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NOTES

A NOTE ON ECONOMIC GROWTH WITH SUBSISTENCE CONSUMPTION

HOLGER STRULIK
University of Hannover

It is well known that the performance of simple models of economic growth improves substantially through the introduction of subsistence consumption. How to compute subsistence needs, however, is a difficult and controversial issue. Here, I reconsider the linear (Ak) growth model with subsistence consumption and show that the evolution of savings rates and economic growth rates over time is independent of the size of subsistence needs. The model is thus more general and less subject to arbitrariness than might have been thought initially. Quantitatively, it is shown that, although there is no degree of freedom to manipulate transitional dynamics, the model approximates the historical evolution of savings rates and growth rates reasonably well.

Keywords: Economic Growth, Saving, Subsistence Needs

1. INTRODUCTION

The historical economic development of the Western world was characterized by a slow and gradual takeoff out of poverty accompanied by slowly increasing savings rates and slowly increasing economic growth rates. For England, for example, GDP per capita growth was 0.0% from year 1 to 1000, 0.1% from 1000 to 1500, 0.2% from 1500 to 1700, 0.3% from 1700 to 1820, 1.3% from 1820 to 1870, 1.0% from 1870 to 1913, 1.2% from 1913 to 1960, and 2.1% from 1960 to 2000 [inferred from the Maddison (2001) data]. Thus, during the industrial revolution, that is, at the time when there was the greatest change of growth rates, growth itself was high compared to what it had been so far but it was low from today's perspective.

Similarly, the savings rate (investment rate) was rising from 3% to 6% in 1688 [Deane and Cole (1969)] to 8% for 1761–1770, and 14% for 1791–1800 [Feinstein (1981)]. These historical observations are consistent with the empirical literature showing that savings rates are increasing in income across individuals [e.g., Dynan et al. (2004)] and across countries [Loayza et al. (2000)].

Unfortunately, simple models of economic growth with endogenous savings rates have problems in getting this adjustment dynamics right. Whereas the neoclassical growth model predicts that growth rates and savings rates are falling

Address correspondence to: Holger Strulik, Department of Economics, University of Hannover, Koenigsworther Platz 1, 30167 Hannover, Germany; e-mail: strulik@vwl.uni-hannover.de.

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as an economy gets richer [for reasonable choices of parameters, see Barro and Sala-i-Martin (2003)], the linear Ak growth model predicts that growth rates and savings rates are constant over time.

The simplest cure for these shortcomings is to introduce subsistence needs $\bar{c}$ into the utility function, that is, to consider utility of the Stone–Geary form. In that case the elasticity of marginal utility is declining in the level of consumption, which yields the result that savings rates and economic growth rates are jointly rising with economic development. In the Ak case, both rates approach positive constants when the economy converges toward its balanced growth path. This has been shown in detail by Steger (2000).[1]

The augmented Ak model provides transitional dynamics from the subsistence level toward balanced growth. In calibrating the model, there is comparatively little controversy as to how to find parameters that describe an economy along the balanced growth path reasonably well. But what about transitional dynamics? Will it depend crucially on the specification of subsistence needs? This could be a problem because uncertainty about the true value of $\bar{c}$, that is, about how to conceptualize subsistence needs, permits a degree of freedom. It opens the possibility of specifying $\bar{c}$ ad libitum, which could make the model, in principle, unfalsifiable.[2]

Fortunately, with respect to the model's key variables, this is not the case. In the next section I show that the transitional dynamics of the rate of economic growth and the savings rate are independent of the size of $\bar{c}$. In fact, the specification of economic growth and savings *along the balanced growth path* is sufficient to determine how these variables evolve over time in general.

Not having the degree of freedom for designing adjustment dynamics by "appropriate choice" of $\bar{c}$, it could be that the augmented Ak model is refuted by the empirical facts; for example, adjustment dynamics as predicted by the model's steady state could be too fast or too slow vis a vis the real data. With respect to the Western world this is, fortunately, not the case. In Section 3 I consider a model calibration and conclude that the augmented Ak growth model describes the historical evolution of growth and savings over time, as observed for England from the year 1200 to the year 2000, reasonably well.

2. THE MODEL

The description of the setup of the model can be brief since it has been discussed in great detail by Steger (2000). Here, I will solve the problem differently in order to provide the result of invariance with respect to subsistence needs. Consider a representative individual who derives intertemporal utility from consumption c whereby instantaneous utility is of the Stone–Geary form

$$\max_{c} \int_0^\infty \frac{(c-\bar{c})^{1-\theta}}{1-\theta} e^{-\rho t}\, dt. \tag{1}$$

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The parameter $\bar{c}$ is the level of subsistence consumption, ρ is the time preference rate, and θ is the ultimate elasticity of marginal utility, which is revealed when consumption goes to infinity.

That the elasticity of marginal utility is not constant in general but decreasing in the deviation of c from $\bar{c}$ is the crucial feature that provides interesting adjustment dynamics. As the distance of c from $\bar{c}$ gets larger, subsistence needs become less pressing, the effective rate of time preference decreases, and people save a larger share of their income, an effect that increases the distance of c from $\bar{c}$ even further.

Output is produced using capital k by a linear production function with productivity A. Thus capital evolves according to

$$\dot{k} = Ak - c. \tag{2}$$

We assume $A > \rho$ in order to allow for positive balanced growth.

The first-order conditions for a solution of (1) and (2) are $(c - \bar{c})^{-\theta}$ and $\lambda A = \lambda\rho - \dot{\lambda}$, where λ is the co-state variable of the associated current-value Hamiltonian. Log-differentiating the first condition with respect to time and inserting it into the second condition eliminates λ and provides $(A - \rho)(c - \bar{c}) = \theta\dot{c}$. Noting that $\dot{c} = c'(k)\dot{k}$ and using (2), this can be written as

$$(A - \rho)(c - \bar{c}) = \theta c'(k)(Ak - c).$$

An explicit solution of this differential equation is obtained using the method of undetermined coefficients:

$$c = \frac{(A - \rho)\bar{c}}{\theta A} + \left[\frac{(\theta - 1)}{\theta}A + \frac{\rho}{\theta}\right]k. \tag{3}$$

The expression in square brackets is the familiar term from the standard Ak growth model according to which it is optimal to consume a constant fraction of capital (and thus income). The first term modifies this result and introduces a kind of "Engel's law." With rising income, per capita consumption expenditure increases, but the expenditure share of consumption decreases, implying an increasing savings rate.

Dynamics of the economy can be most conveniently analyzed by introducing the consumption capital ratio $x = c/k$, which evolves according to $\dot{x} = (\dot{c}k + c\dot{k})/k^2 = \dot{k}[c'(k) - x]/k$. Insert $c'(k)$, obtained from (3), and $\dot{k}$, from (2), to get the economy represented by a single differential equation in x:

$$\dot{x} = (A - x)\left[\frac{(\theta - 1)}{\theta}A + \frac{\rho}{\theta} - x\right]. \tag{4}$$

Inspecting (4), we get the following results.

PROPOSITION 1. *There exists a unique steady state of stagnation at $x = A$. There exists a unique balanced growth path along which the economy grows at* *rate* $(A - \rho)/\theta$ *and where*

$$x = x^* = \frac{(\theta - 1)}{\theta} A + \frac{\rho}{\theta}. \quad \textbf{(5)}$$

The steady state of stagnation is unstable and the balanced growth path is locally stable.

Proof. Inspect (4) to see that the steady state of stagnation is where the term in parenthesis equals zero, that is, where $x = A$, implying that $Ak = c = \bar{c}$. Observe the second steady state x^* by setting the term in square brackets to zero. Insert x^* into $\dot{k}/k = A - x$, as obtained from (2), to get the balanced growth rate. Compute $\partial \dot{x}/\partial x = -[x^* - x] - (A - x)$. At the equilibrium where $x = A$, we have $x > x^*$, and thus the equilibrium is unstable. At the equilibrium where $x = x^*$, we have $x < A$, and thus the equilibrium is locally stable. Locally means that any economy starting at an arbitrarily small positive distance from subsistence arrives at the balanced growth path. ■

Note that the evolution of the economic system as specified by (4) is independent of the size of subsistence consumption $\bar{c}$. Transitional dynamics is obtained by starting the economy close to the equilibrium stagnation and solving (4). Once the path of $x(t)$ has been found, we can infer the path of income per capita growth $g_y(t) = A - x(t)$ and the path of the savings rate $s(t) = 1 - c(t)/y(t) = 1 - x(t)/A$. Both paths are invariant to the specification of subsistence needs $\bar{c}$.

There exists an even stronger invariance result. Along the balanced growth path, consumption grows at rate $g_c^* = A - x^*$, implying that $x^* = A - g_c^*$. Furthermore, the savings rate along the balanced growth path is $s^* = 1 - x^*/A$, implying that $A = g_c^*/s^*$. Using these values and (5), the dynamics of system (4) can be rewritten as

$$\dot{x} = \left(\frac{g_c^*}{s^*} - x\right)\left[g_c^*\left(\frac{1}{s^*} - 1\right) - x\right]. \quad \textbf{(6)}$$

This implies that the evolution of x over time is completely determined by the specification of the growth rate and the savings rate that we assume to hold along the balanced growth path. Once we have computed the path of $x(t)$, we can recover the path of savings, $s(t) = 1 - x(t)/A$, and the path of growth, $g_y(t) = A - x$. In other words, whereas A is implied by the choice of g_c^* and s^*, we can leave θ and ρ unspecified. This is a convenient result because there exists some uncertainty about the "true" values of these parameters of the utility function as well. The following proposition summarizes the results.

PROPOSITION 2. *Adjustment dynamics for the savings rate and the rate of economic growth from stagnation to balanced growth as implied by the augmented Ak model specified in (1) and (2) is independent of the size of subsistence consumption* $\bar{c}$.

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Adjustment dynamics is also independent of the size of the time preference rate ρ and the ultimate elasticity of marginal utility θ (given that their numerical specification supports a certain growth rate and savings rate along the steady state).

For an intuition of what causes the invariance with respect to subsistence needs, note that any change of $\bar{c}$ entails a change of the capital stock that supports the equilibrium of stagnation. From $x = A$, at the steady state of stagnation, we have $k = \bar{c}/A$, implying that dynamics *with respect to the reference point* $(\bar{c}, \bar{c}/A)$ remains unchanged. To see this clearly, consider the transformation of variables $\tilde{c} = c - \bar{c}$, $\tilde{k} = k - \bar{c}/A$. Problem (1)–(2) in the new notation reads

$$\max_{\tilde{c}} \int_0^\infty \frac{\tilde{c}^{1-\theta}}{1-\theta} e^{-\rho t}\, dt \quad \text{such that} \quad \dot{\tilde{k}} = A\tilde{k} - \tilde{c}.$$

This problem is isomorphic to the setup of the standard Ak model [see, for example, Barro and Sala-i-Martin (2003)]. From the first-order conditions we get the well-known Euler equation and policy function $\tilde{c}(\tilde{k})$:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{A-\rho}{\theta} \quad \Rightarrow \quad \tilde{c}(\tilde{k}) = \left[\frac{(\theta-1)}{\theta}A + \frac{\rho}{\theta}\right]\tilde{k}.$$

Note that dynamics in the $(\tilde{c}, \tilde{k})$ space is independent of the choice of $\bar{c}$. A change of $\bar{c}$ affects "only" the locus of origin of the $(\tilde{c}, \tilde{k})$ space in units of c and k. As a consequence, the shape of the adjustment path for economic rates, such as the consumption capital ratio, the savings rate, and the rate of economic growth, is independent of $\bar{c}$. The choice of $\bar{c}$ determines of course the evolution of levels, such as the level of consumption and income per capita. Usually in growth theory, however, we are not interested in absolute levels so much as in getting the evolution of economic rates right.

An alternative way to disentangle the effect of $\bar{c}$ is to begin by noting that every economy has to develop along the *unique* path $x(t)$ that solves (4). If an economy starts with a given $k(0) = k_1$ and subsistence consumption $\bar{c} = \bar{c}_1$, the optimal consumption choice (3) implies

$$x(0) = \frac{c(0)}{k(0)} = \frac{(A-\rho)}{\theta A}\frac{\bar{c}_1}{k_1} + \left[\frac{(\theta-1)}{\theta}A + \frac{\rho}{\theta}\right],$$

which clearly depends on the choice $\bar{c} = \bar{c}_1$. Adjustment dynamics for an alternative economy with subsistence needs $\bar{c}_2 > \bar{c}_1$ will be different. But note from the above equation that adjustment dynamics of the two economies are identical when the second economy starts at k_2 such that $\bar{c}_1/k_1 = \bar{c}_2/k_2$. Both economies start with the same $x(0)$ and obey (4); that is, they are identical everywhere. In other words, the choice of $\bar{c}$ determines for a given $k(0)$ at which point $x(0)$ the economy starts on the $x(t)$ trajectory and thus the time t at which any $x(t)$ is attained, but it does not affect the *shape* of the trajectory, which is invariant to $\bar{c}$.

The obtained invariance result is helpful because subsistence needs—although exogenous in the Ak growth context—are actually endogenous [Dalgaard and Strulik (2010)]. Metabolic needs are, for example, determined by the available diet, body size, and ambient temperature [West and Brown (2005)]. The augmented Ak model predicts that we must not care about the country- and individual-specific subsistence needs as long as we are interested in saving rates and rates of economic growth, because the shape of the adjustment path for these variables is independent of the specification of subsistence needs.

3. THE SLOW TRANSITION TOWARD MODERN GROWTH: A CALIBRATION STUDY

The invariance result implies that we have one parameter less to experiment with in order to fit the model to data. In fact, inspection of (6) shows that once we have decided about the parameter values that support growth and savings along the balanced growth path, we have no possibility at all of manipulating transitional dynamics. A reasonable specification of the balanced growth path may thus simultaneously imply implausible adjustment dynamics.

With respect to the shape of adjustment paths, we can eliminate this concern immediately. To see this, note that (4), or (6), respectively, is the generalized logistic equation. It has an explicit solution in the form of an S-shaped adjustment path for savings and thus for growth. In accordance with the historical observation (and in contrast to other simple models of economic growth), the model thus predicts irrespective of its numerical specification that economic change (i.e., the speed of change of economic rates) gets the highest momentum (say, an industrial revolution) when savings rate and growth rate are around half of their final steady-state values.

Nevertheless, the steady-state specification could fail to predict a reasonable adjustment speed. For example, if the gap between stagnation and balanced growth were closed within hours or days, nothing would be gained by augmenting the standard Ak growth model with subsistence needs. Fortunately, this is not the case. