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Multiple Solutions for a Class of Neumann Elliptic Problems on Compact Riemannian Manifolds with Boundary

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Abstract. We study a semilinear elliptic problem on a compact Riemannian manifold with boundary, subject to an inhomogeneous Neumann boundary condition. Under various hypotheses on the nonlinear terms, depending on their behaviour in the origin and infinity, we prove multiplicity of solutions by using variational arguments.

1 Introduction

Let (M, g) be a smooth, connected, compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. For $\lambda > 0$ and $\mu > 0$, we consider the following inhomogeneous Neumann boundary value problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\triangle u + k(x)u = \lambda K(x) f(u(x)) & \text{for } x \in M, \\ \dfrac{\partial u}{\partial n} = \mu D(x) h(u(x)) & \text{for } x \in \partial M, \end{cases}$$

where $k, K\colon M \to \mathbb{R}$, $D\colon \partial M \to \mathbb{R}$ are positive continuous functions, Δ denotes the Laplace-Beltrami operator in the metric g, $\frac{\partial}{\partial n}$ is the normal derivative with respect to the outward normal n on ∂M in the metric g.

Problems like $(P_{\lambda,\mu})$ arise in various contexts, motivated by certain physical phenomena; see for example [1, 6] and references therein. On the other hand, when $f(s) = |s|^{\frac{4}{n-2}} s$ and $g(s) = |s|^{\frac{2}{n-2}} s$, the problem of the existence of a positive solution for $(P_{\lambda,\mu})$ is equivalent to the classical problem of finding a conformal metric g' on M with the prescribed scalar curvature K on M and the mean curvature D on ∂M, see [3–5]. For the quasilinear extension, we refer the reader to [2].

The purpose of this paper is to provide multiple solutions for problem $(P_{\lambda,\mu})$ when the nonlinearities f and h have various growth conditions. Note that if $K(x)/k(x) = \lambda_0$ =constant on M and $\mathrm{Fix}(\lambda\lambda_0 f) \cap h^{-1}(0) \neq \varnothing$, then $(P_{\lambda,\mu})$ has at least one solution for every $\mu > 0$, where $\mathrm{Fix}(\lambda\lambda_0 f)$ is the fixed point set of the function $s \mapsto \lambda\lambda_0 f(s)$. Indeed, the constant function $u(x) = c \in \mathrm{Fix}(\lambda\lambda_0 f) \cap h^{-1}(0)$ verifies both equations in $(P_{\lambda,\mu})$. Clearly, we are interested not only in this particular case when K/k is constant on M. Due to this fact, we assume that the continuous function $f\colon \mathbb{R} \to \mathbb{R}$ verifies the following:

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(f_1) $\lim_{s\to 0} \frac{f(s)}{s} = 0$;
(f_2) $\lim_{|s|\to +\infty} \frac{f(s)}{s} = 0$;
(f_3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)dt$.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. For every $q \in [1, \frac{n}{n-2})$, we introduce the assumption

(h_q) $\sup_{s\in\mathbb{R}} \frac{|h(s)|}{1+|s|^q} < \infty$.

Theorem 1.1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function that fulfills the assumptions (f_1)–(f_3). Then, there exist a number σ and a non-degenerate compact interval $A \subset [0, +\infty)$ such that for every $\lambda \in A$ and every continuous function $h\colon \mathbb{R} \to \mathbb{R}$ fulfilling (h_q) for some $q \in [1, \frac{n}{n-2})$, there exists $\delta_{\lambda,h} > 0$ with the property that for each $\mu \in (0, \delta_{\lambda,h})$, the problem $(P_{\lambda,\mu})$ has at least three weak solutions which are in norm less than σ.*

We return to the case where K/k is constant on M in order to state our second result. For a fixed $\lambda > 0$, we assume the following.

(f_λ) $\frac{K(x)}{k(x)} = \lambda_0$ for every $x \in M$, and the set of all global minima of $t \mapsto \tilde{F}_\lambda(t) := \frac{1}{2}t^2 - \lambda\lambda_0 F(t)$ has at least $m \geq 2$ connected components.

