Combinatorics, Probability and Computing (2013) 22, 433–454. © Cambridge University Press 2013
doi:10.1017/S0963548313000047

The Size of the Largest Part of Random Weighted Partitions of Large Integers

LJUBEN MUTAFCHIEV

American University in Bulgaria, 2700 Blagoevgrad, Bulgaria
and
Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences
(e-mail: ljuben@aubg.bg)

Received 25 July 2011; revised 22 January 2013; first published online 21 February 2013

We consider partitions of the positive integer n whose parts satisfy the following condition. For a given sequence of non-negative numbers $\{b_k\}_{k\geqslant 1}$, a part of size k appears in exactly b_k possible types. Assuming that a weighted partition is selected uniformly at random from the set of all such partitions, we study the asymptotic behaviour of the largest part X_n. Let $D(s) = \sum_{k=1}^{\infty} b_k k^{-s}, s = \sigma + iy$, be the Dirichlet generating series of the weights b_k. Under certain fairly general assumptions, Meinardus (1954) obtained the asymptotic of the total number of such partitions as $n \to \infty$. Using the Meinardus scheme of conditions, we prove that X_n, appropriately normalized, converges weakly to a random variable having Gumbel distribution (*i.e.*, its distribution function equals $e^{-e^{-t}}, -\infty < t < \infty$). This limit theorem extends some known results on particular types of partitions and on the Bose–Einstein model of ideal gas.

AMS 2010 *Mathematics subject classification*: Primary 05A17
Secondary 60C05, 60F05

1. Introduction and statement of the result

A weighted partition of the positive integer n is a multiset of size n whose decomposition into a union of disjoint components (parts) satisfies the following condition: for a given sequence of non-negative numbers $\{b_k\}_{k\geqslant 1}$, a part of size k appears in exactly one of b_k possible types. For more details on properties of multisets, we refer the reader to [3], for example. Weighted partitions are also associated with the generalized Bose–Einstein model of ideal gas, where $n(= E)$ is interpreted as the total energy of the system of particles. The weights $b_k, k \geqslant 1$, are viewed as counts of the distinct positions of the particles in the state space, where a particle in a given position has (rescaled) energy k (for more details on the relationship between combinatorial partitions and various models of ideal gas, see [21]). From combinatorial point of view, it is fairly natural to assume that $b_k, k \geqslant 1$, are integers (see, *e.g.*, the 'money changing problem' discussed in detail in [24, § 3.15]). On the

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

other hand, it turns out that this requirement is not necessary for the analytical approach used in this paper. That is why we assume that $b_k, k \geqslant 1$, are real non-negative numbers.

For a given sequence $b = \{b_k, k \geqslant 1\}$, let $\mathcal{P}_b(n)$ be the set of all weighted partitions of the positive integer n and let $p_b(n) = |\mathcal{P}_b(n)|$ be its cardinality. It is known that the generating function $f_b(x)$ of the numbers $p_b(n)$ is of Euler's type, namely,

$$f_b(x) = 1 + \sum_{n=1}^{\infty} p_b(n)x^n = \prod_{k=1}^{\infty}(1 - x^k)^{-b_k}, \quad |x| < 1 \tag{1.1}$$

(see [24, § 3.14]). We introduce the uniform probability measure $\mathbb{P} = \mathbb{P}_{n,b}$ on the set of weighted partitions of n assuming that the probability $1/p_b(n)$ is assigned to each n-partition with weight sequence b. In this paper we focus on the size of the largest part X_n of a random weighted partition of n. With respect to the probability measure $\mathbb{P}$, X_n becomes a random variable, defined on the set $\mathcal{P}_b(n)$. It is also well known that

$$f_{m,b}(x) = 1 + \sum_{n=1}^{\infty} p_b(n)\mathbb{P}(X_n \leqslant m)x^n = \prod_{k=1}^{m}(1 - x^k)^{-b_k}, \quad m \geqslant 1 \tag{1.2}$$

(see [24, § 3.15]).

The asymptotic behaviour of the combinatorial numbers $p_b(n)$ (the Taylor coefficients in (1.1)) will play an important role in our further analysis. A fairly general scheme of assumptions on the parametric sequence b was proposed by Meinardus [13] (see also [2, Ch. 6]), who found an asymptotic expansion for the numbers $p_b(n)$ as $n \to \infty$. The same asymptotic was also obtained in [10], where one of the Meinardus conditions was weakened. The Meinardus approach is based on considering two generating series:

$$D(s) = \sum_{k=1}^{\infty} b_k k^{-s}, \quad s = \sigma + iy, \tag{1.3}$$

and

$$G(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |z| \leqslant 1. \tag{1.4}$$

Below we give the Meinardus scheme of conditions. Throughout the paper we use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary part, respectively, of the complex number z.

(M_1) The Dirichlet series (1.3) converges in the half-plane $\sigma > \rho > 0$, and there is a constant $C_0 \in (0, 1)$ such that the function $D(s)$ has an analytic continuation to the half-plane $\{s : \sigma \geqslant -C_0\}$ on which it is analytic, except for the simple pole at $s = \rho$ with residue $A > 0$.

(M_2) There exists a constant $C_1 > 0$ such that

$$D(s) = O(|y|^{C_1}), \quad |y| \to \infty,$$

uniformly for $\sigma \geqslant -C_0$.

(M_3) There are constants $\epsilon > 0$ and $C_2(= C_2(\epsilon) > 0)$ such that the function $g(\tau) = G(e^{-\tau})$, $\tau = \alpha + 2\pi iu$, u real and $\alpha > 0$ (see (1.4)), satisfies

$$\Re(g(\tau)) - g(\alpha) \leqslant -C_2\alpha^{-\epsilon}, \quad |\arg(\tau)| > \pi/4, \quad 0 \neq |u| \leqslant 1/2,$$

for sufficiently small values of α.

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

The first assumption (M_1) specifies the domain, say $\mathcal{H}$, in which $D(s)$ has an analytic continuation. The second is related to the asymptotic behaviour of $D(s)$, whenever $|\Im(s)| \to \infty$. Functions which are bounded by $O(|\Im(s)|^q), 0 < q < \infty$, in a certain domain, as $|\Im(s)| \to \infty$, are called functions of finite order. It is known that the sum of the Dirichlet series in (1.3) satisfies the finite order property in a closed half-plane contained in the half-plane of convergence $\sigma > \rho$ (see, *e.g.*, [20, § 9.3.3]). The Meinardus second condition requires that the same holds for the analytic continuation of $D(s)$ in the whole domain $\mathcal{H}$. Finally, the Meinardus third condition implies a bound on $\Re(G(e^{-\tau}))$ (see (1.4)) for certain specific complex values of τ. In some cases its verification is technically complicated. Since

$$\Re(g(\tau)) - g(\alpha) = -2\sum_{k=1}^{\infty} b_k e^{-k\alpha} \sin^2(\pi k u) \tag{1.5}$$

and the inequality $|\arg(\tau)| > \pi/4$ implies that $\tan(|\arg(\tau)|) = 2\pi|u|/\alpha > 1$, condition ($M_3$) can also be reformulated as follows:

$$S_n := \sum_{k=1}^{\infty} b_k e^{-k\alpha} \sin^2(\pi k u) \geqslant C_2 \alpha^{-\epsilon}, \quad 0 < \frac{\alpha}{2\pi} < |u| \leqslant 1/2, \tag{1.6}$$

for sufficiently small α and some constants $C_2, \epsilon > 0$ ($C_2 = C_2(\epsilon)$) [10, p. 310].

Moreover, Granovsky, Stark and Erlihson [10, Lemma 1] proved that this inequality holds for any sequence $b_k, k \geqslant 1$, satisfying the inequality $b_k \geqslant Ck^{\nu-1}, k \geqslant k_0$, for some $k_0 \geqslant 1$ and $C, \nu > 0$. We notice that if

$$b_k = Ck^{\nu-1}, \quad k \geqslant 1, \tag{1.7}$$

then $D(s) = C\zeta(s - \nu + 1)$, where ζ denotes the Riemann zeta function. Therefore, $D(s)$ has a single pole at $s = \nu$ with residue $C > 0$ and a meromorphic analytic continuation to the whole complex plane [23, § 13.13]. These facts show that conditions (M_1)–(M_3) are satisfied by the weights (1.7) with $\rho = \nu$ and $A = C$.

Throughout the paper we assume that conditions (M_1)–(M_3) are satisfied. Our aim is to determine asymptotically, as $n \to \infty$, the distribution of the maximal part size X_n. Recalling (1.2), we also point out that our results may be interpreted in terms of the asymptotic of the combinatorial counts of partitions whose part sizes are $\leqslant m$, where the range of the values of m is specified by the weak convergence of the random variable X_n to a non-degenerate random variable. In the brief review given below we summarize some known results on the limiting behaviour of the random variable X_n.

First consider the classical case of linear integer partitions, where the weights satisfy $b_k = 1, k \geqslant 1$. This kind of partition has been generally studied by many authors in many respects. Their graphical representations by Ferrers diagrams show that their total number of parts and their maximal part size X_n are identically distributed for all n (see [2, § 1.3]). Erdős and Lehner [6] were apparently the first to apply a probabilistic approach to the study of integer partitions. In fact they found an appropriate normalization for X_n in this case and showed that $\pi X_n/(6n)^{1/2} - \log((6n)^{1/2}/\pi)$ converges weakly, as $n \to \infty$, to a random variable having the extreme value (Gumbel) distribution. The local version of their theorem was derived later by Auluck, Chowla and Gupta [4]. Fristedt [9] studied linear

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

integer partitions using a transfer method to functionals of independent and geometrically distributed random variables. Among other results, he obtained the limiting distribution of the kth largest part size whenever k is fixed. Finally, we notice that among weighted integer partitions only the linear ones possess the property that the number of parts and the maximum part size are identically distributed. The limiting distribution of the number of parts in the general case of random weighted partitions under the Meinardus scheme of conditions is studied in [15]. It turns out that the limiting distribution laws depend on particular ranges in which the parameter ρ varies (see condition (M_1)).

