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Global well-posedness and decay estimates of strong solutions to the nonhomogeneous Boussinesq equations for magnetohydrodynamics convection*

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We deal with an initial boundary value problem of nonhomogeneous Boussinesq equations for magnetohydrodynamics convection in two-dimensional domains. We prove that there is a unique global strong solution. Moreover, we show that the temperature converges exponentially to zero in H^1 as time goes to infinity. In particular, the initial data can be arbitrarily large and vacuum is allowed. Our analysis relies on energy method and a lemma of Desjardins (Arch. Rational Mech. Anal. 137:135–158, 1997).

Keywords: Nonhomogeneous Boussinesq-MHD equations; global well-posedness; decay estimates; large initial data; vacuum

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, we study the following nonhomogeneous Boussinesq system for magnetohydrodynamic convection (Boussinesq-MHD) in Ω:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla P = \mathbf{b}\cdot\nabla\mathbf{b} + \rho\theta\mathbf{e}_2, \\ (\rho\theta)_t + \operatorname{div}(\rho\theta\mathbf{u}) - \kappa\Delta\theta = 0, \\ \mathbf{b}_t - \nu\Delta\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{u} = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{b} = 0, \end{cases} \tag{1.1}$$

with the initial condition

$$(\rho, \rho\mathbf{u}, \rho\theta, \mathbf{b})(x, 0) = (\rho_0, \rho_0\mathbf{u}_0, \rho_0\theta_0, \mathbf{b}_0)(x), \quad x \in \Omega, \tag{1.2}$$

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and the boundary condition

$$\mathbf{u} = \mathbf{0},\ \theta = 0,\ \mathbf{b} = \mathbf{0},\ x \in \partial\Omega,\ t \geqslant 0. \tag{1.3}$$

Here $\rho = \rho(x,t)$, $\mathbf{u} = (u^1, u^2)(x,t)$, $\theta = \theta(x,t)$, $\mathbf{b} = (b^1, b^2)(x,t)$, and $P = P(x,t)$ denote the density, velocity, absolute temperature, magnetic field, and pressure of the fluid, respectively. $\mu > 0$ stands for the viscosity constant and $\kappa > 0$ is the heat conductivity coefficient. $\nu > 0$ is the magnetic diffusion coefficient. $\mathbf{e}_2 := (0,1)^\top$. The forcing term $\rho\theta\mathbf{e}_2$ in the momentum equation describes the action of the buoyancy force on fluid motion.

The Boussinesq-MHD system is a combination of the Boussinesq equations of fluid dynamics and Maxwell's equations of electromagnetism, where the displacement current can be neglected. It models the convection of an incompressible flow driven by the buoyant effect of a thermal or density field, and the Lorenz force, generated by the magnetic field of the fluid. Specifically, it closely relates to a natural type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below, with the presence of a magnetic field. For more physics background, one may refer to [**19**, **20**] and references therein.

When ρ is constant, the system (1.1) is so-called the homogeneous Boussinesq-MHD system. Recently, the well-posedness issues of such model and its variants have attracted much attention. Bian [**2**] studied the initial boundary value problem (IBVP) of two-dimensional (2D) viscous Boussinesq-MHD system without thermal conductivity and obtained a unique classical solution for H^3 initial data. Without smallness assumption on the initial data, Bian and Gui [**3**] proved the global unique solvability of 2D Boussinesq-MHD system with the temperature-dependent viscosity, thermal diffusivity and electrical conductivity. Later, the authors [**4**] established the global existence of weak solutions with H^1 initial data. By imposing a higher regularity assumption on the initial data, they also obtained a unique global strong solution. In [**14**], Larios and Pei proved the local well-posedness of solutions to the fully dissipative 3D Boussinesq-MHD system, and also the fully inviscid, irresistive, non-diffusive Boussinesq-MHD system. Moreover, they also provided a Prodi-Serrin-type global regularity condition for the 3D Boussinesq-MHD system without thermal diffusion, in terms of only two velocity and two magnetic components. By Fourier localization techniques, Zhai and Chen [**24**] investigated well-posedness to the Cauchy problem of the Boussinesq-MHD system with the temperature-dependent viscosity in Besov spaces. Very recently, Liu *et al.* [**18**] showed the global existence and uniqueness of strong and smooth large solutions to the 3D Boussinesq-MHD system with a damping term. Meanwhile, Bian and Pu [**5**] proved global axisymmetric smooth solutions for the 3D Boussinesq-MHD equations without magnetic diffusion and heat convection.

If the fluid is not affected by the Lorentz force (i.e., $\mathbf{b} = \mathbf{0}$), then the system (1.1) becomes the nonhomogeneous Boussinesq system. The authors [**10**, **25**] studied regularity criteria for 3D case, while Qiu and Yao [**21**] showed the local existence and uniqueness of strong solutions of multi-dimensional nonhomogeneous incompressible Boussinesq equations in Besov spaces. A blow-up criterion was also obtained in [**21**]. We should point out here that the results in [**10**, **21**, **25**] always require the initial density is bounded away from zero. For the initial density allowing vacuum states, Zhong [**26**] recently showed global existence of strong solutions of the Cauchy problem in $\mathbb{R}^2$ by making use of weighted energy estimate techniques. Meanwhile, he [**27**] also proved local well-posedness to the nonhomogeneous Boussinesq equations with zero heat diffusion and large initial data. It should be noted that if it is not assumed that the density is bounded away from zero, then the analysis gets wilder, since the system degenerates (in vacuum regions, the term $\rho\mathbf{u}_t$ in the momentum equation vanishes). There are also very interesting investigations about the global regularity and stability problems for the 2D homogeneous Boussinesq equations, especially those with partial dissipation, please refer to [**6**, **7**, **13**, **16**] and references therein.

In this paper, we aim at establishing the global existence and decay estimates of strong solutions to the problem (1.1)–(1.3) with general large initial date. In particular, the initial density is allowed to vanish.

Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote by

$$\int \cdot \mathrm{d}x = \int_\Omega \cdot \mathrm{d}x.$$

For $1 \leqslant p \leqslant \infty$ and integer $k > 0$, we use $L^p = L^p(\Omega)$ and $W^{k,p} = W^{k,p}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces, respectively. When $p = 2$, we use $H^k = W^{k,2}(\Omega)$. The space $H^1_{0,\sigma}$ stands for the closure in H^1 of the space $C^\infty_{0,\sigma} := \{\boldsymbol{\phi} \in C_0^\infty(\Omega) |\operatorname{div}\boldsymbol{\phi} = 0\}$.

Our main result reads as follows:

Theorem 1.1. *For constant* $q \in (2, \infty)$, *assume that the initial data* $(\rho_0 \geqslant 0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ *satisfies*

$$\rho_0 \in W^{1,q}(\Omega),\ (\mathbf{u}_0, \mathbf{b}_0) \in H^1_{0,\sigma}(\Omega),\ \theta_0 \in H^1_0(\Omega). \tag{1.4}$$

Then the system (1.1)–(1.3) *has a unique global strong solution* $(\rho \geqslant 0, \mathbf{u}, P, \theta, \mathbf{b})$ *such that for* $\tau > 0$ *and* $2 \leqslant r < q$,

$$\begin{cases} \rho_t \in L^\infty(0,\infty;L^r),\ \rho \in C([0,\infty);W^{1,q}),\\ \nabla\mathbf{u} \in L^\infty(0,\infty;L^2) \cap C([\tau,\infty);H^1) \cap L^2(\tau,\infty;H^2),\\ \nabla P \in L^\infty(\tau,\infty;L^2) \cap L^2(\tau,\infty;H^1),\\ \nabla\theta \in L^\infty(0,\infty;L^2) \cap C([\tau,\infty);H^1) \cap L^2(\tau,\infty;H^2),\\ \mathbf{b} \in L^\infty(0,\infty;H^1) \cap C([\tau,\infty);H^2) \cap L^2(\tau,\infty;H^3),\\ \rho\mathbf{u}, \rho\theta, \mathbf{b} \in C([0,\infty);L^2),\\ \sqrt{\rho}\mathbf{u}, \sqrt{t}\sqrt{\rho}\mathbf{u}_t, \sqrt{t}\mathbf{b}_t, \sqrt{t}\sqrt{\rho}\theta_t \in L^\infty(0,\infty;L^2),\\ \Delta\mathbf{b}, \nabla\mathbf{u}_t, \nabla\mathbf{b}_t, \nabla\theta_t \in L^2(0,\infty;L^2),\\ e^{\frac{\sigma}{2}t}\nabla\theta, e^{\frac{\sigma}{2}t}\sqrt{\rho}\theta_t \in L^2(0,\infty;L^2), \end{cases} \tag{1.5}$$

where $\sigma = \kappa/(d^2\|\rho_0\|_{L^\infty})$ *with* d *the diameter of* Ω. *Moreover, for any positive integer* m, *there exists a positive constant* C *depending only on* Ω, μ, $\nu, \kappa, \|\rho_0\|_{L^\infty}$,

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$\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{b}_0\|_{L^2}$, $\|\nabla\theta_0\|_{L^2}$, q, *and* m *such that for* $t \geqslant 1$,

$$\begin{cases} \|\sqrt{\rho}\theta(\cdot,t)\|_{L^2}^2 \leqslant C\mathrm{e}^{-2\sigma t}, \ \|\nabla\theta(\cdot,t)\|_{L^2}^2 \leqslant C\mathrm{e}^{-\sigma t}, \\ \|\mathbf{u}(\cdot,t)\|_{H^2}^2 + \|\mathbf{b}(\cdot,t)\|_{H^2}^2 + \|\nabla^2\theta(\cdot,t)\|_{L^2}^2 + \|\nabla P(\cdot,t)\|_{L^2}^2 \leqslant Ct^{-m}, \\ \|\sqrt{\rho}\mathbf{u}_t(\cdot,t)\|_{L^2}^2 + \|\mathbf{b}_t(\cdot,t)\|_{L^2}^2 + \|\sqrt{\rho}\theta_t(\cdot,t)\|_{L^2}^2 \leqslant Ct^{-m}. \end{cases} \tag{1.6}$$

Remark 1.2. It should be noted that our theorem 1.1 holds for arbitrarily large initial data.

