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MICROLOCAL DEFECT MEASURES FOR A DEGENERATE THERMOELASTICITY SYSTEM

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Abstract In this paper, we study a system of thermoelasticity with a degenerate second-order operator in the heat equation. We analyze the evolution of the energy density of a family of solutions. We consider two cases: when the set of points where the ellipticity of the heat operator fails is included in a hypersurface and when it is an open set. In the first case, and under special assumptions, we prove that the evolution of the energy density is that of a damped wave equation: propagation along the rays of the geometric optic and damping according to a microlocal process. In the second case, we show that the energy density propagates along rays which are distortions of the rays of the geometric optic.

Keywords: thermoelasticity equations; energy density; microlocal defect measures

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1. Introduction

We consider Ω an open subset of $\mathbf{R}^d$ and the following system of thermoelasticity:

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla\cdot(\gamma(x)\theta) = 0, & (t,x)\in\mathbf{R}^+\times\Omega,\\ \partial_t\theta - \nabla\cdot(B(x)\nabla\theta) + \gamma(x)\cdot\nabla\partial_t u = 0, & (t,x)\in\mathbf{R}^+\times\Omega,\\ u_{|t=0} = u_0 \quad \text{and} \quad \partial_t u_{|t=0} = u_1 \quad \text{in } \Omega \\ \theta_{|t=0} = \theta_0 \quad \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \quad \partial_t u_{|\partial\Omega} = 0 \quad \text{and} \quad \theta_{|\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where u and θ are scalar real-valued functions. The matrix-valued function $x\mapsto B(x)$ and the vector-valued function $x\mapsto\gamma(x)$ are supposed to be defined on $\mathbf{R}^d$ (and thus on $\overline{\Omega}$) and to depend smoothly on the variable $x\in\mathbf{R}^d$. The matrix $B(x)$ is assumed to be symmetric, non-negative: there exists $C_2>0$ such that

$$\forall\,(x,\xi)\in\overline{\Omega}\times\mathbf{R}^d,\quad 0\leqslant B(x)\xi\cdot\xi\leqslant C_2|\xi|^2. \tag{1.2}$$

Note that we may have $\det B(x)=0$.

System (1.1) has an energy

$$E(u,\theta,t) := \int_\Omega |\partial_t u(t,x)|^2 dx + \int_\Omega |\nabla u(t,x)|^2 dx + \int_\Omega \theta^2(t,x)dx$$

which decreases in time according to

$$E(u,\theta,t) - E(u,\theta,0) = -2\int_0^t \int_\Omega B(x)\nabla\theta(s,x)\cdot\nabla\theta(s,x)dsdx \leqslant 0. \tag{1.3}$$

Equation (1.3) gives *a priori* estimates for initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, and $\theta_0 \in L^2(\Omega)$, and yields by classical arguments the existence of a unique solution $(u,\theta) \in \mathcal{C}^0(\mathbf{R}^+, H_0^1(\Omega)) \cap \mathcal{C}^1(\mathbf{R}^+, L^2(\Omega)) \times \mathcal{C}^0(\mathbf{R}^+, L^2(\Omega))$. We are interested in characterizing the way the energy decays: our aim is to describe the weak limits of the energy densities

$$e_n(t,x) = |\partial_t u_n(t,x)|^2 + |\nabla_x u_n(t,x)|^2 + (\theta_n(t,x))^2, \quad n \in \mathbf{N},$$

associated with families of solutions $(u_n,\theta_n)_{n\in\mathbf{N}}$ of (1.1) corresponding to families of initial data $(u_{0,n})_{n\in\mathbf{N}}$ on the one hand and $(u_{1,n})_{n\in\mathbf{N}}$, $(\theta_{0,n})_{n\in\mathbf{N}}$ on the other hand, uniformly bounded in $H^1(\Omega)$ and $L^2(\Omega)$, respectively. Without loss of generality, we suppose that $(u_1^n)_{n\in\mathbf{N}}$ and $(\theta_0^n)_{n\in\mathbf{N}}$ go to zero weakly in $L^2(\Omega)$ and that $(u_0^n)_{n\in\mathbf{N}}$ goes to zero weakly in $H^1(\Omega)$.

The main results on the thermoelasticity system are devoted to the situation where the matrix B is positive. It is known since the work of Dafermos [9] that, for $B(x) = \mathrm{Id}$ and $\gamma(x) = (1,\ldots,1)$, the energy decays to 0 if and only if there does not exist non-zero function $\varphi \in H_0^1(\Omega)$ satisfying, for some $\alpha \in \mathbf{R}$,

$$-\Delta\varphi = \alpha^2\varphi \quad \text{in } \Omega, \quad \operatorname{div}\varphi = 0 \quad \text{in } \Omega, \quad \varphi_{\partial\Omega} = 0.$$

Under this assumption, the description of this decay has been the subject of several contributions. In particular, in [21], for a system where the wave equation of (1.1) is replaced by a Lamé system, Lebeau and Zuazua [21] have proved that, for a large class of domains, the decay rate is not uniform. More precisely, they give a sufficient condition on the geometry of the domain Ω ($d = 2$ or 3) that guarantees that the decay rate is not uniform (Theorem 1.2 in [21]). They also derive, in some cases, sufficient conditions for uniform decay (Theorem 2.1 in [21]). They crucially use a result of Henry *et al.* [18] which shows that the semigroup associated to (1.1) is equal up to a compact operator to the semi-group of a system consisting of a damped wave equation coupled with a heat equation on the temperature θ (see the Appendix for details). The behavior of the energy density $|u_n(t,x)|^2$ associated with families of solutions of this damped wave equation (for initial data $(u_{0,n})_{n\in\mathbf{N}}$ and $(u_{1,n})_{n\in\mathbf{N}}$ uniformly bounded in $H^1(\Omega)$ and $L^2(\Omega)$, respectively) can be studied in the same manner as in the papers of Lebeau [19, 20] (see also the survey of Burq [3]). One obtains that the energy propagates along the rays of the geometric optic associated with the wave operator $\partial_t^2 - \Delta$ with a damping depending simultaneously on the position and the speed of the trajectory. We implement this strategy in the Appendix and obtain results on the rate of exponential decay of the energy. The method is based on the use of microlocal defect measures, and similar works have been achieved for the Lamé system in [7], for the equations of magnetoelasticity in [10], and for the equation of viscoelastic waves by the authors [2].

Of course, the strategy that we have just described fails if the kernel of B does not reduce to zero, and our main concern in this contribution is to analyze situations where the set

$$\Lambda = \{(x, \omega) \in \Omega \times \mathbf{S}^{d-1}, B(x)\omega = 0\}$$

is not empty. Then, the operator $\nabla \cdot (B(x)\nabla \cdot)$ is no longer elliptic, and something else has to be done. The method we use to treat the coupling between the temperature θ and the amplitude u is mainly inspired by the analysis of semiclassical systems performed in [17]. Of course, we recover the result sketched in the previous paragraph when $\Lambda = \emptyset$, and we are also able to extend it to situations where $\Lambda \neq \emptyset$ provided that a *weak degeneracy assumption* stated below holds (see Assumption 2.2). This assumption consists first in a geometric assumption: the projection of Λ on Ω is included in a hypersurface Σ, and, for $(x, \omega) \in \Lambda$, the vector ω is transverse to Σ at the point x. Then, Assumption 2.2 contains a compatibility relation between the vector $\gamma(x)$ and the matrix $B(x)$: $\gamma(x) \in \operatorname{Ran} B(x)$. With these assumptions, we are able to prove that the energy density is still damped along the rays of the geometric optic even though they pass through Σ. In contrast, if $B(x) = 0$ in an open subset $\widetilde{\Omega}$ of Ω, then the damping disappears, and we have transport of the energy along rays which are distortions of the rays observed before. Precise statements of our results are given in § 2, and the organization of the paper is discussed at the end of this section.

Notation. We will say that a sequence $(u_n)_{n\in\mathbf{N}}$ is u.b. in the functional space F if the sequence $(u_n)_{n\in\mathbf{N}}$ is a uniformly bounded family of F. We denote by $|X|_{\mathbf{C}^{d+2}}$ the Hermitian norm of $X = (X_1, \ldots, X_{d+2}) \in \mathbf{C}^{d+2}$:

$$|X|^2_{\mathbf{C}^{d+2}} = |X_1|^2 + \cdots + |X_{d+2}|^2.$$

Similarly, we will use the notation $(X|Y)$ for the Hermitian scalar product of $\mathbf{C}^{d+2}$: $\forall X = (X_1, \ldots, X_{d+2}) \in \mathbf{C}^{d+2}$, $\forall Y = (Y_1, \ldots, Y_{d+2}) \in \mathbf{C}^{d+2}$,

$$(X|Y)_{\mathbf{C}^{d+2}} = X_1\overline{Y_1} + \cdots + X_{d+2}\overline{Y}_{d+2}.$$

2. Main results

In this section, we present our results which crucially rely on the use of microlocal defect measures that we define in the first subsection. The second subsection is devoted to the analysis of properties of the thermoelasticity operator that are important for our purpose. Then, in the third subsection, we mainly consider the situation where the determinant of B vanishes on points of Ω which are simultaneously included in a hypersurface and in a compact subset of Ω (thus, B is non-negative in a neighborhood of $\partial\Omega$). Finally, in the fourth subsection, we discuss what happens if B vanishes in an open subset of Ω.

2.1. Microlocal defect measures

Microlocal defect measures allow us to treat quadratic quantities like the energy density by taking into account microlocal effects. They describe up to a subsequence the limit of quantities of the form $(a(x,D)f_n, f_n)$, where $a(x,D)$ is a pseudodifferential operator and $(f_n)_{n\in\mathbf{N}}$ a u.b. family of $L^2(\Omega)$ (or, more generally, of $H^s(\Omega)$). Recall that the pseudodifferential operator $a(x,D)$ is characterized by its symbol $a(x,\xi)$, which is a smooth function taken, for example, in the space $\mathcal{A}_i^m$ of symbols of order m: this set contains the functions $a=a(x,\xi)$ of $\mathcal{C}^\infty(\Omega\times\mathbf{R}^d)$ such that a is compactly supported in Ω as a function of x and satisfies

$$\forall \alpha,\beta\in\mathbf{N}^d, \exists C_{\alpha,\beta}>0, \forall (x,\xi)\in\Omega\times\mathbf{R}^d, \quad \left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}.$$

We also assume that if $a\in\mathcal{A}_i^m$, there exists a function $a_\infty(x,\xi)$ homogeneous of degree 0 such that,

$$\forall x\in\Omega, \forall\omega\in\mathbf{S}^{d-1}, \quad \lim_{R\to\infty} R^{-m}a(x,R\omega)=a_\infty(x,\omega). \tag{2.1}$$

Then, the operator $a(x,D)$, defined in the Weyl quantization by

$$\forall f\in L^2(\mathbf{R}^d), \quad a(x,D)f(x)=(2\pi)^{-d}\int a\left(\frac{x+y}{2},\xi\right)f(y)\,\mathrm{e}^{i\xi\cdot(x-y)}dy\,d\xi,$$

maps $H^s(\mathbf{R}^d)$ into $H^{s-m}(\Omega)$ (see [1]). Consider $\varphi\in\mathcal{C}_0^\infty(\Omega)$ such that $0\leqslant\varphi\leqslant 1$ and with $\mathrm{supp}(a)\subset\{\varphi=1\}$; we have $a\partial_x^\alpha\varphi=0$ for all $\alpha\in\mathbf{N}^d$ with $|\alpha|\geqslant 1$ and for $f\in L^2(\mathbf{R}^d)$,

$$\varphi a(x,D)(\varphi f)=a(x,D)f+Kf, \tag{2.2}$$

where K maps $H^s(\mathbf{R}^d)$ into $H^\infty(\mathbf{R}^d)$. Note that the operator $\varphi a(x,D)\varphi$ acts on functions of $L^2(\Omega)$. In what follows, we shall denote by $a(x,D)$ the operator $\varphi a(x,D)\varphi$ for some function φ as above: this operator maps $H^s(\Omega)$ into $H^{s-m}(\Omega)$.

The symbols of $\mathcal{A}_i^m$ are called interior symbols because they are compactly supported inside Ω. Such symbols are of no help for studying the behavior of $(f_n)_{n\in\mathbf{N}}$ close to $\partial\Omega$: one then uses tangential symbols which are defined in § 6. It is easy to convince oneself that the limits of quadratic quantities $(a(x,D)f_n, f_n)$ (for $a\in\mathcal{A}_i^0$ and $(f_n)_{n\in\mathbf{N}}$ u.b. in $L^2(\Omega)$ going weakly to 0 in $L^2(\Omega)$) only depend on the function a_∞. Then, following [15, 23], it is possible to prove that these limits are characterized by a positive Radon matrix-valued measure μ on $\Omega\times\mathbf{S}^{d-1}$ such that, up to the extraction of a subsequence,

$$(a(x,D)f_n,\ f_n)\xrightarrow[n\to\infty]{}\langle a_\infty,\mu\rangle.$$

Such a measure μ is called a microlocal defect measure of the family $(f_n)_{n\in\mathbf{N}}$. Note that the measure μ does not depend on the choice of φ in (2.2): if φ and $\tilde\varphi$ can be associated with a, the operators $\varphi a(x,D)\varphi$ and $\tilde\varphi a(x,D)\tilde\varphi$ differ by a smoothing operator K, and we have $(Kf_n,\ f_n)\xrightarrow[n\to\infty]{} 0$ as the weak limit of f_n is 0.

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Let us come back to the sequences $(\partial_t u_n)_{n\in\mathbf{N}}$, $(\nabla_x u_n)_{n\in\mathbf{N}}$, and $(\theta_n)_{n\in\mathbf{N}}$ that we need to study simultaneously. For that purpose, we set

$$U_n = \begin{pmatrix} \partial_t u_n \\ \nabla_x u_n \\ \theta_n \end{pmatrix}, \tag{2.3}$$

and the energy density e_n is

$$e_n(t,x) = |U_n(t,x)|^2_{\mathbf{C}^{d+2}}.$$

We shall consider quadratic quantities $(a(x,D)U_n(t), U_n(t))$, where the symbol

$$a(x,\xi) = \left(a_{i,j}(x,D)\right)_{i,j}$$

is a $(d+2)\times(d+2)$ matrix of interior symbols of $\mathcal{A}^m_i$. Then, a microlocal defect measure of $(U_n(t))_{n\in\mathbf{N}}$ will be matrix valued and will describe, up to the extraction of a subsequence, the limits of the quantities $(a(x,D)U_n(t),\ U_n(t))$. Of course, the t-dependence of these measures is an issue by itself. Therefore, since $U_n \in L^2_{loc}(\mathbf{R}, L^2(\Omega))$, we test $(a(x,D)U_n(t), U_n(t))$ against smooth compactly supported functions of the variable t and consider the limits of

$$I(a,\chi) := \int \chi(t)(a(x,D)U_n(t), U_n(t))dt.$$

A microlocal defect measure M of $(U_n(t))_{n\in\mathbf{N}}$ is a positive Radon measure on the set $\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}$ such that, up to a subsequence, we have the following: $\forall\chi\in\mathcal{C}^\infty_0(\mathbf{R})$

$$I(a,\chi) \xrightarrow[n\to+\infty]{} \int_{\mathbf{R}\times\Omega\times\mathbf{S}^{d-1}} \chi(t)\mathrm{tr}\left(a_\infty(x,\omega)M(dt,dx,d\omega)\right).$$

In particular, $\forall\chi\in\mathcal{C}^\infty_0(\mathbf{R})$, $\forall\phi\in\mathcal{C}^\infty_0(\Omega)$,

$$\int \chi(t)\phi(x)e_n(t,x)dxdt \xrightarrow[n\to\infty]{} \int_{\mathbf{R}\times\Omega\times\mathbf{S}^{d-1}} \chi(t)\phi(x)\mathrm{tr}\, M(dt,dx,d\omega).$$

The matrix-valued measure $M(t,x,\omega)$ is positive in the sense that its diagonal components $m_{i,i}$ are positive Radon measures and its off-diagonal components $m_{i,j}$ are absolutely continuous with respect to $m_{i,i}$ and $m_{j,j}$. Using the special form of the components of the vector $U_n(t)$, one can write

$$M(t,x,\omega) = \begin{pmatrix} m_1(t,x,\omega) & m_{0,1}(t,x,\omega)\omega & m_{0,2}(t,x,\omega) \\ \overline{m_{0,1}}(t,x,\omega)^t\omega & m_0(t,x,\omega)\omega\otimes\omega & m_{1,2}(t,x,\omega)\omega \\ \overline{m_{0,2}}(t,x,\omega) & \overline{m_{1,2}}(t,x,\omega)^t\omega & \nu_0(t,x,\omega) \end{pmatrix},$$

where

- $m_1(t,x,\omega)$, $\nu_0(t,x,\omega)$ are the microlocal defect measures of $(\partial_t u_n)_{n\in\mathbf{N}}$ and of $(\theta_n)_{n\in\mathbf{N}}$, respectively,
- $m_0(t,x,\omega)\omega_i\omega_j$ is the joint measure of $(\partial_{x_i}u_n)_{n\in\mathbf{N}}$ and $(\partial_{x_j}u_n)_{n\in\mathbf{N}}$,
- $m_{0,1}(t,x,\omega)\omega_j$ is the joint measure of $(\partial_t u_n)_{n\in\mathbf{N}}$ and $(\partial_{x_j}u_n)_{n\in\mathbf{N}}$,

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- $m_{0,2}(t,x,\omega)$ is the joint measure of $(\theta_n)_{n\in\mathbf{N}}$ and $(\partial_t u_n)_{n\in\mathbf{N}}$,
- $m_{1,2}(t,x,\omega)\omega_j$ is the joint measure of $(\theta_n)_{n\in\mathbf{N}}$ and $(\partial_{x_j} u_n)_{n\in\mathbf{N}}$.