Another important particular case of weighted partitions arises whenever $b_k = k, k \geqslant 1$. It turns out that in this case the generating function $f_b(x)$ (see (1.1)) enumerates the plane partitions. A plane partition of $n \geqslant 1$ is a matrix of non-negative integers arranged in non-increasing order from left to right and from top to bottom, so that the double sum of its elements equals n. Together with the largest part size X_n, consider also the counts of the non-zero rows and columns of the matrix of a plane partition. It turns out that these three quantities measure the sizes of the corresponding solid diagram of a plane partition in the $3D$ space. (The solid diagram is a heap of n unit cubes placed in the first octant of a coordinate system in a $3D$ space whose columns composed by stacked cubes have non-increasing heights along the x- and y-axis; the height of this heap along the z-axis is just X_n, the largest part size.) Similarly to Ferrers diagrams for linear integer partitions, the three sizes of this heap appear to be identically distributed for all $n \geqslant 1$ (for more details, see [19, p. 371]). Their joint limiting distribution was found in [17]. The marginal limiting distributions (including the limiting distribution of X_n) were obtained in [14]. For more details on various properties of plane partitions and their applications to combinatorics and analysis of algorithms, we refer the reader to [2, Ch. 11], [16, Ch. 11] and [19, Ch. 7].

Our study is also closely related to some recent results on the maximal particle energy in the Bose–Einstein model of ideal gas. The general setting and the probabilistic frame of problems from statistical mechanics and their relationship with enumerative combinatorics were given by Vershik [21]. In the context of the infinite product formula (1.1), he studied the Bose–Einstein model by a family of probability measures $\mu_v, v \in (0,1)$, defined on the set of all b-weighted partitions $\mathcal{P}_b = \cup_{n\geqslant 0}\mathcal{P}_b(n)$. So, for a partition $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{P}, \lambda_1 \geqslant \cdots \geqslant \lambda_l > 0$, let $r_k(\lambda) = \{j : \lambda_j = k\}$ denote the number of parts of λ that are equal to $k \geqslant 1$. Then μ_v is defined by

$$\mu_v(\{\lambda \in \mathcal{P} : r_k(\lambda) = j\}) = \binom{b_k + j - 1}{j} v^{kj}(1 - v^k)^{b_k}, \quad 0 < v < 1.$$

The key feature in the study of this type of measure is the fact that a kind of conditional probability measure on $\mathcal{P}_b(n)$ turns out to be independent of v for all n and coincides with the uniform probability measure $\mathbb{P} = \mathbb{P}_{n,b}$ (for more details, see [21]). In [22] Vershik and Yakubovich studied the limiting distribution of the maximal particle energy $X(\lambda)$, or, equivalently, the largest part size $X(\lambda), \lambda \in \mathcal{P}$, with respect to the measure μ_v, as $v \to 1^-$. In particular, under the assumption that the weight sequence b satisfies (1.7), they proved

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

that

$$\lim_{v\to 1^-} \mu_v(\{\lambda \in \mathcal{P} : (1-v)X(\lambda) - \nu|\log(1-v)| - (\nu-1)\log|\log(1-v)| - (\nu-1)\log\nu - \log C \leqslant t\}) = e^{-e^{-t}}, \quad -\infty < t < \infty. \tag{1.8}$$

As mentioned earlier, the weight sequence (1.7) satisfies conditions (M_1)–(M_3). Vershik and Yakubovich [22] also studied a more realistic model of quantum ideal gas in an N-dimensional space, for which the weights satisfy $\sum_{j=1}^k b_j = K_N k^{N/2} + O(k^{\kappa_N})$ as $k \to \infty$ (K_N and $\kappa_N < N/2$ are computable constants).

The main result of this paper is obtained in terms of the uniform probability measure $\mathbb{P} = \mathbb{P}_{n,b}$ on the set $\mathcal{P}_b(n)$. Before stating it, for the sake of brevity, we introduce the following notation:

$$a(n) = a(n;\rho, A) = \left(\frac{A\Gamma(\rho+1)\zeta(\rho+1)}{n}\right)^{\frac{1}{\rho+1}}, \quad n \geqslant 1, \tag{1.9}$$

where the constants ρ and A are defined by condition (M_1).

Theorem 1.1. *If the weight sequence b, satisfies conditions (M_1)–(M_3), then, for all real t, the limiting distribution of the largest part size X_n is given by*

$$\lim_{n\to\infty} \mathbb{P}(a(n)X_n + \rho\log a(n) - (\rho-1)\log|\log a(n)| - (\rho-1)\log\rho - \log A \leqslant t) = e^{-e^{-t}}. \tag{1.10}$$

Remark. One can easily compare (1.10) with (1.8), setting in the latter $v = 1 - a(n)$, $\nu = \rho$ and $C = A$, and observe the coinciding normalizations. We also notice that the limiting results for linear and plane partitions (see [6] and [14]) follow from (1.10) with $A = 1$ and $\rho = 1$ and 2, respectively.

The method of our proof combines Hayman's theorem for estimating coefficients of admissible power series [11] (see also [7, § VIII.5]), a generalization of the Perron formula, which yields the expression for partial sums of a Dirichlet series by a complex integral of the inverse Mellin transform applied to the Dirichlet series itself (see Theorem 3.1 from [18, Supplement]) and some Mellin transform computations.

We organize our paper as follows. Section 2 includes some auxiliary facts that we will need later. Some proofs are omitted since they are given in [10] and [13]. In Section 3 we present the proof of Theorem 1.1. The Appendix contains some technical details related to the application of the generalized Perron formula [18].

2. Preliminary results

We start with a lemma establishing an asymptotic estimate for infinite product representation (1.1) of the generating function $f_b(x)$. It has been proved by Meinardus [13] (see also [2, Lemma 6.1]).

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Lemma 2.1. *Suppose that sequence b is such that the associated Dirichlet series* (1.3) *satisfies conditions* (M_1) *and* (M_2). *If $\tau = \alpha + i\theta$, then*

$$f_b(e^{-\tau}) = \exp\left(A\Gamma(\rho)\zeta(\rho+1)\tau^{-\rho} - D(0)\log\tau + D'(0) + O(\alpha^{C_0})\right)$$

as $\alpha \to 0^+$, uniformly for $|\theta| \leqslant \pi$ and $|\arg\tau| \leqslant \pi/4$.

Our first goal is to show that the Meinardus conditions (M_1)–(M_3) imply that the generating function $f_b(x)$ possesses Hayman's admissibility properties [11] (see also [7, § VIII.5]) in the unit disc. For $0 < r < 1$, we introduce the functions

$$F_b(r) = \log f_b(r) = -\sum_{k=1}^{\infty} b_k \log(1 - r^k), \tag{2.1}$$

$$\mathcal{A}_b(r) = rF_b'(r) = r\frac{f_b'(r)}{f_b(r)}, \tag{2.2}$$

$$\mathcal{B}_b(r) = r^2 F_b''(r) + rF_b'(r) = r\frac{f_b'(r)}{f_b(r)} + r^2\frac{f_b''(r)}{f_b(r)} - r^2\left(\frac{f_b'(r)}{f_b(r)}\right)^2. \tag{2.3}$$

Furthermore, setting $r = e^{-\alpha}$ in (2.1)–(2.3), we shall obtain their asymptotic expansions as $\alpha \to 0^+$. For the sake of convenience, we also set

$$h = h(\rho, A) = A\Gamma(\rho+1)\zeta(\rho+1). \tag{2.4}$$

The proof of the next lemma is contained in [10, Lemma 2].

Lemma 2.2. *Conditions* (M_1) *and* (M_2) *imply the following asymptotic expansions:*

$$\mathcal{A}_b(e^{-\alpha}) = h\alpha^{-\rho-1} + D(0)\alpha^{-1} + D'(0) + O(\alpha^{C_0-1}), \tag{2.5}$$

$$\mathcal{B}_b(e^{-\alpha}) = \frac{d}{d\alpha}(-\mathcal{A}_b(e^{-\alpha})) = h(\rho+1)\alpha^{-\rho-2} + D(0)\alpha^{-2} + O(\alpha^{C_0-2}), \tag{2.6}$$

$$F_b'''(e^{-\alpha}) = O(\alpha^{-\rho-3}), \tag{2.7}$$

as $\alpha \to 0^+$, where $F_b, \mathcal{A}_b$ and $\mathcal{B}_b$ are defined by (2.1)–(2.3)*, respectively. Moreover, from* (2.5) *it follows that the equation*

$$\mathcal{A}_b(e^{-\alpha}) = n, \quad n \geqslant 1, \tag{2.8}$$

has a unique solution $\alpha = \alpha_n$, such that $\alpha_n \to 0$ as $n \to \infty$. An asymptotic expansion of this solution, as $n \to \infty$, is given by

$$\alpha_n = a(n) + \frac{D(0)}{(\rho+1)n} + O(n^{-1-\beta}), \tag{2.9}$$

where $\beta = \min\left(\frac{C_0}{\rho+1}, \frac{\rho}{\rho+1}\right)$ and $a(n)$ are the normalizing constants given by (1.9).

