensor g_{ik}).

We always assume V together with its boundary (if this exists) to be sufficiently regular so that those theorems from the theory of eigenvalue-problems which are required are valid.

The aim of this note is to study the analytic continuations in the s-plane of the Dirichlet's series (summation from $m = 0$ to $+\infty$ or from $m = 1$ to $+\infty$ according as $\lambda_0 >$ or $= 0$)

$$\sum \frac{\phi_m(P)\phi_m(Q)}{\lambda_m{}^s}, \tag{4}$$

and

$$\sum \frac{\phi_m{}^2(P)}{\lambda_m{}^s}, \tag{5}$$

where if V has a boundary, P and Q shall be interior points of V. In the case where V is a bounded two-dimensional Euclidean domain, the series (5) was first studied by Carleman [1]. Later in the case where V is a bounded Euclidean domain of arbitrary dimension N it was shown by Minakshisundaram [7] by a method different from Carleman's that (4) is an entire function of s with zeros at negative integers and that (5) is a meromorphic function with simple pole at $s = \frac{1}{2}N$ and zeros at negative integers. The method here developed is a generalization of Carleman's.

Even though our results are valid to a certain extent under less restrictive regularity conditions it is convenient to state them here for an analytic manifold V. If $\lambda_0 > 0$ (the formulation of the results is only slightly different in the case when $\lambda_0 = 0$) we find *that both the series* (4), (5) *can be continued arbitrarily far to the left of their abscissas of convergence. The continuation of* (4) *is an entire function with zeros at non-positive integers while* (5) *represents a function analytic except for simple poles at*

$$s = \tfrac{1}{2}N - \nu,\ \nu = 0, 1, 2, \ldots . \qquad \textit{if } N \textit{ is odd},$$

and at

$$s = \tfrac{1}{2}N,\ \tfrac{1}{2}N - 1,\ \tfrac{1}{2}N - 2, \ldots 2, 1 \qquad \textit{if } N \textit{ is even}.$$

The residue at the poles can be determined in terms of the g_{ik}. If N is odd the function defined by (5) *has zeros at non-positive integers and if N is even its values in these points can be explicitly determined from the metric tensor of V. By Ikehara's theorem* (*see* [14], *p.* 44) *we obtain as a corollary the relation*

$$\sum_{\lambda_m < T} \phi_m{}^2(P) \sim \frac{T^{N/2}}{(2\sqrt{\pi})^N \Gamma\left(\dfrac{N}{2} + 1\right)},$$

where the sign $\sim$ indicates that the quotient of both the sides tends to 1 *when T tends to $+\infty$; see* [1].

In the case of a closed manifold the series

$$\sum_{m=1}^{\infty} \lambda_m{}^{-s} \tag{6}$$

is easily seen to have properties similar to those stated for (5) and by the help of Ikehara's theorem we obtain immediately the asymptotic distribution of the eigenvalues. In the case when V has a boundary our method does not give such complete results concerning (6). It is possible by generalizing Carleman's method to deduce the asymptotic eigenvalue-distribution also in this case, but it seems as if this could be done more easily by already available methods (see [3] and [13]). The analytic continuation of (4), (5) and (6) in the case of a sphere was previously studied by Minakshisundaram [8].

1. Construction of a parametrix. We introduce normal coordinates in the neighbourhood of an inner point P of V. If r_{PQ} denotes the geodesic distance from P to Q and $\frac{d}{dr}$ differentiation along a geodesic from P, the normal coordinates (y^i) of Q are defined by

$$y^i = r_{PQ}\left(\frac{dx^i}{dr}\right)_{Q=P}.$$

If Φ is a function of $r = r_{PQ}$ only and U is an arbitrary function we observe that

$$\Delta\Phi U = U\left(\frac{d^2\Phi}{dr^2} + \frac{N-1}{r}\frac{d\Phi}{dr} + \frac{d\log\sqrt{g}}{dr}\frac{d\Phi}{dr}\right) + 2\frac{dU}{dr}\frac{d\Phi}{dr} + \Phi\Delta U \tag{7}$$

on using the well-known formulae

$$r_{PQ}^2 = g_{ik}(P)y^iy^k,$$
$$g_{ik}(Q)y^k = g_{ik}(P)y^k,$$

where the fundamental tensor is determined with respect to the normal coordinates and (y^i) are the coordinates of the point Q.