By 'joint measure' of two u.b. families of $L^2(\Omega)$, $(f_n)_{n\in\mathbf{N}}$ and $(g_n)_{n\in\mathbf{N}}$, we mean a measure which describes the limit up to extraction of a subsequence of quantities $(a(x,D)f_n, g_n)_{n\in\mathbf{N}}$ for symbols a of order 0. In the following, we assume that $(U_n(0))_{n\in\mathbf{N}}$ has only one microlocal defect measure $M(0,x,\omega)$.

2.2. Analysis of the thermoelasticity operator

Let us now come back to our system (1.1), which we rewrite as

$$i\partial_t U_n = P(x,D)U_n,$$

where U_n is defined in (2.3) and

$$P(x,D) = \begin{pmatrix} 0 & i^t\nabla & -i\nabla\cdot(\gamma\cdot) \\ i\nabla & 0 & 0 \\ -i\gamma\cdot\nabla & 0 & i\nabla\cdot(B(x)\nabla\cdot) \end{pmatrix}.$$

We first study the eigenspaces of the matrix $P(x,\xi)$, which is not self-adjoint if $B\neq 0$. However, we have the following proposition.

Proposition 2.1. *There exists $R_0>0$ such that, for $|\xi|\geqslant R_0$, the matrix $P(x,\xi)$ has a kernel of dimension greater or equal to $d-1$ and three smooth eigenvalues $\lambda_0(x,\xi)$, $\lambda_+(x,\xi)$, and $\lambda_-(x,\xi)$, with smooth eigenvectors $V_0(x,\xi)$, $V_+(x,\xi)$, and $V_-(x,\xi)$.*

Denote by $\Pi^k(x,\xi)$ the matrices: $\Pi^k(x,\xi)=V_k(x,\xi)\otimes V_k(x,\xi)$ for $k\in\{0,+,-\}$. Then, for all $R>R_0$ and for $\chi\in\mathcal{C}^\infty(\mathbf{R}^d)$ such that $\chi(\xi)=0$ for $|\xi|\leqslant 1/2$, $\chi(\xi)=1$ for $|\xi|\geqslant 1$, and $0\leqslant\chi\leqslant 1$, we have, in $\mathcal{D}'(\Omega)$,

$$|U_n(t,x)|^2_{\mathbf{C}^{d+2}} = \sum_{k\in\{0,-,+\}} \left|\Pi^k(x,D)\chi\left(\frac{D}{R}\right)U_n(t,x)\right|^2_{\mathbf{C}^{d+2}} + o(1). \tag{2.4}$$

This proposition is a consequence of Propositions 3.1 and 3.2 below, where we study the asymptotics of $P(x,\xi)$ for large ξ (which differ whether $(x,\frac{\xi}{|\xi|})\in\Lambda$ or not). Each matrix $\Pi^k(x,\xi)$ $(k=0,-,+)$ of $P(x,\xi)$ characterizes a mode, and for each mode we will analyze the microlocal defect measure of the component $(\Pi^k(x,D)U_n)_{n\in\mathbf{N}}$.

2.3. Propagation and damping for weakly degenerate (B,γ)

Let us first state our assumptions.

Assumption 2.2. We say that the pair (B,γ) is weakly degenerate if B and γ satisfy the following conditions.

(1) There exists a hypersurface Σ of $\mathbf{R}^d$ such that $\{\det B(x)=0\}\subset\Sigma$ and, for all $(x,\omega)\in\Lambda$, the vector ω is transverse to Σ in x.

(2) There exists $\tilde\gamma\in\mathcal{C}^\infty(\Omega,\mathbf{R}^d)$ such that $\gamma(x)=B(x)\tilde\gamma(x)$.

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Example 2.3. Suppose that $d=2$, $\Omega=\mathbf{R}$, $\gamma(x)=e_2$, and $B(x)=\begin{pmatrix} b(x_1) & 0\\ 0 & 1\end{pmatrix}$. Suppose that $b(y)=0$ if and only if $y=0$. Then,

$$\Lambda=\{(0,y),(\pm 1,0)),y\in\mathbf{R}\},$$

and one can check that (B,γ) is weakly degenerate.

The sequences

$$v_{n,0}^{\pm}=\frac{1}{\sqrt{2}}\left(u_{n,1}\pm i|D|u_{n,0}\right)$$

have only one microlocal defect measure $\mu_0^{\pm}$ given by

$$\mu_0^{\pm}(x,\omega)=m_1(0,x,\omega)+m_0(0,x,\omega)\pm \mathrm{Re}\left(m_{0,1}(0,x,\omega)\right).$$

Theorem 2.4. *Suppose that* (B,γ) *is weakly degenerate,* $\mathrm{Supp}(\mu_0^{\pm})\subset\Lambda^c$, *and that there exists* $\tau_0\in\mathbf{R}^{*+}$ *such that, for* $(x,\omega)\in\mathrm{Supp}(\mu_0^{\pm})$, *and for* $t\in[0,\tau_0]$, $x\pm t\omega\in\Omega$. *Then, for* $\chi\in\mathcal{C}_0^{\infty}([0,\tau_0])$ *and* $\phi\in\mathcal{C}_0^{\infty}(\{\det B(x)\neq 0\})$,

$$\lim_{n\to+\infty}\int_{\mathbf{R}^+\times\Omega}\chi(t)\phi(x)e_n(t,x)dtdx=\int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)\phi(x)d\mu_t^{+}(x,\omega)dt + \int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)\phi(x)d\mu_t^{-}(x,\omega)dt,$$

where, for all $a\in\mathcal{C}_0^{\infty}(\Omega\times\mathbf{S}^{d-1})$,

$$\langle a,\mu_t^{\pm}\rangle=\int_{\Omega\times\mathbf{S}^{d-1}}a\left(x\pm t\omega,\omega\right)\mathrm{Exp}\left[-\int_0^t\frac{(\gamma(x\pm\sigma\omega)\cdot\omega)^2}{B(x\pm\sigma\omega)\omega\cdot\omega}d\sigma\right]d\mu_0^{\pm}(x,\omega). \tag{2.5}$$

Remark 2.5. (1) Even though the support of ϕ does not intersect $\{\det B(x)=0\}$, the trajectories $x\pm t\omega$ which reach the support of ϕ for $t\in[0,\tau_0]$ may pass through it.

(2) Because of (1) in Assumption 2.2, a trajectory $x+s\omega$ crosses Σ at a finite number of times $0<t_1<\cdots<t_N\leqslant\tau_0$. Assuming that $B(x+t_j\omega)\omega=0$, by (2) of Assumption 2.2, for $j\in\{1,\dots,N\}$, there exists $c_j\in\mathbf{R}$ such that

$$\frac{(\gamma(x+s\omega)\cdot\omega)^2}{\omega\cdot B(x+s\omega)\omega}\sim c_j(s-t_j)\quad\text{as } s\sim t_j.$$

This implies that the integral in (2.5) is well defined.

This result is proved in § 4.2. The measures $\mu_t^{\pm}$ contain the part of the energy corresponding to the projection of U_n on the $\pm$-mode. There is no contribution of the 0-mode (which corresponds to the temperature) because of the smoothing effect of the heat equation. Note finally that the damping in (2.5) can be 0 if, for all times $t\in[0,\tau_0]$, we have $\gamma(x\mp t\omega)\cdot\omega=0$. We refer to § 4.2 for a discussion of what happens when (2) fails in Assumption 2.2: all the energy may be damped in finite time (see Remark 4.3).

Finally, in § 6, we briefly discuss what happens close to the boundary under the following assumptions.

Assumption 2.6. (1) The rays of geometric optics have no contact of infinite order with the tangent to $\partial\Omega$.

(2) There exists a compact K such that $\{\det B(x) = 0\} \subset K \subset \Omega$.

(3) For all $x \in \partial\Omega$, $\gamma(x) \in T_x(\partial\Omega)$.

If Assumption 2.6 holds, then one can use the methods developed in [16] and the papers [3], [7], or [20] for the analysis at the boundary of microlocal measures of a family of solutions to a damped wave equation (see also [4, 5]). We explain in § 6 why no new phenomena occurs on the boundary when one has Assumption 2.6 (comparatively with a wave equation). In particular, since B is supposed to be non-degenerate close to the boundary, one can conjecture that the same phenomenon occurs as when B is non-degenerate: propagation of the energy along the generalized bicharacteristic curves as defined in [22] (see § 6 for details). However, this requires further work, and we do not prove this conjecture here.

Note that, if one has this complete description of the evolution of microlocal defect measures of families of solutions to (1.1) (propagation and damping along the rays of geometric optics with reflexions on the boundary), then one can characterize the decay rate of the energy as in [19, 20], and the results of Proposition A.3, which are stated in Appendix for non-degenerate matrices B, extend to weakly degenerate (B, γ), provided that the conjectured behavior at the boundary is proved.

2.4. Distorted propagation in an open set included in $\{B(x) = 0\}$

We suppose now that $B(x) = 0$ in $\widetilde{\Omega}$, an open subset of Ω. Then, the symbol $P(x, \xi)$ is self-adjoint on $\widetilde{\Omega} \times \mathbf{R}^d$, and the method of [17] can be adapted with straightforward modifications. Note first that, in $\widetilde{\Omega}$, the function $\partial_t u_n$ satisfies the wave equation

$$\begin{cases} \partial_t(\partial_t u_n) - \Delta(\partial_t u_n) - \nabla\left(\gamma(x)(\gamma(x) \cdot \nabla \partial_t u_n)\right) = 0, \\ (\partial_t u_n)_{|t=0} = u_{n,1}. \end{cases}$$

This equation induces the wave operator $\partial_t^2 - c(x, D) \circ c(x, D)$, with

$$c(x, \xi) = \sqrt{(\gamma(x) \cdot \xi)^2 + |\xi|^2},$$

which will play an important role in the following. We introduce $\tilde{\mu}_0^{\pm}$ and $\tilde{\nu}_0$, the microlocal defect measures of the sequences

$$\widetilde{v}_n^{\pm} = \frac{1}{\sqrt{2}}\left(\pm u_{n,1} - iW(x, D)|D|u_{n,0} + N(x, D)\theta_{n,0}\right)$$
$$\widetilde{\theta}_n = iN(x, D)|D|u_{n,0} + W(x, D)\theta_{n,0},$$

where $W(x, D)$ and $N(x, D)$ are pseudodifferential operators of order 0 of symbols

$$W(x, \xi) = \frac{|\xi|}{c(x, \xi)}\,\chi(\xi/R_0) \quad \text{and} \quad N(x, \xi) = \frac{\gamma(x) \cdot \xi}{c(x, \xi)}\,\chi(\xi/R_0);$$

the function χ is smooth and satisfies $\chi(\xi) = 0$ for $|\xi| \leqslant 2$ and $\chi(\xi) = 1$ for $|\xi| \geqslant 4$; besides, R_0 is chosen as in Proposition 2.1. Note that the measures $\tilde{\mu}_0^{\pm}$ and $\tilde{\nu}_0$ do not depend on the cut-off function χ, and they satisfy

$$\tilde{\nu}_0 = \frac{1}{c(x,\omega)^2}\left(M(0,x,\omega)\begin{pmatrix} 0 \\ (\gamma(x)\cdot\omega)\ \omega \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ (\gamma(x)\cdot\omega)\ \omega \\ 1 \end{pmatrix}\right)_{\mathbf{C}^{d+2}},$$

$$\tilde{m}_0^{\pm} = \frac{1}{2c(x,\omega)^2}\left(M(0,x,\omega)\begin{pmatrix} \pm c(x,\omega) \\ -\omega \\ \gamma(x)\cdot\omega \end{pmatrix} \middle| \begin{pmatrix} \pm c(x,\omega) \\ -\omega \\ \gamma(x)\cdot\omega \end{pmatrix}\right)_{\mathbf{C}^{d+2}}.$$

Before stating the result, let us introduce some notation. Define

$$H_c(x,\xi) = \nabla_\xi c(x,\xi)\cdot\nabla_x - \nabla_x c(x,\xi)\cdot\nabla_\xi,$$

to be the Hamiltonian vector field associated with c, and H_c^∞ to be the vector field induced by H_c on $S^*\Omega$.

Theorem 2.7. *Suppose that* $\mathrm{Supp}(\tilde{\nu}_0) \subset \widetilde{\Omega}$, $\mathrm{Supp}(\widetilde{\mu}_0^{\pm}) \subset \widetilde{\Omega}$, *and that there exists* $\tau_0 \in \mathbf{R}^{*+}$ *such that, for all* $x \in \widetilde{\Omega}$ *and* $\omega \in \mathbf{S}^{d-1}$, *the projection on* $\widetilde{\Omega}$ *of the integral curve of* H_c^∞ *issued from* (x,ω) *stays in* $\widetilde{\Omega}$ *on the time interval* $[0,\tau_0]$. *Then, for* $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$ *and* $\phi \in \mathcal{C}_0^\infty(\widetilde{\Omega})$,

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbf{R}^+\times\Omega}\chi(t)\phi(x)e_n(t,x)dtdx = &\int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)\phi(x)d\widetilde{\mu}_t^+(x,\omega)dt \\ &+\int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)\phi(x)d\widetilde{\mu}_t^-(x,\omega)dt \\ &+\int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)\phi(x)d\tilde{\nu}_0(x,\omega)dt,\end{aligned}$$

where $\widetilde{\mu}_t^{\pm}$ *satisfy the transport equations* $\partial_t\widetilde{\mu}_t^{\pm} \mp H_c^\infty(x,\omega)\widetilde{\mu}_t^{\pm} = 0$ *with initial data* $\widetilde{\mu}_0^{\pm}$.

Remark 2.8. Let us call distorted bicharacteristic curves the trajectories of $S^*\Omega$ associated with H_c^∞: the transport equation for $\widetilde{\mu}^{\pm}$ implies that the energy propagates along these trajectories.

Note that, inside $\widetilde{\Omega}$, one cannot separate the contribution to the energy density of (u_n) and of (θ_n). The measures $\widetilde{\mu}_t^{\pm}$ depend on the value at time $t = 0$ of both quantities (u_n) and (θ_n).

These results call for further work: it would be interesting to know what happens at the boundary of $\widetilde{\Omega}$ and how transitions occur between the two regimes. It would be also interesting to know whether this result in $\{B(x) = 0\}$ extends to the set Λ provided that the distorted rays issued from points of Λ are included in Λ. The following example show such a situation.