We notice that (2.5), (2.6) and (2.9) imply that

$$\mathcal{A}_b(e^{-\alpha_n}) \to \infty, \quad \mathcal{B}_b(e^{-\alpha_n}) \to \infty, \quad n \to \infty, \tag{2.10}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

that is, Hayman's 'capture' condition [7, p. 565] is satisfied with $r = r_n = e^{-\alpha_n}$. Our next step is to establish Hayman's 'locality' condition, which implies the asymptotic behaviour of $f_b(x)$ in a suitable neighbourhood of $x = 1$.

Lemma 2.3. *Suppose that the weight sequence b satisfies conditions* (M_1) *and* (M_2)*, and* α_n *is the solution of* (2.8) *given by* (2.9)*. Let*

$$\delta_n = \alpha_n^{1+\rho/3}/\omega(n), \quad n \geqslant 1, \tag{2.11}$$

where $\omega(n) \to \infty$ *as* $n \to \infty$ *arbitrarily slowly. Then*

$$e^{-i\theta n}\frac{f_b(e^{-\alpha_n+i\theta})}{f_b(e^{-\alpha_n})} = e^{-\theta^2\mathcal{B}_b(e^{-\alpha_n})/2}(1 + O(1/\omega^3(n))) \tag{2.12}$$

uniformly for $|\theta| \leqslant \delta_n$.

Proof. Applying Lemma 2.1, we observe that

$$e^{-i\theta n}\frac{f_b(e^{-\alpha_n+i\theta})}{f_b(e^{-\alpha_n})} = \exp\left(\frac{h}{\rho}((\alpha_n - i\theta)^{-\rho} - \alpha_n^{-\rho}) - D(0)\log\left(1 - \frac{i\theta}{\alpha_n}\right) - i\theta n + O(\alpha_n^{C_0})\right), \tag{2.13}$$

where h is given by (2.4). Expanding $(\alpha_n - i\theta)^{-\rho}$ and $\log(1 - i\theta/\alpha_n)$ by Taylor's formula and using (2.5), (2.6) and (2.8), we obtain

$$\begin{aligned}
&\frac{h}{\rho}((\alpha_n - i\theta)^{-\rho} - \alpha_n^{-\rho}) - D(0)\log\left(1 - \frac{i\theta}{\alpha_n}\right) - i\theta n \\
&\quad = i\theta(h\alpha_n^{-\rho-1} + D(0)\alpha_n^{-1} + D'(0) - n) - \frac{\theta^2}{2}h(\rho+1)\alpha_n^{-\rho-2} - \frac{D(0)\theta^2}{2\alpha_n^2} \\
&\qquad - i\theta D'(0) + O(|\theta|^3\alpha_n^{-3-\rho}) = i\theta(\mathcal{A}_b(e^{-\alpha_n}) - n + O(\alpha_n^{C_0-1})) \\
&\qquad - \frac{\theta^2}{2}(\mathcal{B}_b(e^{-\alpha_n}) + O(\alpha_n^{C_0-2})) + O(|\theta|) + O(|\theta|^3\alpha_n^{-3-\rho}) \\
&\quad = -\frac{\theta^2}{2}\mathcal{B}_b(e^{-\alpha_n}) + O(\delta_n\alpha_n^{C_0-1}) + O(\delta_n^2\alpha_n^{C_0-2}) + O(\delta_n) + O(\delta_n^3\alpha_n^{-3-\rho}).
\end{aligned}$$

Substituting this into (2.13) and taking into account (2.11), we obtain (2.12). □

To study the behaviour of $f_b(e^{-\alpha_n+i\theta})$ outside the range $-\delta_n < \theta < \delta_n$ we need the Wiener–Ikehara Tauberian theorem on Dirichlet series. It tells us how condition (M_1) implies an asymptotic estimate for the partial sums $\sum_{k=1}^n b_k$.

Wiener–Ikehara theorem (see [12], Theorem 2.2, p. 122]). Suppose that the Dirichlet series $\tilde{D}(s) = \sum_{k=1}^\infty c_k k^{-s}$ is such that the function $\tilde{D}(s) - \frac{c}{s-1}$ has an analytic continuation to the closed half-plane $\Re(s) \geqslant 1$. Then

$$\sum_{k=1}^n c_k \sim cn, \quad n \to \infty. \tag{2.14}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

We also denote by $\{\gamma\}$ the fractional part of the real number γ, and by $\|\gamma\|$ the distance from γ to the nearest integer, so that

$$\|\gamma\| = \begin{cases} \{\gamma\} & \text{if } \{\gamma\} \leqslant 1/2, \\ 1-\{\gamma\} & \text{if } \{\gamma\} > 1/2. \end{cases} \tag{2.15}$$

It is not difficult to show that

$$\sin^2(\pi\gamma) \geqslant 4\|\gamma\|^2 \tag{2.16}$$

(see [8, p. 272]). Now we are ready to prove that Hayman's last ('decay') condition [7, p. 565] is also valid.

Lemma 2.4. *Suppose that $f_b(x)$ satisfies conditions (M_1)–(M_3). Then, for sufficiently large n,*

$$|f_b(e^{-\alpha_n+i\theta})| \leqslant f_b(e^{-\alpha_n})e^{-C_3\alpha_n^{-\epsilon_1}}$$

uniformly for $\delta_n \leqslant |\theta| < \pi$, where C_3 and ϵ_1 are positive constants.

Proof. First, we notice that

$$\frac{|f_b(e^{-\alpha_n+i\theta})|}{f_b(e^{-\alpha_n})} = \exp(\Re(\log f_b(e^{-\alpha_n+i\theta})) - \log f_b(e^{-\alpha_n})). \tag{2.17}$$

Then, setting $\theta = 2\pi u$, for $\alpha_n/2\pi \leqslant |u| = |\theta|/2\pi < 1/2$, we almost repeat the argument from [10, p. 324]:

$$\begin{aligned} &\Re(\log f_b(e^{-\alpha_n+i\theta})) - \log f_b(e^{-\alpha_n}) \\ &= \Re\left(-\sum_{k=1}^{\infty} b_k \log\left(\frac{1-e^{-k\alpha_n+2\pi iuk}}{1-e^{-k\alpha_n}}\right)\right) \\ &= -\frac{1}{2}\sum_{k=1}^{\infty} b_k \log\left(\frac{1-2e^{-k\alpha_n}\cos(2\pi uk)+e^{-2\alpha_n k}}{(1-e^{-\alpha_n k})^2}\right) \\ &= -\frac{1}{2}\sum_{k=1}^{\infty} b_k \log\left(1+\frac{4e^{-\alpha_n k}\sin^2(\pi uk)}{(1-e^{-\alpha_n k})^2}\right) \\ &\leqslant -\frac{1}{2}\sum_{k=1}^{\infty} b_k \log(1+4e^{-\alpha_n k}\sin^2(\pi uk)) \\ &\leqslant -\frac{\log 5}{2}\sum_{k=1}^{\infty} b_k e^{-\alpha_n k}\sin^2(\pi uk) \\ &= -\frac{\log 5}{2} S_n \leqslant -\frac{\log 5}{2} C_2\alpha_n^{-\epsilon}, \end{aligned} \tag{2.18}$$

where the last two inequalities follow from the fact that $\log(1+y) \geqslant \left(\frac{\log 5}{4}\right)y$ $(0 \leqslant y \leqslant 4)$ and (1.6), respectively. Thus, the required inequality is proved for $\alpha_n \leqslant |\theta| < \pi$. It remains to consider the interval $\delta_n \leqslant |\theta| < \alpha_n$. Equation (1.5) implies that we now have to find a lower bound for the sum S_n in (1.6) if $\delta_n/2\pi \leqslant |u| < \alpha_n/2\pi$ (*i.e.*, if $\delta_n \leqslant |\theta| < \alpha_n$). We shall

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

apply the Wiener–Ikehara Tauberian theorem setting, $c_k = k^{-\rho+1}b_k, k \geqslant 1$, and

$$\tilde{D}(s) = \sum_{k\geqslant 1} b_k k^{-s-\rho+1} = D(s+\rho-1), \quad s = \sigma + iy. \tag{2.19}$$

Since both $C_0, \rho > 0$ from condition (M_1), the function $\tilde{D}(s)$ satisfies the condition of the Wiener–Ikehara theorem with $c = A$. Moreover, since by (2.9) $\alpha_n^{-1} \to \infty$ as $n \to \infty$, we can apply (2.14) in the form

$$\sum_{1\leqslant k\leqslant K/\alpha_n} k^{-\rho+1}b_k \sim AK/\alpha_n, \quad n \to \infty, \tag{2.20}$$

where the constant $K > 0$ will be specified later. Our next argument will be similar to that given in [8, Lemma 7]. First, using (2.15), we observe that $\|uk\| = uk$ if $|u|k < 1/2$. This implies that, for $1 \leqslant k \leqslant \pi/\alpha_n$, $\|uk\|$ can be replaced by $|u|k$. Recalling that $|u| \geqslant \delta_n/2\pi$ and applying (2.16) and (2.11), we obtain

$$\begin{aligned}
S_n &= \sum_{k=1}^{\infty} b_k e^{-k\alpha_n} \sin^2(\pi uk) \geqslant 4\sum_{k=1}^{\infty} b_k e^{-k\alpha_n}\|uk\|^2 \\
&\geqslant 4u^2 \sum_{1\leqslant k\leqslant \pi/\alpha_n} k^2 b_k e^{-k\alpha_n} \geqslant (\delta_n/\pi)^2 \sum_{1\leqslant k\leqslant \pi/\alpha_n} k^2 b_k e^{-k\alpha_n} \\
&= \frac{\alpha_n^{2+2\rho/3}}{(\pi\omega(n))^2} \sum_{1\leqslant k\leqslant \pi/\alpha_n} k^{\rho+1}\frac{k^2 b_k}{k^{\rho+1}} e^{-k\alpha_n} \\
&= \frac{\alpha_n^{2+2\rho/3}}{(\pi\omega(n))^2} \sum_{1\leqslant k\leqslant \pi/\alpha_n} (k^{\rho+1}e^{-k\alpha_n})(b_k k^{-\rho+1}).
\end{aligned}$$