We define in a neighbourhood of P

$$H_n(P, Q; t) = \frac{1}{(2\sqrt{\pi})^N} e^{-\frac{r^2}{4t}} t^{-\frac{N}{2}} (U_0 + U_1 t + \dots U_n t^n)$$

by choosing $U_\nu(P, Q)$, $\nu = 0, 1, 2, \dots n$, independent of t and solutions of the differential equations $(U_{-1} \equiv 0)$

$$r\frac{dU_\nu}{dr} + \frac{r}{2}\frac{d\log\sqrt{g}}{dr}U_\nu + \nu U_\nu = \Delta U_{\nu-1}.$$

The functions $U_\nu(P, Q)$ are uniquely determined by the conditions that they shall be finite for $P = Q$ and by the normalizing condition $U_0(P, P) = 1$. It is apparent that the choice of the integer n is limited by the regularity of V. *However, we shall assume V so regular that n can be chosen* $> \frac{1}{2}N - 2$ (see Sec. **2**). We find

$$U_0(P, Q) = \left(\frac{g(P)}{g(Q)}\right)^{\frac{1}{4}}$$

and for $\nu > 0$

$$U_\nu(P, Q) = \frac{U_0(P, Q)}{r_{PQ}^\nu}\int_P^Q \frac{r_{P\Pi}^{\nu-1}\Delta_\Pi U_{\nu-1}(P, \Pi)}{U_0(P, \Pi)}\, dr_{P\Pi}.$$

By the help of (7) we find, on account of the definition of U_ν, that

$$(8) \qquad \left(\Delta - \frac{\partial}{\partial t}\right) H_n = \frac{\Delta U_n}{(2\sqrt{\pi})^N} e^{-\frac{r^2}{4t}} t^{-\frac{N}{2}+n}.$$

For $Q = P, t = 0$ the singularity of $H_n(P, Q; t)$ coincides with the singularity of a fundamental solution of the heat-equation

$$\Delta u - \frac{\partial u}{\partial t} = 0.$$

$H_n(P, Q; t)$ is a *parametrix* of this equation.

By use of a Laplace-transformation we obtain from $H_n(P, Q; t)$ the function

$$K_n(P, Q; -\xi) = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{n} U_\nu \int_0^\infty e^{-\xi t - \frac{r^2}{4t}} t^{-\frac{N}{2}+\nu} dt.$$

The singularity of this function for $Q = P$ coincides with the singularity of a fundamental solution of the equation

$$(9) \qquad \Delta u - \xi u = 0.$$

From (8) it follows that

$$(10) \qquad (\Delta - \xi) K_n(P, Q; -\xi) = \frac{\Delta U_n}{(2\sqrt{\pi})^N} \int_0^\infty e^{-\xi t - \frac{r^2}{4t}} t^{-\frac{N}{2}+n} dt.$$

The function $K_n(P, Q; -\xi)$ is a *parametrix* of (9).

The construction of H_n (and K_n) is analogous to Hadamard's construction of a fundamental solution of the General Wave Equation (see [6]). But by comparing our construction with Hadamard's we see that Hadamard's proof of the convergence of the infinite series he considers cannot be used for the infinite series we obtain from $H_n(P, Q; t)$ by letting n tend to infinity. The same is seen to be true *a fortiori* for the infinite series obtained from

$$K_n(P, Q; -\xi).$$

As a *parametrix in the large* we consider

$$\Gamma_n(P, Q; -\xi) = \eta_R(r_{PQ}) K_n(P, Q; -\xi)$$

where $\eta_R(r)$ is a continuous function of r satisfying

$$\eta_R(r) = \begin{cases} 1 \text{ when } r \leq \dfrac{R}{2}, \\ 0 \text{ when } r \geq R, \end{cases}$$

and having continuous derivatives of order one and two. In the interval $\frac{1}{2}R \leq r \leq R$ the function $\eta_R(r)$ can be chosen as a polynomial satisfying inequalities of the form

$$\left|\eta_R\right| \leq 1, \quad \left|\frac{d\eta_R}{dr}\right| \leq \frac{\text{const.}}{R}, \quad \left|\frac{d^2\eta_R}{dr^2}\right| \leq \frac{\text{const.}}{R^2}.$$

R shall be chosen so small that the geodesic sphere round P with radius R is contained in the neighbourhood of P where the construction of $K_n(P, Q; -\xi)$ is valid. In the case of a closed manifold V we can choose R independent of P.

In the case of a manifold with a boundary the choice of R must depend on the distance from P to this boundary.

2. The Green's function. With the aid of the parametrix $\Gamma_n(P, Q; -\xi)$ we may express the Green's function of (9) in the form

$$G(P, Q; -\xi) = \Gamma_n(P, Q; -\xi) - \gamma_n(P, Q; -\xi). \tag{11}$$

The integer n shall be chosen so large, $n > \frac{1}{2}N - 2$, that all singularities of $G(P, Q; -\xi)$ are contained in $\Gamma_n(P, Q; -\xi)$ and $\gamma_n(P, P; -\xi)$ is finite. As a function of the point Q the "regular part" of the Green's function, viz., $\gamma_n(P, Q; -\xi)$ satisfies the equation

$$(\Delta - \xi)_Q \gamma_n(P, Q; -\xi) = (\Delta - \xi)_Q \Gamma_n(P, Q; -\xi).$$

If V has a boundary, $G(P, Q; -\xi)$ shall satisfy the prescribed boundary condition (2) or (3). On account of the vanishing of Γ_n and $\frac{\partial \Gamma_n}{\partial n}$ on S the function $\gamma_n(P, Q; -\xi)$ satisfies the same boundary condition as the Green's function itself.