Example 2.9. Suppose that $d = 2$, $\Omega = \mathbf{R}$, $\gamma(x) = e_1$, and $B(x) = \begin{pmatrix} b(x_2) & 0 \\ 0 & 1\end{pmatrix}$. Suppose that $b(y) = 0$ if and only if $y = 0$. Then,

$$\Lambda = \{(x,\omega), \exists y \in \mathbf{R}, x = (y,0), \omega = (\pm 1, 0)\}$$

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is invariant and the distorted bicharacteristics curves issued from points of Λ are included in Λ:

$$\left(x_s^{\pm}, \xi_s^{\pm}\right) = ((y \pm s\sqrt{2}, 0), (\pm 1, 0)), \quad s \in \mathbf{R}.$$

Note that, in that case, the distorted trajectories issued from points of Λ coincide with the usual ones; however, they are described with different speed. Note also that (2) of Assumption 2.2 is not satisfied here.

2.5. Organization of the paper

The main part of the article consists in the analysis of the microlocal defect measures associated with the families $\left(\Pi^k(x, D)U_n\right)$ (for $k \in \{0, +, -\}$), where the functions Π^k are defined in Proposition 2.1. We begin in § 3 by studying the symbol $P(x, \xi)$, which allows us to prove Proposition 2.1. Then, in § 4, we prove Theorems 2.4 and 2.7; they rely on the analysis of the propagation of the microlocal defect measures associated with the sequences $\left(\Pi^{\pm}(x, D)U_n\right)$, which is the object of § 5. Finally, § 6 is devoted to a discussion of the reflexion of the measures on the boundary. In the Appendix, we present another proof of the result of Theorem 2.4 when the matrix B is non-degenerate. It is also in the Appendix that we explain how the analysis of microlocal defect measures can give information on the rate of the decay of the energy.

3. Analysis of the symbol of $P(x, D)$

In this section, we analyze the properties of the matrix $P(x, \xi)$. The main interest of Weyl quantization is that the symbol of a self-adjoint operator is real valued. We denote by $\sigma(A)$ the symbol of an operator A, and we have in particular

$$\sigma(\nabla) = i\xi,$$

$$\sigma(\gamma(x) \cdot \nabla) = i\gamma(x) \cdot \xi - \frac{1}{2}\nabla \cdot \gamma(x), \tag{3.1}$$

$$\sigma(\nabla \cdot (B(x)\nabla)) = -B(x)\xi \cdot \xi + b_0(x), \tag{3.2}$$

where

$$b_0(x) = -\frac{1}{4} \sum_{1 \leqslant j,k \leqslant d} \partial^2_{x_j, x_k} B_{jk}(x). \tag{3.3}$$

Observe that, if $d = 1$, the function b_0 has a sign on Λ. Indeed, if $d = 1$, the points $(x, \omega) \in \Lambda$ correspond to values x which are minima of $B(x)$, and in this case $b_0 \leqslant 0$. However, in higher dimension, the function $b_0(x)$ can be positive or negative indifferently, as the following example shows. Choose $d = 2$ and $B(x)$ such that $\Lambda = \{((0, y), (\pm 1, 0)), y \in \mathbf{R}\}$, with

$$B(x) = \begin{pmatrix} x_1^4 & 0 \\ 0 & 1 + x_2^3 \end{pmatrix} + O(|x|^5)$$

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close to (0, 0). Then we have

$$b_0(x) = -\frac{1}{4}(12x_1^2 + 6x_2) + O(|x|^3).$$

Therefore, $b_0(0, y) = -\frac{3}{2}y + O(|y|^3)$ and the sign of b_0 changes on Λ. Note also that, if $B(x) = 0$ in an open subset $\widetilde{\Omega}$ of Ω, then $b_0 = 0$ in $\widetilde{\Omega}$.

The eigenvalues of the matrix $P(x, \xi)$ satisfy the following proposition.

Proposition 3.1. *There exists $R_0 > 0$ such that, for $|\xi| > R_0$, the following facts hold. The matrix $P(x, \xi)$ has a kernel of dimension greater than or equal to $d - 1$ and*

$$\{(0, e, 0), e \cdot \xi = 0\} \subset \operatorname{Ker} P(x, \xi). \tag{3.4}$$

Moreover, $P(x, \xi)$ has three smooth eigenvalues λ_-, λ_0, and λ_+ with the following properties.

(1) *If $\left(x, \frac{\xi}{|\xi|}\right) \notin \Lambda$, then $\dim[\operatorname{Ker} P(x, \xi)] = d - 1$ and*

$$\lambda_0(x, \xi) = -i\xi \cdot B(x)\xi + ib_0(x) + i\frac{(\gamma(x) \cdot \xi)^2}{\xi \cdot B(x)\xi} + O(|\xi|^{-1}), \tag{3.5}$$
$$\lambda_\pm(x, \xi) = \pm\beta(x, \xi) + i\alpha(x, \xi),$$

with

$$\beta(x, \xi) = |\xi| + O(1), \quad \alpha(x, \xi) = -\frac{1}{2}\frac{(\gamma(x) \cdot \xi)^2}{\xi \cdot B(x)\xi} + O(|\xi|^{-1}). \tag{3.6}$$

(2) *If $\left(x, \frac{\xi}{|\xi|}\right) \in \Lambda$, then, if $b_0(x) = 0$, $\dim[\operatorname{Ker} P(x, \xi)] = d$ and, if $b_0(x) \neq 0$, $\dim[\operatorname{Ker} P(x, \xi)] = d - 1$. Moreover,*

$$\lambda_0(x, \xi) = ib_0(x)\frac{|\xi|^2}{c(x, \xi)^2} + O(|\xi|^{-1}), \tag{3.7}$$
$$\lambda_\pm(x, \xi) = \pm\beta(x, \xi) + i\alpha(x, \xi),$$

with

$$\beta(x, \xi) = c(x, \xi) + O(1), \quad \alpha(x, \xi) = \frac{1}{2}b_0(x)\frac{(\gamma(x) \cdot \xi)^2}{c(x, \xi)^2} + O(|\xi|^{-1}), \tag{3.8}$$

and $c(x, \xi) = \sqrt{(\gamma(x) \cdot \xi)^2 + |\xi|^2}$.

The modes $\pm$ (corresponding to the eigenvalues $\lambda_\pm$) give the wave feature of the equation. The speed of propagation is characterized by the function β and, in the first case, the function α corresponds to the damping. Note that, in the second case, the speed of propagation is distorted in comparison with the initial wave operator $\partial_t^2 - \Delta$. Outside Λ, the eigenvalue λ_0 encounters the heat aspect.

Proof. We write $P(x, \xi) = iQ(x, \xi)$, and for simplicity we work with $Q(x, \xi)$. For $p, q \in \mathbf{N}^*$, we denote by $0_{p,q}$ the $p \times q$ matrix with all coefficients equal to 0. We

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have

$$Q(x,\xi)=\begin{pmatrix} 0 & {}^t i\xi & -\overline{k(x,\xi)} \\ i\xi & 0_{d,d} & 0_{d,1} \\ k(x,\xi) & 0_{1,d} & b(x,\xi) \end{pmatrix},$$

where, in view of (3.1) and (3.2), $b=b_2+b_0$ with $b_2=-B(x)\xi\cdot\xi$ real valued, and $k=ik_1+k_0$ with

$$k_1=-\gamma(x)\cdot\xi,\quad k_0=\frac{1}{2}\nabla\cdot\gamma(x).$$

The vector $(x,Y,y)\in\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}$ is an eigenvector of Q for the eigenvalue ν if and only if

$$\begin{cases} (1) \quad \nu x - i\xi\cdot Y + \bar{k}y = 0, \\ (2) \quad ix\xi - \nu Y = 0, \\ (3) \quad kx + (b-\nu)y = 0. \end{cases}$$

Let us suppose first that $\nu=0$; then, for $\xi\neq 0$, equation (2) gives $x=0$. For $b\neq 0$, equation (3) then implies $y=0$, and (1) gives $\xi\cdot Y=0$. Therefore, for large ξ, the kernel of $Q(x,\xi)$ is of dimension at least $d-1$, and we have (3.4).

Let us now suppose that $\nu\neq 0$. Equation (2) implies that Y is colinear to ξ and $x=-i\nu r$, where $Y=r\xi$. Equations (1) and (3) become

$$\begin{cases} \bar{k}y - i(\nu^2+|\xi|^2)r = 0, \\ (b-\nu)y - i\nu kr = 0. \end{cases}$$

Therefore, the non-zero eigenvalues are the roots of the real-valued polynomial

$$f(X)=-X^3+X^2b-X(|\xi|^2+|k|^2)+b|\xi|^2.$$

It is easy to see that, for large ξ, f has one real-valued root ν_0 (with $\nu_0\neq 0$ for $b_0\neq 0$) and two conjugated complex-valued roots ν_+ and ν_-. These three roots of the polynomial f are simple and thus smooth; they give three smooth eigenvalues of Q. Consider three associated eigenvectors V_0, V_+, and V_-; they are independent from the vectors $V_j(\xi)$ defined above. Therefore, we are left with a basis of eigenvectors: the matrix $Q(x,\xi)$ diagonalizes.

Let us now study more precisely the asymptotics of the eigenvalues.

(1) Suppose that $(x,\xi)\notin\Lambda$, ξ large, and denote by $X_1<X_2<0$ the two negative roots of $f'(X)$. We have $X_1=\frac{2}{3}b_2+O(|\xi|)$ and $X_2=O(|\xi|)$. Then, using $f'(X_2)=0$, we obtain $f(X_2)=b(X_2^2+|\xi|^2)<0$ and deduce that f has only one real-valued root ν_0 with $\nu_0<X_1$. We set $\nu_0=\phi b$ with $\phi\geqslant\frac{2}{3}+o(1)$ and

$$0=f(\nu_0)=b^3\phi^2(1-\phi)-\phi b(|k|^2+|\xi|^2)+b|\xi|^2. \tag{3.9}$$

Necessarily, $\phi=1+r$ with $r=-k_1^2/b_2^2+O\left(|\xi|^{-3}\right)$, whence $\nu_0=b-\frac{k_1^2}{b_2}+O\left(|\xi|^{-1}\right)$ and (3.5).

Let $\nu_\pm$ be the two other (non-real-valued) roots; we set $\nu_\pm = \alpha \mp i\beta$. We observe that

$$\begin{cases} \nu_+ + \nu_- = 2\alpha = b - \nu_0, \\ \nu_0(\nu_+ + \nu_-) + \nu_+\nu_- = 2\alpha\nu_0 + \alpha^2 + \beta^2 = |k|^2 + |\xi|^2, \end{cases} \tag{3.10}$$

whence $\alpha = \frac{k_1^2}{2b_2} + O(|\xi|^{-1})$ and $\beta^2 = |\xi|^2 + O(|\xi|)$. This implies (3.6).
(2) Suppose now that $(x, \xi) \in \Lambda$; then $b(x, \xi) = b_0(x)$. The polynomial function $f'(X)$ has no real-valued root and $f(X)$ has only one real-valued root ν_0. Since $f(0)f(b_0) < 0$, the function $\nu_0(x, \xi) = O(1)$ as ξ grows and

$$\nu_0(|\xi|^2 + |k|^2) = b_0|\xi|^2 + O(1),$$

whence (3.7). Besides, denoting as before by $\nu_\pm = \alpha \mp i\beta$ the two other roots, (3.10) gives

$$\alpha = \frac{b_0}{2}\frac{k_1^2}{k_1^2 + |\xi|^2} + O(|\xi|^{-1}) \quad \text{and} \quad \beta^2 = k_1^2 + |\xi|^2 + O(1),$$

whence (3.8). □

Let us now describe the eigenspaces of $P(x, \xi)$ for large ξ.

Proposition 3.2. *Let $R_0 > 0$ be sufficiently large. There exist smooth vector-valued functions V_0, V_+, and V_- defined on $\Omega \times \{|\xi| > R_0\}$ such that*

$$P(x, \xi)V_k(x, \xi) = \lambda_k(x, \xi)V_k(x, \xi) \quad \textit{for } \in \{0, +, -\}.$$

Besides, we have the following expansions.

(1) *If* $\left(x, \frac{\xi}{|\xi|}\right) \notin \Lambda$,

$$V_\pm = \frac{i}{\sqrt{2}}\left(\mp 1, \frac{\xi}{|\xi|}, 0\right) + O(|\xi|^{-1}), \tag{3.11}$$

$$V_0 = (0, 0, 1) + O(|\xi|^{-2}). \tag{3.12}$$

(2) *If* $\left(x, \frac{\xi}{|\xi|}\right) \in \Lambda$,

$$V_\pm = \frac{1}{\sqrt{2}}\left(-1, \pm\frac{\xi}{c(x, \xi)}, \mp\frac{\gamma(x)\cdot\xi}{c(x, \xi)}\right) + O(|\xi|^{-1}), \tag{3.13}$$

$$V_0 = \left(0, \frac{\gamma(x)\cdot\xi}{c(x, \xi)}\frac{\xi}{|\xi|}, \frac{|\xi|}{c(x, \xi)}\right) + O(|\xi|^{-1}). \tag{3.14}$$

Note that there exist smooth eigenvectors, but their asymptotics are discontinuous; similarly, their asymptotics are orthogonal while the original vectors are not. When $b_0 = 0$, there is an eigenvalue crossing between the eigenspace for λ_0 which merges into the kernel of P. However, close to a point of Ω, there still exist smooth eigenvectors, and we will take advantage of this fact in the following sections. Besides, we have the following remark.

Remark 3.3. Assuming (2) of Assumption 2.2, we have, for $(x, \omega) \in \Lambda$,

$$\gamma(x)\cdot\omega = \widetilde{\gamma}(x)\cdot(B(x)\omega) = 0.$$

Therefore, if $\left(x, \frac{\xi}{|\xi|}\right) \in \Lambda$, $c(x,\xi) = |\xi|$ and

$$V_\pm = \frac{1}{\sqrt{2}}\left(-1, \pm\frac{\xi}{|\xi|}, 0\right) + O(|\xi|^{-1}),$$
$$V_0 = (0,0,1) + O(|\xi|^{-1}).$$

Let us now prove Proposition 3.2.

Proof. In the proof of Proposition 3.1, we have seen that the eigenvectors of $Q(x,\xi)$ associated with ν_0, ν_+, and ν_- are of the form $(x, r\xi, y)$, with $x = -i\nu r$ and

$$\begin{cases} (b-\nu)y & - & i\nu k r & = & 0, \\ \bar{k}y & - & i(\nu^2 + |\xi|^2)r & = & 0. \end{cases}$$

Let us consider first the $\pm$-modes. We have $\nu_\pm = O(|\xi|)$. Therefore, $b - \nu_\pm \neq 0$ for large ξ independently of the fact that $(x, \xi/|\xi|) \in \Lambda$ or not. The vectors

$$V_\pm(x,\xi) = \tilde{r}\left(\nu_\pm(\nu_\pm - b), i(\nu_\pm - b)\xi, k\nu_\pm\right),$$

with $\tilde{r} = \left(|\nu_\pm - b|^2(|\nu_\pm|^2 + |\xi|^2) + |k|^2|\nu_\pm|^2\right)^{-1/2}$ are smooth non-zero eigenvectors associated with $\nu_\pm$. In view of the asymptotics of Proposition 3.1, we obtain asymptotics for $V_\pm$.

(1) In Λ^c, $\nu_\pm = \mp i|\xi| + O(1)$, $k = -i\gamma(x)\cdot\xi + O(1)$ and $b = -B(x)\xi\cdot\xi + O(1)$. Therefore

$$\tilde{r} = \frac{1}{\sqrt{2}}\left(|\xi| B(x)\xi\cdot\xi\right)^{-1}\left(1 + O(|\xi|^{-1})\right),$$
$$V_\pm(x,\xi) = \tilde{r}\left[\,(\mp i|\xi|(B(x)\xi\cdot\xi), i(B(x)\xi\cdot\xi)\,\xi, \mp|\xi|\gamma(x)\cdot\xi) + O(|\xi|^2)\right],$$

whence (3.11).

(2) In Λ, $\nu_\pm = \mp ic(x,\xi) + O(1)$, $b = O(1)$, and we still have $k = -i\gamma(x)\cdot\xi + O(1)$. Therefore

$$\tilde{r} = 1/(\sqrt{2}\,c(x,\xi)^2)(1 + O(1)),$$
$$V_\pm(x,\xi) = \tilde{r}\left[\left(-c(x,\xi)^2, \pm c(x,\xi)\,\xi, \mp c(x,\xi)\gamma(x)\cdot\xi\right) + O(|\xi|)\right],$$

whence (3.13).

Let us consider now the 0-mode. We have $\nu_0 = O(1)$ in Λ and $\nu_0 + O(1) \in \mathbf{R}$ in Λ^c. Therefore, $\nu_0^2 + |\xi|^2 \neq 0$ for large ξ, and the vector

$$V_0(x,\xi) = r_0\left(-\nu_0\bar{k}, -i\bar{k}\xi, \nu_0^2 + |\xi|^2\right)$$
$$\text{with} \quad r_0 = \left(|\nu_0|^2|k|^2 + |\xi|^2|k|^2 + \left(\nu_0^2 + |\xi|^2\right)^2\right)^{-1/2}$$

is a smooth eigenvector associated with the eigenvalue ν_0. We are now left with a smooth basis of eigenvectors. Let us now study the asymptotics of this vector.

(1) In Λ^c, $\nu_0 = -B(x)\xi\cdot\xi + O(1)$; therefore

$$r_0 = (B(x)\xi\cdot\xi)^{-2}\left(1 + O(|\xi|^{-1})\right) \quad \text{and} \quad V_0(x,\xi) = (0,0,1) + O(|\xi|^{-2}).$$

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(2) In Λ, $\nu_0 = O(1)$, whence

$$r_0 = 1/(c(x,\xi)|\xi|)(1 + O(|\xi|^{-1}),$$
$$V_0(x,\xi) = r_0 \left[\left(0, (\gamma(x)\cdot\xi)\,\xi, |\xi|^2\right) + O(|\xi|) \right],$$

which gives (3.14). □

Before concluding this section, we point out that these asymptotics imply Proposition 2.1. Indeed, we have obtained the existence of R_0 such that, for $|\xi| > R_0$,

$$P(x,\xi) = \lambda_0(x,\xi)\Pi^0(x,\xi) + \lambda_+(x,\xi)\Pi^+(x,\xi) + \lambda_-(x,\xi)\Pi^-(x,\xi)$$

where the smooth eigenprojectors are asymptotically orthogonal with

$$\Pi^k(x,\xi) = V_k(x,\xi) \otimes \overline{V_k(x,\xi)} + o(1).$$

Since $U_n(t,x)$ goes weakly to 0 in $L^2(\Omega)$, we have

$$|U_n(t,x)|^2_{\mathbf{C}^{d+2}} = \left| \chi\left(\frac{D}{R}\right) U_n(t,x) \right|^2 + o(1) \quad \text{in } \mathcal{D}'(\Omega).$$

Besides, for $R > R_0$, we observe that $\Pi^k(x,D)\chi(D/R)U_n(t,x)$ is well defined and we have, in $\mathcal{D}'(\Omega)$,

$$\left| \Pi^k(x,D)\chi\left(\frac{D}{R}\right) U_n(t,x) \right|^2_{\mathbf{C}^{d+2}} = \left| \left(V_k(x,D) | \chi\left(\frac{D}{R}\right) U_n(t,x) \right)_{\mathbf{C}^{d+2}} \right|^2 + o(1).$$

We can now use the asymptotics of Proposition 3.2 which, combined with the weak convergence to 0 of $U_n(t,x)$, gives

$$\sum_{k\in\{0,-,+\}} \left| \left(V_k(x,D) | \chi\left(\frac{D}{R}\right) U_n(t,x) \right)_{\mathbf{C}^{d+2}} \right|^2$$
$$= (\partial_t u_n(t,x))^2 + |\nabla u_n(t,x)|^2 + (\theta_n(t,x))^2 + o(1) \quad \text{in } \mathcal{D}'(\Omega),$$

whence Proposition 2.1.

4. Proof of the main results

The proofs of Theorems 2.4 and 2.7 are inspired by the method developed in [17] for analyzing semi-classical measures associated with solutions of a system of partial differential equations. The proof of Theorem 2.7 is a direct adaptation of the results of [17] in the microlocal defect measures setting, while the proof of Theorem 2.4 requires non-trivial adaptations due to the fact that $P(x,\xi)$ is not self-adjoint and that one of its eigenvalues is a symbol of order 2. Therefore, we focus on the proof of Theorem 2.4, and we leave to the reader the simple adaptation of these arguments to prove Theorem 2.7. Theorem 2.4 relies on Propositions 4.1 and 4.2 stated in § 4.1; then the proof of Theorem 2.4 is performed in § 4.2.

4.1. Preliminaries

We state technical results (Propositions 4.1 and 4.2) that we will use in the next subsection. The proof of Proposition 4.1 is done at the end of this section, while that of Proposition 4.2 is postponed to § 5.

The first result describes the evolution of the temperature θ_n.

Proposition 4.1. *Let ν be a microlocal defect measure of the sequence $(\theta_n)_{n\in\mathbf{N}}$ of $L^2_{loc}(\mathbf{R}, L^2(\mathbf{R}^d))$; then, as a measure on $\mathbf{R}^+ \times \Lambda^c$, $\nu = 0$.*

Our second result concerns the contribution to the energy density of the sequences $\left(\Pi^{\pm}(x, D)\chi(D/R)u_n\right)_{n\in\mathbf{N}}$ for $R > R_0$ and χ as in Proposition 2.1. We set

$$U^{\pm}_{n,R}(t, x) = \Pi_{\pm}(x, D)\chi\left(\frac{D}{R}\right) U_n.$$

The sequences $(U^{\pm}_{n,R})_{n\in\mathbf{N}}$ are uniformly bounded in $L^2(\Omega)$, and their microlocal defect measures are matrix-valued measures independent of $R > 0$. Besides, by the definition of $U^{\pm}_{n,R}$, these measures are of the form $\mu_{\pm}(t, x, \omega)\Pi^{\pm}(x, \omega)$, where the measures $\mu^{\pm}(t, x, \omega)$ can be understood as the traces of $\mu_{\pm}(t, x, \omega)\Pi^{\pm}(x, \omega)$. We prove the following result.

Proposition 4.2. *Assume that condition (2) of Assumption 2.2 is satisfied in $\Omega_1 \subset \Omega$, and let $T > 0$. There exist a subsequence n_k and a continuous map $t \mapsto \mu_{\pm}(t)$ from $[0, T]$ into the set of positive Radon measures on $\Omega \times \mathbf{S}^{d-1}$ such that, for all $t \in [0, T]$ and for all scalar symbols $a \in \mathcal{A}^0_i$, we have*

$$\left(a(x, D_x)U^{\pm}_{n_k,R}(t)|U^{\pm}_{n_k,R}(t)\right) \xrightarrow[k\to+\infty]{} \int a_\infty(x, \omega)d\mu_{\pm}(t, x, \omega). \tag{4.1}$$

Moreover, in $\mathcal{D}'(\{t \geqslant 0\} \times \Omega_1 \times \mathbf{S}^{d-1})$, we have

$$\partial_t\mu_{\pm} \pm H_\beta(\mu_{\pm}) - 2\alpha_\infty\mu_{\pm} = \nu_{\pm}, \tag{4.2}$$

where $\nu_{\pm}$ is a measure supported on Λ absolutely continuous with respect to $\mu_{\pm}$, and where, for all $a \in \mathcal{A}^0_i$,

$$\int a_\infty(x, \omega)\, d\left(H_\beta(\mu_{\pm})\right) = -\int (H_\beta a)_\infty\, d\mu_{\pm} = \int (\{a, \beta\})_\infty\, d\mu_{\pm}.$$

Proposition 4.2 is proved in § 5 below.

Let us now prove Proposition 4.1.

Proof. It is enough to show that, if q is a symbol of order 0 such that $q(x, \omega) \in \mathcal{C}^\infty_0(\Lambda^c)$, then $q(x, D)\theta_n$ goes to 0 in $L^2_{loc}(\mathbf{R}, L^2(\Omega))$. We observe that we only need to consider large values of ξ. Indeed, if $\chi \in \mathcal{C}^\infty(\mathbf{R}^d)$, $\chi(\xi) = 0$ for $|\xi| \leqslant 1$, and $\chi(\xi) = 1$ for $|\xi| \geqslant 2$, with $0 \leqslant \chi \leqslant 1$, we have

$$q(x, D)\theta_n = q(x, D)\chi\left(\frac{D}{R}\right)\theta_n + o(1)$$

in $L^2(\Omega)$, because θ_n goes weakly to 0 and the operator $1-\chi\left(\frac{D}{R}\right)$ is compact. We write

$$q(x,\xi)\chi\left(\frac{\xi}{R}\right)=\frac{1}{R}Q_R(x,\xi)\cdot\sqrt{B(x)}\xi,$$

where Q_R is the vector-valued symbol of order -1:

$$\begin{aligned}Q_R(x,\xi)&=R\chi\left(\frac{\xi}{R}\right)\frac{q(x,\xi)}{B(x)\xi\cdot\xi}\sqrt{B(x)}\xi\\&=\widetilde{\chi}\left(\frac{\xi}{R}\right)\frac{|\xi|q(x,\xi)}{B(x)\xi\cdot\xi}\sqrt{B(x)}\xi,\end{aligned}$$

with $\widetilde{\chi}(u)=|u|^{-1}\chi(u)$. Note that Q_R is smooth, since $q=0$ in a neighborhood of Λ, and because $\widetilde{\chi}\in\mathcal{C}_0^\infty(\mathbf{R}^d)$. Note also that Q_R satisfies symbols estimates uniformly with respect to R. Therefore, there exists a constant $C>0$ such that, for all $f\in L^2(\Omega)$,

$$\|Q_R(x,D)f\|_{L^2(\Omega)}\leqslant C\|f\|_{L^2(\Omega)}.$$

We write

$$q(x,D)\theta_n=R^{-1}Q_R(x,D)\sqrt{B(x)}\nabla_x\theta_n+o(1)$$

in $L^2(\Omega)$, where the $o(1)$ is uniform in $R>1$ when n goes to $+\infty$. As a consequence, if $f\in L^2(\Omega)$ and $\psi\in\mathcal{C}_0^\infty(\mathbf{R})$, we have

$$\begin{aligned}\left|\int\psi(t)\,(q(x,D)\theta_n(t)|f)\,dt\right|&=\frac{1}{R}\left|\int\psi(t)\left(\sqrt{B(x)}\nabla\theta_n(t)|Q_R(x,D)^*f\right)dt\right|+o(1)\\&\leqslant\frac{C}{R}\,\|f\|_{L^2(\Omega)}\int|\psi(t)|\,\|\sqrt{B(x)}\nabla\theta_n(t)\|_{L^2(\Omega)}dt+o(1)\end{aligned}$$

for some constant $C>0$, and where the $o(1)$ is uniform in $R>1$ as n goes to $+\infty$. We observe that the energy equality (1.3) gives that the family $\left(\sqrt{B(x)}\nabla_x\theta_n(t)\right)$ is u.b. in $L^2\left([0,T]\times\Omega,\mathbf{R}^d\right)$ for all $T>0$. Therefore, letting n and R go to ∞, we obtain the result. □

4.2. Proof of Theorem 2.4

We suppose that (B,γ) is weakly degenerate; that is, that the conditions (1), (2), and (3) of Assumption 2.2 are satisfied. Let χ and ϕ be as in Theorem 2.4. We observe that the energy density $e_n(t,x)$ is

$$e_n(t,x)=|\theta_n(t,x)|^2+\frac{1}{2}\left|\partial_t u_n-\frac{D}{|D|}\cdot\nabla u_n\right|^2+\frac{1}{2}\left|\partial_t u_n+\frac{D}{|D|}\cdot\nabla u_n\right|^2.$$

Therefore, it remains to analyze the limit of each of these terms for ϕ supported outside $\{\det B(x)=0\}$.

First, we observe that, by Proposition 4.1, if $\phi\in\mathcal{C}_0^\infty(\{\det B(x)\neq 0\})$,

$$\lim_{n\to+\infty}\int_{\mathbf{R}}\int_\Omega\chi(t)\phi(x)|\theta_n(t,x)|^2dtdx=\int_{\mathbf{R}}\int_{\Lambda^c}\chi(t)\phi(x)d\nu(t,x,\omega)=0.$$

Therefore, the weak limit of the energy density expresses only in terms of the sequences $\partial_t u_n\pm\frac{D}{|D|}\cdot\nabla u_n$, of which the microlocal defect measures are $\mu_\pm(t,x,\omega)$ by Remark 3.3.

By Proposition 4.2, the measures $\mu_\pm$ satisfy

$$\partial_t \mu_\pm \pm \omega \nabla_x \mu_\pm = -\mathbf{1}_{\Lambda^c} \frac{(\gamma(x)\cdot\omega)^2}{B(x)\omega\cdot\omega}\mu_\pm + \nu_\pm,$$

where we have used $\alpha_\infty(x,\omega)\mathbf{1}_\Lambda = \frac{1}{2}b_0(x)\frac{(\gamma(x)\cdot\omega)^2}{c(x,\omega)^2} = 0$ by Remark 3.3.

Let us now prove that the fact that ω is transverse to the hypersurface Σ implies that $\mu_\pm \mathbf{1}_\Sigma = 0$. Let $f(x) = 0$ be a local equation of Σ in a subset Ω_2 of Ω, and let a be a symbol supported in $\Omega_2 \times \mathbf{R}^d$ and such that $a \geqslant 0$ and $\xi \cdot \nabla f(x) > 0$ for $(x,\xi) \in \operatorname{Supp} a$, $\xi \neq 0$. We choose a function $\chi \in \mathcal{C}_0(\mathbf{R})$ such that $\chi'(0) = 1$, and we use the test symbol

$$b_\delta(x,\xi) = \delta\, a(x,\xi)\chi\left(\frac{f(x)}{\delta}\right),$$

where $\delta > 0$. The transport equations for $\mu_\pm$ imply that

$$\int a_\infty(x,\omega)\,\omega\cdot\nabla f(x)\chi'\left(\frac{f(x)}{\delta}\right) d\mu_\pm(x,\omega) = O(\delta).$$

By letting δ go to 0, we obtain $\int_{f(x)=0} a_\infty(x,\omega)\omega\cdot\nabla f(x) d\mu_\pm(x,\omega) = 0$, whence $\mu_\pm \mathbf{1}_\Sigma = 0$ on the support of a. In this way (inspired from [11]), we finally obtain $\mu_\pm \mathbf{1}_\Lambda = 0$, since $\Lambda \subset \Sigma \times \mathbf{R}^d$.

Besides, since the measure $\nu_\pm$ is supported on Λ and absolutely continuous with respect to $\mu_\pm$, we deduce that $\nu_\pm = 0$.

Finally, we observe that the function $F(x,\omega) = -\frac{(\gamma(x)\cdot\omega)^2}{B(x)\omega\cdot\omega}$ extends continuously to the set $\Omega \times \mathbf{S}^{d-1}$ with $F(x,\omega) = 0$ on Λ (since $\gamma(x)\cdot\omega = \tilde{\gamma}(x)\cdot(B(x)\omega) = O(|B(x)\omega|)$ by (2) of Assumption 2.2). Therefore, we can write

$$\partial_t \mu_\pm \pm \omega \nabla_x \mu_\pm = -\frac{(\gamma(x)\cdot\omega)^2}{B(x)\omega\cdot\omega}\mu_\pm.$$

As a conclusion, we obtain Theorem 2.4. Indeed, take $a \in \mathcal{C}_0^\infty(\Omega \times \mathbf{S}^{d-1})$, $0 \leqslant s \leqslant t \leqslant \tau_0$, and set

$$a_t(s,x,\omega) = a\left(x \pm (t-s)\omega, \omega\right) \operatorname{Exp}\left[\int_0^{t-s} F(x \pm \sigma\omega, \omega) d\sigma\right];$$

then $a_t(t,x,\omega) = a(x,\omega)$ and

$$\frac{d}{ds}\langle a_t(s), \mu^\pm(s)\rangle = 0,$$

whence formula (2.5).

Let us conclude this section by a remark.

Remark 4.3. If (2) of Assumption 2.2 fails, then all the energy may be damped in finite time.