In particular, (f_λ) implies that the function $t \mapsto \tilde{F}_\lambda(t)$ has at least $m-1$ local maxima. Thus, Card(Fix$(\lambda\lambda_0 f)$)$\geq 2m-1$. Therefore, if an element from Fix$(\lambda\lambda_0 f)$ belongs to $h^{-1}(0)$, it may be considered as a constant solution for problem $(P_{\lambda,\mu})$ for every $\mu > 0$.

Theorem 1.2 *Let $f, h\colon \mathbb{R} \to \mathbb{R}$ be two continuous functions that fulfill the assumptions (f_2), (f_λ) for some $\lambda > 0$ fixed, and (h_1), respectively. Then there exists a number $\delta_\lambda > 0$ such that for every $\mu \in (0, \delta_\lambda)$, problem $(P_{\lambda,\mu})$ has at least $m + 1$ distinct weak solutions.*

The proof of Theorems 1.1 and 1.2 are based on two recent results of Ricceri [7,8]. In the next section, we recall some notions and results that we will use in the sequel. In Sections 3 and 4 our main results are proved.

2 Preliminaries

We denote by $2^\star = 2n/(n-2)$ and $\overline{2}^\star = (2n-2)/(n-2)$ the critical Sobolev exponents for the embedding $W_1^2(M) \hookrightarrow L^{2^\star}(M)$ and the trace-embedding $W_1^2(M) \hookrightarrow L^{\overline{2}^\star}(\partial M)$, respectively. Here, and in the sequel, (M, g) is a compact Riemannian manifold with boundary and $W_1^2(M)$ is the standard Sobolev space equipped with the norm

$$\|u\| = \left(\int_M |\nabla u|^2 d\mu_g + \int_M u^2 d\mu_g \right)^{\frac{1}{2}}.$$

In the sequel we use the notations:

$$k_m = \min_M k, \quad k_M = \max_M k; \quad D_{\partial M} = \max_{\partial M} D, \quad K_M = \max_M K.$$

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It is easy to see that the norm

$$\|u\|_k = \left(\int_M |\nabla u|^2 d\mu_g + \int_M k(x) u^2 d\mu_g \right)^{\frac{1}{2}}$$

is equivalent to the norm $\|\cdot\|$ defined above, *i.e.*, $a_k\|u\| \leq \|u\|_k \leq b_k\|u\|$, where $a_k = \min\{1, \sqrt{k_m}\}$ and $b_k = \max\{1, \sqrt{k_M}\}$.

It is well known that the embedding $W_1^2(M) \hookrightarrow L^r(M)$ is compact for $r \in [1, 2^\star)$ and the trace-embedding $W_1^2(M) \hookrightarrow L^s(\partial M)$ for $s \in [1, \overline{2}^\star)$, respectively. We denote by $C_{M,r}$ the Sobolev embedding constant of $W_1^2(M) \hookrightarrow L^r(M)$ and by $C_{\partial M,s}$ the embedding $W_1^2(M) \hookrightarrow L^s(\partial M)$.

Let $f, h\colon \mathbb{R} \to \mathbb{R}$ be two continuous functions, and let

$$F(s) = \int_0^s f(t)dt, \qquad H(s) = \int_0^s h(t)dt. \tag{2.1}$$

We introduce the energy functional $\mathcal{E}_{\lambda,\mu}\colon W_1^2(M) \to \mathbb{R}$ given by

$$\mathcal{E}_{\lambda,\mu}(u) = \mathcal{A}(u) - \lambda \mathcal{F}(u) + \mu \mathcal{H}(u),$$

where

$$\mathcal{A}(u) = \tfrac{1}{2}\|u\|_k^2, \quad \mathcal{F}(u) = \int_M K(x) F(u(x)) d\mu_g,$$

and

$$\mathcal{H}(u) = -\int_{\partial M} D(x) H(u(x)) d\nu_g. \tag{2.2}$$

Under the hypotheses of our main theorems, a standard argument shows that the functional $\mathcal{E}_{\lambda,\mu}\colon W_1^2(M) \to \mathbb{R}$ is of class C^1 and that its critical points are exactly the weak solutions of $(P_{\lambda,\mu})$. Therefore, it is enough to show the existence of multiple critical points of $\mathcal{E}_{\lambda,\mu}$ for the parameters λ, μ specified in our results. Before concluding this section, we recall two recent critical point results which are used in order to prove Theorems 1.1 and 1.2, respectively.

Theorem 2.1 ([7, Theorem 1]) *Let X be a reflexive real Banach space, $\mathbb{I} \subset \mathbb{R}$ an interval, and $\Phi\colon X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on X^*. Assume Φ is bounded on each bounded subset of X, and $J\colon X \to \mathbb{R}$ is a C^1 functional with compact derivative. Assume that*

$$\lim_{\|x\|\to\infty} (\Phi(x) + \lambda J(x)) = +\infty$$

for all $\lambda \in \mathbb{I}$ and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in\mathbb{I}} \inf_{x\in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x\in X} \sup_{\lambda\in\mathbb{I}} (\Phi(x) + \lambda(J(x) + \rho)). \tag{2.3}$$

Then there exists a nonempty open set $A \subseteq \mathbb{I}$ and a real number $\sigma > 0$ such that, for each $\lambda \in A$ and every C^1 functional $\Psi\colon X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) + \lambda J'(x) + \mu\Psi'(x) = 0$$

has at least three solutions in X whose norms are less than σ.

Theorem 2.2 ([8, Theorem 5]) *Let X be a separable and reflexive real Banach space, and let $\mathcal{N}, \mathcal{H}\colon X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functionals, with $\mathcal{N}$ coercive. Assume that the functional $\mathcal{N} + \mu\mathcal{H}$ satisfies the (PS)-condition for every $\mu > 0$ small enough and that the set of all global minima of $\mathcal{N}$ has at least m connected components in the weak topology, with $m \geq 2$.*

Then, there exists $\overline{\mu} > 0$ such that for every $\mu \in (0, \overline{\mu})$, the functional $\mathcal{N} + \mu\mathcal{H}$ has at least $m + 1$ critical points.

3 Proof of Theorem 1.1

Throughout this section we suppose that the hypotheses of Theorem 1.1 are fulfilled.

Lemma 3.1 $\lim_{t\to 0^+} \sup\{\mathcal{F}(u) : \mathcal{A}(u) < t\}/t = 0.$

Proof Due to (f_1), for an arbitrarily small $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(s)| < \frac{\varepsilon}{2}|s| \quad \text{for all} \quad |s| < \delta.$$

Using the above inequality and (f_2), we obtain

$$|F(s)| \leq \varepsilon s^2 + K(\delta)|s|^r \quad \text{for all} \quad s \in \mathbb{R}, \tag{3.1}$$

where $r \in (2, 2^\star)$ is fixed and $K(\delta) > 0$ does not depend on s. For $t > 0$ and $\alpha = a_k^{-2}$, define the sets

$$S_t^1 = \{u \in W_1^2(M) : \mathcal{A}(u) < t\}, \quad S_t^2 = \{u \in W_1^2(M) : \|u\|^2 < 2\alpha t\}.$$

It is easy to see that $S_t^1 \subset S_t^2$. Relation (3.1) yields that

$$\mathcal{F}(u) \leq \varepsilon K_M\|u\|^2 + K(\delta)K_M C_{M,r}^r\|u\|^r \quad \text{for all} \quad u \in W_1^2(M). \tag{3.2}$$

Using (3.2), we obtain

$$0 \leq \frac{\sup_{u\in S_t^1} \mathcal{F}(u)}{t} \leq \frac{\sup_{u\in S_t^2} \mathcal{F}(u)}{t} \leq 2\alpha K_M\varepsilon + (2\alpha)^{r/2}K(\delta)K_M C_{M,r}^r t^{\frac{r}{2}-1}.$$

Since $\varepsilon > 0$ is arbitrary and $t \to 0^+$, we get the desired limit. ∎

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Let us define the function $\beta(t) = \sup\{\mathcal{F}(u) : \mathcal{A}(u) < t\}$. For $t > 0$, we have that $\beta(t) \geq 0$, and Lemma 3.1 yields

$$\lim_{t\to 0^+} \frac{\beta(t)}{t} = 0. \tag{3.3}$$

We consider the constant function $u_0(x) = s_0$ for every $x \in M$, s_0 being from (f_3). Note that $s_0 \neq 0$ (since $F(0) = 0$). Moreover, $\mathcal{F}(u_0) > 0$ and $\mathcal{A}(u_0) > 0$. Therefore, it is possible to choose a number $\eta > 0$ such that

$$0 < \eta < \mathcal{F}(u_0)[\mathcal{A}(u_0)]^{-1}.$$

By (3.3) we get the existence of a number $t_0 \in (0, \mathcal{A}(u_0))$ such that $\beta(t_0) < \eta t_0$. Thus

$$\beta(t_0) < [\mathcal{A}(u_0)]^{-1}\mathcal{F}(u_0)t_0. \tag{3.4}$$

Due to the choice of t_0 and using (3.4), we conclude that there exists $\rho_0 > 0$ such that

$$\beta(t_0) < \rho_0 < \mathcal{F}(u_0)[\mathcal{A}(u_0)]^{-1}t_0 < \mathcal{F}(u_0). \tag{3.5}$$

Now define the function $\mathcal{G}\colon W_1^2(M) \times \mathbb{I} \to \mathbb{R}$ by $\mathcal{G}(u,\lambda) = \mathcal{A}(u) - \lambda\mathcal{F}(u) + \lambda\rho_0$, where $\mathbb{I} = [0, +\infty)$.

Lemma 3.2 $\sup_{\lambda\in\mathbb{I}} \inf_{u\in W_1^2(M)} \mathcal{G}(u,\lambda) < \inf_{u\in W_1^2(M)} \sup_{\lambda\in\mathbb{I}} \mathcal{G}(u,\lambda)$.

Proof The function

$$\mathbb{I} \ni \lambda \mapsto \inf_{u\in W_1^2(M)} [\mathcal{A}(u) + \lambda(\rho_0 - \mathcal{F}(u))]$$

is obviously upper semicontinuous on $\mathbb{I}$. It follows from (3.5) that

$$\lim_{\lambda\to+\infty} \inf_{u\in W_1^2(M)} \mathcal{G}(u,\lambda) \leq \lim_{\lambda\to+\infty} [\mathcal{A}(u_0) + \lambda(\rho_0 - \mathcal{F}(u_0))] = -\infty.$$

Thus we find an element $\overline{\lambda} \in \mathbb{I}$ such that

$$\sup_{\lambda\in\mathbb{I}} \inf_{u\in W_1^2(M)} \mathcal{G}(u,\lambda) = \inf_{u\in W_1^2(M)} [\mathcal{A}(u) + \overline{\lambda}(\rho_0 - \mathcal{F}(u))]. \tag{3.6}$$

Since $\beta(t_0) < \rho_0$, it follows from the definition of β that for all $u \in W_1^2(M)$ with $\mathcal{A}(u) < t_0$, we have $\mathcal{F}(u) < \rho_0$. Hence,

$$t_0 \leq \inf\{\mathcal{A}(u) : \mathcal{F}(u) \geq \rho_0\}. \tag{3.7}$$

On the other hand,

$$\begin{aligned}\inf_{u\in W_1^2(M)} \sup_{\lambda\in\mathbb{I}} \mathcal{G}(u,\lambda) &= \inf_{u\in W_1^2(M)} \Big[\mathcal{A}(u) + \sup_{\lambda\in\mathbb{I}}\big(\lambda(\rho_0 - \mathcal{F}(u))\big)\Big] \\ &= \inf_{u\in W_1^2(M)} \{\mathcal{A}(u) : \mathcal{F}(u) \geq \rho_0\}.\end{aligned}$$

Thus, inequality (3.7) is equivalent to

$$t_0 \leq \inf_{u \in W_1^2(M)} \sup_{\lambda \in \mathbb{I}} \mathcal{G}(u, \lambda). \tag{3.8}$$

We consider the following two cases:
(I) If $0 \leq \overline{\lambda} < \frac{t_0}{\rho_0}$, then we have

$$\inf_{u \in W_1^2(M)} [\mathcal{A}(u) + \overline{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \mathcal{G}(0, \overline{\lambda}) = \overline{\lambda}\rho_0 < t_0.$$

Combining this inequality with (3.6) and (3.8) we obtain the desired inequality.