Remark 1.3. After the completion of this work, it came to our attention that Fan *et al.* [**11**] studied the similar problem. However, these two works are independent of each other. On one hand, compared with [**11**], although the equations (1.1) degenerate near vacuum, there is no need to impose compatibility conditions for the initial data by using time weighted estimates. On the other hand, our proof is different from that of in [**11**]. Precisely, we will apply a lemma due to Desjardins (see lemma 2.4) to handle the right-hand side of (3.24), while Fan *et al.* [**11**] used the following critical Sobolev inequality of logarithmic type

$$\|f\|_{L^\infty(\Omega)} \leqslant C\left(1 + \|f\|_{H^1(\Omega)} \log^{1/2}(e + \|f\|_{W^{1,p}(\Omega)})\right) \quad \text{for any} \quad 2 < p < \infty. \tag{1.7}$$

Finally, we emphasize that the decay rate (1.6) is a new result. Consequently, we improve the main result of [**11**].

Remark 1.4. We should point out that the methods used in the present paper depend heavily on the boundedness of the domains and it seems difficult to show decay-in-time and time-independent estimates for solutions to the Cauchy problem of (1.1) in the whole space $\mathbb{R}^2$. Nevertheless, we can establish the global existence of strong solutions for the system (1.1) in $\mathbb{R}^2$ even in the case of $\kappa = 0$ (see [**29**]).

Remark 1.5. Very recently, we [**28**] established the global existence and uniqueness of strong solutions to the IBVP of nonhomogeneous MHD equations (i.e., (1.1)–(1.3) with $\theta = 0$) with vacuum and large initial data in two-dimensional bounded domains. Meanwhile, the corresponding strong solution admits the exponential decay-in-time property which is quite different from theorem 1.1 for the related MHD equations. Hence the temperature acts as some significant roles on the large time behaviours of the velocity and the magnetic field.

We now make some comments on the key ingredients of the analysis in this paper. The local existence and uniqueness of strong solutions to the problem (1.1)–(1.3) follows from the works in literature such as [**15**, **22**] (see lemma 2.1). Thus our efforts are devoted to establishing global a priori estimates on strong solutions to the system (1.1) in suitable higher-order norms. It should be pointed out that compared with the related works in literature, the proof of theorem 1.1 is much more involved due to the absence of the positive lower bound for the initial density as well as the absence of the smallness and the compatibility conditions for the initial data. Consequently, some new ideas are needed to overcome these difficulties.

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First, applying the upper bounds on the density (see (3.2)) and the Poincaré inequality, we have the following key observation:

$$\|\sqrt{\rho}\theta\|_{L^2}^2 \leqslant \|\rho\|_{L^\infty}\|\theta\|_{L^2}^2 \leqslant C\|\nabla\theta\|_{L^2}^2,$$

which implies that $\|\sqrt{\rho}\theta(t)\|_{L^2}^2$ decays with the rate of $\mathrm{e}^{-2\sigma t}$ for some $\sigma > 0$ depending only on $\kappa, \|\rho_0\|_{L^\infty}$, and the diameter of the Ω (see (3.9)). With the help of this key exponential decay-in-time rate, we can show that $\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2$ decays with the rate of $(1+t)^{-m}$ for any positive integer m (see lemma 3.2 for details). Next, we need to derive the bound of $\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2$. However, it prevents us to achieve this goal due to the presence of vacuum. To overcome this difficulty, we make use of an inequality for $\mathbf{u}$ with degenerate weight $\sqrt{\rho}$ (see lemma 2.4) to obtain time-weighted estimate on the $L^\infty(0,T;L^2)$-norm of the gradients of the velocity and the magnetic field (see (3.21)). Indeed, the time-weighted estimate is crucial in dropping the compatibility condition on the initial data (see [**15**, **22**] for example). On the other hand, (3.3) allows us to derive exponential decay-in-time rate of $\|\nabla\theta\|_{L^2}^2$ (see (3.22)). With these time-weighted estimates at hand, we can then obtain that $\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2$ decays as t^{-m} for large time (see (3.48)). In fact, all these exponential decay-in-time rates and the time-weighted estimate play a crucial role in obtaining the desired uniform bound (with respect to time) on the $L^1(0,T;L^\infty)$-norm of $\nabla\mathbf{u}$ (see (3.71)). Finally, using these a priori estimates, we establish the time-independent higher order estimates on the solution $(\rho, \mathbf{u}, \theta, \mathbf{b})$ (see lemmas 3.5 and 3.6 for details), and thus claims the proof of theorem 1.1.

The rest of this paper is organized as follows. In § 2, we collect some elementary facts and inequalities that will be used later. § 3 is devoted to the a priori estimates. Finally, we will give the proof of theorem 1.1 in § 4.

2. Preliminaries

In this section, we will recall some known facts that will be used later.

We begin with the local existence and uniqueness of strong solutions whose proof can be performed by using standard procedures (see e.g., [**15**, **22**]).

LEMMA 2.1. *Assume that* $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ *satisfies* (1.4)*, then there exist a small time* $T > 0$ *and a unique strong solution* $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ *to the problem* (1.1)–(1.3) *in* $\Omega \times (0, T)$*.*

Next, the following Gagliardo–Nirenberg inequality (see [**12**, theorem 10.1, p. 27]) will be useful in the next section.

LEMMA 2.2 Gagliardo-Nirenberg. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded smooth domain. Assume that* $1 \leqslant q, r \leqslant \infty$*, and* j, m *are arbitrary integers satisfying* $0 \leqslant j < m$*. If* $v \in W^{m,r}(\Omega) \cap L^q(\Omega)$*, then we have*

$$\|D^j v\|_{L^p} \leqslant C\|v\|_{L^q}^{1-a}\|v\|_{W^{m,r}}^a,$$

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where

$$-j+\frac{2}{p}=(1-a)\frac{2}{q}+a\left(-m+\frac{2}{r}\right),$$

and

$$a\in\begin{cases}[\frac{j}{m},1), & \text{if } m-j-\frac{2}{r} \text{ is a nonnegative integer},\\ [\frac{j}{m},1], & \text{otherwise.}\end{cases}$$

The constant C depends only on m, j, q, r, a, and Ω. In particular, we have

$$\|v\|_{L^4}^4 \leqslant C\|v\|_{L^2}^2\|v\|_{H^1}^2, \tag{2.1}$$

which will be used frequently in the next section.

Next, we give some regularity properties for the following Stokes system:

$$\begin{cases}-\mu\Delta\mathbf{u}+\nabla P=\mathbf{F}, & x\in\Omega,\\ \operatorname{div}\mathbf{u}=0, & x\in\Omega,\\ \mathbf{u}=\mathbf{0}, & x\in\partial\Omega.\end{cases} \tag{2.2}$$

LEMMA 2.3. *Suppose that $\mathbf{F}\in L^r(\Omega)$ with $1<r<\infty$. Let $(\mathbf{u},P)\in H_0^1\times L^2$ be the unique weak solution to the problem (2.2), then $(\mathbf{u},P)\in W^{2,r}\times W^{1,r}$ and there exists a constant C depending only on Ω and r such that*

$$\|\mathbf{u}\|_{W^{2,r}}+\|P\|_{W^{1,r}/\mathbb{R}}\leqslant C\|\mathbf{F}\|_{L^r}.$$

Proof. See [**1**, proposition 4.3]. □

Finally, by zero extension of $\mathbf{u}$ outside Ω, we can derive the following lemma due to Desjardins (see [**8**, lemma 1]), which plays a key role in the proof of lemma 3.3 in the next section.

LEMMA 2.4. *Let $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ be a strong solution to the system (1.1)–(1.3) on $(0,T)$. Suppose that $0\leqslant\rho\leqslant\bar{\rho}$, then we have*

$$\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\leqslant C(\bar{\rho},\Omega)(1+\|\sqrt{\rho}\mathbf{u}\|_{L^2})\|\nabla\mathbf{u}\|_{L^2}\sqrt{\log(2+\|\nabla\mathbf{u}\|_{L^2}^2)}. \tag{2.3}$$

3. A priori estimates

In this section, we will establish some necessary a priori bounds for strong solutions $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ to the problem (1.1)–(1.3) to extend the local strong solution guaranteed by lemma 2.1. In what follows, we will use $C(f)$ to emphasize the dependence on f.