Proof. Suppose that $\mu_0^- = 0$ and $\mu_0^+ = a_0\delta(x - x_0) \otimes \delta(\omega - \omega_0)$ for $(x_0,\omega_0) \notin \Lambda$, and suppose that there exists τ_0 such that

$$\forall t \in [0, t_0[, \quad (x_0 + t\omega_0, \omega_0) \notin \Lambda \quad \text{and} \quad (x_0 + t_0\omega_0, \omega_0) \in \Lambda.$$

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Set $y_0 = x_0 + t_0\omega_0$. One first observes that, in view of $B(x)\omega_0 \cdot \omega_0 \geqslant 0$, the point y_0 is a minimum of the function $x \mapsto B(x)\omega_0 \cdot \omega_0$ and $dB(y_0)\omega_0 \cdot \omega_0 = 0_{\mathbf{R}^d}$. Therefore, for s close to t_0, there exists $A_0 \geqslant 0$ such that we have

$$B(x_0 + s\omega_0)\omega_0 \cdot \omega_0 = A_0(t_0 - s)^2 + o((t_0 - s)^2).$$

Let us assume that $A_0 \neq 0$.

Using Theorem 2.4 on $[0, t]$ for all $t \in [0, t_0)$, we obtain, for $\chi \in \mathcal{C}_0^\infty(\Omega)$,

$$\int_\Omega e_n(t, x)\chi(x)dx \xrightarrow[n\to+\infty]{} \int \chi(x + s\omega)d\mu_t^+(x, \omega)$$
$$= a_0\chi(x_0 + t\omega_0)\mathrm{Exp}\left[-\int_0^t \frac{(\gamma(x_0 + s\omega_0)\cdot\omega_0)^2}{B(x_0 + s\omega_0)\omega_0\cdot\omega_0}ds\right].$$

Since

$$\mathrm{Exp}\left[-\int_0^t \frac{(\gamma(x_0 + s\omega_0)\cdot\omega_0)^2}{B(x_0 + s\omega_0)\omega_0\cdot\omega_0}ds\right] \xrightarrow[t\to t_0]{} 0,$$

all the energy is damped between times 0 and t_0. □

5. Propagation of microlocal defect measures

In this section, we prove Proposition 4.2. We consider the + mode; the − mode can be treated in the same way. We proceed in three steps.

- First, we analyze the time derivative of $\left(a(x, D)U_{n,R}^+(t)|U_{n,R}^+(t)\right)$ for scalar-valued symbols $a \in \mathcal{A}_i^0$, and prove that there exists a symbol $T \in \mathcal{A}_i$ such that, uniformly in R as n goes to $+\infty$,

$$\frac{d}{dt}\left(a(x, D)U_{n,R}^+|U_{n,R}^+\right) = -\left(\{a, \beta\}(x, D)U_{n,R}^+|U_{n,R}^+\right) + 2\left((a\alpha)(x, D)U_{n,R}^+|U_{n,R}^+\right)$$
$$+ \left(T(x, D)\chi\left(\frac{D}{R}\right)U_n \,|\, \chi\left(\frac{D}{R}\right)U_n\right) + o(1). \quad (5.1)$$

- We calculate precisely the symbol $T(x, \xi)$, and show that $T \in \mathcal{A}_i^0$. Therefore, the quantity $\left(\frac{d}{dt}\left(a(x, D)U_{n,R}^\pm|U_{n,R}^\pm\right)\right)_{n\in\mathbf{N}}$ is uniformly bounded, and by considering a dense subset of $\mathcal{A}_0^i$, the Ascoli theorem yields the existence of the continuous map $t \mapsto \mu_\pm(t)$ satisfying (4.1).
- Finally, we prove that, for all $\psi \in \mathcal{C}_0^\infty(\mathbf{R})$,

$$\lim_{n\to+\infty}\int \psi(t)\left(T(x, D)\chi\left(\frac{D}{R}\right)U_n(t)|\chi\left(\frac{D}{R}\right)U_n(t)\right)dt$$
$$= \int \psi(t)a_\infty(x, \omega)d\nu_\pm(x, \omega)dt, \quad (5.2)$$

  where $\nu_\pm$ is a measure supported on Λ and absolutely continuous with respect to $\mu_\pm$.

At the end of these three steps, we have obtained Proposition 4.2. We now detail each of these steps.

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5.1. First step: proof of (5.1)

The family of functions $U^+_{n,R}$ satisfies the equation

$$i\partial_t U^+_{n,R}(t,x) = \Pi^+(x,D)\chi\left(\frac{D}{R}\right)P(x,D)U_n. \tag{5.3}$$

Since $\nabla\chi$ is compactly supported, we have

$$\chi\left(\frac{D}{R}\right)P(x,D) = P(x,D)\chi\left(\frac{D}{R}\right) + K_R(x,D),$$

where K_R is a compact operator. Moreover, we have

$$\Pi^+(x,D)P(x,D) = \lambda_+(x,D)\Pi^+(x,D) + R_1(x,D),$$

where the symbol R_1 will be precisely calculated in the next subsection. So (5.3) becomes

$$\begin{aligned} i\partial_t U^+_{n,R}(t,x) &= \big(\lambda_+(x,D)\Pi^+(x,D) + R_1(x,D)\big)\,\chi\left(\frac{D}{R}\right)U_n + \Pi^+(x,D)K_R(x,D)U_n \\ &= \lambda_+(x,D)U^+_{n,R} + R_1(x,D)\chi\left(\frac{D}{R}\right)U_n + \Pi^+(x,D)K_R(x,D)U_n \\ &= \lambda_+(x,D)U^+_{n,R} + F^+_{n,R}(x,D), \end{aligned} \tag{5.4}$$

where

$$F^+_{n,R}(x,D) = R_1(x,D)\chi\left(\frac{D}{R}\right)U_n + \Pi^+(x,D)K_R(x,D)U_n. \tag{5.5}$$

For a real-valued symbol $a\in\mathcal{A}^0_i$, (5.4) implies that $\frac{d}{dt}(a(x,D)U^+_{n,R}|U^+_{n,R}) = I_1 + I_2$, with

$$\begin{aligned} I_1 &= \frac{1}{i}(a(x,D)\lambda_+(x,D)U^+_{n,R}|U^+_{n,R}) - \frac{1}{i}(a(x,D)U^+_{n,R}|\lambda_+(x,D)U^+_{n,R}), \\ I_2 &= \frac{1}{i}(a(x,D)F^+_{n,R}(x,D)|U^+_{n,R}) - \frac{1}{i}(a(x,D)U^+_{n,R}|F^+_{n,R}). \end{aligned}$$

The term I_1 will give the transport by the vector field H_β and the damping by α. The term I_2 is a rest term, and its main contribution will be described by a symbol T.

Let us study I_1.

$$I_1 = \frac{1}{i}\big((a(x,D)\lambda_+(x,D) - (\lambda_+(x,D))^* a(x,D))U^+_{n,R}|U^+_{n,R}\big).$$

We recall that $\lambda_+(x,D) = \beta(x,D) + i\alpha(x,D)$, so, since we use the Weyl quantification for the symbols, we have $(\lambda_+(x,D))^* = \beta(x,D) - i\alpha(x,D)$, and

$$\begin{aligned} I_1 &= \frac{1}{i}\big([a(x,D),\beta(x,D)]U^+_{n,R}|U^+_{n,R}\big) + 2\big((a\alpha)(x,D)U^+_{n,R}|U^+_{n,R}\big) \\ &\quad + \big(r_{-1}(x,D)U^+_{n,R}|U^+_{n,R}\big) \\ &= -\big(\{a,\beta\}(x,D)U^+_{n,R}|U^+_{n,R}\big) + 2\big((a\alpha)(x,D)U^+_{n,R}|U^+_{n,R}\big) + \big(r_{-1}(x,D)U^+_{n,R}|U^+_{n,R}\big), \end{aligned}$$

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where $r_{-1}(x, \xi)$ will denote from now on a generic symbol in $\mathcal{A}_i^{-1}$. Using (3.6), we then obtain that, if n and R tend to infinity,

$$I_1 \to -\int \{a, \beta\}_\infty(x, \omega) d\mu_+(x, \omega) + 2\int a_\infty(x, \omega)\alpha_\infty(x, \omega) d\mu_+(x, \omega). \tag{5.6}$$

Note that, for these terms, we do not need to integrate in time to get the convergence.

Let us now study I_2. We set $U_{n,R} = \chi\left(\frac{D}{R}\right) U_n$, and, by (5.5), we obtain $I_2 = I_{2,1} + I_{2,2}$, with

$$\begin{aligned} I_{2,1}(t) &= \frac{1}{i}\big(a(x, D)\Pi^+(x, D)K_R U_n(t) | \Pi^+(x, D)U_{n,R}(t)\big) \\ &\quad - \frac{1}{i}\big(a(x, D)\Pi^+(x, D)U_{n,R}(t) | \Pi^+(x, D)K_R U_n(t)\big) \\ I_{2,2}(t) &= \frac{1}{i}\big(a(x, D)R_1(x, D)U_{n,R}(t) | \Pi^+(x, D)U_{n,R}(t)\big) \\ &\quad - \frac{1}{i}\big(a(x, D)\Pi^+(x, D)U_{n,R}(t) | R_1(x, D)U_{n,R}(t)\big). \end{aligned}$$

Since $(U_n)_{n\in\mathbf{N}}$ goes weakly to 0 as n goes to $+\infty$ and K_R is a compact operator, the sequence $(K_R U_n)_{n\in\mathbf{N}}$ goes strongly to 0. Therefore,

$$\forall t \in \mathbf{R}^+, \quad I_{2,1}(t) \to 0. \tag{5.7}$$

Besides, $I_{2,2}(t) = \big(T(x, D)U_{n,R} | U_{n,R}\big)$, with

$$T(x, D) := \frac{1}{i}\left(\Pi^+(x, D)^* a(x, D)R_1(x, D) - R_1(x, D)^* a(x, D)\Pi^+(x, D)\right).$$

5.2. Second step: the symbol of the rest term

In the following, rest terms in the symbol class $\mathcal{A}_i^{-k}$ for $k \in \mathbf{N}$ will be denoted by r_{-k}. We set

$$K = \begin{pmatrix} 0_{d+1,d+1} & 0_{d+1,1} \\ 0_{1,d+1} & 1 \end{pmatrix},$$

and we prove the following lemma.

Lemma 5.1. *Suppose that Ω_1 is like in Proposition 4.2. Then, in Ω_1, $T \in \mathcal{A}_i^0$ and*

$$T(x, \xi) = T_1(x, \xi) + T_2(x, \xi) + r_{-1}(x, \xi),$$

with

$$\begin{aligned} T_1 &= -\frac{a}{2}\Big(\Pi^+\{\Pi^+, \beta\} + \{\Pi^+, \beta\}\Pi^+ + \Pi^+\{\Pi^+, \beta\}\Pi^- \\ &\quad + \Pi^-\{\Pi^+, \beta\}\Pi^+\Big) \end{aligned} \tag{5.8}$$

$$T_2 = \frac{a}{2i}\left(\Pi^+\{\Pi^+, b\}K - K\{\Pi^+, b\}\Pi^+\right). \tag{5.9}$$

Besides, $T_1\Pi_0$ and $\Pi_0 T_1$ are symbols of $\mathcal{A}_i^{-1}$.

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Note that $T_2 \in \mathcal{A}_i^0$, because $\Pi^+K,\ K\Pi^+ \in \mathcal{A}_i^{-1}$ by Remark 3.3.

Proof. We first calculate R_1. Recall that

$$R_1(x,D) = \Pi^+(x,D)P(x,D) - \lambda_+(x,D)\Pi^+(x,D).$$

We write $P(x,\xi) = P_1(x,\xi) + ib(x,\xi)K$, with

$$P_1(x,\xi) = \begin{pmatrix} 0 & -{}^t\xi & \gamma(x)\cdot\xi \\ -\xi & 0_{d,d} & 0_{d,1} \\ \gamma(x)\cdot\xi & 0_{1,d} & 0 \end{pmatrix}.$$

Since $b(x,\xi)$ is of order 2, R_1 is a priori of order 1, and, in view of $\Pi^+P = \lambda_+\Pi^+$, we have

$$R_1 = \frac{1}{2i}\left(\{\Pi^+, P\} - \{\lambda_+, \Pi^+\}\right) + QK + r_{-1},$$

where the matrix-valued symbol Q is in $\mathcal{A}_i^0$ and is the sum of second derivatives of Π^+ multiplied by second derivatives of b. More precisely, and after ordering the terms of higher degrees, we write

$$R_1 = \frac{1}{2}\{\Pi^+,\ b\}K - \frac{i}{2}\left(\{\Pi^+,\beta\} + \{\Pi^+, P_1\}\right) + QK + r_{-1},$$

where we have used that $\lambda_+ = \beta + i\alpha$ and $\{\alpha, \Pi^+\} \in \mathcal{A}_i^{-1}$. However, since $\gamma(x) \in \operatorname{Ran} B(x)$, $\gamma(x)\cdot\xi = 0$ for $\left(x, \frac{\xi}{|\xi|}\right) \in \Lambda$, and, by Remark 3.3,

$$K\Pi^+,\ \Pi^+K \in \mathcal{A}_i^{-1}. \tag{5.10}$$

Derivations of these relations imply that $\{\Pi^+, b\}K \in \mathcal{A}_i^0$ and $QK \in \mathcal{A}_i^{-1}$. Finally, we find that

$$R_1 = \frac{1}{2}\{\Pi^+, b\}K - \frac{i}{2}\left(\{\Pi^+,\beta\} + \{\Pi^+, P_1\}\right) + r_{-1} \in \mathcal{A}_i^0,$$

whence $T \in \mathcal{A}_i^0$, with

$$\begin{aligned} T &= \frac{1}{i}\left(\Pi^+ a R_1 - R_1^* a \Pi^+\right) + r_{-1} \\ &= \frac{a}{2i}\left(\Pi^+\{\Pi^+, b\}K - K\{\Pi^+, b\}\Pi^+\right) \\ &\quad - \frac{a}{2}\left(\Pi^+\{\Pi^+,\beta\} + \{\Pi^+,\beta\}\Pi^+ + \Pi^+\{\Pi^+, P_1\} - \{P_1, \Pi^+\}\Pi^+\right) + r_{-1}, \end{aligned}$$

where we have used $(\Pi^+)^* = \Pi^+ + r_{-1}$ and

$$R_1^* = \frac{1}{2}K\{\Pi^+, b\} + \frac{i}{2}\left(\{\Pi^+,\beta\} - \{P_1, \Pi^+\}\right) + r_{-1}.$$

We now transform the expression of the principal symbol of T. We write

$$P_1 = P - ibK = \lambda_+\Pi^+ + \lambda_-\Pi^- + \lambda_0\Pi^0 - ibK.$$

By Propositions 3.1 and 3.2 and Remark 3.3, we notice that

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- in Λ^c, $\Pi^0 = K + O(|\xi|^{-2})$ and $\lambda_0 = ib + O(1)$,
- in Λ, $\Pi^0 = K + O(|\xi|^{-1})$, $\lambda_0 = O(1)$ and $b = O(1)$.

Therefore,

$$P_1 = \lambda_+ \Pi^+ + \lambda_- \Pi^- + r_0.$$

In view of (5.9), we deduce that $T = T_2 + \widetilde{T}$, with

$$\begin{aligned}\widetilde{T} &= -\frac{a}{2}\Big(\big(\Pi^+\{\Pi^+, \beta\} + \{\Pi^+, \beta\}\Pi^+\big) \\ &\quad + \sum_{\ell\in\{+,-\}} \Big(\Pi^+\{\Pi^+, \lambda_\ell \Pi^\ell\} - \{\bar{\lambda}_\ell \Pi^\ell, \Pi^+\}\Pi^+\Big)\Big) + r_{-1} \\ &= -\frac{a}{2}\Big(\big(\Pi^+\{\Pi^+, \beta\} + \{\Pi^+, \beta\}\Pi^+\big) \\ &\quad + \sum_{\ell\in\{+,-\}} \Big(\Pi^+\{\Pi^+, \beta\Pi^\ell\} - \{\beta\Pi^\ell, \Pi^+\}\Pi^+\Big)\Big) + r_{-1}.\end{aligned}$$

Observing that $\{\Pi^\pm, \Pi^+\}, \{\Pi^+, \Pi^\pm\} \in \mathcal{A}_i^{-1}$, by Remark 3.3 and equation (5.8), we obtain

$$\begin{aligned}\widetilde{T} &= -\frac{a}{2}\Big(\Pi^+\{\Pi^+, \beta\} + \{\Pi^+, \beta\}\Pi^+ + \sum_{\ell\in\{+,-\}} \Big(\Pi^+\{\Pi^+, \beta\}\Pi^\ell - \Pi^\ell\{\beta, \Pi^+\}\Pi^+\Big)\Big) + r_{-1}, \\ &= T_1 + r_{-1}\end{aligned}$$

where we have used $\Pi^+\{\beta, \Pi^+\}\Pi^+ = 0$ (which comes from $(\Pi^+)^2 = \Pi^+$, whence $\{\beta, \Pi^+\} = \Pi^+\{\beta, \Pi^+\} + \{\beta, \Pi^+\}\Pi^+$ and, multiplying by Π^+ on both sides, we obtain $\Pi^+\{\beta, \Pi^+\}\Pi^+ = 0$). Notice that $\{\beta, \Pi^+\}\Pi^0 \in \mathcal{A}_i^{-2}$ and $\Pi^0\{\beta, \Pi^+\} \in \mathcal{A}_i^{-2}$, whence $\Pi^0 T,\ T\Pi^0 \in \mathcal{A}_i^{-1}$. □

5.3. Third step: passing to the limit in the rest term

We use the following lemma.

Lemma 5.2. *Consider A a smooth symbol of order 0 supported in Ω_1 and such that $A = \Pi^+ A \Pi^\ell$ or $A = \Pi^\ell A \Pi^+$, where $\Pi^\ell = \Pi^-$ or $\Pi^\ell = \Pi^j$, $1 \leqslant j \leqslant d-1$. Then, if $\psi \in \mathcal{C}_0^\infty(\mathbf{R})$,*

$$\int \psi(t)\, (A(x, D)U_n(t),\ U_n(t))\, dt \xrightarrow[n\to+\infty]{} 0.$$

Since the symbol T_1 is the sum of terms of the form $\Pi^+\{\Pi^+, \beta\}\Pi^\ell$ or $\Pi^\ell\{\Pi^+, \beta\}\Pi^+$ with $\ell = -$ or $\ell = j$, $1 \leqslant j \leqslant d-1$, we can apply the lemma, and we obtain

$$\begin{aligned}&\int \psi(t) \left(T(x, D)\chi\left(\frac{D}{R}\right) U_n(t) | \chi\left(\frac{D}{R}\right) U_n(t)\right) dt \\ &\quad = \int \psi(t) \left(T_2(x, D)\chi\left(\frac{D}{R}\right) U_n(t) | \chi\left(\frac{D}{R}\right) U_n(t)\right) dt + o(1)\end{aligned}$$

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as n goes to $+\infty$. In view of (5.9), we have

$$\limsup_{n\to+\infty} \int \psi(t) \left(T_2(x,D)\chi\left(\frac{D}{R}\right) U_n(t) | \chi\left(\frac{D}{R}\right) U_n(t)\right) dt$$
$$\xrightarrow[R\to+\infty]{} \int \chi(t)(T_2)_\infty(x,\omega) d\widetilde{\nu}_+(t,x,\omega),$$

where $\widetilde{\nu}_+$ is the joint measure of $KU_n = \theta_n$ and of $U^+_{n,R}$. The measure ν_+ is absolutely continuous with respect to μ_+ and ν. By Proposition 4.1, $\widetilde{\nu}_+$ is supported on Λ, and we obtain (5.2), with $\nu_+ = (T_2)_\infty \widetilde{\nu}_+$, which is absolutely continuous with respect to μ_+ and supported on Λ. We now focus on the proof of Lemma 5.2.

Proof. Let Π^ℓ be one of the projectors Π^- or Π^j, $1 \leqslant j \leqslant d-1$, and let us consider a term of the form $\Pi^+ A \Pi^\ell$; the proof is similar for the other terms. We have

$$[\Pi^+ A \Pi^\ell,\ P] = (\lambda_\ell - \lambda_+) \Pi^+ A \Pi^\ell.$$

Let us denote by C^ℓ the symbol of $\mathcal{A}_i^{-1}$:

$$C^\ell = (\lambda_\ell - \lambda_+)^{-1} \Pi^+ A \Pi^\ell.$$

We have $C^\ell \in \mathcal{A}_i^{-1}$, and, by Remark 3.3, we have $C^\ell K, KC^\ell \in \mathcal{A}_i^{-2}$. We write

$$\int \psi(t) \left((\Pi^+ A \Pi^\ell)(x,D) U_{n,R} | U_{n,R} \right)_{L^2(\mathbf{R}^d)} \mathrm{d}t$$
$$= \int \psi(t) \left(([C^\ell, P])(x,D) U_{n,R} | U_{n,R} \right)_{L^2(\mathbf{R}^d)} \mathrm{d}t.$$

Besides,

$$[C^\ell, P](x,D) = [C^\ell(x,D), -i\partial_t + P(x,D)] - \frac{1}{2i}\left(\{C^\ell, P\} - \{P, C^\ell\} \right)(x,D)$$
$$+ r_{-1}(x,D).$$

Using $P = P_1 + ibK$ and $\{C^\ell, P_1\},\ \{P_1, C^\ell\} \in \mathcal{A}_i^{-1}$, we obtain

$$\{C^\ell, P\} - \{P, C^\ell\} = i\{C^\ell, b\}K - iK\{b, C^\ell\} + r_{-1}.$$

Therefore, we have

$$[C^\ell, P](x,D) = [C^\ell(x,D), -i\partial_t + P(x,D)] - \frac{1}{2}\left(\{C^\ell, b\}K - K\{b, C^\ell\} \right)(x,D)$$
$$+ r_{-1}(x,D)$$
$$= [C^\ell(x,D), -i\partial_t + P(x,D)] + r_{-1}(x,D),$$

where we have used $\{C^\ell, b\}K = \{C^\ell K, b\} \in \mathcal{A}_i^{-1}$ and $K\{C^\ell, b\} = \{KC^\ell, b\} \in \mathcal{A}_i^{-1}$ (as a consequence of $KC^\ell, C^\ell K \in \mathcal{A}_i^{-2}$). In view of $P^*(x,D) = P(x,D) - 2ib(x,D)K$, we can write

$$[C^\ell, P](x,D) = C^\ell(x,D)\left(-i\partial_t + P(x,D)\right) - \left(-i\partial_t + P^*(x,D)\right) C^\ell(x,D)$$
$$- 2ib(x,D)KC^\ell(x,D) + r_{-1}(x,D).$$

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Finally, using $(i\partial_t - P(x,D))U_{n,R} = K_R U_n$, with K_R compact, we obtain

$$\begin{aligned}&\int \chi(t)\left((\Pi^+ A \Pi^\ell)(x,D)U_{n,R}|U_{n,R}\right)_{L^2(\mathbf{R}^d)} \mathrm{d}t\\ &= -i\int \chi'(t)\left(C^\ell(x,D)U_{n,R}|U_{n,R}\right)_{L^2(\mathbf{R}^d)} \mathrm{d}t\\ &\quad - 2i\int \chi(t)\left(b(x,D)KC^\ell(x,D)U_{n,R}|KU_{n,R}\right)_{L^2(\mathbf{R}^d)} \mathrm{d}t + o(1).\end{aligned}$$

Since C^ℓ is of order lower or equal to -1, we have

$$\left(C^\ell(x,D)U_{n,R}|U_{n,R}\right)_{L^2(\mathbf{R}^d)} \xrightarrow[n,R\to+\infty]{} 0.$$

We use

$$b(x,D)KU_{n,R} = \left(0,\ldots,0,\chi\left(\frac{D}{R}\right)b(x,D)\theta_n\right),$$

with $b(x,D)\theta_n \in L^2_{loc}(\mathbf{R}, H^1(\Omega))$ by Proposition 4.1. Since, moreover, bKC^ℓ is of order 0 in Ω_1, we have

$$\limsup_{n\to+\infty}\int \psi(t)\left(b(x,D)KC^\ell(x,D)U_{n,R}|KU_{n,R}\right)_{L^2(\mathbf{R}^d)} \mathrm{d}t \xrightarrow[R\to+\infty]{} 0,$$

whence

$$\limsup_{n\to+\infty}\int \psi(t)\left((\Pi^+ A\Pi^\ell)(x,D)U_{n,R}|U_{n,R}\right)_{L^2(\mathbf{R}^d)} \mathrm{d}t \xrightarrow[R\to+\infty]{} 0. \qquad \square$$

6. Analysis on the boundary

In this section, we investigate what happens close to the boundary when one has Assumption 2.6 and how one reduces to the analysis of a wave equation. We briefly recall the arguments of [7] and explain how they apply to our setting. In all this section, we work in a neighborhood Ω_1 of a point of $\partial\Omega$ where B is non-degenerate. We first recall in the first subsection the definition of a Melrose–Sjöstrand compressed bundle (see [22] and the survey [3]) and of the generalized bicharacteristics. Then, in the second subsection, we will link [7]'s approach and ours (we also refer to [3, 10]). The last subsection will be devoted to the proof of the main statement of this section.

6.1. Melrose–Sjöstrand compressed bundle and the generalized bicharacteristics

We work in space–time variables, and set $L = \mathbf{R}_t \times \Omega$. We denote by (z,ζ) the points of T^*L: $z = (t,x)$ and $\zeta = (\tau,\xi)$. Then, the Melrose–Sjöstrand compressed bundle to L is given by

$$T_b^*L = \left(T^*L \setminus \{0\}\right) \cup \left(T^*\partial L \setminus \{0\}\right).$$

Quotienting by the action of $\mathbf{R}^+$ through homotheties, one obtains the normal compressed bundle to Ω:

$$S_b^*L = T_b^*L/\mathbf{R}^+.$$

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The projection

$$\pi : \left(T^*\mathbf{R}^{d+1}\right)_{|\overline{L}} \to T_b^*L$$

induces a topology on T_b^*L. On T^*L, we denote by p_0 the symbol of the wave operator and by Σ_0 the projection on T_b^*L of its characteristic set:

$$p_0(z,\zeta) = \tau^2 - |\xi|^2, \quad \Sigma_0 = \pi\left(\{(z,\zeta), p_0(z,\zeta) = 0\}\right). \tag{6.1}$$

Locally, near a point of ∂L, we use normal geodesic coordinates

$$(y,\eta) = (y_0, y_1, \dots, y_d; \eta_0, \eta_1, \dots, \eta_d) \in \mathbf{R}^{2d+2}$$

in an open set V of $T^*\mathbf{R}^{d+1}$ such that $L \cap V = \{y_d > 0\} \cap V$. Therefore, the wave operator $\partial_t^2 - \Delta$ is $-\partial_{y_d}^2 - R(y, D_{y'})$, where the symbol $R(y,\eta')$ is a homogeneous polynomial of degree 2 in η'; we denote by $r(y,\eta')$ its principal symbol. We can now distinguish between different sorts of points of $T^*\partial L \setminus \{0\}$: those which are not in Σ_0 and those which are in Σ_0 depending whether $R(y,\eta') < 0$ or not. In the case where $\rho \in \Sigma_0$, there exist at most two points of $\{\tau^2 = |\xi|^2\} = \{\eta_d^2 = R(y,\eta')\}$ which are in $\pi^{-1}(\{\rho\})$; they correspond to the two roots of the equation $\eta_d^2 = R(y_n, y', \eta')$. Consider $\rho \in T^*\partial L \setminus \{0\}$.

- If $\rho \notin \Sigma_0$, one says that ρ is elliptic.
- If $\mathrm{Card}\left(\pi^{-1}(\{\rho\}) \cap \{\tau^2 = |\xi|^2\}\right) = 2$, ρ is said to be hyperbolic.
- If $\mathrm{Card}\left(\pi^{-1}(\{\rho\}) \cap \{\tau^2 = |\xi|^2\}\right) = 1$, ρ is said to be glancing.

We denote by $\mathcal{H}$ (respectively, $\mathcal{G}$) the hyperbolic (respectively, glancing) points of ∂L. We say that $\rho \in \mathcal{G}$ is

- non-strictly gliding if $\partial_{y_d} r(\rho) \geqslant 0$,
- strictly gliding if $\partial_{y_d} r(\rho) < 0$,
- diffractive if $\partial_{y_d} r(\rho) > 0$,
- gliding of order k if

$$H^j_{r(y',0,\eta')}(\partial_{y_d} r_{|y_d=0})(\rho) = 0, \quad 0 \leqslant j < k-2 \quad \text{and} \quad H^{k-2}_{r(y',0,\eta')}(\partial_{y_d} r_{|y_d=0})(\rho) \neq 0.$$

We denote by $\mathcal{G}_k$ (respectively, $\mathcal{G}_d$) the set of points which are gliding of order k (respectively, diffractive). The assumption that Ω has no contact of infinite order with its tangents ((1) in Assumption 2.6) consists in assuming that

$$\mathcal{G} = \bigcup_{k \in \mathbf{N}^*} \mathcal{G}_k. \tag{6.2}$$

The generalized bicharacteristics are defined by taking the Hamiltonian trajectory of p_0 inside Ω and by specifying how the connection is made on the boundary. The only problematic points are the glancing ones where the trajectory arrives tangentially to the boundary. Indeed, for $\rho \in \mathcal{G}$, we have $y_d = \eta_d = r(y,\eta') = 0$. Recall that the geodesic trajectories are generated by the Hamiltonian flow H_{p_0}, which in coordinates (y,η) is

$$H_{p_0} = \left(-\nabla_{\eta'} r(y,\eta'), 2\eta_d, \nabla_{y'} r(y',\eta'), \partial_{y_d} r(y,\eta')\right).$$

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If $\rho \in \mathcal{G}_d$, $\partial_{y_d} r(y,\eta') > 0$; therefore η_d increases on the trajectory and changes sign at ρ; thus, y_d decreases before ρ and increases after, which implies that this trajectory remains in Ω. In contrast, if $\rho \notin \mathcal{G}_d$, the coordinate on ∂_{η_d} is non-positive and the trajectory will leave Ω: if η_d decreases after passing in ρ, then η_d and y_d become non-positive. To overcome this difficulty, one uses the vector

$$\tilde{H}(\rho) = H_{p_0}(\rho) - \partial_{y_d} r(y,\eta')\partial_{\eta_d}.$$

This vector has a coordinate on ∂_{η_d} which is 0. One then defines the generalized bicharacteristic as follows (see [22] or [3]).

Definition 1. A generalized bicharacteristic is a continuous map $\Phi : \mathbf{R} \to T_b^*L$ such that there exists a set I of isolated points with

- $\Phi(s) \in T^*L \cup \mathcal{G}$ for $s \notin I$ *and* $\Phi(s) \in \mathcal{H}$ for $s \in I$;
- for $s \notin I$, Φ is differentiable with

$$\begin{cases} \dot{\Phi}(s) = H_{p_0}(\Phi(s)) & \text{if } \Phi(s) \in T^*L \cup \mathcal{G}_d, \\ \dot{\Phi}(s) = \tilde{H}(\rho) & \text{if } \Phi(s) \in \mathcal{G} \setminus \mathcal{G}_d. \end{cases}$$

It is proved in [22] that these definitions are intrinsic and that, if assumption (6.2) is satisfied, then, for $\rho_0 \in T_b^*L \cap \Sigma_0$, there exists a unique generalized bicharacteristic curve such that $\Phi(0) = \rho_0$.