(II) If $\frac{t_0}{\rho_0} \leq \overline{\lambda}$, then from (3.4) and (3.5), it follows that

$$\begin{aligned}\inf_{u \in W_1^2(M)} \left[\mathcal{A}(u) + \overline{\lambda}(\rho_0 - \mathcal{F}(u))\right] &\leq \mathcal{A}(u_0) + \overline{\lambda}(\rho_0 - \mathcal{F}(u_0)) \\ &\leq \mathcal{A}(u_0) + \frac{t_0}{\rho_0}(\rho_0 - \mathcal{F}(u_0)) < t_0.\end{aligned}$$

Now, we apply (3.8) again. ■

Proof of Theorem 1.1 Let us choose $X = W_1^2(M)$, $\mathbb{I} = [0, +\infty)$, $\Phi = \mathcal{A}$ and $J = -\mathcal{F}$ in Theorem 2.1. Since the embedding $W_1^2(M) \hookrightarrow L^2(M)$ is compact, the compactness of $J' = -\mathcal{F}'$ trivially holds. Because of Lemma 3.2, the minimax inequality (2.3) holds too, by choosing $\rho = \rho_0$.

It remains to prove the coercivity of $\Phi + \lambda J = \mathcal{A} - \lambda\mathcal{F}$ for every $\lambda \in \mathbb{I}$. Fix $\lambda \in \mathbb{I}$ arbitrarily. By (f_2), there exists $\delta = \delta(\lambda) > 0$ such that

$$|f(s)| \leq c_k K_M^{-1}(1 + \lambda)^{-1}|s| \quad \text{for all} \quad |s| \geq \delta, \tag{3.9}$$

where $c_k = a_k^2 = \min\{1, k_m\}$. Integrating the above inequality we get that

$$|F(s)| \leq \frac{1}{2} c_k K_M^{-1}(1 + \lambda)^{-1}s^2 + \max_{|t| \leq \delta} |f(t)||s| \quad \text{for all} \quad s \in \mathbb{R}.$$

Thus, for every $u \in W_1^2(M)$, we have

$$|\mathcal{F}(u)| \leq \frac{1}{2} c_k (1 + \lambda)^{-1}\|u\|^2 + K_M\sqrt{\mathrm{vol}_g(M)}\|u\| \max_{|t| \leq \delta} |f(t)|, \tag{3.10}$$

where $\mathrm{vol}_g(M)$ denotes the Riemann–Lebesgue volume of M in the metric g. Using (3.10), we obtain

$$\begin{aligned}\mathcal{A}(u) - \lambda\mathcal{F}(u) &\geq \mathcal{A}(u) - \lambda|\mathcal{F}(u)| \\ &\geq \frac{1}{2}\frac{c_k}{1 + \lambda}\|u\|^2 - \lambda K_M\sqrt{\mathrm{vol}_g(M)}\|u\| \max_{|t| \leq \delta} |f(t)|.\end{aligned}$$

Therefore, when $\|u\| \to \infty$, then $\mathcal{A}(u) - \lambda\mathcal{F}(u) \to +\infty$ as well, *i.e.*, $\Phi + \lambda J = \mathcal{A} - \lambda\mathcal{F}$ is coercive.

Now, fix a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ fulfilling (h_q) for some $q \in [1, \frac{n}{n-2})$, and use the notations from (2.1) and (2.2). We clearly have that $\Psi = \mathcal{H}$ has a compact derivative, due to the compact embedding $W_1^2(M) \hookrightarrow L^{q+1}(\partial M)$.

Consequently, Theorem 2.1 assures the existence of a nonempty open set $A \subset [0, +\infty)$ and a number $\sigma > 0$ such that for every $\lambda \in A$, there exists $\delta_{\lambda,h} > 0$ with the property that for each $\mu \in (0, \delta_{\lambda,h})$, the equation $\mathcal{A}'(u) - \lambda\mathcal{F}'(u) + \mu\mathcal{H}'(u) = 0$ has at least three solutions which are in norm less than σ. This completes the proof. ∎

4 Proof of Theorem 1.2

We assume the hypotheses of Theorem 1.2 are fulfilled. Using the notation from the previous sections, we define the functional $\mathcal{N}_\lambda\colon W_1^2(M) \to \mathbb{R}$ by

$$\mathcal{N}_\lambda(u) = \mathcal{A}(u) - \lambda\mathcal{F}(u) = \mathcal{A}(u) - \lambda\lambda_0 \int_M k(x)F(u(x))d\mu_g, \quad u \in W_1^2(M),$$

where λ and λ_0 are from hypothesis (f_λ).