It is easy to check that the sequence $k^{\rho+1}e^{-k\alpha_n}, k \geqslant 1$, is non-increasing if $k \geqslant (\rho+1)/\alpha_n - 1/2 + O(\alpha_n)$. Hence, for $\rho + 1 < \pi$, (2.20) with $K = \pi, \rho + 1$ implies that

$$\begin{aligned}
S_n &\geqslant \frac{\alpha_n^{2+2\rho/3}}{(\pi\omega(n))^2} \sum_{(\rho+1)/\alpha_n\leqslant k\leqslant \pi/\alpha_n} (k^{\rho+1}e^{-k\alpha_n})(b_k k^{-\rho+1}) \\
&\geqslant \frac{\alpha_n^{2+2\rho/3}}{(\pi\omega(n))^2}(\pi/\alpha_n)^{\rho+1}e^{-\pi} \sum_{(\rho+1)/\alpha_n\leqslant k\leqslant \pi/\alpha_n} b_k k^{-\rho+1} \\
&= \frac{\pi^{\rho-1}e^{-\pi}\alpha_n^{1-\rho/3}}{\omega^2(n)}\left(\sum_{1\leqslant k\leqslant \pi/\alpha_n} b_k k^{-\rho+1} - \sum_{1\leqslant k<(\rho+1)/\alpha_n} b_k k^{-\rho+1}\right) \\
&\sim A\pi^{\rho-1}(\pi-\rho-1)e^{-\pi}\alpha_n^{-\rho/3}/\omega^2(n).
\end{aligned} \tag{2.21}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

If $\pi \leqslant \rho+1$, then $k^{\rho+1}e^{-k\alpha_n}$ is a non-decreasing sequence for $1 \leqslant k \leqslant \pi/\alpha_n$. Then, for some $l \in (0,\pi)$, we have $k^{\rho+1}e^{-k\alpha_n} \geqslant l^{\rho+1}e^{-l\alpha_n}$, and in the same way we observe that

$$\begin{aligned} S_n &\geqslant \frac{\alpha_n^{2+2\rho/3}}{(\pi\omega(n))^2} \sum_{l/\alpha_n \leqslant k \leqslant \pi/\alpha_n} (k^{\rho+1}e^{-k\alpha_n})(b_k k^{-\rho+1}) \\ &\geqslant \frac{\alpha_n^{2+2\rho/3} l^{\rho+1} e^{-l}}{(\pi\omega(n))^2 \alpha_n^{\rho+1}} \sum_{l/\alpha_n \leqslant k \leqslant \pi/\alpha_n} b_k k^{-\rho+1} \\ &\sim \frac{A l^{\rho+1} e^{-l}(\pi - l)\alpha_n^{-\rho/3}}{(\pi\omega(n))^2}(1+o(1)). \end{aligned} \tag{2.22}$$

Consequently, (2.18), (2.21) and (2.22) imply that there are two constants $C_3, \epsilon_1 > 0$, such that

$$S_n \geqslant C_3 \alpha_n^{-\epsilon_1} \tag{2.23}$$

uniformly for $\delta_n/2\pi \leqslant |u| < 1/2$. Moreover, $\epsilon_1 \leqslant \min(\epsilon, \rho/3)$ since $\omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. Hence, noting that $u = \theta/2\pi$, we obtain that the required inequality holds uniformly for $\delta_n \leqslant |\theta| < \pi$. □

We now recall (2.6) from Lemma 2.2. It implies that

$$\mathcal{B}_b^{1/2}(e^{-\alpha_n}) \sim (h(\rho+1))^{1/2}\alpha_n^{-1-\rho/2}, \quad n \to \infty.$$

Combining this asymptotic equivalence with the result of Lemma 2.4, we obtain Hayman's 'decay' condition [7, p. 565], namely,

$$|f_b(e^{-\alpha_n + i\theta})| = o(f_b(e^{-\alpha_n})/\mathcal{B}_b^{1/2}(e^{-\alpha_n})), \quad n \to \infty, \tag{2.24}$$

uniformly for $\delta_n \leqslant |\theta| < \pi$.

Equations (2.10), (2.12) and (2.24) show that the function $f_b(x)$ is admissible in the sense of Hayman. Therefore, we can apply Theorem VIII.4 of [7] for its coefficients. We state this result in the next lemma.

Lemma 2.5. *Suppose that the weight sequence b satisfies Meinardus conditions* (M_1)–(M_3). *Then the asymptotic for the total number of weighted partitions is given by*

$$p_b(n) \sim \frac{e^{n\alpha_n} f_b(e^{-\alpha_n})}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}} \tag{2.25}$$

as $n \to \infty$, *where* α_n *is the unique solution of* (2.8) *whose asymptotic expansion is given by* (2.9) *and* $\mathcal{B}_b(e^{-\alpha_n})$ *is defined by* (2.6).

Remark. The asymptotic equivalence (2.25) is in fact the Meinardus asymptotic formula [13] for the number of weighted partitions of n. Here we give the formula in a slightly different form, which is more convenient for our further asymptotic analysis. One can easily show the coincidence of (2.25) with Meinardus's original formula, applying the result of Lemma 2.1 to $f_b(e^{-\alpha_n})$ and replacing α_n and $\mathcal{B}_b(e^{-\alpha_n})$ by (2.9) and (2.6), respectively.

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Further, we also need a bound on the rate of growth of the weights b_k, as $k \to \infty$. Using the Wiener–Ikehara Tauberian theorem, Granovsky, Stark and Erlihson [10, p. 310] showed that $b_k = o(k^\rho)$ as $k \to \infty$. We need this bound in a different, slightly more precise form.

Lemma 2.6. *If the sequence of weights b satisfies conditions (M_1) and (M_2), then there is a sequence of numbers $L_k, k \geqslant 1$, satisfying $\lim_{k\to\infty} L_k = 0$ and such that*

$$b_k = (L_k - L_{k-1})k^\rho + (A + L_{k-1})k^{\rho-1}, \quad k \geqslant 2, \quad L_1 = b_1 - A, \tag{2.26}$$

where A is the constant defined in condition (M_1).

Proof. As in [10, p. 310], we rewrite (2.14) in the following way:

$$\frac{1}{k}\sum_{j=1}^{k} c_j = \frac{1}{k}c_k + \frac{1}{k}\sum_{j=1}^{k-1} c_j = c + L_k, \quad k \geqslant 2,$$

where $\lim_{k\to\infty} L_k = 0$. We also set $L_1 = c_1 - c$. Then, for $k \geqslant 2$, we have

$$\frac{1}{k}c_k = L_k - L_{k-1} + \frac{1}{k}(c + L_{k-1}).$$

To obtain (2.26) it remains to set $c_k = k^{-\rho+1}b_k, k \geqslant 1, c = A$ and recall that the weights b_k satisfy conditions (M_1) and (M_2) with $C_0, \rho > 0$. □

Now, we also recall formula (1.2) for the truncated products $f_{m,b}(x), m \geqslant 1$. Similarly to (2.1), we set

$$F_{m,b}(x) = \log f_{m,b}(x) = -\sum_{k=1}^{m} b_k \log(1 - x^k). \tag{2.27}$$

(Here we consider the main branch of the logarithmic function, assuming that $\log y < 0$ for $0 < y < 1$.) Further on, when computing the derivatives of (2.1) and (2.27), we shall write

$$F_b^{(j)}(e^{-\alpha_n}) = F_b^{(j)}(x)\,|_{x=e^{-\alpha_n}}, \quad F_{m,b}^{(j)}(e^{-\alpha_n}) = F_{m,b}^{(j)}(x)\,|_{x=e^{-\alpha_n}}, \ j = 1, 2, 3.$$

Our next lemma establishes estimates on the tails $F_b^{(j)}(e^{-\alpha_n}) - F_{m,b}^{(j)}(e^{-\alpha_n})$ for some specific values of m.

Lemma 2.7. *Suppose that the weight sequence b satisfies conditions (M_1) and (M_2) and that $\alpha_n, n \geqslant 1$ is defined by (2.9). Moreover, let $m = m(n)$ be a sequence of integers satisfying*

$$m \sim \rho\alpha_n^{-1} \log \alpha_n^{-1}, \quad n \to \infty. \tag{2.28}$$

Then

$$F_b^{(j)}(e^{-\alpha_n}) - F_{m,b}^{(j)}(e^{-\alpha_n}) = O(\alpha_n^{-j} \log^{\rho+j} \alpha_n^{-1}), \quad j = 1, 2, 3.$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Proof. We shall consider only the case $j = 1$. The other two cases are studied in a similar way.