Now $\gamma_n(P, Q; -\xi)$ can be obtained as a solution of the variational problem: *to search for the minimum of* (ξ *is supposed to be real and positive*)

$$E(u; \Gamma_n) = \int_V \left(g^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k} + \xi u^2 + 2Fu \right) dV$$

where $F(Q) = (\Delta - \xi)_Q \Gamma_n(P, Q; -\xi)$. The admissible functions u are assumed to be continuous with piece-wise, continuous, first-order derivatives in the open kernel of V. The integral

$$\int_V \left(g^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k} + \xi u^2 \right) dV$$

shall be finite. If the boundary condition $u = 0$ on S is considered, the admissible functions shall also satisfy this condition (see [4], p. 482).

The minimizing function $\gamma_n(P, Q; -\xi)$ *satisfies the equation* (*see* [10])

$$\gamma_n(P, P; -\xi) = E(\gamma_n; \Gamma_n) - \int_V \Gamma_n(\Delta\Gamma_n - \xi\Gamma_n) dV \tag{12}$$

from which we observe that on the one hand

$$\gamma_n(P, P; -\xi) \leq - \int_V \Gamma_n(\Delta\Gamma_n - \xi\Gamma_n)\, dV,$$

and on the other

$$\gamma_n(P, P; -\xi) \geq - \frac{1}{\xi} \int_V F^2 dV - \int_V \Gamma_n(\Delta\Gamma_n - \xi\Gamma_n)\, dV$$

$$= - \frac{1}{\xi} \int_V (\Delta\Gamma_n - \xi\Gamma_n)^2 dy - \int_V \Gamma_n(\Delta\Gamma_n - \xi\Gamma_n)\, dV.$$

The first inequality follows from the fact that $u \equiv 0$ is an admissible function in our minimum-problem, the second is obtained by forming a complete square under the integral-sign in $E(u; \Gamma_n)$. Thus an estimate for $\gamma_n(P, P; -\xi)$ for large positive values of ξ can be deduced from the estimate for

$$\int_V \Gamma_n(\Delta\Gamma_n - \xi\Gamma_n)\, dV \tag{13}$$

and

$$\frac{1}{\xi}\int_V (\Delta\Gamma_n - \xi\Gamma_n)^2 dV. \tag{14}$$

By help of the estimations for (13), (14) *obtained in the next paragraph we find*

$$|\gamma_n(P, P; -\xi)| \leq \text{const.}\ \xi^{\frac{N}{2}-n-2} \tag{15}$$

where the constant depends on R.

3. Auxiliary estimations. On account of the inequalities

$$|\eta_R| \leq 1,\ \frac{1}{2}\left(\xi t + \frac{r^2}{4t}\right) \geq \frac{r\sqrt{\xi}}{2}$$

and

$$C^{-\frac{\xi t}{2}} \sum_{\nu=0}^{n} |U_\nu|\, t^\nu \leq \text{const.}, \ (\xi \geq \xi_0 > 0), \tag{16}$$

it follows when $N > 2$ that

$$|\Gamma_n(P, Q; -\xi)| \leq \text{const.}\ C^{-\frac{r\sqrt{\xi}}{2}} \int_0^\infty C^{-\frac{r^2}{8t}}\, t^{-\frac{N}{2}} dt = \text{const.}\ \frac{C^{-\frac{r\sqrt{\xi}}{2}}}{r^{N-2}}\,. \tag{17}$$

When $N = 2$ we use instead of (16) the inequality

$$C^{-\frac{\xi t}{2}} \sum_{\nu=0}^{n} |U_\nu|\, t^\nu \leq \text{const.}\ C^{-a\xi t},\ (\xi \geq \xi_0 > 0;\ 0 < a < \tfrac{1}{2}),$$

and by the help of well-known properties of the Bessel-function (see [12], pp. 183, 80, 202)

$$K_0(z) = \frac{1}{2}\int_0^\infty e^{-\tau-\frac{z^2}{4\tau}}\, \tau^{-1} d\tau$$

for z tending to 0 and to $+\infty$ we obtain in this case

$$|\Gamma_n(P, Q; -\xi)| \leq \text{const.}\ e^{-\frac{r\sqrt{\xi}}{2}} \log(r\sqrt{\xi}). \tag{18}$$

For (10) we see that for $r \leq \frac{1}{2}R$

$$|\Delta\Gamma_n - \xi\Gamma_n| \leq \text{const.}\ e^{-\frac{r\sqrt{\xi}}{2}} \int_0^\infty e^{-\frac{\xi t}{2}}\, t^{-\frac{N}{2}+n} dt = \text{const.}\ \xi^{\frac{N}{2}-n-1}\, e^{-\frac{r\sqrt{\xi}}{2}} \tag{19}$$

provided $-\frac{1}{2}N + n > -1$. When $-\frac{1}{2}N + n \leq -1$ we use the method which gave us (17) and (18) and deduce in this way:

(20) $$|\Delta\Gamma_n - \xi\Gamma_n| \leq \text{const.}\, r^{-N+2n+2}\, e^{-\frac{r\sqrt{\xi}}{2}} \text{ when } -\tfrac{1}{2}N + n < -1,$$

and

(21) $$|\Delta\Gamma_n - \xi\Gamma_n| \leq \text{const.}\, e^{-\frac{r\sqrt{\xi}}{2}} \log (r\sqrt{\xi}) \text{ when } -\tfrac{1}{2}N + n = -1.$$

In the interval $\frac{1}{2}R \leq r \leq R$ we find inequalities for $|\Delta\Gamma_n - \xi\Gamma_n|$ in which the majorizing expressions contain the factor $e^{-\frac{R\sqrt{\xi}}{2}}$. This fact and the inequality $r \geq \frac{1}{2}R$ make it possible to give to these inequalities the same forms as (19), (20), (21) the constants being now dependent on R.