Lemma 4.1 *The set of all global minima of the functional $\mathcal{N}_\lambda$ has at least m connected components in the weak topology on $W_1^2(M)$.*

Proof First, for every $u \in W_1^2(M)$ we have

$$\begin{aligned}\mathcal{N}_\lambda(u) &= \frac{1}{2}\|u\|_k^2 - \lambda\lambda_0 \int_M k(x)F(u(x))d\mu_g \\ &= \frac{1}{2}\int_M |\nabla u|^2 d\mu_g + \int_M k(x)\tilde{F}_\lambda(u(x))d\mu_g \\ &\geq \|k\|_1 \inf_{t\in\mathbb{R}} \tilde{F}_\lambda(t).\end{aligned}$$

Moreover, if we consider $u(x) = u_{\tilde{t}}(x) = \tilde{t}$ for almost every $x \in M$, where $\tilde{t} \in \mathbb{R}$ is a minimum point of the function $t \mapsto \tilde{F}_\lambda(t)$, then we have equality in the previous estimation. Thus,

$$\inf_{u\in W_1^2(M)} \mathcal{N}_\lambda(u) = \|k\|_1 \inf_{t\in\mathbb{R}} \tilde{F}_\lambda(t).$$

Moreover, if $u \in W_1^2(M)$ is not a constant function, then $|\nabla u|^2 = g^{ij}\partial_i u \partial_j u > 0$ on a set of positive measure of the manifold M. In this case, we have

$$\mathcal{N}_\lambda(u) = \frac{1}{2}\int_M |\nabla u|^2 d\mu_g + \int_M k(x)\tilde{F}_\lambda(u(x))d\mu_g > \|k\|_1 \inf_{t\in\mathbb{R}} \tilde{F}_\lambda(t).$$

Thus, there is a one-to-one correspondence between the sets

$$\operatorname{Min}(\mathcal{N}_\lambda) = \{u \in W_1^2(M)\colon \mathcal{N}_\lambda(u) = \inf_{u\in W_1^2(M)} \mathcal{N}_\lambda(u)\}$$

and

$$\mathrm{Min}(\tilde{F}_\lambda) = \{t \in \mathbb{R} \colon \tilde{F}_\lambda(t) = \inf_{t\in\mathbb{R}} \tilde{F}_\lambda(t)\}$$

Indeed, let θ be the function that associates with every number $t \in \mathbb{R}$ the equivalence class of those functions that are almost everywhere equal to t in the whole manifold M. Then $\theta \colon \mathrm{Min}(\tilde{F}_\lambda) \to \mathrm{Min}(\mathcal{N}_\lambda)$ is actually a homeomorphism between $\mathrm{Min}(\tilde{F}_\lambda)$ and $\mathrm{Min}(\mathcal{N}_\lambda)$, where the set $\mathrm{Min}(\mathcal{N}_\lambda)$ is considered with the relativization of the weak topology on $W_1^2(M)$. Because of the hypothesis (f_λ), the set $\mathrm{Min}(\tilde{F}_\lambda)$ contains at least $m \geq 2$ connected components. Therefore, the same is true for the set $\mathrm{Min}(\mathcal{N}_\lambda)$, which completes the proof. ■

Lemma 4.2 *For arbitrarily $\lambda > 0$ and $\mu > 0$ small enough, the functional $\mathcal{E}_{\lambda,\mu} = \mathcal{N}_\lambda + \mu\mathcal{H}$ satisfies the (PS)-condition.*

Proof Hypothesis (h_1) implies that

$$|H(s)| \leq \frac{c_h}{2}s^2 + c_h|s| \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.1}$$

Inequality (4.1) yields

$$|\mathcal{H}(u)| \leq \frac{c_h}{2} D_{\partial M} C_{\partial M,2}^2 \|u\|^2 + c_h D_{\partial M} C_{\partial M,2} \sqrt{\mathrm{area}_g(\partial M)}\|u\|, \tag{4.2}$$

where $\mathrm{area}_g(\partial M)$ denotes the area of ∂M in the metric g.