Before proceeding, we rewrite another equivalent form of the system (1.1) as the following

$$\begin{cases} \rho_t + \mathbf{u}\cdot\nabla\rho = 0, \\ \rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \mathbf{b}\cdot\nabla\mathbf{b} + \rho\theta\mathbf{e}_2, \\ \rho\theta_t + \rho\mathbf{u}\cdot\nabla\theta - \kappa\Delta\theta = 0, \\ \mathbf{b}_t - \nu\Delta\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{u} = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{b} = 0. \end{cases} \tag{3.1}$$

First, due to $(3.1)_1$, we have the following well-known estimate on the $L^\infty(0,T;L^\infty)$-norm of the density (see [**17**, theorem 2.1]).

LEMMA 3.1. *It holds that*

$$\sup_{0\leqslant t\leqslant T}\|\rho\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty}. \tag{3.2}$$

Next, the following exponential decay estimate of $\|\sqrt{\rho}\theta\|_{L^2}^2$ is crucial to obtain the time-independent estimates on the gradients of the velocity and the magnetic field.

LEMMA 3.2. *For any positive integer m, there exists a positive constant C depending only on Ω, μ, ν, κ, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}$, $\|\mathbf{b}_0\|_{L^2}$, $\|\sqrt{\rho_0}\theta_0\|_{L^2}$, and m such that*

$$\sup_{0\leqslant t\leqslant T}\left(\mathrm{e}^{2\sigma t}\|\sqrt{\rho}\theta\|_{L^2}^2\right) + \kappa\int_0^T \mathrm{e}^{\sigma t}\|\nabla\theta\|_{L^2}^2\,\mathrm{d}t \leqslant \frac{3}{2}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2, \tag{3.3}$$

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^T \left(\mu\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\|_{L^2}^2\right)\mathrm{d}t \\ &\leqslant \left(\frac{1}{2} + \frac{1}{2\sigma}\mathrm{e}^{\frac{1}{\sigma}} + \mathrm{e}^{\frac{1}{\sigma}}\right)\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2\right) + \left(\frac{1}{2\sigma^2}\mathrm{e}^{\frac{1}{\sigma}} + \frac{1}{\sigma}\mathrm{e}^{\frac{1}{\sigma}} + \frac{1}{2\sigma}\right)\|\sqrt{\rho_0}\theta_0\|_{L^2}^2, \end{aligned} \tag{3.4}$$

and

$$\sup_{0\leqslant t\leqslant T} t^m\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^T t^m\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right)\mathrm{d}t \leqslant C, \tag{3.5}$$

where $\sigma = \kappa/(d^2\|\rho_0\|_{L^\infty})$ with d the diameter of Ω.

Proof. 1. Multiplying $(3.1)_3$ by 2θ, and then integration by parts over Ω, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta\|_{L^2}^2 + 2\kappa\|\nabla\theta\|_{L^2}^2 = 0. \tag{3.6}$$

Since $\theta|_\Omega = 0$, it follows from the Poincaré inequality (see [**23**, (A.3), p. 266]) that

$$\|\theta\|_{L^2}^2 \leqslant d^2\|\nabla\theta\|_{L^2}^2, \tag{3.7}$$

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where d is the diameter of Ω. As a consequence, we infer from (3.2) and (3.7) that

$$\|\sqrt{\rho}\theta\|_{L^2}^2 \leqslant d^2\|\rho\|_{L^\infty}\|\nabla\theta\|_{L^2}^2 \leqslant d^2\|\rho_0\|_{L^\infty}\|\nabla\theta\|_{L^2}^2. \tag{3.8}$$

Inserting (3.8) into (3.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta\|_{L^2}^2 + \frac{2\kappa}{d^2\|\rho_0\|_{L^\infty}}\|\sqrt{\rho}\theta\|_{L^2}^2 \leqslant 0,$$

which yields immediately that

$$\sup_{0\leqslant t\leqslant T}\left(\mathrm{e}^{2\sigma t}\|\sqrt{\rho}\theta\|_{L^2}^2\right) \leqslant \|\sqrt{\rho_0}\theta_0\|_{L^2}^2, \quad \text{for } \ \sigma = \frac{\kappa}{d^2\|\rho_0\|_{L^\infty}}. \tag{3.9}$$

Multiplying (3.6) by $\mathrm{e}^{\sigma t}$ and using (3.9), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\sigma t}\|\sqrt{\rho}\theta\|_{L^2}^2\right) + 2\kappa\mathrm{e}^{\sigma t}\|\nabla\theta\|_{L^2}^2 = \sigma\mathrm{e}^{\sigma t}\|\sqrt{\rho}\theta\|_{L^2}^2 \leqslant \sigma\mathrm{e}^{-\sigma t}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2,$$

which integrated in time over $[0,T]$ implies

$$2\kappa\int_0^T \mathrm{e}^{\sigma t}\|\nabla\theta\|_{L^2}^2\,\mathrm{d}t \leqslant \|\sqrt{\rho_0}\theta_0\|_{L^2}^2. \tag{3.10}$$

Hence, the desired (3.3) follows from (3.9) and (3.10).

2. Multiplying $(3.1)_2$ by $\mathbf{u}$, $(3.1)_4$ by $\mathbf{b}$, and integrating by parts, we get from (3.9) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + 2\mu\|\nabla\mathbf{u}\|_{L^2}^2 + 2\nu\|\nabla\mathbf{b}\|_{L^2}^2 \\
&= 2\int \rho\theta\mathbf{u}\cdot\mathbf{e}_2\,\mathrm{d}x \\
&\leqslant 2\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\sqrt{\rho}\theta\|_{L^2} \\
&\leqslant \mathrm{e}^{-\sigma t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \mathrm{e}^{\sigma t}\|\sqrt{\rho}\theta\|_{L^2}^2 \\
&\leqslant \mathrm{e}^{-\sigma t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \mathrm{e}^{-\sigma t}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2.
\end{aligned} \tag{3.11}$$

Thus, Gronwall's inequality leads to

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) \\
&\leqslant \exp\left(\int_0^T \mathrm{e}^{-\sigma t}\,\mathrm{d}t\right)\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2 + \int_0^T \mathrm{e}^{-\sigma t}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2\,\mathrm{d}t\right) \\
&\leqslant \mathrm{e}^{\frac{1}{\sigma}}\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2 + \sigma^{-1}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2\right).
\end{aligned} \tag{3.12}$$

Consequently, integrating (3.11) over $[0,T]$ together with (3.12) yields

$$\begin{aligned}
&\int_0^T \left(2\mu\|\nabla\mathbf{u}\|_{L^2}^2+2\nu\|\nabla\mathbf{b}\|_{L^2}^2\right)\mathrm{d}t\\
&\leqslant \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2+\|\mathbf{b}_0\|_{L^2}^2+\sup_{0\leqslant t\leqslant T}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\int_0^T \mathrm{e}^{-\sigma t}\,\mathrm{d}t+\|\sqrt{\rho_0}\theta_0\|_{L^2}^2\int_0^T \mathrm{e}^{-\sigma t}\,\mathrm{d}t\\
&\leqslant \left(1+\frac{1}{\sigma}\mathrm{e}^{\frac{1}{\sigma}}\right)\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2+\|\mathbf{b}_0\|_{L^2}^2\right)+\left(\frac{1}{\sigma^2}\mathrm{e}^{\frac{1}{\sigma}}+\frac{1}{\sigma}\right)\|\sqrt{\rho_0}\theta_0\|_{L^2}^2. \qquad (3.13)
\end{aligned}$$

This along with (3.12) implies the desired (3.4).

3. We prove (3.5) by induction. Multiplying (3.11) by t, we obtain from (3.2) and the Poincaré inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+t\|\mathbf{b}\|_{L^2}^2\right)+2\mu t\|\nabla\mathbf{u}\|_{L^2}^2+2\nu t\|\nabla\mathbf{b}\|_{L^2}^2\\
&\leqslant \mathrm{e}^{-\sigma t}(t\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2)+t\mathrm{e}^{-\sigma t}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2\\
&\leqslant \mathrm{e}^{-\sigma t}(t\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2)+t\mathrm{e}^{-\sigma t}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2+\|\rho_0\|_{L^\infty}d^2\|\nabla\mathbf{u}\|_{L^2}^2+d^2\|\nabla\mathbf{b}\|_{L^2}^2. \qquad (3.14)
\end{aligned}$$

Thus, Gronwall's inequality implies

$$\sup_{0\leqslant t\leqslant T}\left(t\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+t\|\mathbf{b}\|_{L^2}^2\right)+\int_0^T t\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)\mathrm{d}t\leqslant C, \qquad (3.15)$$

due to (3.13) and the following fact

$$\int_0^T t\mathrm{e}^{-\sigma t}\,\mathrm{d}t=\frac{1}{\sigma^2}-\frac{1}{\sigma^2\mathrm{e}^{\sigma T}}-\frac{T}{\sigma\mathrm{e}^{\sigma T}}\leqslant C(\sigma).$$

Assume (3.5) holds for m. That is,

$$\sup_{0\leqslant t\leqslant T}\left(t^m\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+t^m\|\mathbf{b}\|_{L^2}^2\right)+\int_0^T t^m\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)\mathrm{d}t\leqslant C. \qquad (3.16)$$

Consider $m+1$. Multiplying (3.11) by t^{m+1}, we obtain from (3.2) and the Poincaré inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{m+1}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+t^{m+1}\|\mathbf{b}\|_{L^2}^2\right)+2\mu t^{m+1}\|\nabla\mathbf{u}\|_{L^2}^2+2\nu t^{m+1}\|\nabla\mathbf{b}\|_{L^2}^2\\
&\leqslant \mathrm{e}^{-\sigma t}(t^{m+1}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2)+t^{m+1}\mathrm{e}^{-\sigma t}\|\sqrt{\rho_0}\theta_0\|_{L^2}^2\\
&\quad+(m+1)t^m\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+(m+1)t^m\|\mathbf{b}\|_{L^2}^2\\
&\leqslant \mathrm{e}^{-\sigma t}(t^{m+1}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2)+Ct^{m+1}\mathrm{e}^{-\sigma t}+Ct^m\|\nabla\mathbf{u}\|_{L^2}^2+Ct^m\|\nabla\mathbf{b}\|_{L^2}^2, \qquad (3.17)
\end{aligned}$$

which combined with Gronwall's inequality, (3.16), and the following

$$I_{m+1}:=\int_0^T t^{m+1}\mathrm{e}^{-\sigma t}\,\mathrm{d}t=-\frac{T^{m+1}}{\sigma\mathrm{e}^{\sigma T}}+\frac{m+1}{\sigma}I_m\leqslant C(m,\sigma) \qquad (3.18)$$

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yields

$$\sup_{0\leqslant t\leqslant T} t^{m+1}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2\right)+\int_0^T t^{m+1}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)\,\mathrm{d}t\leqslant C. \tag{3.19}$$

This finishes the proof of lemma 3.2.

□

The following lemma concerns the time-weighted estimates on the $L^\infty(0;T;L^2)$-norm of the gradients of the velocity, the magnetic field and the temperature.

LEMMA 3.3. *Let m and σ be as in lemma 3.2, then there exists a positive constant C depending only on Ω, μ, ν, κ, $\|\rho_0\|_{L^\infty}$, $\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{b}_0\|_{L^2}$, $\|\nabla\theta_0\|_{L^2}$, and m such that*

$$\sup_{0\leqslant t\leqslant T}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\Delta\mathbf{b}\|_{L^2}^2+\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2\right)\,\mathrm{d}t\leqslant C, \tag{3.20}$$

$$\sup_{0\leqslant t\leqslant T} t^m\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)+\int_0^T t^m\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\Delta\mathbf{b}\|_{L^2}^2+\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2\right)\,\mathrm{d}t\leqslant C, \tag{3.21}$$

and

$$\sup_{0\leqslant t\leqslant T}\left(\mathrm{e}^{\sigma t}\|\nabla\theta\|_{L^2}^2\right)+\int_0^T \mathrm{e}^{\sigma t}\|\sqrt{\rho}\theta_t\|_{L^2}^2\,\mathrm{d}t\leqslant C. \tag{3.22}$$

Proof. 1. Multiplying $(3.1)_2$ by $\mathbf{u}_t$ and integrating the resulting equation over Ω, we get

$$\begin{aligned}\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\mathbf{u}|^2\,\mathrm{d}x+\int\rho|\mathbf{u}_t|^2\,\mathrm{d}x&=-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x\\&\quad+\int\theta\rho\mathbf{e}_2\cdot\mathbf{u}_t\,\mathrm{d}x+\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t\,\mathrm{d}x.\end{aligned} \tag{3.23}$$

By Hölder's inequality and (2.1), we have

$$\begin{aligned}\left|-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x\right|&\leqslant\frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\&\leqslant\frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}.\end{aligned} \tag{3.24}$$

By Cauchy–Schwarz inequality and (3.9), we find that

$$\left|\int\theta\rho\mathbf{e}_2\cdot\mathbf{u}_t\,\mathrm{d}x\right|\leqslant\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\theta\|_{L^2}\leqslant\frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho_0}\theta_0\|_{L^2}^2\mathrm{e}^{-2\sigma t}. \tag{3.25}$$

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Integration by parts together with the conditions $\operatorname{div}\mathbf{b} = 0$ in Ω and $\mathbf{b} = \mathbf{0}$ on $\partial\Omega$, we arrive at

$$\begin{aligned}\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t\,\mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x + \int \mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x + \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_t\,\mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x + \int(\Delta\mathbf{b}-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x \\ &\quad + \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot(\Delta\mathbf{b}-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u})\,\mathrm{d}x \\ &\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x + \frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2 + C\|\mathbf{b}\|_{L^6}^6 + C\|\nabla\mathbf{u}\|_{L^3}^3 \\ &\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x + \frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2 + C\|\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}^4 \\ &\quad + C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{H^2}. \end{aligned} \tag{3.26}$$

Hence, substituting (3.24)–(3.26) into (3.23), we derive from (3.4) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\int|\nabla\mathbf{u}|^2\,\mathrm{d}x + \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x\right) + \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \\ &\leqslant C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1} + Ce^{-2\sigma t} + \frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2 \\ &\quad + C\|\nabla\mathbf{b}\|_{L^2}^4 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{H^2}. \end{aligned} \tag{3.27}$$

2. Multiplying $(3.1)_4$ by $\Delta\mathbf{b}$ and integrating the resulting equality over Ω, it follows from Hölder's and Gagliardo–Nirenberg inequalities that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\mathbf{b}|^2\,\mathrm{d}x + \nu\int|\Delta\mathbf{b}|^2\,\mathrm{d}x \leqslant C\int|\nabla\mathbf{u}||\nabla\mathbf{b}|^2\,\mathrm{d}x + C\int|\nabla\mathbf{u}||\mathbf{b}||\Delta\mathbf{b}|\,\mathrm{d}x \\ &\leqslant C\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{b}\|_{L^2}^{\frac{4}{3}}\|\Delta\mathbf{b}\|_{L^2}^{2/3} + C\|\nabla\mathbf{u}\|_{L^3}\|\mathbf{b}\|_{L^6}\|\Delta\mathbf{b}\|_{L^2} \\ &\leqslant C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2} + C(1+\|\mathbf{b}\|_{L^2}^2)\|\nabla\mathbf{b}\|_{L^2}^4 + \frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2, \end{aligned} \tag{3.28}$$

which together with (3.27) and (3.4) gives rise to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 + \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x\Big) + \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{\nu}{2}\|\Delta\mathbf{b}\|_{L^2}^2 \\ &\leqslant C\left(\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2 + \|\nabla\mathbf{u}\|_{L^2}\right)\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^2} + C\|\nabla\mathbf{b}\|_{L^2}^4 + C\mathrm{e}^{-2\sigma t}. \end{aligned} \tag{3.29}$$

3. Recall that $(\mathbf{u}, P)$ satisfies the following Stokes system

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla P = -\rho\mathbf{u}_t - \rho\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \rho\theta\mathbf{e}_2, & x\in\Omega, \\ \operatorname{div}\mathbf{u} = 0, & x\in\Omega, \\ \mathbf{u} = \mathbf{0}, & x\in\partial\Omega. \end{cases} \tag{3.30}$$

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Applying lemma 2.3 with $\mathbf{F} = -\rho\mathbf{u}_t - \rho\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \rho\theta\mathbf{e}_2$, we obtain from (3.2) and (3.9) that

$$\begin{aligned}
\|\mathbf{u}\|_{H^2} &\leqslant C\left(\|\rho\mathbf{u}_t\|_{L^2} + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2} + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2} + \|\rho\theta\|_{L^2}\right)\\
&\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4} + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}\\
&\quad + \|\rho_0\|_{L^\infty}^{1/2}\|\sqrt{\rho_0}\theta_0\|_{L^2}\mathrm{e}^{-\sigma t}\\
&\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{H^1}^{1/2} + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2} + C\mathrm{e}^{-\sigma t}\\
&\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2} + \frac{1}{2}\|\mathbf{u}\|_{H^2} + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2} + C\mathrm{e}^{-\sigma t},
\end{aligned}$$

and thus

$$\|\mathbf{u}\|_{H^2} \leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2} + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2} + C\mathrm{e}^{-\sigma t}. \tag{3.31}$$

Inserting (3.31) into (3.29) and applying Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}B(t) + \frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{\nu}{2}\|\Delta\mathbf{b}\|_{L^2}^2\\
&\leqslant C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^4 + C\|\nabla\mathbf{b}\|_{L^2}^4 + \varepsilon\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2 + C\mathrm{e}^{-2\sigma t},
\end{aligned} \tag{3.32}$$

where

$$B(t) := \frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 + \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x \tag{3.33}$$

satisfies

$$\frac{\mu}{4}\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 - C_1\|\mathbf{b}\|_{L^4}^4 \leqslant B(t) \leqslant C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{b}\|_{L^2}^2, \tag{3.34}$$

owing to Gagliardo–Nirenberg inequality, (3.4), and the following estimate

$$\int |\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}|\,\mathrm{d}x \leqslant \frac{\mu}{4}\|\nabla\mathbf{u}\|_{L^2}^2 + C_1\|\mathbf{b}\|_{L^4}^4. \tag{3.35}$$