6.2. Propagation near the boundary

Working in space–time variables, one first defines tangential symbols by use of the system of local normal geodesic coordinates: in an open set $\mathcal{O}$ where we have such coordinates (y,η), the function $a(y,\eta') \in \mathcal{C}^\infty(\overline{L} \times \mathbf{R}^d)$ is said to be a symbol of $\mathcal{A}_b^m$ if a is compactly supported in $\mathcal{O}$ in the variable y and satisfies $\forall \alpha,\ \beta \in \mathbf{N}^d$, $\exists C_{\alpha,\beta} > 0$,

$$\forall (y,\eta) \in \overline{L} \times \mathbf{R}^{d+1}, \qquad \left|\partial_y^\alpha \partial_{\eta'}^\beta \left(a(y,\eta')\right)\right| \leqslant C_{\alpha,\beta}(1+|\eta'|)^{m-|\beta|}. \tag{6.3}$$

This definition implies that the sets $\mathcal{A}_b^m$ depend on the choice of the open sets $(\mathcal{O}_j)_{j\in J}$. Then, one defines

$$\mathcal{A}^m = \{q \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbf{R}^d), \exists q^i \in \mathcal{A}_i^m, \exists q^b \in \mathcal{A}_b^m, q = q^b + q^i\},$$

and one considers elements $q \in \mathcal{A}^m$ which have a principal symbol: there exists a function q_∞ homogeneous of degree 0 such that, for all $(z,\zeta) \in S^d(2)$, the sphere of radius 2 of $\mathbf{R}^{d+1}$, we have

$$q_\infty(z,\zeta) = \lim_{R\to+\infty} R^{-m} q(z,R\zeta).$$

Then, one associates with the sequence $u_n(t,x)$ its H^1 space–time microlocal defect measure μ by the following: up to extraction of a subsequence, for all $q = q_b + q_i \in \mathcal{A}^2$ admitting a principal symbol,

$$(q(z,D_z)u_n, u_n) \xrightarrow[n\to+\infty]{} \int_{\overline{L}\times S^d(2)} q_\infty(z,\zeta)\, d\mu(z,\zeta).$$

Note that, on $S^*\Omega$, the measure μ is the usual microlocal defect measure (or H-measure) as introduced in [15] and [23]. The link between this measure μ and the measures $\mu^\pm$ of Theorem 2.4 is described in the following proposition.

Proposition 6.1. *In* $\mathbf{R} \times \Omega_1 \times S^d(2)$*, we have*

$$\mu(t, x, \tau, \xi) = \frac{1}{2}\,\delta(\tau + 1) \otimes \mu_t^+(x, \xi) \otimes dt + \frac{1}{2}\,\delta(\tau - 1) \otimes \mu_t^-(x, \xi) \otimes dt. \tag{6.4}$$

$$H_{p_0}\mu = 4\tau\alpha_\infty\mu \quad \textit{with } \alpha_\infty(x, \omega) = -\frac{1}{2}\frac{(\gamma(x)\omega)^2}{B(x)\omega\cdot\omega}.$$

This proposition is proved in § 6.3 below.

If one wants to have a complete description of the behavior of the measure μ close to the boundary, the next step should consist in proving the reflexion of μ close to the boundary, that is, in proving the analog of Theorem A.1 in [20]: μ-a.e., the generalized bicharacteristic map $\Phi(s)$ is well defined and satisfies $\Phi(s) \circ \Phi(-s) = \mathrm{Id}$; besides, for all Borelian $\omega \subset \Sigma_b$,

$$\mu\left(\Phi(s)\omega\right) = \int_{\omega\cap\{p_0=0\}} \exp\left[-\int_0^s \tau\tilde{a}\left(\Phi(\sigma, \rho)\right) d\sigma\right] d\mu(\rho). \tag{6.5}$$

We conjecture that (6.5) is true under Assumptions 2.2 and 2.6 and, to support this conjecture, we point out that most of the results of Theorem 15 in [3] for a damped wave equation also hold in our setting. Recall that, if $\rho \in T^*\partial L$ is a hyperbolic or a glancing point, then there exist two points $\rho^\pm$ such that $\pi(\rho^\pm) = \rho$ and $\rho^\pm \in \{\tau^2 = |\xi|^2\}$. These two points are equal if $\rho \in \mathcal{G}$. When $\rho \in \mathcal{H}$, they differ by their ξ-component, which we will denote by $\xi^\pm$. Besides, as in [7], because of the equation satisfied by $(u_n)_{n\in\mathbf{N}}$, the sequence of the normal derivatives $(\partial_N u_n)_{n\in\mathbf{N}}$ is a uniformly bounded family of $L^2_{loc}(\mathbf{R}, L^2(\partial\Omega))$, and we denote by λ its microlocal defect measure (we suppose that we have extracted a subsequence so that λ is uniquely determined).

Proposition 6.2. *Let $N(x)$ be the exterior normal vector to $\partial\Omega$; then*

$$H_{p_0}\mu - 4\tau\,\alpha_\infty\,\mu = \frac{\delta(\xi - \xi^+) - \delta(\xi - \xi^-)}{(\xi_+ - \xi_-)\cdot N(x)}\,\lambda\,\mathbf{1}_{\mathcal{H}\cup\mathcal{G}}.$$

Moreover, μ has no mass above hyperbolic points of $T^*\partial L$ and λ has no mass above non-strictly gliding points.

Proof. Let us calculate $\ell := H_{p_0}\mu - 4\tau\,\alpha_\infty\,\mu$. We already know that, if q is an interior symbol, $\langle q, \ell\rangle = 0$. For analyzing the action of ℓ on tangential symbols, one computes the quantity

$$I_n(t) = \left(\left[\partial_{y_d}^2 + R(y, D_{y'}), q_b(y, D_{y'})\right] u_n(t),\ u_n(t)\right)_{L^2(\{y_d>0\})}$$

(for q_b a tangential symbol of order 1) by use of integration by parts. These integration by parts generate terms on the boundary: some of them are the same than in [7] and generate a distribution ℓ_2 in the limit $n \to +\infty$; some others are new and produce a distribution that we denote ℓ_1. References [3] (pages 14–15) and [7] give

$$\ell_2 = \frac{\delta(\xi - \xi^+) - \delta(\xi - \xi^-)}{(\xi_+ - \xi_-) \cdot N(x)} \lambda \, \mathbf{1}_{\mathcal{H} \cup \mathcal{G}}.$$

On the other hand, the distribution ℓ_1 comes from the analysis of

$$I_n^1(t) = \big(q_b(y, D_{y'})u_n(t), \tilde{\gamma}(y) \cdot \nabla\theta_n(t)\big) - \big(q_b(y, D_{y'})\tilde{\gamma}(y) \cdot \nabla\theta_n(t), u_n(t)\big) + o(1),$$

where $\tilde{\gamma}(y) \cdot \eta$ is the principal symbol of the operator Γ which arises when writing $\nabla \cdot (\gamma \cdot)$ in the normal geodesic coordinates. On the one hand, if q_b is an interior symbol, the limit of I_n^1 as n goes to $+\infty$ is described by the joint measure of θ_n and ∇u_n (close to the boundary); this joint measure is absolutely continuous with respect to μ. On the other hand, let us take $\delta > 0$ and $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$ such that $\chi(t) = 1$ for $|t| < 1/2$ and $\chi(t) = 0$ for $|t| > 1$ with $0 \leqslant \chi \leqslant 1$, and study

$$J_{n,\delta}(t) = \left(\chi\left(\frac{y_d}{\delta}\right) q_b(y, D_{y'})u_n, \tilde{\gamma}(y) \cdot \nabla\theta_n\right).$$

The fact that γ is tangent to the boundary implies that

$$\tilde{\gamma}(y', 0) = \big(\tilde{\gamma}'(y', 0), 0\big);$$

therefore, we can write $\tilde{\gamma} = (\tilde{\gamma}', y_d\tilde{\gamma}_d)$. Then, the worst term to estimate – which is the one which involves ∂_{y_d} derivatives of θ_n – is

$$\tilde{J}_{n,\delta}(t) = -\left(\chi\left(\frac{y_d}{\delta}\right) q_b(y, D_{y'})y_d\tilde{\gamma}_d(y)\partial_{y_d}\theta_n(t), u_n(t)\right),$$

and we observe that

$$\begin{aligned}\left|\tilde{J}_{n,\delta}(t)\right| &= \left|\int_0^{+\infty} \chi\left(\frac{y_d}{\delta}\right) \big(q_b(y, D_{y'})y_d\tilde{\gamma}_d(y)\partial y_d\theta_n(t), u_n(t)\big)_{L^2(\mathbf{R}_{y'}^{d-1})} dy_d\right| \\ &\leqslant \delta \|u_n(t)\|_{H^1(\Omega)} \|\partial_y\theta_n(t)\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, for any $\Theta \in \mathcal{C}_0^\infty(\mathbf{R})$, there exists a constant C such that

$$\left|\limsup_{n\to+\infty} \int \Theta(t)J_{n,\delta}(t)dt\right| \leqslant C\,\delta.$$

Finally, letting δ go to 0, we obtain that this term has no contribution on the boundary. As a conclusion, ℓ_1 is a measure absolutely continuous with respect to μ.

Then, we can argue like in [7]: using that $\ell\mathbf{1}_{\mathcal{H}}$ is a measure, an argument similar to the one of § 4.2 gives that $\mu\mathbf{1}_{\mathcal{H}} = 0$, whence $\ell_1\mathbf{1}_{\mathcal{H}} = 0$. Besides, close to glancing points of the boundary, by using an adapted test function (see page 15 of [3]), one can prove that λ (and thus ℓ_2 too) has no mass on the set of non-strictly gliding points of $T^*\partial L$. □

Unfortunately, we are not able to deduce a full description of the measure μ form this proposition. Heuristically, following a bicharacteristic which reaches $\partial\Omega$ in a hyperbolic point, the measure is totally restituted (since $\mu = 0$ above $\mathcal{H}$) and the restitution is described by $\ell_2\mathbf{1}_{\mathcal{H}}$, which means propagation along the reflected curve. Besides, close to

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glancing points of the boundary, the right-hand side term of the equation of the measure is 0 above those points, which also means propagation along these trajectories (again heuristically). Therefore, if Ω is the exterior of a bounded convex obstacle, there are no strictly gliding points, and the heuristic argument sketched before can be made correct (see [14], Section II.3 and [16], Section 2, Theorem 2.3). Note that, in that setting, the set Ω is no longer supposed to be bounded; then one has to avoid losses of mass at infinity in order to deduce results about the energy. In other situations (like ours), there are strictly gliding points. For the damped wave equation, the only existing proof is the one of section A.3 in [20] (see Theorem A.1). It is likely that it adapts to our context, but it requires further works. Note that the analysis of a Lamé system leads to similar difficulties which are overcome in [7] in a different way (in particular, the authors use the polarization aspects that the microlocal measure presents, since the wave equation is replaced by a system). It is also interesting to keep in mind that the transport equation of Proposition 6.2 gives information about the support of the microlocal defect measure, which is enough for applications in control theory (see [10] for results in the framework of magnetoelasticity).

6.3. The link between the measure μ and the measures $\mu_t^\pm$

In this section, we prove Proposition 6.1.

Note first that the second assertion is a simple consequence of the first one. Suppose that we have (6.4) where, by Proposition 4.2, the measures $\mu_t^\pm$ satisfy

$$\partial_t \mu_t^\pm \pm \xi \cdot \nabla_x \mu_t^\pm = 2\alpha_\infty \mu_t^\pm, \quad \text{in } \Omega_1 \times \mathbf{S}^{d-1}.$$

We obtain

$$\begin{aligned} H_{p_0}\mu &= 2\left(\tau \partial_t \mu - \xi \cdot \nabla_x \mu\right) \\ &= \left(-\partial_t \mu_t^+ - \xi \cdot \nabla_x \mu_t^+\right) \otimes \delta(\tau+1) \otimes \mathrm{d}t + \left(\partial_t \mu_t^- - \xi \cdot \nabla_x \mu_t^-\right) \otimes \delta(\tau-1) \otimes \mathrm{d}t \\ &= -2\alpha_\infty \mu_t^+(x,\xi) \otimes \delta(\tau+1) \otimes \mathrm{d}t + 2\alpha_\infty \mu_t^-(x,\xi) \otimes \delta(\tau-1) \otimes \mathrm{d}t \\ &= 4\tau\, \alpha_\infty\, \mu. \end{aligned}$$

Let us now prove (6.4). We take $q \in \mathcal{A}_i^0$ and apply $q(z, D_z)$ to the first equation of (1.1). We get

$$0 = \left(q(z, D_z)\left(\partial_t^2 u_n - \Delta u_n + \nabla \cdot (\gamma(x)\theta_n)\right), u_n\right).$$

Using that $(\theta_n)_{n\in\mathbf{N}}$ is u.b. in $L^2_{loc}\left(\mathbf{R}, H^1(\Omega)\right)$ and that u_n goes to 0 weakly in $H^1(\Omega)$, we obtain, passing to the limit,

$$\int_{\mathbf{R}\times\Omega_1\times S^d(2)} q(z,\zeta)(|\xi|^2 - \tau^2)d\mu(z,\zeta) = 0.$$

Therefore, on the support of μ, we have $\tau^2 = |\xi|^2$. Since, moreover,

$$|\zeta|^2 = \tau^2 + |\xi|^2 = 2 \quad \text{on } S^d(2),$$

we obtain that $\tau^2 = |\xi|^2 = 1$ on the support of μ, whence the existence of two measures $\tilde{\mu}_\pm$ on $\mathbf{R} \times \Omega_1 \times S^{d-1}$ such that

$$\mu(t, x, \tau, \xi) = \delta(\tau + 1) \otimes \tilde{\mu}_+(t, x, \xi) + \delta(\tau - 1) \otimes \tilde{\mu}_-(t, x, \xi) \quad \text{in } \mathcal{D}'\big(\mathbf{R} \times \Omega_1 \times S^d(2)\big). \tag{6.6}$$

Let us now link the measure $\tilde{\mu}_\pm$ with the measures $\mu^\pm$ of § 4.2 inside Ω. Let us consider, as in § 4.2,

$$f_{\pm,n}(t, x) = \frac{1}{\sqrt{2}} \left(i\partial_t u_n(t, x) \pm |D_x| u_n(t, x)\right).$$

These families are uniformly bounded in $L^2(\Omega)$ for all $t \in \mathbf{R}$, and their microlocal defect measures are the measures $\mu^\pm$. Besides, by the definition of μ, for $q \in \mathcal{A}_i^0$, we have

$$\begin{aligned}\big(q(z, D_z) f_{\pm,n}, f_{\pm,n}\big) &= \frac{1}{2}\left((i\partial_t \pm |D|) q(z, D_z)(i\partial_t \pm |D|) u_n, u_n\right) \\ &\xrightarrow[n\to+\infty]{} \frac{1}{2} \int_{M \times S^d(2)} q_\infty(z, \zeta)(-\tau \pm |\xi|)^2 d\mu(z, \zeta) \\ &= 2 \int_{\mathbf{R} \times \Omega \times S^{d-1}} q_\infty(t, x, \mp 1, \xi) d\tilde{\mu}_\pm(t, x, \xi),\end{aligned}$$

where we have used (6.6) for the last equality. Let us choose $q(z, \zeta) = \chi(t) b(x, \xi)$ with $b(x, \xi)$ a symbol of order 0 compactly supported in Ω. We obtain on the one hand

$$\int_{\mathbf{R} \times \Omega \times S^{d-1}} q_\infty(t, x, \mp 1, \xi) d\tilde{\mu}_\pm(t, x, \xi) = \int_{\mathbf{R} \times \Omega \times S^{d-1}} \chi(t) b(x, \xi) d\tilde{\mu}_\pm(t, x, \xi).$$

On the other hand, we have, for all $t \in \mathbf{R}$,

$$(b(x, D) f_{\pm,n}(t), f_{\pm,n}) \xrightarrow[n\to+\infty]{} \int_{\Omega \times S^{d-1}} q(x, \omega) \mathrm{d}\mu_t^\pm(x, \omega),$$

whence $\tilde{\mu}_\pm(t, x, \omega) = \frac{1}{2} \mu_t^\pm(x, \omega) \otimes \mathrm{d}t$.

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Appendix. The non-degenerate case

In this appendix, we focus on the non-degenerate situation, when the matrix B satisfies

$$\exists c_0 > 0, \quad B(x) \geqslant c_0 \, \mathrm{Id}. \tag{A 1}$$

We show by classical methods how one can describe the behavior of microlocal defect measures of families of solutions to (1.1). This provides an alternative proof of Theorem 2.4 in the non-degenerate setting. We also discuss how the analysis of microlocal defect measures allows one to study the existence of a lower bound for the exponential rate of the energy by following the strategy introduced by Lebeau in [19] and [20]. These arguments apply in both situations: for the weakly degenerate pair (B, γ) as in § 2.3 (see the last paragraph therein) and for the non-degenerate case.