By introducing (17) and (19) in the expressions (13) and (14) we find

$$\left| \int_V \Gamma_n(\Delta\Gamma_n - \xi\Gamma_n)\, dV \right| \leq \text{const.}\ \xi^{\frac{N}{2}-n-2},$$

and

(22) $$\left| \frac{1}{\xi} \int_V (\Delta\Gamma_n - \xi\Gamma_n)^2 dV \right| \leq \text{const.}\ \xi^{\frac{N}{2}-2n-3}$$

on account of the fact that dV can approximately be substituted by the Euclidean volume-element $r^{N-1}dr\, d\Omega$. Observing (see the beginning of Sec. **2**) that $-N + 2n + 4$ is a positive integer it is easily seen that all combinations of the inequalities (17), (18) with (19), (20), (21) give the same result (22).

4. A fundamental formula. Starting from the relation

$$G(P, Q; -\xi) - G(P, Q; -\xi_0) = -(\xi - \xi_0) \int_V G(P, \Pi\,; -\xi)\, G(\Pi, G; -\xi_0)\, dV_\Pi$$

we obtain by repeated application

(23) $$G(P, Q; -\xi) - \sum_{\nu=0}^{p} (-1)^\nu (\xi - \xi_0)^\nu G^{(\nu)}(P, Q; -\xi_0)$$
$$= (-1)^{p+1}(\xi - \xi_0)^{p+1} \int_V G(P, \Pi; -\xi)\, G^{(p)}(\Pi, Q; -\xi_0)\, dV_\Pi,$$

where $$G^{(0)}(P, Q; -\xi_0) = G(P, Q; -\xi_0),$$
$$G^{(k+1)}(P, Q; -\xi_0) = \int_V G(P, \Pi; -\xi_0)\, G^{(k)}(\Pi, Q; -\xi_0)\, dV_\Pi .$$

We assume without detailed discussion that the integral

$$\int_V (G^{(q)}(P, Q; -\xi_0))^2\, dV_Q$$

is finite when[2] $q \geq \left[\frac{N}{4}\right]$ and P is an interior point of V. In the case when V is a bounded Euclidean domain this follows from the inequality

$$|G(P, Q; -\xi_0)| \leq \frac{\text{const.}}{r_{PQ}^{N-2}}.$$

[2] $[a]$ = integer, $a - 1 < [a] \leq a$.

https://doi.org/10.4153/CJM-1949-021-5 Published online by Cambridge University Press

For other cases we may refer to the works of Giraud [5] and de Rham [11] (for the case when V is closed). It follows that for $p \geq 2\left[\frac{N}{4}\right]$ the right side of (23) can be developed into a convergent series of eigenfunctions, and we have

$$(24) \qquad \Gamma_n(P, Q; -\xi) - \gamma_n(P, Q; -\xi) - \sum_{\nu=0}^{p} (-1)^\nu (\xi - \xi_0)^\nu G^{(\nu)}(P, Q; -\xi_0)$$
$$= (-1)^{p+1} (\xi - \xi_0)^{p+1} \sum_{m=0}^{\infty} \frac{\phi_m(P)\phi_m(Q)}{(\lambda_m + \xi)(\lambda_m + \xi_0)^{p+1}}.$$

Since the right side of this relation is finite for $Q = P$ the singular parts of $\Gamma_n(P, Q; -\xi)$ must cancel the singular parts of the sum

$$(25) \qquad \sum_{\nu=0}^{p} (-1)^\nu (\xi - \xi_0)^\nu G^{(\nu)}(P, Q; -\xi_0).$$

What remains after a transition to the limit, $Q \to P$, is the "finite part" of $\Gamma_n(P, Q; -\xi)$ for $Q = P$ minus $\gamma_n(P, P; -\xi)$ minus a polynomial in ξ of degree p contributed by the finite part for $Q = P$ of the sum (25). In order to calculate the finite part of $\Gamma_n(P, Q; -\xi)$ for $Q = P$ we have to consider the finite contributions for $r = 0$ from

$$\int_0^\infty e^{-\xi t - \frac{r^2}{4t}} t^{-\frac{N}{2}+\nu} dt = 2^{\frac{N}{2}-\nu} \left(\frac{\sqrt{\xi}}{r}\right)^{\frac{N}{2}-\nu-1} K_{\frac{N}{2}-\nu-1}(r\sqrt{\xi}),$$

where $K_\zeta(z)$ denotes the Bessel K-function of order ζ (see [12], p. 183). If N is odd, $2\zeta = N - 2\nu - 2$ is odd and ([12], p. 78)

$$K_\zeta(z) = \frac{\pi}{2 \sin \pi\zeta} (I_{-\zeta}(z) - I_\zeta(z)).$$

By this formula and by well-known developments of the Bessel functions $I_{-\zeta}(z)$ and $I_\zeta(z)$ we find that the finite contribution from $\Gamma_n(P, Q; -\xi)$ for $Q = P$ is, when N is odd

$$(26) \qquad M_n(\xi, P) = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{n} \Gamma\left(-\frac{N}{2} + \nu + 1\right) \xi^{\frac{N}{2}-\nu-1} U_\nu(P, P).$$