4. Multiplying $(3.1)_4$ by $|\mathbf{b}|^2\mathbf{b}$ and integrating the resulting equality by parts over Ω, we infer from Gagliardo–Nirenberg inequality and (3.4) that

$$\begin{aligned}
\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{b}\|_{L^4}^4 + \||\nabla\mathbf{b}||\mathbf{b}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathbf{b}|^2\|_{L^2}^2 &\leqslant C\|\nabla\mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^4}^2\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^2}\|\nabla|\mathbf{b}|^2\|_{L^2}\\
&\leqslant \frac{1}{4}\|\nabla|\mathbf{b}|^2\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^4 + C\|\nabla\mathbf{b}\|_{L^2}^4.
\end{aligned} \tag{3.36}$$

Now, adding (3.36) multiplied by $4(C_1+1)$ to (3.32) and choosing ε suitably small, we obtain after using (2.3) and (3.4) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(B(t)+(C_1+1)\|\mathbf{b}\|_{L^4}^4\right)+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\nu\|\Delta\mathbf{b}\|_{L^2}^2+\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2\\
&\leqslant C\|\nabla\mathbf{b}\|_{L^2}^4+C\|\nabla\mathbf{u}\|_{L^2}^4+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\nabla\mathbf{u}\|_{L^2}^2\\
&\leqslant C\|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\\
&\quad+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\log(2+\|\nabla\mathbf{u}\|_{L^2}^2)+C\mathrm{e}^{-2\sigma t}. \qquad (3.37)
\end{aligned}
$$

Set

$$
f(t):=2+B(t)+(C_1+1)\|\mathbf{b}\|_{L^4}^4,\ \ g(t)\triangleq\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2+\mathrm{e}^{-2\sigma t},
$$

then we deduce from (3.37) and (3.34) that

$$
f'(t)\leqslant Cg(t)f(t)+Cg(t)f(t)\log f(t),
$$

which yields

$$
(\log f(t))'\leqslant Cg(t)+Cg(t)\log(f(t)). \qquad (3.38)
$$

We thus infer from (3.38), (3.4), Gronwall's inequality and (3.34) that

$$
\sup_{0\leqslant t\leqslant T}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\|_{L^4}^4\right)\leqslant C. \qquad (3.39)
$$

Integrating (3.37) with respect to t together with (3.39) and (3.4) leads to

$$
\int_0^T\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\Delta\mathbf{b}\|_{L^2}^2+\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2\right)\,\mathrm{d}t\leqslant C. \qquad (3.40)
$$

This along with (3.39) gives the desired (3.20).

5. Multiplying (3.37) by t^m, we then infer from (3.39), (3.34) and Sobolev's inequality that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(t^mB(t)+(C_1+1)t^m\|\mathbf{b}\|_{L^4}^4\right)+t^m\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\nu\|\Delta\mathbf{b}\|_{L^2}^2+\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2\right)\\
&\leqslant Ct^m\|\nabla\mathbf{b}\|_{L^2}^4+Ct^m\|\nabla\mathbf{u}\|_{L^2}^4+Ct^m\mathrm{e}^{-2\sigma t}+mt^{m-1}B(t)\\
&\quad+m(C_1+1)t^{m-1}\|\mathbf{b}\|_{L^4}^4\\
&\leqslant Ct^m\|\nabla\mathbf{b}\|_{L^2}^4+Ct^m\|\nabla\mathbf{u}\|_{L^2}^4+Ct^m\mathrm{e}^{-2\sigma t}+Ct^{m-1}\|\nabla\mathbf{u}\|_{L^2}^2\\
&\quad+Ct^{m-1}\|\nabla\mathbf{b}\|_{L^2}^2+Ct^{m-1}\|\nabla\mathbf{b}\|_{L^2}^4\\
&\leqslant C\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)\left(t^m\|\nabla\mathbf{u}\|_{L^2}^2+t^m\|\nabla\mathbf{b}\|_{L^2}^2\right)+Ct^m\mathrm{e}^{-2\sigma t}\\
&\quad+Ct^{m-1}\|\nabla\mathbf{u}\|_{L^2}^2+Ct^{m-1}\|\nabla\mathbf{b}\|_{L^2}^2, \qquad (3.41)
\end{aligned}
$$

which combined with Gronwall's inequality, (3.34), (3.5) and (3.18) leads to the desired (3.21).

6. Multiplying $(3.1)_3$ by θ_t and integrating the resulting equation over Ω, we get

$$\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\theta|^2\,\mathrm{d}x+\int\rho\theta_t^2\,\mathrm{d}x=-\int\theta_t\rho\mathbf{u}\cdot\nabla\theta\,\mathrm{d}x. \tag{3.42}$$

By Hölder's inequality and (2.1), we obtain from Sobolev's inequality, Gagliardo–Nirenberg inequality and (3.20) that for any $\delta>0$,

$$\begin{aligned}\left|-\int\theta_t\rho\mathbf{u}\cdot\nabla\theta\,\mathrm{d}x\right|&\leqslant\frac{1}{2}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\theta\|_{L^4}^2\\&\leqslant\frac{1}{2}\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{H^1}\\&\leqslant\frac{1}{2}\|\sqrt{\rho}\theta_t\|_{L^2}^2+C(\delta)\|\nabla\theta\|_{L^2}^2+\delta\|\nabla^2\theta\|_{L^2}^2.\end{aligned} \tag{3.43}$$

Hence, substituting (3.43) into (3.42) yields

$$\kappa\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\leqslant C\|\nabla\theta\|_{L^2}^2+2\delta\|\nabla^2\theta\|_{L^2}^2. \tag{3.44}$$

It follows from $(3.1)_3$, Hölder's inequality, (2.1), and (3.20) that

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}^2&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\||\mathbf{u}||\nabla\theta|\|_{L^2}^2\\&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\mathbf{u}\|_{L^4}^2\|\nabla\theta\|_{L^4}^2\\&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{3.45}$$

which combined with (3.44) and choosing δ suitably small imply that

$$\kappa\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\leqslant C\|\nabla\theta\|_{L^2}^2. \tag{3.46}$$

Multiplying (3.46) by $\mathrm{e}^{\sigma t}$ leads to

$$\kappa\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\sigma t}\|\nabla\theta\|_{L^2}^2)+\mathrm{e}^{\sigma t}\|\sqrt{\rho}\theta_t\|_{L^2}^2\leqslant C\mathrm{e}^{\sigma t}\|\nabla\theta\|_{L^2}^2. \tag{3.47}$$

Integrating (3.47) in time over $[0,T]$ together with (3.10) leads to the desired (3.22).

□

As an application of lemmas 3.2 and 3.3, we have the following time-weighted estimates on $\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2$, $\|\mathbf{b}_t\|_{L^2}^2$, and $\|\sqrt{\rho}\theta_t\|_{L^2}^2$, which play an important role in deriving the uniform-in-time bound of $\int_0^T\|\nabla\mathbf{u}\|_{L^\infty}\,\mathrm{d}t$.

Lemma 3.4. *Let m be as in lemma 3.2, then there exists a positive constant C depending only on Ω, μ, ν, κ, $\|\rho_0\|_{L^\infty}$, $\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{b}_0\|_{L^2}$, $\|\nabla\theta_0\|_{L^2}$, and m such that*

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}t^m\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\big)\\&+\int_0^T t^m\left(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{b}_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\mathrm{d}t\leqslant C.\end{aligned} \tag{3.48}$$

Proof. 1. Differentiating $(3.1)_2$ with respect to t, we arrive at

$$\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \mu\Delta\mathbf{u}_t = -\nabla P_t - \rho_t\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} + \mathbf{b}_t\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{b}_t + (\rho\theta)_t\mathbf{e}_2. \tag{3.49}$$

Multiplying (3.49) by $\mathbf{u}_t$ and integrating (by parts) over Ω and using $(1.1)_1$ yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\mathbf{u}_t|^2\,\mathrm{d}x + \mu\int|\nabla\mathbf{u}_t|^2\,\mathrm{d}x\\ &= \int\mathrm{div}(\rho\mathbf{u})|\mathbf{u}_t|^2\,\mathrm{d}x + \int\mathrm{div}(\rho\mathbf{u})\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x - \int\rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x\\ &\quad + \int\mathbf{b}_t\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t\,\mathrm{d}x + \int\mathbf{b}\cdot\nabla\mathbf{b}_t\cdot\mathbf{u}_t\,\mathrm{d}x + \int(\rho\theta)_t\mathbf{e}_2\cdot\mathbf{u}_t\,\mathrm{d}x =: \sum_{k=1}^{6}J_k.\end{aligned} \tag{3.50}$$

By virtue of Hölder's inequality, Gagliardo–Nirenberg inequality, Sobolev's inequality, (3.2), and (3.20), we find that

$$\begin{aligned}|J_1| &= \left|-\int\rho\mathbf{u}\cdot\nabla|\mathbf{u}_t|^2\,\mathrm{d}x\right|\\ &\leqslant 2\|\rho\|_{L^\infty}^{1/2}\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\\ &\leqslant \frac{\mu}{12}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\\ &\leqslant \frac{\mu}{12}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2;\end{aligned}$$

$$\begin{aligned}|J_2| &= \left|-\int\rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t)\,\mathrm{d}x\right|\\ &\leqslant \int\left(\rho|\mathbf{u}||\nabla\mathbf{u}|^2|\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\right)\,\mathrm{d}x\\ &\leqslant C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^4}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\\ &\quad + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\\ &\leqslant C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^2}\|\nabla^2\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\\ &\quad + C\|\mathbf{u}\|_{H^2}\|\nabla\mathbf{u}_t\|_{L^2}\\ &\leqslant \frac{\mu}{12}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2;\end{aligned}$$

$$\begin{aligned}|J_3| &\leqslant C\|\nabla\mathbf{u}\|_{L^4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\mathbf{u}_t\|_{L^4}\\ &\leqslant C\|\mathbf{u}\|_{H^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\\ &\leqslant \frac{\mu}{12}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2;\end{aligned}$$

$$|J_4| = \left|-\int\mathbf{b}_t\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}\,\mathrm{d}x\right|$$