First, we choose a sequence of integers $m_1(n)$ that satisfies the asymptotic equivalence

$$m_1 = m_1(n) \sim (\rho + 1)\alpha_n^{-1} \log \alpha_n^{-1}, \tag{2.29}$$

and decompose the difference of the first derivatives in the following way:

$$F_b'(e^{-\alpha_n}) - F_{m,b}'(e^{-\alpha_n}) = \sum_{k=m+1}^{\infty} \frac{kb_k e^{-(k-1)\alpha_n}}{1 - e^{-k\alpha_n}} = \Sigma_1 + \Sigma_2, \tag{2.30}$$

where

$$\Sigma_1 = \sum_{k=m+1}^{m_1} \frac{kb_k e^{-(k-1)\alpha_n}}{1 - e^{-k\alpha_n}}, \quad \Sigma_2 = \sum_{k=m_1+1}^{\infty} \frac{kb_k e^{-(k-1)\alpha_n}}{1 - e^{-k\alpha_n}}.$$

We also notice that (2.28) and (2.29) imply that

$$e^{-m\alpha_n} \sim \alpha_n^{\rho}, \quad e^{-m_1\alpha_n} \sim \alpha_n^{\rho+1}, \quad n \to \infty. \tag{2.31}$$

Hence, applying the result of Lemma 2.6, for Σ_1 we obtain

$$\begin{aligned}\Sigma_1 &= O\left(\alpha_n^{\rho} \sum_{k=m+1}^{m_1} kb_k\right)\\ &= O\left(\alpha_n^{\rho} \sum_{k=m+1}^{m_1} k^{\rho+1}(L_k - L_{k-1})\right) + O\left(\alpha_n^{\rho} \sum_{k=m+1}^{m_1} k^{\rho}\right).\end{aligned} \tag{2.32}$$

The second summand in (2.32) can be approximated by a Riemann integral using (2.28) and (2.29). We have

$$\sum_{k=m+1}^{m_1} k^{\rho} = m_1^{\rho+1} \sum_{\frac{\rho}{\rho+1} < \frac{k}{m_1} \leqslant 1} \left(\frac{k}{m_1}\right)^{\rho} \frac{1}{m_1} \sim m_1^{\rho+1} \int_{\frac{\rho}{\rho+1}}^{1} u^{\rho}\, du = O(m_1^{\rho+1}).$$

Therefore

$$O\left(\alpha_n^{\rho} \sum_{k=m+1}^{m_1} k^{\rho}\right) = O(\alpha_n^{\rho} m_1^{\rho+1}) = O(\alpha_n^{-1} \log^{\rho+1} \alpha_n^{-1}). \tag{2.33}$$

The first sum in (2.32) can also be rewritten as

$$\begin{aligned}\sum_{k=m+1}^{m_1} k^{\rho+1}(L_k - L_{k-1}) &= -L_m(m+1)^{\rho+1} + L_{m_1} m_1^{\rho+1}\\ &\quad + \sum_{j=1}^{m_1-m-1} L_{m+j}((m+j)^{\rho+1} - (m+j+1)^{\rho+1}).\end{aligned} \tag{2.34}$$

We recall that by Lemma 2.6, (2.28) and (2.29), $L_m = o(1)$ and $L_{m_1} = o(1)$. Hence

$$(-L_m(m+1)^{\rho+1} + L_{m_1} m_1^{\rho+1})\alpha_n^{\rho} = o(\alpha_n^{-1} \log^{\rho+1} \alpha_n^{-1}). \tag{2.35}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

The last sum in (2.34) is estimated again using an approximation by an integral. First, applying a binomial expansion, we get

$$\begin{aligned}(m+j)^{\rho+1}-(m+j+1)^{\rho+1} &= (m+j)^{\rho+1}\left(1-\left(1+\frac{1}{m+j}\right)^{\rho+1}\right)\\ &= (m+j)^{\rho+1}\left(-\frac{\rho+1}{m+j}+O((m+j)^{-2})\right)\\ &= -(\rho+1)(m+j)^{\rho}+O((m+j)^{\rho-1}).\end{aligned}$$

Then, from (2.28), (2.29) and the fact that L_{m+j} are bounded for $1 \leqslant j \leqslant m_1 - m - 1$, we obtain

$$\begin{aligned}&\sum_{j=1}^{m_1-m-1} L_{m+j}((m+j)^{\rho+1}-(m+j+1)^{\rho+1})\\ &= -(\rho+1)\sum_{j=1}^{m_1-m-1} L_{m+j}((m+j)^{\rho}+O((m+j)^{\rho-1}))\\ &= -(\rho+1)m_1^{\rho+1}\sum_{\frac{1}{m_1}\leqslant\frac{j}{m_1}<1-\frac{m}{m_1}} L_{m+j}\left(\frac{1}{m_1}\right)\left(\left(\frac{m+j}{m_1}\right)^{\rho}+O\left(\frac{1}{m_1}\left(\frac{m+j}{m_1}\right)^{\rho-1}\right)\right)\\ &= O\left(m_1^{\rho+1}\int_0^{1-\frac{\rho}{\rho+1}}\left(\left(\frac{\rho}{\rho+1}+u\right)^{\rho}+O\left(\frac{1}{m_1}\right)\right)du\right) = O(m_1^{\rho+1}).\end{aligned} \tag{2.36}$$

Combining (2.29) with (2.32)–(2.36), we obtain

$$\Sigma_1 = O(\alpha_n^{-1}\log^{\rho+1}\alpha_n^{-1}). \tag{2.37}$$

Σ_2 can be estimated in a similar way. We have

$$\begin{aligned}\Sigma_2 &= O\left(\sum_{k=m_1+1}^{\infty}\frac{k^{\rho+1}e^{-k\alpha_n}}{1-e^{-k\alpha_n}}\right) = O\left(\sum_{k=m_1+1}^{\infty}k^{\rho+1}e^{-k\alpha_n}\right)\\ &= O\left(\alpha_n^{-\rho-2}\int_{m_1\alpha_n}^{\infty}u^{\rho+1}e^{-u}du\right) = O(\alpha_n^{-\rho-2}(m_1\alpha_n)^{\rho+1}e^{-m_1\alpha_n})\\ &= O(\alpha_n^{-1}\log^{\rho+1}\alpha_n^{-1}),\end{aligned} \tag{2.38}$$

where the last equality follows from (2.29) and (2.31), while in the previous one we used the asymptotic behaviour of the incomplete gamma function $\Gamma(a,z)$ as $z\to\infty$ (see [1, § 6.5]). The required estimate now follows from (2.30), (2.37) and (2.38). □

Our final lemma will supply us with integral representations for $F_b(e^{-\alpha})$ and $F_{m,b}(e^{-\alpha})$, $\alpha > 0$, using the Dirichlet series (1.3) and its partial sums

$$D_m(s) = \sum_{k=1}^{m} b_k k^{-s}, \quad s=\sigma+iy, \quad m\geqslant 1. \tag{2.39}$$

The proof is based on a Mellin transform technique and can be found in [13], [10, Lemma 2(ii)] and [2, § 6.2].

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Lemma 2.8. *For any $\alpha, \Delta > 0$, we have*

$$F_{m,b}(e^{-\alpha}) = \frac{1}{2\pi i}\int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha^{-s}\Gamma(s)\zeta(s+1)D_m(s)ds \tag{2.40}$$

and

$$F_b(e^{-\alpha}) = \frac{1}{2\pi i}\int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha^{-s}\Gamma(s)\zeta(s+1)D(s)ds, \tag{2.41}$$

where $D_m(s)$ and $D(s)$ are defined by (2.39) *and* (1.3), *respectively.*

3. Proof of the main result

We first apply the Cauchy coefficient formula to (1.2) using the circle $x = e^{-\alpha_n+i\theta}, \pi < \theta \leqslant \pi$, as a contour of integration (α_n is determined by (2.9)). We obtain

$$p_b(n)\mathbb{P}(X_n \leqslant m) = \frac{e^{n\alpha_n}}{2\pi}\int_{-\pi}^{\pi} f_{m,b}(e^{-\alpha_n+i\theta})e^{-i\theta n}d\theta.$$

Then, we break up the range of integration as follows:

$$p_b(n)\mathbb{P}(X_n \leqslant m) = J_1(m,n) + J_2(m,n), \tag{3.1}$$

where

$$\begin{aligned} J_1(m,n) &= \frac{e^{n\alpha_n}}{2\pi}\int_{-\delta_n}^{\delta_n} f_{m,b}(e^{-\alpha_n+i\theta})e^{-i\theta n}d\theta \\ &= \frac{e^{n\alpha_n+F_{m,b}(e^{-\alpha_n})}}{2\pi}\int_{-\delta_n}^{\delta_n} \frac{f_{m,b}(e^{-\alpha_n+i\theta})}{f_{m,b}(e^{-\alpha_n})}e^{-i\theta n}d\theta, \end{aligned} \tag{3.2}$$

$$J_2(m,n) = \frac{e^{n\alpha_n+F_{m,b}(e^{-\alpha_n})}}{2\pi}\int_{\delta_n<|\theta|\leqslant\pi} \frac{f_{m,b}(e^{-\alpha_n+i\theta})}{f_{m,b}(e^{-\alpha_n})}e^{-i\theta n}d\theta \tag{3.3}$$

(δ_n and $F_{m,b}(x)$ are defined by (2.11) and (2.27), respectively).

We start with an estimate for $J_1(m,n)$, expanding the integrand of (3.2) by Taylor's formula:

$$\begin{aligned} \frac{f_{m,b}(e^{-\alpha_n+i\theta})}{f_{m,b}(e^{-\alpha_n})} &= \exp\{(e^{i\theta}-1)e^{-\alpha_n}F'_{m,b}(e^{-\alpha_n}) \\ &\quad + \frac{1}{2}(e^{i\theta}-1)^2e^{-2\alpha_n}F''_{m,b}(e^{-\alpha_n}) + O(|\theta|^3F'''_{m,b}(e^{-\alpha_n}))\}. \end{aligned}$$

Hence, we can rewrite (3.2) as follows:

$$J_1(m,n) = \frac{e^{n\alpha_n}}{\sqrt{2\pi}}e^{F_{m,b}(e^{-\alpha_n})}I_n, \tag{3.4}$$

where

$$\begin{aligned} I_n = \frac{1}{\sqrt{2\pi}}\int_{-\delta_n}^{\delta_n} &\exp\{(e^{i\theta}-1)e^{-\alpha_n}F'_{m,b}(e^{-\alpha_n}) \\ &+ \frac{1}{2}(e^{i\theta}-1)^2e^{-2\alpha_n}F''_{m,b}(e^{-\alpha_n}) + O(|\theta|^3F'''_{m,b}(e^{-\alpha_n})) - i\theta n\}d\theta. \end{aligned}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Lemma 2.7 shows that, for those integers m satisfying (2.28), we can replace the derivatives $F_{m,b}^{(j)}(e^{-\alpha_n})$ by $F_b^{(j)}(e^{-\alpha_n}), j = 1,2,3,$ at the expense of a negligible error term. In fact, combining Lemma 2.7 with (2.11), we have

$$\begin{aligned}(e^{i\theta}-1)^j F_{m,b}^{(j)}(e^{-\alpha_n}) &= (e^{i\theta}-1)^j F_b^{(j)}(e^{-\alpha_n}) + O(\delta_n^j \alpha_n^{-j}\log^{\rho+j}\alpha_n^{-1})\\ &= (e^{i\theta}-1)^j F_b^{(j)}(e^{-\alpha_n}) + O(\alpha_n^{\rho j/3}\log^{\rho+j}\alpha_n^{-1}/\omega^j(n)), \quad j=1,2,\end{aligned}$$

and

$$O(|\theta|^3 F_{m,b}'''(e^{-\alpha_n})) = O(|\theta|^3 F_b'''(e^{-\alpha_n})) + O(\alpha_n^{\rho}(\log^{\rho+3}\alpha_n^{-1})/\omega^3(n)),$$

where the function $\omega(n)\to\infty$ as $n\to\infty$ arbitrarily slowly. Since all error terms above tend to 0, we obtain

$$\begin{aligned}I_n &= \frac{1+o(1)}{\sqrt{2\pi}}\int_{-\delta_n}^{\delta_n}\exp\{F_b(e^{-\alpha_n+i\theta}) - F_b(e^{-\alpha_n}) - i\theta n\}d\theta\\ &= \frac{1+o(1)}{\sqrt{2\pi}}\int_{-\delta_n}^{\delta_n}\frac{f_b(e^{-\alpha_n+i\theta})}{f_b(e^{-\alpha_n})}e^{-i\theta n}d\theta,\end{aligned}$$

where the last equality follows from a similar Taylor expansion for $F_b(e^{-\alpha_n+i\theta})$ and the fact that $O(|\theta|^3F_b'''(e^{-\alpha_n})) = O(\delta_n^3 F_b'''(e^{-\alpha_n})) = O(1/\omega^3(n)) = o(1)$ (see (2.7) of Lemma 2.2 and (2.11)). Now, from Lemma 2.3 it follows that

$$\begin{aligned}I_n &\sim \frac{1}{\sqrt{2\pi}}\int_{-\delta_n}^{\delta_n} e^{-\theta^2\mathcal{B}_b(e^{-\alpha_n})/2}d\theta\\ &= \frac{1}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}}\int_{-\delta_n\sqrt{\mathcal{B}_b(e^{-\alpha_n})}}^{\delta_n\sqrt{\mathcal{B}_b(e^{-\alpha_n})}} e^{-y^2/2}dy \sim \frac{1}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}}\int_{-\infty}^{\infty}e^{-y^2/2}dy\\ &= \frac{1}{\sqrt{\mathcal{B}_b(e^{-\alpha_n})}}, \quad n\to\infty.\end{aligned}$$

The last asymptotic equivalence follows from (2.6) of Lemma 2.2, which implies that

$$\delta_n\sqrt{\mathcal{B}_b(e^{-\alpha_n})} \sim \frac{\alpha_n^{-\rho/6}}{\omega(n)}\sqrt{h(\rho+1)} \to \infty,$$

if $\omega(n)\to\infty$ slower than $\alpha_n^{-\rho/6}$.

Substituting the asymptotic equivalence for I_n into (3.4), we conclude that

$$J_1(m,n) \sim \frac{e^{n\alpha_n}}{\sqrt{2\pi\mathcal{B}(e^{-\alpha_n})}}e^{F_{m,b}(e^{-\alpha_n})} \tag{3.5}$$

if m satisfies (2.28) as $n\to\infty$.

For the estimate of $J_2(m,n)$, we recall (3.3) and the proof of (2.18). Thus, for any real u, we obtain

$$\begin{aligned}\Re(F_{m,b}(e^{-\alpha_n+2\pi iu})) - F_{m,b}(e^{-\alpha_n}) &\leqslant -\frac{\log 5}{2}\sum_{k=1}^{m} b_k e^{-\alpha_n k}\sin^2(\pi uk)\\ &= -\frac{\log 5}{2}\left(\sum_{k=1}^{\infty} b_k e^{-\alpha_n k}\sin^2(\pi uk) - \sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k}\sin^2(\pi uk)\right).\end{aligned} \tag{3.6}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Consider again the sequences $m = m(n)$ and $m_1 = m_1(n)$ defined by (2.28) and (2.29), respectively. We have

$$\sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k} \sin^2(\pi k u) \leqslant \sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k} = \sum_{k=m+1}^{m_1} b_k e^{-\alpha_n k} + \sum_{k=m_1+1}^{\infty} b_k e^{-\alpha_n k}. \tag{3.7}$$

These sums can be estimated using the argument given in the proof of Lemma 2.7 (see (2.32)–(2.38)). We obtain in the same way that

$$\sum_{k=m+1}^{m_1} b_k e^{-\alpha_n k} = O\left(\alpha_n^\rho \sum_{k=m+1}^{m_1} b_k\right) = O((\alpha_n m_1)^\rho) = O(\log^\rho \alpha_n^{-1}), \tag{3.8}$$

$$\sum_{k=m_1+1}^{\infty} b_k e^{-\alpha_n k} = O\left(\sum_{k=m_1+1}^{\infty} k^\rho e^{-\alpha_n k}\right) = O(\log^\rho \alpha_n^{-1}) \tag{3.9}$$

and thus (3.7)–(3.9) imply that

$$\sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k} \sin^2(\pi k u) = O(\log^\rho \alpha_n^{-1}).$$

Replacing the second term of the right-hand side of (3.6) by the last O-estimate and applying inequality (2.23) (see also (1.6)) to its first term, for $\delta_n/2\pi \leqslant |u| < 1/2$, we obtain

$$\Re(F_{m,b}(e^{-\alpha_n+2\pi i u})) - F_{m,b}(e^{-\alpha_n}) \leqslant -C_3\alpha_n^{-\epsilon_1} + O(\log^\rho \alpha_n^{-1}).$$

Now we are ready to compare the growth of (3.3) with that of (3.5) whenever m satisfies (2.28). We have

$$\begin{aligned}
|J_2(m,n)| &\leqslant \exp(n\alpha_n + F_{m,b}(e^{-\alpha_n})) \\
&\quad \times \int_{\frac{\delta_n}{2\pi} < |u| \leqslant \frac{1}{2}} |f_{m,b}(e^{-\alpha_n+2\pi i u})/f_{m,b}(e^{-\alpha_n})| du \\
&= \exp(n\alpha_n + F_{m,b}(e^{-\alpha_n})) \int_{\frac{\delta_n}{2\pi} < |u| \leqslant \frac{1}{2}} (\Re(F_{m,b}(e^{-\alpha_n+2\pi i u}) - F_{m,b}(e^{-\alpha_n}))du \\
&= O(\exp(n\alpha_n + F_{m,b}(e^{-\alpha_n}) - C_3\alpha_n^{-\epsilon_1} + O(\log^\rho \alpha_n^{-1}))) \\
&= O(e^{-C_3\alpha_n^{-\epsilon_1}} \sqrt{2\pi \mathcal{B}_b(e^{-\alpha_n})} J_1(m,n)) = o(J_1(m,n)),
\end{aligned} \tag{3.10}$$

where for the last o-estimate we have used (2.6). It is now clear that (3.1), (3.5) and (3.10) imply that

$$p_b(n)\mathbb{P}(X_n \leqslant m) \sim \frac{e^{n\alpha_n}}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}} e^{F_{m,b}(e^{-\alpha_n})}, \quad n \to \infty.$$

Subsequent application of the asymptotic equivalence (2.25) from Lemma 2.5 implies that

$$\mathbb{P}(X_n \leqslant m) \sim \exp\{F_{m,b}(e^{-\alpha_n}) - F_b(e^{-\alpha_n})\}, \tag{3.11}$$

where α_n and m satisfy (2.9) and (2.28), respectively.

Further on we shall study the asymptotic behaviour of the exponent in (3.11). Our analysis will be based on a generalization of the Perron formula that expresses the partial sums of a Dirichlet series as complex integrals of the inverse Mellin-type transforms

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

applied to the Dirichlet series itself. We shall use it in the form given in [18, § 3, Supplement]. So, first we represent $F_{m,b}(e^{-\alpha_n})$ using (2.40) of Lemma 2.8, and then we apply the Perron formula to the partial sum $D_m(s)$ of the Dirichlet series $D(s)$ (recall also (2.39) and (1.3)). In this way we arrive at the following complex integral representation: for any $\Delta > 1$, we have

$$F_{m,b}(e^{-\alpha_n}) = \frac{1}{2\pi i}\int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha_n^{-s}\Gamma(s)\zeta(s+1)\left(\frac{A(m+1)^{\rho-s}}{\rho-s} + D(s) + \Omega_m\right)ds, \tag{3.12}$$

where $\Omega_m = o(1), m \to \infty$. Furthermore, (3.11) and (3.12) imply that

$$\mathbb{P}(X_n \leqslant m) \sim \exp\left\{-\frac{A\alpha_n^{-\rho}}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty} ((m+1)\alpha_n)^{-s}\Gamma(s+\rho)\zeta(s+\rho+1)\frac{ds}{s}\right\}. \tag{3.13}$$

The proofs of (3.12) and (3.13) contain some technical details that will be given in the Appendix.

We continue with the computation of the complex integral in the exponent of (3.13). The sequence $m = m(n)$ will be specified later in a more precise way. At this moment we only assume that it satisfies (2.28). We set in the integral of (3.13)

$$u = u_n = (m+1)\alpha_n, \tag{3.14}$$

and consider it as a function of u. First we shall obtain its explicit form and then we shall estimate it as $u \to \infty$ (see (2.28) and (3.14)). Clearly, we can consider this integral as the inverse Mellin transform of the function $\Gamma(s+\rho)\zeta(s+\rho+1)/s$. For the sake of convenience, we set

$$H(u) = \frac{1}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty} u^{-s}\Gamma(s+\rho)\zeta(s+\rho+1)\frac{ds}{s}. \tag{3.15}$$

It is known that, for $\Re(s) > 0$, $g_1(s) = 1/s$ is the Mellin transform of the (Heaviside-like) step function

$$H_1(u) = \begin{cases} 1 & \text{if } 0 \leqslant u < 1, \\ 0 & \text{if } u > 1, \end{cases}$$

while $g_2(s) = \Gamma(s)\zeta(s+1)$ is the Mellin transform of

$$H_2(u) = \sum_{j=1}^{\infty}\frac{e^{-ju}}{j} = -\log(1-e^{-u}) \tag{3.16}$$

(see, *e.g.*, [7, Appendix B.7]). Next, for $\Delta > 1$, we apply formula (6.1.14) from [5] with $\alpha = 0$ and $\beta = \rho - 1$. We obtain

$$\begin{aligned} H(u) &= u^{\alpha}\int_0^{\infty} y^{\beta}H_1(u/y)H_2(y)dy = -\int_u^{\infty} y^{\rho-1}\log(1-e^{-y})dy \\ &= \int_u^{\infty} y^{\rho-1}e^{-y}dy + R(u) = \Gamma(\rho,u) + R(u), \end{aligned} \tag{3.17}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

where $\Gamma(\rho, u)$ denotes the incomplete gamma function, while $R(u)$ is the error term given by

$$R(u) = \int_u^\infty y^{\rho-1}\left(\sum_{j=2}^{\infty}\frac{e^{-jy}}{j}\right)dy.$$

It is easily estimated as follows:

$$R(u) \leqslant \frac{1}{2(1-e^{-u})}\int_u^\infty y^{\rho-1}e^{-2y}dy = O(e^{-u}\Gamma(\rho,u)), \quad u \to \infty.$$

Combining this estimate with (3.13)–(3.15) and (3.17) and applying the asymptotic $\Gamma(\rho, u_n) = u_n^{\rho-1}e^{-u_n}(1+O(1/u_n))$ of the incomplete gamma function (see again [1, § 6.5]), we obtain

$$\begin{aligned}\mathbb{P}(X_n \leqslant m) &\sim \exp\{-A\alpha_n^{-\rho}(u_n^{\rho-1}e^{-u_n}(1+O(1/u_n)) + O(u_n^{\rho-1}e^{-2u_n}))\}\\ &= \exp\{-A\alpha_n^{-1}m^{\rho-1}e^{-m\alpha_n}(1+O(1/m\alpha_n))\}\\ &= \exp\{-A\alpha_n^{-1}m^{\rho-1}e^{-m\alpha_n}(1+O(1/\log\alpha_n^{-1}))\},\end{aligned} \tag{3.18}$$

where for the last equality we have used again (2.28). It is now clear that $\mathbb{P}(X_n \leqslant m)$ converges to the distribution function $e^{-e^{-t}}, -\infty < t < \infty$, if $m = m(n)$ satisfies

$$-m\alpha_n + (\rho-1)\log m + \log(A\alpha_n^{-1}) = -t + o(1)$$

as $n \to \infty$. From this we deduce

$$m = \alpha_n^{-1}\log\alpha_n^{-1} + (\rho-1)\alpha_n^{-1}\log m + (\log A + t)\alpha_n^{-1} + o(\alpha_n^{-1}), \tag{3.19}$$

which in turn implies that

$$\begin{aligned}\log m &= \log(\alpha_n^{-1}\log\alpha_n^{-1})\\ &\quad + \log\left(1 + \frac{\log A + t}{\log\alpha_n^{-1}} + (\rho-1)\frac{\log m}{\log\alpha_n^{-1}} + o(1/\log\alpha_n^{-1})\right)\\ &= \log\alpha_n^{-1} + \log\log\alpha_n^{-1}\\ &\quad + \log\left(1 + \frac{\log A + t}{\log\alpha_n^{-1}} + (\rho-1)\frac{\log\alpha_n^{-1} + \log\log\alpha_n^{-1} + O(1)}{\log\alpha_n^{-1}}\right)\\ &= \log\alpha_n^{-1} + \log\log\alpha_n^{-1} + \log\left(1 + (\rho-1) + O\left(\frac{\log\log\alpha_n^{-1}}{\log\alpha_n^{-1}}\right)\right)\\ &= \log\alpha_n^{-1} + \log\log\alpha_n^{-1} + \log\rho + O\left(\frac{\log\log\alpha_n^{-1}}{\log\alpha_n^{-1}}\right).\end{aligned}$$

Hence, (3.19) becomes

$$\begin{aligned}m &= \rho\alpha_n^{-1}\log\alpha_n^{-1} + (\rho-1)\alpha_n^{-1}\log\log\alpha_n^{-1}\\ &\quad + \alpha_n^{-1}(\rho-1)\log\rho + (\log A + t)\alpha_n^{-1} + O\left(\alpha_n^{-1}\frac{\log\log\alpha_n^{-1}}{\log\alpha_n^{-1}}\right).\end{aligned}$$

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Now replacing this value of m into (3.18) and using the continuity of the distribution function $e^{-e^{-t}}, -\infty < t < \infty$, we obtain

$$
\begin{aligned}
&\mathbb{P}(X_n \leqslant m) \\
&\quad = \mathbb{P}(\alpha_n X_n - \rho \log \alpha_n^{-1} - (\rho - 1)\log\log \alpha_n^{-1} - (\rho - 1)\log \rho - \log A + o(1) \leqslant t) \\
&\quad \to e^{-e^{-t}}, \quad n \to \infty. \qquad (3.20)
\end{aligned}
$$

To complete the proof of the theorem, it remains to justify the normalization for X_n stated in (1.10). We have to show that the sequence $\alpha_n, n \geqslant 1$, in (3.20) can be replaced by $a(n) = a(n;\rho, A), n \geqslant 1$ (see (1.9)). So, we first recall (2.9), and notice that by taking logarithms from both sides we easily obtain

$$
\begin{aligned}
\log \alpha_n^{-1} &= |\log a(n)| + O(n^{-\frac{\rho}{\rho+1}}), \\
\log\log \alpha_n^{-1} &= \log|\log a(n)| + O(n^{-\frac{\rho}{\rho+1}} / \log n). \qquad (3.21)
\end{aligned}
$$

Next, we set $Z_n := \alpha_n X_n + z_n$, $Y_n := a(n)X_n + y_n$, where

$$
\begin{aligned}
z_n &:= \rho \log \alpha_n - (\rho - 1)\log\log \alpha_n^{-1} - (\rho - 1)\log \rho - \log A, \\
y_n &:= \rho \log a(n) - (\rho - 1)\log|\log a(n)| - (\rho - 1)\log \rho - \log A,
\end{aligned}
$$

for $n \geqslant 1$. Furthermore, (3.20) can be written more concisely as follows:

$$
G_{n,Z}(t) := \mathbb{P}(Z_n \leqslant t) \to e^{-e^{-t}}, \quad n \to \infty. \qquad (3.22)
$$

So, we have to prove the same convergence for $G_{n,Y}(t) := \mathbb{P}(Y_n \leqslant t), n \geqslant 1$. It is easy to verify that the above representations for Y_n and Z_n imply that

$$
Y_n = (1 + \eta_n)Z_n + e_n, \qquad (3.23)
$$

where $\eta_n := a(n)/\alpha_n - 1, e_n := y_n - (a(n)/\alpha_n)z_n, n \geqslant 1$. From (1.9), (2.9) and (3.21) we obtain the estimates $\eta_n = O(n^{-\frac{\rho}{\rho+1}})$ and $e_n = O(n^{-\frac{\rho}{\rho+1}} \log n)$ as $n \to \infty$. Moreover, (3.23) implies that

$$
\begin{aligned}
G_{n,Y}(t) &= \mathbb{P}\left(Z_n \leqslant \frac{t - e_n}{1 + \eta_n}\right) = \mathbb{P}\left(Z_n \leqslant t - \frac{t\eta_n + e_n}{1 + \eta_n}\right) \\
&= G_{n,Z}\left(t - \frac{t\eta_n + e_n}{1 + \eta_n}\right), \quad n \geqslant 1.
\end{aligned}
$$

Now taking an arbitrary $\eta > 0$ and n sufficiently large that $-\eta < (t\eta_n + e_n)/(1 + \eta_n) < \eta$, for fixed t, we obtain

$$
G_{n,Z}(t - \eta) \leqslant G_{n,Y}(t) \leqslant G_{n,Z}(t + \eta).
$$

Letting $n \to \infty$ in the above inequalities, from (3.22) we find that

$$
e^{-e^{-(t-\eta)}} \leqslant \liminf_{n\to\infty} G_{n,Y}(t) \leqslant \limsup_{n\to\infty} G_{n,Y}(t) \leqslant e^{-e^{-(t+\eta)}}
$$

for all $\eta > 0$. Letting now $\eta \to 0^+$, the required result stated in (1.10) follows from the continuity of the distribution function $e^{-e^{-t}}$.

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

Appendix

Proof of (3.12). First, we recall (2.40) of Lemma 2.8. Our goal is to represent the mth partial sum $D_m(s)$, defined by (2.39), using the inversion formula given by Theorem 3.1 in [18, Supplement] (see also formula (3.4) there). Instead of $D(s)$, we shall now consider the Dirichlet series $\tilde{D}(s) = D(s+\rho-1)$ (see (2.19) and condition (M_1)). It converges absolutely for $\Re(s) = \sigma > 1$. Lemma 2.6 implies that the coefficients of $D(s+\rho-1)$ satisfy

$$b_k k^{-\rho+1} = o(k^\rho)k^{-\rho+1} = o(k) < \tilde{c}k,$$

for some constant $\tilde{c} > 0$ and all k (in other words, $\Phi(x) = x$ in Theorem 3.1 of [18, Supplement]). Furthermore, from condition (M_1) it follows that

$$\sum_{k=1}^{\infty} b_k k^{-\rho+1} k^{-\sigma} = \frac{A}{\sigma-1} + \phi(s),$$

where $\phi(s)$ denotes a function which is analytic for $\sigma \geqslant -C_0$. Hence

$$\sum_{k=1}^{\infty} b_k k^{-\rho+1} k^{-\sigma} = O((\sigma-1)^{-1}), \quad \sigma \to 1^+.$$

So, the conditions of Theorem 3.1 [18, Supplement] are satisfied and by its second part we conclude that, for sufficiently large $T > 0$, $\Delta > 1$ and $d > 0$, we have

$$\begin{aligned} D_m(w+\rho-1) &+ \frac{1}{2} b_{m+1}(m+1)^{-\rho+1}(m+1)^{-w} \\ &= \frac{1}{2\pi i}\int_{d-iT}^{d+iT} D(w+z+\rho-1)\frac{(m+1)^z}{z}dz + O\left(\frac{m^d}{(\Delta+d)T}\right) \\ &\quad + O\left(\frac{m^{1-\Delta}\log m}{T}\right), \quad w = 1+\Delta+iy, -\infty < y < \infty. \end{aligned} \tag{A.1}$$

Lemma 2.6 implies that the second term on the left-hand side of (A.1) is $o(m^{-\Delta})$ as $m \to \infty$. To compute the integral on the right-hand side of (A.1), we set $d = 1/\log m, m \geqslant 2$, and use a contour integral around the rectangle $d-iT, d+iT, -C_0-\rho-\Delta+iT, -C_0-\rho-\Delta-iT$. Using condition (M_2), we estimate the integral over the end segment $(-C_0-\rho-\Delta+iT, -C_0-\rho-\Delta-iT)$ by $O(T^{C_1+1}m^{-C_0-\rho-\Delta})$. Hence, it tends to 0 as $m, T \to \infty$, provided $T = o(m^{(C_0+\rho+\Delta)/(C_1+1)})$. The integrals on the segments $(-C_0-\rho-\Delta+iT, d+iT)$ and $(-C_0-\rho-\Delta-iT, d-iT)$ are easily estimated. By condition (M_2) and the choice of d, both are of order

$$O\left(T^{C_1-1}\int_{-C_0-\rho-\Delta}^{1/\log m}(m+1)^\sigma d\sigma\right) = O\left(\frac{T^{C_1-1}m^{1/\log m}}{\log m}\right) = O\left(\frac{T^{C_1-1}}{\log m}\right).$$

Therefore, we conclude that T should satisfy

$$T = \begin{cases} o(m^{(C_0+\rho+\Delta)/(C_1+1)}) & \text{if } C_1 \leqslant 1, \\ o((\log m)^{1/(C_1-1)}) & \text{if } C_1 > 1. \end{cases} \tag{A.2}$$

Furthermore we shall assume that $T = T(m) \to \infty$ as $m \to \infty$ and m and T satisfy (A.2), where the constants C_0, ρ and C_1 are defined by conditions (M_1) and (M_2), and $\Delta > 1$ is

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

fixed. Thus, all integrals on the end segments except the integral on $(d - iT, d + iT)$ are close to 0 for sufficiently large m and T. The same obviously holds for both O-estimates on the right-hand side of (A.1). So, we can compute $D_m(w + \rho - 1)$ summing up the residues of the integrand in (A.1). Inside the contour of integration it has only two simple poles: at $z = 1 - w$ and $z = 0$. Thus, we obtain

$$D_m(w + \rho - 1) = \frac{A(m+1)^{1-w}}{1 - w} + D(w + \rho - 1) + \Omega_m, \qquad \text{(A.3)}$$
$$w = 1 + \Delta + iy, -\infty < y < \infty,$$

where Ω_m equals the sum of all negligible terms described above. Clearly,

$$\Omega_m \to 0, \quad m \to \infty, \qquad \text{(A.4)}$$

for m and T satisfying (A.2). Setting in $w = s - \rho + 1$ (A.3) and substituting this expression into (2.40), we arrive at (3.12). □

Proof of (3.13). (3.12) and (2.41) imply that

$$F_{m,b}(e^{-\alpha_n}) = \frac{A}{2\pi i} \int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha_n^{-s} \Gamma(s)\zeta(s+1) \frac{(m+1)^{\rho-s}}{\rho - s} ds$$
$$+ F_b(e^{-\alpha_n}) + \frac{A\Omega_m}{2\pi i} \int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha_n^{-s} \Gamma(s)\zeta(s+1) ds.$$

The last integral represents an inverse Mellin transform whose original (see (3.16) and [7, Appendix B.7]) is $H_2(\alpha_n) = -\log(1 - e^{-\alpha_n}) = O(-\log \alpha_n)$, as $n \to \infty$. Hence assumption (2.28) and (A.4) imply that

$$F_{m,b}(e^{-\alpha_n}) = \frac{A}{2\pi i} \int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha_n^{-s} \Gamma(s)\zeta(s+1) \frac{(m+1)^{\rho-s}}{\rho - s} ds$$
$$+ F_b(e^{-\alpha_n}) + o(-\log \alpha_n).$$

To obtain (3.13) it is sufficient to replace this expression into (3.11), change the variable s in the above integral by $s + \rho$ and observe that $-\log \alpha_n = o(\alpha_n^{-\rho})$. □

Acknowledgements

The author would like to thank the referee for valuable comments, and especially for indicating some defects contained in previous versions of the paper.

References

[1] Abramovitz, M. and Stegun, I. A. (1965) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover.

[2] Andrews, G. E. (1976) *The Theory of Partitions*, Vol. 2 of *Encyclopedia of Mathematics and its Applications*, Addison-Wesley.

[3] Arratia, R., Barbour, A. D. and Tavaré, S. (2003) *Logarithmic Combinatorial Structures: A Probabilistic Approach*, European Mathematical Society.

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press

[4] Auluck, F. C., Chowla, S. and Gupta, H. (1942) On the maximum value of the number of partitions of n into k parts. *J. Indian Math. Soc.* **6** 105–112.
[5] Bateman, H. and Erdélyi, A. (1954) *Tables of Integral Transforms*, McGraw-Hill.
[6] Erdős, P. and Lehner, J. (1941) The distribution of the number of summands in the partition of a positive integer. *Duke Math. J.* **8** 335–345.
[7] Flajolet, P. and Sedgewick, R. (2009) *Analytic Combinatorics*, Cambridge University Press.
[8] Freiman, G. A. and Granovsky, B. L. (2002) Asymptotic formula for a partition function of reversible coagulation–fragmentation processes. *Israel J. Math.* **130** 259–279.
[9] Fristedt, B. (1993) The structure of random partitions of large integers. *Trans. Amer. Math. Soc.* **337** 703–735.
[10] Granovsky, B., Stark, D. and Erlihson, M. (2008) Meinardus theorem on weighted partitions: Extensions and a probabilistic proof. *Adv. Appl. Math.* **41** 307–328.
[11] Hayman, W. K. (1956) A generalization of Stirling's formula. *J. Reine Angew. Math.* **196** 67–95.
[12] Korevaar, J. (2004) *Tauberian Theory*, Springer.
[13] Meinardus, G. (1954) Asymptotische Aussagen über Partitionen. *Math. Z.* **59** 388–398.
[14] Mutafchiev, L. (2006) The size of the largest part of random plane partitions of large integers. *Integers: Electron. J. Combin. Number Theory* **6** #A13.
[15] Mutafchiev, L. (2011) Limit theorems for the number of parts in a random weighted partition. *Electron. J. Combin.* **18** #P206.
[16] Nijenhuis, A. and Wilf, H. (1978) *Combinatorial Algorithms*, second edition, Academic Press.
[17] Pittel, B. (2005) On dimensions of a random solid diagram. *Combin. Probab. Comput.* **14** 873–895.
[18] Prachar, K. (1957) *Primzahlverteilung*, Springer.
[19] Stanley, R. P. (1999) *Enumerative Combinatorics 2*, Vol. 62 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press.
[20] Titchmarsh, E. C. (1939) *The Theory of Functions*, Oxford University Press.
[21] Vershik, A. (1996) Statistical mechanics of combinatorial partitions and their limit configurations. *Funct. Anal. Appl.* **30** 90–105.
[22] Vershik, A. and Yakubovich, Y. (2006) Fluctuations of the maximal particle energy of the quantum ideal gas and random partitions. *Comm. Math. Phys.* **261** 759–769.
[23] Whittaker, E. T. and Watson, G. N. (1927) *A Course of Modern Analysis*, Cambridge University Press.
[24] Wilf, H. (1994) *generatingfunctionology*, Academic Press.

https://doi.org/10.1017/S0963548313000047 Published online by Cambridge University Press