When N is even we make use of similar considerations but now based on the formula (ζ is a positive integer; see [12], pp. 79 and 80)

$$K_{\pm\zeta}(z) = \frac{1}{2}\left(\frac{2}{z}\right)^\zeta \sum_{m=0}^{\zeta-1} (-1)^m \frac{\Gamma(\zeta - m)}{m!}\left(\frac{z}{2}\right)^{2m} + (-1)^{\zeta+1}\left(\frac{z}{2}\right)^\zeta$$
$$\cdot \sum_{m=\zeta}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma(\zeta + m + 1)}\left(\log\frac{z}{2} - \frac{1}{2}\psi(m+1) - \frac{1}{2}\psi(\zeta + m + 1)\right)$$

where

$$\psi(k+1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \gamma,$$

$$\psi(1) = -\gamma,$$

$$\gamma = \text{Euler's constant.}$$

The finite contribution from $\Gamma_n(P, Q; -\xi)$ for $Q = P$ obtained in this case is

$$(27) \quad M_n(\xi, P) = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{\frac{N}{2}-1} (-1)^{\frac{N}{2}-\nu} \frac{\xi^{\frac{N}{2}-\nu-1}}{\Gamma\left(\frac{N}{2}-\nu\right)} \left(\log \xi - 2 \log 2 - \psi(1) - \psi\left(\frac{N}{2}-\nu\right)\right) + \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=\frac{N}{2}}^{n} \Gamma\left(-\frac{N}{2}+\nu+1\right) \xi^{\frac{N}{2}-\nu-1}$$

So we have from (24)

$$(28) \quad M_n(\xi, P) - \gamma_n(P, P; -\xi) + A_p(\xi, \xi_0, P) = (-1)^{p+1} (\xi - \xi_0)^{p+1} \sum_{m=0}^{\infty} \frac{\phi_m^2(P)}{(\lambda_m + \xi)(\lambda_m + \xi_0)^{p+1}},$$

where $A_p(\xi, \xi_0, P)$ is a polynomial of degree p in ξ with coefficients depending on ξ_0 and P and where $M_n(\xi, P)$ is equal to (26) or (27) according as N is odd or even. In the case when $\lambda_0 > 0$ we obtain by performing the transition to the limit $\xi_0 \rightarrow 0$ that

$$(29) \quad M_n(\xi, P) - \gamma_n(P, P; -\xi) + \sum_{\nu=0}^{p} A_\nu(P) \xi^\nu = (-1)^{p+1} (\xi - \xi_0)^{p+1} \sum_{m=0}^{\infty} \frac{\phi_m^2(P)}{(\lambda_m + \xi) \lambda_m^{p+1}}.$$

When $\lambda_0 = 0$ we transfer the first term of the series on the right in (28) to the left, take it together with $A_n(\xi, \xi_0, P)$ and note that since the other terms of the equation remain finite for $\xi_0 \rightarrow 0$, the value of

$$A_p(\xi, \xi_0, P) - (-1)^{p+1} (\xi - \xi_0)^{p+1} \frac{\phi_0^2(P)}{\xi \xi_0^{p+1}}$$

remains finite and gives an expression of the form

$$\sum_{\nu=0}^{p} A_\nu(P) \xi^\nu - \frac{1}{V\xi}$$

V being the volume of the manifold. We obtain in the case when $\lambda_0 = 0$ the formula

(30)
$$M_n(\xi, P) - \gamma_n(P, P; -\xi) + \sum_{\nu=0}^{p} A_\nu(P)\xi^\nu - \frac{1}{V\xi}$$
$$= (-1)^{p+1}\xi^{p+1} \sum_{m=1}^{\infty} \frac{\phi_m^2(P)}{(\lambda_m + \xi)\lambda_m^{p+1}}.$$

5. Analytic continuation of (5). We suppose first $\lambda_0 > 0$ and multiply both sides of (29) by
$$\frac{1}{2\pi i(-\xi)^s} = \frac{1}{2\pi i} e^{-s \log|\xi| - is(\arg \xi - \pi)}$$
and integrate along the following contour in the complex ξ-plane with a cut along the real positive axis. From $+\infty$ to a (a real, $0 < a < \lambda_0$) along the lower part of the cut, from a to a along a circle round the origin and then from a to $+\infty$ along the upper part of the cut. We obtain in this way when the real part $\Re s$ of s is sufficiently large

(31)
$$\sum_{m=0}^{\infty} \frac{\phi_m^2(P)}{\lambda_m^s} = \frac{\sin \pi s}{\pi}\left(- \int_a^\infty \frac{M_n(\xi, P)}{\xi^s} d\xi + \int_a^\infty \frac{\gamma_n(P, P; -\xi)}{\xi^s} d\xi + \sum_{\nu=0}^{p} \frac{A_\nu(P)a^{\nu-s+1}}{s-\nu-1}\right) - \frac{a^{1-s}e^{is\pi}}{2\pi}\int_0^{2\pi} F(ae^{i\theta}, P)\, e^{i\theta(1-s)}\, d\theta$$

where
$$F(\xi, P) = (-1)^{p+1}\xi^{p+1} \sum_{m=0}^{\infty} \frac{\phi_m^2(P)}{(\lambda_m + \xi)\lambda_m^{p+1}}.$$

The last integral in (31) is an entire function of s vanishing when s equals a non-positive integer and so is the expression
$$\frac{\sin \pi s}{\pi} \sum_{\nu=0}^{p} \frac{A_\nu(P)a^{\nu-s+1}}{s-\nu-1}.$$

On account of our estimation for $\gamma_n(P, P; -\xi)$, (15), the integral
$$\int_a^\infty \frac{\gamma_n(P, P; -\xi)}{\xi^s} d\xi$$
is a regular function in the half-plane $\Re s > \frac{1}{2}N - n - 2$. In the case when N is odd the first integral in (31) is equal to the following expression (see(26))

(32)
$$-\int_a^\infty \frac{M_n(\xi, P)}{\xi^s} d\xi = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{n} \Gamma\left(-\frac{N}{2} + \nu + 1\right) \frac{a^{\frac{N}{2}-\nu-s}}{s - \frac{N}{2} + \nu} U_\nu(P, P)$$

and in the case when N is even (see (27))

$$
-\int_a^\infty \frac{M_n(\xi, P)}{\xi^s}\, d\xi = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{\frac{N}{2}-1} \frac{(-1)^{\frac{N}{2}-\nu} a^{\frac{N}{2}-\nu-s}}{\Gamma\left(\frac{N}{2}-\nu\right)\left(s-\frac{N}{2}+\nu\right)}
$$

$$
(33) \qquad \cdot\left(\log a + \frac{1}{s-\frac{N}{2}+\nu} - 2\log 2 - \psi(1) - \psi\left(\frac{N}{2}-\nu\right)\right) U_\nu(P, P)
$$

$$
+ \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=\frac{N}{2}}^{n} \Gamma\left(-\frac{N}{2}+\nu+1\right) \frac{a^{\frac{N}{2}-\nu-s}}{s-\frac{N}{2}+\nu} U_\nu(P, P).
$$

All taken together we obtain from (31), (32), (33) the following

THEOREM. *The Dirichlet's series*

$$
(34) \qquad \sum_{m=0}^{\infty} \frac{\phi_m^2(P)}{\lambda_m^s}
$$

can be continued to the left of its abscissa of absolute convergence. The function $\zeta(s, P)$ thus obtained can be written

$$
\zeta(s, P) = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{n} \frac{U_\nu(P, P)}{\Gamma\left(\frac{N}{2}-\nu\right)\left(s-\frac{N}{2}+\nu\right)} + R_n(s, P) \text{ if } N \text{ is odd,}
$$

and

$$
\zeta(s, P) = \frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{\frac{N}{2}-1} \frac{U_\nu(P, P)}{\Gamma\left(\frac{N}{2}-\nu\right)\left(s-\frac{N}{2}+\nu\right)} + R_n(s, P) \text{ if } N \text{ is even,}
$$

where in both cases $R_n(s, P)$ is regular in the half-plane $\Re s > \frac{1}{2}N - n - 2$. When s is equal to a non-positive integer ($> \frac{1}{2}N - n - 2$) we have

$$
\zeta(s, P) = 0 \text{ in the case when } N \text{ is odd,}
$$

and

$$
\zeta(s, P) = \frac{\Gamma(1-s)}{(2\sqrt{\pi})^N} U_{\frac{N}{2}-s}(P, P) \text{ when } N \text{ is even.}
$$

In the case when $\lambda_0 = 0$ we have to consider the series

$$
(35) \qquad \sum_{m=1}^{\infty} \frac{\phi_m^2(P)}{\lambda_m^s}
$$

instead of (34) and to use the relation (30) instead of (29). *We obtain essentially* *the same theorem but because of the term* $-\dfrac{1}{V\xi}$ *in* (30) *we arrive at the relations*

$$\zeta(0, P) = -\frac{1}{V} \text{ when } N \text{ is odd,}$$

and

$$\zeta(0, P) = \frac{U_{\frac{N}{2}}(P, P)}{(2\sqrt{\pi})^N} - \frac{1}{V} \text{ when } N \text{ is even,}$$

instead of the corresponding relation in the theorem (*we denote here by* $\zeta(s, P)$ *the analytic continuation of* (35)).