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$$
\begin{aligned}
&\leqslant \|\mathbf{b}_t\|_{L^3}\|\nabla \mathbf{u}_t\|_{L^2}\|\mathbf{b}\|_{L^6} \leqslant C\|\mathbf{b}_t\|_{L^2}^{1/2}\|\mathbf{b}_t\|_{L^6}^{1/2}\|\nabla \mathbf{u}_t\|_{L^2}\|\nabla \mathbf{b}\|_{L^2}\\
&\leqslant \frac{\mu}{12}\|\nabla \mathbf{u}_t\|_{L^2}^2 + C(\delta)\|\mathbf{b}_t\|_{L^2}^2 + \frac{\delta}{4}\|\nabla \mathbf{b}_t\|_{L^2}^2;\\
|J_5| &= \left| -\int \mathbf{b}\cdot\nabla \mathbf{u}_t\cdot \mathbf{b}_t \,\mathrm{d}x \right|\\
&\leqslant \|\mathbf{b}\|_{L^6}\|\nabla \mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^3} \leqslant C\|\nabla \mathbf{b}\|_{L^2}\|\nabla \mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^2}^{1/2}\|\mathbf{b}_t\|_{L^6}^{1/2}\\
&\leqslant \frac{\mu}{12}\|\nabla \mathbf{u}_t\|_{L^2}^2 + C(\delta)\|\mathbf{b}_t\|_{L^2}^2 + \frac{\delta}{4}\|\nabla \mathbf{b}_t\|_{L^2}^2;\\
|J_6| &= \left| \int (-\operatorname{div}(\rho\theta\mathbf{u}) + \kappa\Delta\theta)\mathbf{e}_2\cdot \mathbf{u}_t \,\mathrm{d}x \right|\\
&\leqslant \int \rho\theta|\mathbf{u}||\nabla \mathbf{u}_t| \,\mathrm{d}x + \kappa\int |\nabla\theta|\nabla \mathbf{u}_t| \,\mathrm{d}x\\
&\leqslant C\left(\|\mathbf{u}\|_{L^4}\|\rho\|_{L^\infty}\|\theta\|_{L^4} + \|\nabla\theta\|_{L^2}\right)\|\nabla \mathbf{u}_t\|_{L^2}\\
&\leqslant C\|\nabla\theta\|_{L^2}\|\nabla \mathbf{u}_t\|_{L^2}\\
&\leqslant \frac{\mu}{12}\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2.
\end{aligned}
$$

Substituting the above estimates into (3.50) and applying $(3.1)_4$, we derive that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \mu\|\nabla \mathbf{u}_t\|_{L^2}^2\\
&\leqslant C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2 + \delta\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\|\mathbf{b}_t\|_{L^2}^2\\
&\leqslant C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2 + \delta\|\nabla \mathbf{b}_t\|_{L^2}^2\\
&\quad + C\|\Delta \mathbf{b}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla \mathbf{b}\|_{L^2}^2 + C\|\mathbf{b}\|_{L^4}^2\|\nabla \mathbf{u}\|_{L^4}^2\\
&\leqslant C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2 + \delta\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\|\Delta \mathbf{b}\|_{L^2}^2.
\end{aligned}
\tag{3.51}
$$

Here we have used the following

$$
\begin{aligned}
\|\mathbf{b}_t\|_{L^2}^2 &\leqslant C\|\Delta \mathbf{b}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla \mathbf{b}\|_{L^2}^2\\
&\quad + C\|\mathbf{b}\|_{L^4}^2\|\nabla \mathbf{u}\|_{L^4}^2 \leqslant C\|\Delta \mathbf{b}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{H^1}^2,
\end{aligned}
\tag{3.52}
$$

due to $(3.1)_4$, (3.20), and Sobolev's inequality.

2. Differentiating $(3.1)_4$ with respect to t and multiplying the resulting equations by $\mathbf{b}_t$, we obtain from integration by parts and (3.20) that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{b}_t|^2 \,\mathrm{d}x + \nu\int |\nabla \mathbf{b}_t|^2 \,\mathrm{d}x \leqslant C\left(\||\mathbf{u}_t||\mathbf{b}|\|_{L^2} + \||\mathbf{u}||\mathbf{b}_t|\|_{L^2}\right)\|\nabla \mathbf{b}_t\|_{L^2}\\
&\leqslant C\left(\|\mathbf{u}_t\|_{L^6}\|\mathbf{b}\|_{L^3} + \|\mathbf{u}\|_{L^6}\|\mathbf{b}_t\|_{L^2}^{1/2}\|\mathbf{b}_t\|_{L^6}^{1/2}\right)\|\nabla \mathbf{b}_t\|_{L^2}
\end{aligned}
$$

$$\leqslant C\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}_t\|_{L^2}^{1/2}\|\nabla\mathbf{b}_t\|_{L^2}^{1/2}\right)\|\nabla\mathbf{b}_t\|_{L^2}$$
$$\leqslant \frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{b}_t\|_{L^2}^2,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{b}_t\|_{L^2}^2+\nu\|\nabla\mathbf{b}_t\|_{L^2}^2\leqslant C_2\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{b}_t\|_{L^2}^2. \tag{3.53}$$

Adding (3.51) multiplied by $2C_2$ to (3.53) and then choosing $\delta=\nu/4C_2$, we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(2C_2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+C_2\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2\\&\leqslant C\|\mathbf{u}\|_{H^2}^2\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+C\|\mathbf{u}\|_{H^2}^2+C\|\nabla\theta\|_{L^2}^2+C\|\Delta\mathbf{b}\|_{L^2}^2.\end{aligned} \tag{3.54}$$

From (3.31), (2.3), (3.4), and (3.20), we have

$$\|\mathbf{u}\|_{H^2}^2\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\mathrm{e}^{-2\sigma t}, \tag{3.55}$$

which combined with (3.20) and (3.4) implies that

$$\int_0^T\|\mathbf{u}\|_{H^2}^2\,\mathrm{d}t\leqslant C. \tag{3.56}$$

Multiplying (3.54) by t^m, we get from (3.52) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(2C_2t^m\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+t^m\|\mathbf{b}_t\|_{L^2}^2\right)+C_2t^m\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}t^m\|\nabla\mathbf{b}_t\|_{L^2}^2\\&\leqslant C\|\mathbf{u}\|_{H^2}^2\left(t^m\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+t^m\|\mathbf{b}_t\|_{L^2}^2\right)+Ct^m\|\mathbf{u}\|_{H^2}^2+Ct^m\|\nabla\theta\|_{L^2}^2\\&\quad+Ct^m\|\Delta\mathbf{b}\|_{L^2}^2+Ct^{m-1}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+Ct^{m-1}\|\mathbf{b}_t\|_{L^2}^2\\&\leqslant C\|\mathbf{u}\|_{H^2}^2\left(t^m\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+t^m\|\mathbf{b}_t\|_{L^2}^2\right)+Ct^m\|\mathbf{u}\|_{H^2}^2+Ct^m\|\nabla\theta\|_{L^2}^2\\&\quad+Ct^m\|\Delta\mathbf{b}\|_{L^2}^2+Ct^{m-1}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+Ct^{m-1}\|\Delta\mathbf{b}\|_{L^2}^2+Ct^{m-1}\|\mathbf{u}\|_{H^2}^2,\end{aligned} \tag{3.57}$$

which together with Gronwall's inequality, (3.56), (3.55), (3.5), (3.21), and (3.18) gives

$$\sup_{0\leqslant t\leqslant T}t^m\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\int_0^T t^m\left(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{b}_t\|_{L^2}^2\right)\mathrm{d}t\leqslant C. \tag{3.58}$$

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3. Differentiating $(3.1)_3$ with respect to t shows

$$\rho\theta_{tt} + \rho\mathbf{u}\cdot\nabla\theta_t - \kappa\Delta\theta_t = -\rho_t(\theta_t + \mathbf{u}\cdot\nabla\theta) - \rho\mathbf{u}_t\cdot\nabla\theta. \tag{3.59}$$

Multiplying (3.59) by θ_t and integrating the resulting equality over Ω yield that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\theta_t^2\,\mathrm{d}x + \kappa\int|\nabla\theta_t|^2\,\mathrm{d}x &= -\int\rho_t(\theta_t+\mathbf{u}\cdot\nabla\theta)\theta_t\,\mathrm{d}x\\ &\quad-\int\rho\theta_t\mathbf{u}_t\cdot\nabla\theta\,\mathrm{d}x =: I_1 + I_2.\end{aligned} \tag{3.60}$$

It follows from $(1.1)_1$, integration by parts, Hölder's inequality, Sobolev's inequality, (3.2), (3.20), (3.22), Gagliardo–Nirenberg inequality, and $(3.1)_3$ that

$$\begin{aligned}|I_1| &\leqslant C\int\rho|\mathbf{u}|\left(|\theta_t||\nabla\theta_t| + |\theta_t||\nabla\mathbf{u}||\nabla\theta| + |\theta_t||\mathbf{u}||\nabla^2\theta| + |\nabla\theta_t||\mathbf{u}||\nabla\theta|\right)\mathrm{d}x\\
&\leqslant C\|\rho\|_{L^\infty}^{3/4}\|\mathbf{u}\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^2}^{1/2}\|\theta_t\|_{L^6}^{1/2}\left(\|\nabla\theta_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4}\right)\\
&\quad + C\|\rho\|_{L^\infty}^{3/4}\|\mathbf{u}\|_{L^{12}}^2\|\sqrt{\rho}\theta_t\|_{L^2}^{1/2}\|\theta_t\|_{L^6}^{1/2}\|\nabla^2\theta\|_{L^2}\\
&\quad + C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}^2\|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^{1/2}\|\nabla\theta_t\|_{L^2}^{3/2} + C\|\sqrt{\rho}\theta_t\|_{L^2}^{1/2}\|\nabla\theta_t\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{H^1}^{1/2}\|\nabla\theta\|_{H^1}^{1/2}\\
&\quad + C\|\sqrt{\rho}\theta_t\|_{L^2}^{1/2}\|\nabla\theta_t\|_{L^2}^{1/2}\|\nabla^2\theta\|_{L^2} + C\|\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^2}\|\nabla\theta_t\|_{L^2}\\
&\leqslant \frac{\kappa}{4}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{H^1}^2\\
&\leqslant \frac{\kappa}{4}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C\|\rho\theta_t + \rho\mathbf{u}\cdot\nabla\theta\|_{L^2}^2\\
&\leqslant \frac{\kappa}{4}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2\|\mathbf{u}\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2\\
&\leqslant \frac{\kappa}{4}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2.\end{aligned} \tag{3.61}$$

By Hölder's inequality, (3.2), Sobolev's inequality, and (3.22), we have

$$\begin{aligned}|I_2| &\leqslant C\|\rho\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^4}\|\mathbf{u}_t\|_{L^4} \leqslant C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\\
&\leqslant \frac{\kappa}{4}\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2.\end{aligned} \tag{3.62}$$

Hence, substituting (3.61) and (3.62) into (3.60), we obtain from (3.31) and (3.76) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \kappa\|\nabla\theta_t\|_{L^2}^2 \leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2. \tag{3.63}$$

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Multiplying (3.63) by t^m, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^m\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\kappa t^m\|\nabla\theta_t\|_{L^2}^2\leqslant Ct^m\|\sqrt{\rho}\theta_t\|_{L^2}^2+Ct^m\|\mathbf{u}\|_{H^2}^2+Ct^m\|\nabla\theta\|_{L^2}^2 \\ +Ct^m\|\nabla\mathbf{u}_t\|_{L^2}^2+Ct^{m-1}\|\sqrt{\rho}\theta_t\|_{L^2}^2. \tag{3.64}$$

This along with Gronwall's inequality, (3.22), (3.56), (3.10), and (3.58) yields

$$\sup_{0\leqslant t\leqslant T}\left(t^m\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\int_0^T t^m\|\nabla\theta_t\|_{L^2}^2\,\mathrm{d}t\leqslant C. \tag{3.65}$$

Hence, the desired (3.48) follows from (3.58) and (3.65).

□

LEMMA 3.5. *Let q be as in theorem 1.1 and m be as in lemma 3.2, then there exists a positive constant C depending only on Ω, μ, ν, κ, $\|\rho_0\|_{L^\infty}$, $\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{b}_0\|_{L^2}$, $\|\nabla\theta_0\|_{L^2}$, m, and q such that for $r\in[2,q)$,*

$$\sup_{0\leqslant t\leqslant T}\left(\|\rho\|_{W^{1,q}}+\|\rho_t\|_{L^r}\right)\leqslant C. \tag{3.66}$$

Proof. 1. We infer from Sobolev's embedding theorem, lemma 2.3, (3.2), (3.22), (2.1), and Young's inequality that

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^\infty}&\leqslant C\|\mathbf{u}\|_{W^{2,3}}\\&\leqslant C\left(\|\rho\mathbf{u}_t\|_{L^3}+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^3}+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^3}+\|\rho\theta\|_{L^3}\right)\\&\leqslant C\|\rho\mathbf{u}_t\|_{L^3}+C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^3}+C\|\mathbf{b}\|_{L^{12}}\|\nabla\mathbf{b}\|_{L^4}+C\|\nabla\theta\|_{L^2}\\&\leqslant C\|\rho\mathbf{u}_t\|_{L^3}+C\|\mathbf{u}\|_{H^2}^2+C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}^{1/2}\|\nabla\mathbf{b}\|_{H^1}^{1/2}+C\mathrm{e}^{-(\sigma/2)t}\\&\leqslant C\|\rho\mathbf{u}_t\|_{L^3}+C\|\mathbf{u}\|_{H^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^{3/2}\|\Delta\mathbf{b}\|_{L^2}^{1/2}+C\mathrm{e}^{-(\sigma/2)t}\\&\leqslant C\|\rho\mathbf{u}_t\|_{L^3}+C\|\mathbf{u}\|_{H^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2+C\|\Delta\mathbf{b}\|_{L^2}^2+C\mathrm{e}^{-(\sigma/2)t}\end{aligned} \tag{3.67}$$

By Hölder's inequality, Sobolev's inequality, and (3.2), we have

$$\|\rho\mathbf{u}_t\|_{L^3}\leqslant\|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{1/2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{1/2}\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{1/2}\|\nabla\mathbf{u}_t\|_{L^2}^{1/2},$$

which together with Hölder's inequality implies for any $0\leqslant a<b<\infty$,

$$\begin{aligned}\int_a^b\|\rho\mathbf{u}_t\|_{L^3}\,\mathrm{d}t&\leqslant C\int_a^b t^{-(3/8)}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{1/2}\cdot t^{3/8}\|\nabla\mathbf{u}_t\|_{L^2}^{1/2}\mathrm{d}t\\&\leqslant C\Big[\int_a^b t^{-(1/2)}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{2/3}\mathrm{d}t\Big]^{3/4}\times\Big[\int_a^b t^{3/2}\|\nabla\mathbf{u}_t\|_{L^2}^2\,\mathrm{d}t\Big]^{1/4}.\end{aligned} \tag{3.68}$$

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As a consequence, if $T \leqslant 1$, we obtain from (3.68) and (3.48) that

$$
\begin{aligned}
&\int_0^T \|\rho \mathbf{u}_t\|_{L^3}\,\mathrm{d}t \\
&\leqslant C\left[\int_0^T t^{-\frac{1}{2}}\cdot t^{-\frac{1}{3}}t^{\frac{1}{3}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{2}{3}}\,\mathrm{d}t\right]^{\frac{3}{4}} \times \left[\int_0^T t^{1/2}\|\nabla \mathbf{u}_t\|_{L^2}\cdot t\|\nabla \mathbf{u}_t\|_{L^2}\,\mathrm{d}t\right]^{\frac{1}{4}} \\
&\leqslant C\left(\sup_{0\leqslant t\leqslant T} t\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{1}{4}}\left(\int_0^T t^{-\frac{5}{6}}\,\mathrm{d}t\right)^{\frac{3}{4}} \\
&\quad \times \left(\int_0^T t\|\nabla \mathbf{u}_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{1}{8}}\left(\int_0^T t^2\|\nabla \mathbf{u}_t\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{8}} \\
&\leqslant CT^{\frac{1}{8}} \leqslant C.
\end{aligned} \tag{3.69}
$$

If $T > 1$, one deduces from (3.69), (3.68), and (3.48) that

$$
\begin{aligned}
&\int_0^T \|\rho \mathbf{u}_t\|_{L^3}\,\mathrm{d}t \\
&= \int_0^1 \|\rho \mathbf{u}_t\|_{L^3}\,\mathrm{d}t + \int_1^T \|\rho \mathbf{u}_t\|_{L^3}\,\mathrm{d}t \\
&\leqslant C + C\left[\int_1^T t^{-\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{2}{3}}\,\mathrm{d}t\right]^{\frac{3}{4}} \times \left[\int_1^T t^{1/2}\|\nabla \mathbf{u}_t\|_{L^2}\cdot t\|\nabla \mathbf{u}_t\|_{L^2}\,\mathrm{d}t\right]^{\frac{1}{4}} \\
&\leqslant C + C\left(\sup_{1\leqslant t\leqslant T} t^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{1}{4}}\left(\int_1^T t^{-\frac{1}{2}}\cdot t^{-\frac{2}{3}}\,\mathrm{d}t\right)^{3/4} \\
&\quad \times \left(\int_1^T t\|\nabla \mathbf{u}_t\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{8}}\left(\int_1^T t^2\|\nabla \mathbf{u}_t\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{8}} \\
&\leqslant C + C\left(1 - T^{-\frac{1}{6}}\right)^{\frac{3}{4}} \leqslant C.
\end{aligned} \tag{3.70}
$$

Hence, we derive from (3.67), (3.69), (3.70), (3.56), (3.13), and (3.20) that

$$\int_0^T \|\nabla \mathbf{u}\|_{L^\infty}\,\mathrm{d}t \leqslant C. \tag{3.71}$$

2. Taking spatial derivative ∇ on the transport equation $(3.1)_1$ leads to

$$\partial_t \nabla\rho + \mathbf{u}\cdot\nabla^2\rho + \nabla\mathbf{u}\cdot\nabla\rho = \mathbf{0}.$$

Thus standard energy methods yield for any $q \in (2, \infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q} \leqslant C(q)\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q},$$

which combined with Gronwall's inequality and (3.71) gives

$$\sup_{0\leqslant t\leqslant T}\|\nabla\rho\|_{L^q}\leqslant C. \tag{3.72}$$

Noticing the following fact

$$\|\rho_t\|_{L^r}=\|\mathbf{u}\cdot\nabla\rho\|_{L^r}\leqslant\|\nabla\rho\|_{L^q}\|\mathbf{u}\|_{L^{\frac{qr}{q-r}}}\leqslant\|\nabla\rho\|_{L^q}\|\nabla\mathbf{u}\|_{L^2},$$

which together with (3.72) and (3.20) yields

$$\sup_{0\leqslant t\leqslant T}\|\rho_t\|_{L^r}\leqslant C. \tag{3.73}$$

Thus, the desired (3.66) follows from (3.2), (3.72), and (3.73).

□

LEMMA 3.6. *Let m be as in lemma 3.2 and q be as in theorem 1.1, then there exists a positive constant C depending only on Ω, μ, ν, κ, $\|\rho_0\|_{L^\infty}$, $\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{b}_0\|_{L^2}$, $\|\nabla\theta_0\|_{L^2}$, m, and q such that*

$$\sup_{0\leqslant t\leqslant T}t^m\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2+\|\mathbf{b}\|_{H^2}^2\right) \tag{3.74}$$

$$+\int_0^T t^m\left(\|\mathbf{u}\|_{H^3}^2+\|\nabla P\|_{H^1}^2+\|\theta\|_{H^3}^2+\|\mathbf{b}\|_{H^3}^2\right)\mathrm{d}t$$

$$\leqslant C. \tag{3.75}$$

Proof. 1. It follows from $(3.1)_3$, (3.2), Sobolev's inequality, (2.1), and (3.20) that

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}^2&\leqslant C\|\rho\theta_t\|_{L^2}^2+C\|\rho|\mathbf{u}||\nabla\theta|\|_{L^2}^2\\&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\mathbf{u}\|_{L^4}^2\|\nabla\theta\|_{L^4}^2\\&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{H^1}\\&\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2,\end{aligned}$$

which combined with (3.22) and (3.65) leads to

$$\sup_{0\leqslant t\leqslant T}\left(t^m\|\nabla^2\theta\|_{L^2}^2\right)\leqslant C. \tag{3.76}$$

We derive from the regularity theory of elliptic system, $(3.1)_4$, and (3.20) that

$$\begin{aligned}\|\mathbf{b}\|_{H^2}^2&\leqslant C\left(\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{H^1}^2\right)\\&\leqslant C\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^3}^2+C\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2\\&\leqslant C\|\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^6}+C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}^2\\&\leqslant C\|\mathbf{b}_t\|_{L^2}^2+\frac{1}{2}\|\mathbf{b}\|_{H^2}^2,\end{aligned}$$

which together with (3.58) yields

$$\sup_{0\leqslant t\leqslant T}\left(t^m\|\mathbf{b}\|_{H^2}^2\right)\leqslant C. \tag{3.77}$$

From lemma 2.3, (3.55), (3.77), and Sobolev's inequality, we have

$$\begin{aligned}\|\mathbf{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2 &\leqslant C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\mathrm{e}^{-2\sigma t}\right)\\ &\leqslant C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\mathrm{e}^{-2\sigma t}\right)\\ &\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+\mathrm{e}^{-2\sigma t}.\end{aligned}$$

This along with (3.21) and (3.58) yields

$$\sup_{0\leqslant t\leqslant T} t^m\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2\right)\leqslant C. \tag{3.78}$$

2. We obtain from (3.30), (3.2), (3.9), (3.77), (3.72), and Sobolev's inequality, we have

$$\begin{aligned}&\|\mathbf{u}\|_{H^3}^2+\|\nabla P\|_{H^1}^2\\ &\leqslant C\left(\|\rho\mathbf{u}_t\|_{H^1}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^1}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{H^1}^2+\|\rho\theta\|_{H^1}^2\right)\\ &\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2+C\mathrm{e}^{-2\sigma t}+C\|\nabla(\rho\mathbf{u}_t)\|_{L^2}^2\\ &\quad+C\|\nabla(\rho\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}^2+C\|\nabla(\mathbf{b}\cdot\nabla\mathbf{b})\|_{L^2}^2+C\|\nabla(\rho\theta)\|_{L^2}^2\\ &\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2+C\mathrm{e}^{-2\sigma t}\\ &\quad+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\rho\|_{L^q}^2\|\mathbf{u}_t\|_{L^{\frac{2q}{q-2}}}^2\\ &\quad+C\|\mathbf{u}\|_{H^2}^2+C\|\nabla\rho\|_{L^q}^2\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^{\frac{2q}{q-2}}}^2+C\|\mathbf{b}\|_{H^2}^2\\ &\quad+C\|\nabla\theta\|_{L^2}^2+C\|\nabla\rho\|_{L^q}^2\|\theta\|_{L^{\frac{2q}{q-2}}}^2\\ &\leqslant C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\mathrm{e}^{-2\sigma t}+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2\\ &\quad+C\|\nabla\mathbf{b}\|_{L^2}^2+C\|\Delta\mathbf{b}\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2\\ &\leqslant C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\Delta\mathbf{b}\|_{L^2}^2\right)+C\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)+C\mathrm{e}^{-2\sigma t}\\ &\quad+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2,\end{aligned}$$

which together with (3.21), (3.5), (3.18), (3.58), and (3.3) yields

$$\int_0^T t^m\left(\|\mathbf{u}\|_{H^3}^2+\|\nabla P\|_{H^1}^2\right)\mathrm{d}t\leqslant C. \tag{3.79}$$

Similarly, one gets

$$\int_0^T t^m\|\theta\|_{H^3}^2\,\mathrm{d}t\leqslant C,\quad \int_0^T t^m\|\mathbf{b}\|_{H^3}^2\,\mathrm{d}t\leqslant C. \tag{3.80}$$

Hence, the desired (3.74) follows from (3.76)–(3.80).

□

4. Proof of theorem 1.1

With the a priori estimates in § 3 at hand, we are now in a position to prove theorem 1.1.

By lemma 2.1, there exists a $T_* > 0$ such that the problem (1.1)–(1.3) has a unique local strong solution $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ on $\Omega \times (0, T_*]$. We plan to extend the local solution to all time.

Set

$$T^* = \sup\{T \mid (\rho, \mathbf{u}, \theta, \mathbf{b}) \text{ is a strong solution on } \Omega \times (0, T]\}. \tag{4.1}$$

First, for any $0 < \tau < T_* < T \leqslant T^*$ with T finite, we deduce from (3.48), (3.74), and [**9**, theorem 4, p. 304] that

$$\nabla \mathbf{u},\ \nabla \theta,\ \nabla \mathbf{b} \in C([\tau, T]; H^1). \tag{4.2}$$

Moreover, it follows from (3.66) and [**17**, lemma 2.3] that

$$\rho \in C([0, T]; W^{1,q}). \tag{4.3}$$

Owing to (3.2) and (3.20), we get

$$\rho \mathbf{u}_t = \sqrt{\rho} \cdot \sqrt{\rho} \mathbf{u}_t \in L^2(0, T; L^2).$$

From (3.73), (3.4), and Sobolev's inequality, one has

$$\rho_t \mathbf{u} \in L^2(0, T; L^2).$$

Thus, we arrive at

$$(\rho \mathbf{u})_t = \rho \mathbf{u}_t + \rho_t \mathbf{u} \in L^2(0, T; L^2). \tag{4.4}$$

From (3.2) and (3.12), we have

$$\rho \mathbf{u}) = \sqrt{\rho} \cdot \sqrt{\rho} \mathbf{u} \in L^\infty(0, T; L^2),$$

which combined with (4.4) yields

$$\rho \mathbf{u} \in C([0, T]; L^2). \tag{4.5}$$

Similarly, we can derive

$$\rho \theta \in C([0, T]; L^2) \ \text{ and } \ \mathbf{b} \in C([0, T]; L^2). \tag{4.6}$$

Finally, if $T^* < \infty$, it follows from (4.2), (4.3), (3.20), and (3.22) that

$$(\rho, \mathbf{u}, \theta, \mathbf{b})(x, T^*) = \lim_{t \to T^*} (\rho, \mathbf{u}, \theta, \mathbf{b})(x, t)$$

satisfies the initial condition (1.4) at $t = T^*$. Thus, taking $(\rho, \mathbf{u}, \theta, \mathbf{b})(x, T^*)$ as the initial data, lemma 2.1 implies that one can extend the strong solutions beyond T^*. This contradicts the assumption of T^* in (4.1). Furthermore, the other estimates as those in (1.5) and (1.6) follow from lemmas 3.2–3.6. The proof of theorem 1.1 is complete. □

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