Fix $\lambda > 0$ and define $\delta_\lambda^* = \frac{a_k^2}{(1+\lambda)}(c_h D_{\partial M})^{-1} C_{\partial M,2}^{-2}$. Fix also $\mu \in (0, \delta_\lambda^*)$. Using (3.10), (4.2), we get that

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(u) \geq \frac{1}{2}\Big[\frac{a_k^2}{(1+\lambda)} - \mu c_h D_{\partial M} C_{\partial M,2}^2\Big]\|u\|^2 \\ - \lambda K_M \sqrt{\mathrm{vol}_g(M)} \max_{|t|\leq\delta} |f(t)|\|u\| - \mu c_h D_{\partial M} C_{\partial M,2} \sqrt{\mathrm{area}_g(\partial M)}\|u\|,\end{aligned}$$

where $\delta > 0$ appears at (3.9). Consequently, the functional $\mathcal{E}_{\lambda,\mu}$ is coercive.

We prove now that $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS)-condition for λ, μ specified before. For this, let $\{u_n\} \subset W_1^2(M)$ be a (PS)-sequence for the function $\mathcal{E}_{\lambda,\mu}$, *i.e.*, $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded, and $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$ as $n \to \infty$. Since $\mathcal{E}_{\lambda,\mu}$ is coercive, the sequence $\{u_n\}$ is bounded. By passing, if necessary, to a subsequence, we may suppose that $u_n \rightharpoonup u$ weakly in $W_1^2(M)$, $u_n \to u$ strongly in $L^2(M)$,and $u_n \to u$ strongly in $L^2(\partial M)$. We have that

$$\begin{aligned}\langle \mathcal{E}'_{\lambda,\mu}(u_n), u_n - u\rangle + \langle \mathcal{E}'_{\lambda,\mu}(u), u - u_n\rangle = \int_M |\nabla u_n - \nabla u|^2 d\mu_g \\ + \int_M k(x)(u_n - u)^2 d\mu_g \\ - \lambda \int_M K(x)[f(u_n) - f(u)](u_n - u) d\mu_g \\ - \mu \int_{\partial M} D(x)[h(u_n) - h(u)](u_n - u) d\nu_g.\end{aligned}$$

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Consequently,

$$\langle \mathcal{E}'_{\lambda,\mu}(u_n), u_n - u\rangle + \langle \mathcal{E}'_{\lambda,\mu}(u), u - u_n\rangle + \lambda K_M \int_M |f(u_n) - f(u)||u_n - u| d\mu_g \\ + \mu D_{\partial M} \int_{\partial M} |h(u_n) - h(u)||u_n - u| d\nu_g \geq \|u_n - u\|_k^2.$$

Because $\{u_n\}$ is a (PS)-sequence and $u_n \rightharpoonup u$ weakly in $W_1^2(M)$, it follows that $\langle \mathcal{E}'_{\lambda,\mu}(u_n), u_n - u\rangle \to 0$ and $\langle \mathcal{E}'_{\lambda,\mu}(u), u - u_n\rangle \to 0$, respectively.

On the other hand, we have that

$$\int_M |f(u_n) - f(u)||u_n - u| d\mu_g \leq c_f [2\sqrt{\mathrm{vol}_g(M)} + \|u_n\|_2 + \|u\|_2] \|u_n - u\|_2.$$

Since $u_n \to u$ strongly in $L^2(M)$, it follows that

$$\lim_{n\to\infty} \int_M |f(u_n) - f(u)||u_n - u| d\mu_g = 0.$$

In the same way, since $u_n \to u$ strongly in $L^2(\partial M)$, we may prove that

$$\lim_{n\to\infty} \int_{\partial M} |h(u_n) - h(u)||u_n - u| d\nu_g = 0.$$

Hence, $u_n \to u$ strongly in $W_1^2(M)$, *i.e.*, the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS)-condition. ∎

Proof of Theorem 1.2 Taking into account that the embedding $W_1^2(M) \hookrightarrow L^2(M)$ and the trace-embedding $W_1^2(M) \hookrightarrow L^{q+1}(\partial M)$ are compact, standard arguments show the sequentially weakly lower semicontinuouity of $\mathcal{N}_\lambda$ and $\mathcal{H}$. The coercivity of $\mathcal{N}_\lambda$ holds also true. Thus, because of Lemmas 4.1 and 4.2, we may apply Theorem 2.2, concluding the proof of Theorem 1.2. ∎

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