In so far as the Dirichlet's series (34) or (35) with positive terms can be continued to the left of its abscissa of absolute convergence with a simple pole at $s = \frac{1}{2}N$ and residue (we observe that $U_0(P, P) = 1$)

$$\frac{1}{(2\sqrt{\pi})^N \, \Gamma\left(\dfrac{N}{2}\right)}$$

we could apply Ikehara's theorem (see [18], p. 44) and obtain the following asymptotic distribution of the squares of the eigenfunctions as a

COROLLARY. $$\sum_{\lambda_m < T} \phi_m^2(P) \sim \frac{T^{\frac{N}{2}}}{(2\sqrt{\pi})^N \, \Gamma\left(\dfrac{N}{2} + 1\right)}.$$

6. Analytic continuation of (4). Let P and Q be two different (inner) points of V and let R be so chosen that the geodesic spheres round P and Q with radius R have no points in common. From the formula (24) we obtain when $\lambda_0 > 0$

$$(36) \qquad \Gamma_n(P, Q; -\xi) - \gamma_n(P, Q; -\xi) + \sum_{\nu=0}^{p} B_\nu(P, Q)\xi^\nu$$
$$= (-1)^{p+1}\xi^{p+1} \sum_{m=0}^{\infty} \frac{\phi_m(P)\phi_m(Q)}{(\lambda_m + \xi)\lambda_m^{p+1}}.$$

By choice of R the value of $\Gamma_n(P, Q; -\xi)$ is zero and we have only to estimate $\gamma_n(P, Q; -\xi)$ as a function of ξ for $\xi \to +\infty$. Corresponding to (12) we have

$$2\{\gamma_n(P, Q; -\xi) + \gamma_n(Q, P; -\xi)\}$$
$$= E(\gamma_n(P, \Pi; -\xi) + \gamma_n(Q, \Pi; -\xi) \,;\, \Gamma_n(P, \Pi; -\xi) + \Gamma_n(Q, \Pi; -\xi))$$
$$(37) \quad - E(\gamma_n(P, \Pi; -\xi) - \gamma_n(Q, \Pi; -\xi) \,;\, \Gamma_n(P, \Pi; -\xi) - \Gamma_n(Q, \Pi; -\xi))$$
$$+ 2\{\tilde{D}(\Gamma_n(P, \Pi; -\xi), \Gamma_n(Q, \Pi; -\xi)) + \tilde{D}(\Gamma_n(Q, \Pi; -\xi), \Gamma_n(P, \Pi; -\xi))\},$$

where

$$\tilde{D}(u, v) = -\int_V u(\Delta v - \xi v)\, dV,$$

and where Π is the point of integration. (37) can be deduced in a similar way

https://doi.org/10.4153/CJM-1949-021-5 Published online by Cambridge University Press

to (12); see [14]. The last expression on the right of (37) vanishes on account of our choice of R. When

$$u(\Pi) = \gamma_n(P, \Pi; -\xi) + \gamma_n(Q, \Pi; -\xi)$$

and

$$u(\Pi) = \gamma_n(P, \Pi; -\xi) - \gamma_n(Q, \Pi; -\xi),$$

the expressions

$$E(u; \Gamma_n(P, \Pi; -\xi) + \Gamma_n(Q, \Pi; -\xi))$$

and

$$E(u; \Gamma_n(P, \Pi; -\xi) - \Gamma_n(Q, \Pi; -\xi))$$

attain their minimum values. These minimum values can be estimated as in Secs. **2**—**3** and we find (it is easily seen that we need only the estimation for the expression (14))

$$(38) \qquad |\gamma_n(P, Q; -\xi) + \gamma_n(Q, P; -\xi)| \leq \text{const. } \xi^{\frac{N}{2}-2n-3}$$

Interchanging P and Q in (36) and adding we obtain, on using (38)

$$(39) \qquad (-1)^{p+1}\xi^{p+1} \sum_{m=0}^{\infty} \frac{\phi_m(P)\phi_m(Q)}{(\lambda_m + \xi)\lambda_m^{p+1}} = \sum_{\nu=0}^{p} A_\nu(P, Q)\xi^\nu + O(\xi^{\frac{N}{2}-2n-3})$$

From this relation it follows as in Sec. **5** *that the Dirichlet's series*

$$(40) \qquad \sum_{m=0}^{\infty} \frac{\phi_m(P)\phi_m(Q)}{\lambda_m{}^s}$$

which has a finite abscissa of convergence can be continued analytically to the left of this line and represents a regular function $\zeta(s, P, Q)$ *for* $\Re s > \frac{1}{2}N - 2n - 3$. $\zeta(s, P, Q) = 0$ *when* s *is equal to a non-positive integer* ($> \frac{1}{2}N - 2n - 3$).

When $\lambda_0 = 0$ we consider (40) but with summation from $m = 1$. Using instead of (36) the formula (see Sec. 4, formula (30))

$$\Gamma_n(P, Q; -\xi) - \gamma_n(P, Q; -\xi) + \sum_{\nu=0}^{p} B_\nu(P, Q)\xi^\nu - \frac{1}{V\xi}$$

$$= (-1)^{p+1}\xi^{p+1} \sum_{m=1}^{\infty} \frac{\phi_m(P)\phi_m(Q)}{(\lambda_m + \xi)\lambda_m^{p+1}}.$$

We find that the analytic continuation of

$$(41) \qquad \sum_{m=1}^{\infty} \frac{\phi_m(P)\phi_m(Q)}{\lambda_m{}^s}$$

has the same properties as the analytic continuation of (40) *with the only exception that the function* $\zeta(s, P, Q)$ *represented by* (41) *is not zero for* $s = 0$ *but has the value* $-\dfrac{1}{V}$.

7. The series $\sum_{m=1}^{\infty} \lambda_m^{-s}$ in the case of a closed manifold. We add a discussion of the series

$$\sum_{m=1}^{\infty} \lambda_m^{-s}$$

in the case of a closed manifold V. In this case R can be chosen independent of P and we can make use of the inequality (15) with a constant independent of R. Integrating (30) over V we have on account of this inequality

$$(42) \qquad (-1)^{p+1}\xi^{p+1} \sum_{m=1}^{\infty} \frac{1}{(\lambda_m + \xi)\lambda_m^{p+1}} = \int_V M_n(\xi, P) dV$$
$$+ \sum_{\nu=0}^{p} \xi^{\nu} \int_V A_\nu(P) dV - \frac{1}{\xi} + O(\xi^{\frac{N}{2}-n-2}).$$

Using the same method as in Sec. **5** we obtain the

THEOREM. *If V is closed the Dirichlet's series*

$$\sum_{m=1}^{\infty} \lambda_m^{-s}$$

can be continued analytically to the left of its abscissa of convergence and the function thus obtained can be written in the form

$$\frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{n} \frac{\int_V U_\nu(P, P)\, dV}{\Gamma\left(\frac{N}{2} - \nu\right)\left(s - \frac{N}{2} + \nu\right)} + R_n(s) \text{ when } N \text{ is odd,}$$

and

$$\frac{1}{(2\sqrt{\pi})^N} \sum_{\nu=0}^{\frac{N}{2}-1} \frac{\int_V U_\nu(P, P)\, dV}{\Gamma\left(\frac{N}{2} - \nu\right)\left(s - \frac{N}{2} + \nu\right)} + R_n(s) \text{ when } N \text{ is even.}$$

In both cases $R_n(s)$ is regular in the half-plane $\Re s > \frac{1}{2}N - n - 2$. For $s = 0$ the value of the analytic continuation is -1 if N is odd, and

$$\frac{1}{(2\sqrt{\pi})^N} \int_V U_{\frac{N}{2}}(P, P)\, dV - 1$$

if N is even. For s equal to a negative integer the analytic continuation is zero when N is odd, and when N is even its value is

$$\frac{\Gamma(1-s)}{(2\sqrt{\pi})^N} \int_V U_{\frac{N}{2}-s}(P, P)\, dV.$$

https://doi.org/10.4153/CJM-1949-021-5 Published online by Cambridge University Press

Ikehara's theorem gives the asymptotic distribution of the eigenvalues

$$N(\lambda_m < T) \sim \frac{VT^{\frac{N}{2}}}{(2\sqrt{\pi})^N \Gamma\left(\frac{N}{2}+1\right)}$$

where $N(\lambda_m < T)$ *denotes the number of eigenvalues* $< T$.

References

[1] T. Carleman, "Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes," *Skand. Matem. Kongress* (1934).

[2] T. Carleman, "Uber die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen," *Berichte Verhandl. Akad. Leipzig*, vol. 88 (1936).

[3] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik I* (Berlin, 1924).

[4] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik II* (Berlin, 1937).

[5] M. G. Giraud, "Généralisation des problèmes sur les opérations du type elliptique," *Bull. Sci. Math.*, Series 2, vol. 56 (1932).

[6] J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques* (Paris, 1932).

[7] S. Minakshisundaram, "A Generalization of Epstein Zeta Functions," will appear in this Journal.

[8] S. Minakshisundaram, "Zeta-functions on the Sphere," will appear in *J. Indian Math. Soc.*

[9] Å. Pleijel, "Propriétés asymptotiques des fonctions et valeurs propres de certains problèmes de vibrations," *Arkiv Mat., Astr. o. Fys.*, vol. 27 A, no. 13 (1940).

[10] Å. Pleijel, "Asymptotic Properties of the Eigenfunctions of Certain Boundary-Value Problems of Polar Type," will appear in *Amer. J. Math.*

[11] G. de Rham, "Sur la théorie des formes différentielles harmoniques," *Ann. Univ. Grenoble*, Sec. Sci. Math. Phys., (N.S.) vol. 22 (1946).

[12] G. M. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge, 1944).

[13] H. Weyl, "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)," *Math. Ann.*, vol. 71 (1911).

[14] N. Wiener, "Tauberian Theorems," *Ann. of Math.*, vol. 33 (1932).

Andhra University
Waltair, South India