Let us now come back to the non-degenerate case and consider families of solutions to (1.1) $(u_{0,n})_{n\in\mathbf{N}}$ and $(u_{1,n})_{n\in\mathbf{N}}$ uniformly bounded in $H^1(\Omega)$ and $L^2(\Omega)$, respectively. Since the operator $\nabla\cdot(B(x)\nabla)$ is elliptic, by Theorem 3 of Henry *et al.* [18], the semigroup associated to (1.1) is equal up to a compact operator to the semi-group of the system

$$\begin{cases}\partial_t^2\tilde{u}-\Delta\tilde{u}+\Gamma\partial_t\tilde{u}=0, & (t,x)\in\mathbf{R}^+\times\Omega\\ \partial_t\tilde{\theta}-\nabla\cdot(B(x)\nabla\tilde{\theta})+\gamma(x).\nabla\partial_t\tilde{u}=0, & \end{cases} \tag{A 2}$$

where the operator Γ is a damping operator given by

$$\Gamma=G^*Q^{-1}G,\quad G=\gamma(x)\cdot\nabla\quad\text{and}\quad Q=\nabla\cdot(B(x)\nabla\cdot)\,. \tag{A 3}$$

This system consists of a damped wave equation on $\tilde{u}$ coupled with the equation on the temperature $\tilde{\theta}$. Besides, the analog of Proposition 4.1 induces that the measure $|\tilde{\theta}_n(t,x)|^2dtdx$ goes to 0 in $\mathcal{D}'(\mathbf{R}_t\times\Omega)$, and we are reduced to the analysis of the microlocal defect measure of the sequences $(\tilde{u}_n)$. The wave equation satisfied by $\tilde{u}_n$ presents a damping term given by a pseudodifferential operator of order 0: $\Gamma=\tilde{a}(x,D)$ with principal symbol

$$\sigma(\tilde{a})(x,\xi)=\frac{(\gamma(x)\cdot\xi)^2}{B(x)\xi\cdot\xi}.$$

The case of a damping by a function $a(x)$ has been extensively studied in the literature. As long as one uses microlocal methods, they naturally extend to a pseudodifferential damping $\Gamma=\tilde{a}(x,D)$. In the first subsection, we describe the results derived from the works of Lebeau [19, 20] concerning the damped wave equation on $\tilde{u}_n$; we will follow the presentation of the survey of Burq [3]. Then, in the second subsection, we explain how the properties of the microlocal measures of families of solutions to (1.1) allow us to characterize the rate of exponential decay. We do not give the proofs in details and refer to the literature.

A.1. Microlocal defect measures associated with a family of solutions to a damped wave equation

One gets the following. Consider $\mu(t,x,\tau,\omega)$, the microlocal defect measure associated with u_n viewed as a sequence of $L^2_{loc}(\mathbf{R},H^1(\mathbf{R}^d))$. This measure satisfies that, up to a subsequence, for all symbols $a(t,x,\tau,\xi)$ of order 2,

$$(a(t,x,D_t,D_x)u_n,\ u_n)\xrightarrow[n\to+\infty]{}\int a(t,x,\tau,\xi,\omega)\mu(dt,dx,d\tau,d\omega).$$

The analog of Theorem 15 in [3] (which sums up the results on the subject) gives that the generalized bicharacteristic map $\Phi(s)$ is well defined, μ-a.e., and satisfies

$\Phi(s) \circ \Phi(-s) = \mathrm{Id}$, μ-a.e. and for all Borelian $\omega \subset \Sigma_b$,

$$\mu\left(\Phi(s)\omega\right) = \int_{\omega \cap \{p_0=0\}} \exp\left[-\int_0^s \tau \tilde{a}\left(\Phi(\sigma, \rho)\right) d\sigma\right] d\mu(\rho). \tag{A 4}$$

The only difference with Theorem 15 of [3] is in the damping operator: $\Gamma = -2a(x)$ in [3], and here we have a pseudodifferential operator Γ given by (A 3). Of course, this is not a difficulty, since the microlocal defect measures are adapted to pseudodifferential operators. The proof of (A 4) is made by Lebeau in Section 3 of [20]; the argument combines the Melrose–Sjöstrand result about propagation of singularities [22] for identifying the support of μ with the generalized bicharacteristic curves, and energy estimates to gain the quantitative information contained in (A 4).

A.2. Analysis of the decay rate of the energy of the thermoelasticity system

We denote by A the operator of thermoelasticity; with the notation of (A 3), we set

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \Delta & 0 & -\nabla\cdot(\gamma(x)\cdot) \\ 0 & -\gamma(x)\cdot\nabla & \nabla\cdot(B(x)\nabla\cdot) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \Delta & 0 & G^* \\ 0 & -G & Q \end{pmatrix},$$

with domain $D(A) = (H_0^1 \cap H^2) \oplus H_0^1 \oplus (H_0^1 \cap H^2)$. Equation (1.1) becomes $\partial_t V = AV$ with $V = {}^t(u, \partial_t u, \theta)$. We assume that the eigenvalues of A have negative real part:

$$\sup\{-\mathrm{Re}\lambda,\ \lambda \in \mathrm{sp}A\} > 0. \tag{A 5}$$

If B is non-degenerate – that is, if B satisfies (A 1) – then, (A 5) is equivalent to assumption (i) in [7], section 6, namely the following.

Assumption A.1. If ϕ satisfies

$$\Delta\phi + \omega^2\phi = 0 \quad \text{and} \quad \gamma(x)\cdot\nabla\phi = 0$$

in Ω, then $\phi = 0$.

This assumption is known to be satisfied if all the eigenvalues of the Dirichlet problem for the Laplacian in Ω are simple, which is a generic property among smooth domains (see also [9, 18]).

In the general case, when B does not satisfy (A 1), Assumption (A 5) is equivalent to the following spectral property of the operators G and Γ (which is Assumption (A 5) when Q is elliptic).

Assumption A.2. If ϕ satisfies

$$\Delta\phi - G^*G\phi + \omega^2\phi = 0 \quad \text{and} \quad Q^{1/2}G\phi = 0$$

in Ω, then $\phi = 0$.

Proof. The equivalence come from the analysis of (u, v, θ) such that $A(u, v, \theta)^t = i\omega(u, v, \theta)^t$: they satisfy

$$\Delta u + \omega^2 u + G^*\theta = 0, \quad Q\theta - i\omega\theta = i\omega G u, \quad v = i\omega u.$$

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Therefore, $(G^*\theta,\ u) = (\theta, Gu) = \|\nabla u\|^2_{L^2} + \omega^2\|u\|^2 \in \mathbf{R}$ and

$$(Q\theta, \theta) = i\omega\left((Gu, \theta) + |\theta|^2\right) \in i\mathbf{R},$$

whence $Q^{1/2}\theta = 0$ and $\theta = -Gu$. One deduces that u satisfies

$$\Delta u - GG^* u + \omega^2 u = 0 \quad \text{and} \quad Q^{1/2}Gu = 0.$$ □

Property (A 5) (or equivalently, Assumption A.1 if B is non-degenerate or Assumption A.2 if not) implies that the energy of solutions of the thermoelasticity system (1.1) goes to 0: for any solution (u, θ) of (1.1), we have

$$E(u, \theta, t) = \|\nabla u(t)\|^2_{L^2} + \|\partial_t u(t)\|^2_{L^2} + \|\theta(t)\|^2_{L^2} \xrightarrow[t\to+\infty]{} 0. \tag{A 6}$$

The reader can refer to the proof of Theorem 2 in [18] in the non-degenerate case, which generalizes easily to the degenerate situation with Assumption A.2 (see also [6] for similar proofs in the context of the standard damped wave equation). Indeed, following [18], one is reduced to showing that it is not possible to find $(\hat{u}, \hat{\theta})$ satisfying (1.1) with $Q^{1/2}\hat{\theta} = 0$ and with non-zero energy (note that $Q^{1/2}\hat{\theta} = 0$ implies that the energy is constant in time). Actually, set $\hat{v} = \partial_t \hat{u}$; then we have

$$\partial_t^2 \hat{v} - \Delta\hat{v} + G^*G\hat{v} = 0 \quad \text{and} \quad Q^{1/2}G\hat{v} = 0.$$

Decomposing $\hat{v}$ on a basis of eigenfunctions of $-\Delta + G^*G$, the same arguments as in [18] give $\hat{v} = 0$, and hence a contradiction, since this implies that $\hat{u} = 0$ (in view of $u_{|\partial\Omega} = 0$) and, similarly, $\hat{\theta} = 0$.

Besides, we have the following result about the exponential decay of the energy of solutions of the thermoelasticity system (1.1).

Proposition A.3. *Assume that B is non-degenerate (see (A 1)), and set*

$$\mathcal{W} = \{(x, \omega) \in \Omega \times \mathbf{S}^{d-1}, \gamma(x) \cdot \omega > 0\}.$$

We have the following results.

(1) *If Assumption A.1 holds and if there exists $T > 0$ such that any generalized bicharacteristic reaches the set $\mathcal{W} \cap [0, T]$, then there exist $C > 0$ and $\alpha > 0$ such that, for any (u, θ) solution to (1.1) and for all $t \in \mathbf{R}^+$,*

$$E(u, \theta, t) \leqslant C\mathrm{e}^{-\alpha t} E(u, \theta, 0).$$

(2) *If, for all $T > 0$, there exists a generalized bicharacteristic which does not reach the set $\mathcal{W} \cap [0, T]$, then for all $t \in \mathbf{R}^+$,*

$$\sup_{(u,\theta) \text{ solving } (1.1)} \frac{E(u, \theta, t)}{E(u, \theta, 0)} = 1.$$

Note that the proof below shows that the results of Proposition A.3 are true if Assumptions 2.2, 2.6 and A.2 hold and if the conjecture on the behavior close to the boundary is proved.

Proof. (1) The proof relies on a stabilization inequality where one crucially uses our result on microlocal defect measures.

Lemma A.4. *There exist $T > 0$ and $C_0 > 0$ such that any solution (u, θ) of (1.1) with initial data $u_{|t=0} = u_0$, $\partial_t u_{|t=0} = u_1$, and $\theta_{|t=0} = \theta_0$, satisfies*

$$\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \leqslant C_0 \int_0^T \int_\Omega B(x)\nabla_x\theta(t,x)\cdot\nabla_x\theta(t,x)dxdt.$$

Note that the lemma conjugated with the energy estimate (1.3) implies that $E(u, \theta, T) \leqslant c\,E(u, \theta, 0)$ with $0 < c < 1$. By repeating the argument between nT and $(n+1)T$, one gets

$$E(u, \theta, t) \leqslant C\mathrm{e}^{-mt}$$

for some $C, m > 0$. It remains to prove Lemma A.4.

Proof. The proof of this lemma is done by contradiction. If the lemma is false, there exists a sequence (u_n, θ_n) of solutions to (1.1) and such that

$$\forall t \in \mathbf{R}^+, \qquad \int_0^t \int_\Omega B(x)\nabla_x\theta_n(t,x)\cdot\nabla_x\theta_n(t,x)dxdt \leqslant \frac{1}{n}E(u_n, \theta_n, 0).$$

By multiplying u_n and θ_n by a constant if necessary, one can assume that

$$E(u_n, \theta_n, 0) = 1.$$

Then we have

$$\forall t \in R^+, \qquad \int_0^t \int_\Omega B(x)\nabla_x\theta_n(t,x)\cdot\nabla_x\theta_n(t,x)dxdt \xrightarrow[n\to+\infty]{} 0. \tag{A 7}$$

We can consider a weak limit (u, θ) of (u_n, θ_n), up to extraction of a subsequence. Then, (u, θ) is a solution to (1.1) and satisfies $B\nabla_x\theta\cdot\nabla_x\theta = 0$ because of (A 7). In particular, the energy estimate (1.3) gives $E(u, \theta, 0) = E(u, \theta, t)$ for all $t \in \mathbf{R}^+$. Since $E(u, \theta, t) \xrightarrow[t\to+\infty]{} 0$ (see (A 6)), we deduce that $E(u, \theta, 0) = 0$, and thus the weak limit of the sequence (u_n, θ_n) is $(u, \theta) = (0, 0)$.

Therefore, we can consider its microlocal defect measures which obey Theorem 2.4. Consider $\mu_t^\pm$, the measures associated with (u_n, θ_n); for all $t \in \mathbf{R}$,

$$E(u_n, \theta_n, t) \xrightarrow[n\to+\infty]{} \int_\Omega \left(\mu_t^+(dx, d\omega) + \mu_t^-(dx, d\omega)\right).$$

The support of $\mu^\pm$ is the union of bicharacteristic curves $\Phi_{x_0,\omega_0}(s)$ (where $\Phi_{x_0,\omega_0}(0) = (x_0, \omega_0) \in \Omega \times \mathbf{S}^{d-1}$). By assumption, there exist $s_0, \delta_0 > 0$ such that $\Phi_{x_0,\omega_0} \in V$ for all $s \in]s_0 - \delta_0, s_0 + \delta_0[$. Therefore, setting $\Theta(x, \xi) = \frac{(\gamma(x)\cdot\xi)^2}{B(x)\xi\cdot\xi}$, the damping term

$$\kappa(x_0, \omega_0) = \mathrm{Exp}\left[-\int_0^t \Theta(\Phi_{x_0,\omega_0}(\sigma))d\sigma\right] dt$$

is strictly smaller than 1. Since $\overline{\Omega} \times \mathbf{S}^{d-1}$ is compact, there exists κ_0 such that $\kappa(x_0, \omega_0) \leqslant \kappa_0 < 1$ for all the curves included in the support of $\mu^\pm$. As a consequence, we have

$$\lim_{n\to+\infty} E(u_n,\theta_n,t) = \int_\Omega \left(\mu_t^+(dx,d\omega)+\mu_t^-(dx,d\omega)\right) < \int_\Omega \left(\mu_0^+(dx,d\omega)+\mu_0^-(dx,d\omega)\right) = \lim_{n\to+\infty} E(u_n,\theta_n,0) = 1.$$

Then contradiction comes from the fact that (A 7) induces

$$\lim_{n\to+\infty} E(u_n,\theta_n,t) = \lim_{n\to+\infty} E(u_n,\theta_n,0) = 1. \qquad \square$$

(2) Let us now prove (2). We consider a generalized bicharacteristic $\Phi(s)$ issued from $(x_0,\omega_0)\in\Omega\times\mathbf{S}^{d-1}$ which does not reach the set $\mathcal{W}\cap[0,T]$ and initial data $(u_{n,0},\theta_{n,0}=0)$ with an initial microlocal defect measure supported above (x_0,ω_0). We assume for example that $\mu_0^+=\delta_{x_0,\omega_0}$ and $\mu_0^-=0$. Then, in view of Theorem 2.4, for positive times t, any microlocal defect measure of the family of solutions (u_n,θ_n) associated with the data $(u_{n,0},0)$ concentrates on the curve $\Phi(s)$ and its mass is not damped: $\mu_t^-=0$ and $\mu_t^+=\delta_{\Phi(t)}$. Assume moreover that the microlocal defect measure describes all the energy of u_0 at time 0: $E(u_n,\theta_n,0)=1$. Then, one can express the energy in terms of a microlocal defect measure, and we obtain that, for any $\chi\in\mathcal{C}_0^\infty(\mathbf{R})$ with $\int\chi(t)dt=1$,

$$\lim_{n\to+\infty}\int_{\mathbf{R}}\chi(t)E(u_n,\theta_n,t)dt = \int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)d\mu_t^+(x,\omega)dt = \int_{\mathbf{R}^+\times\Omega\times\mathbf{S}^{d-1}}\chi(t)d\mu_0^+(x,\omega)dt = E(u_n,\theta_n,0).$$

Since, moreover, we always have $E(u_n,\theta_n,t)\leqslant E(u_n,\theta_n,0)$, we deduce that it is not possible to have $\lim_{n\to+\infty}E(u_n,\theta_n,t)<E(u_n,\theta_n,0)$ for some $t>0$. $\square$

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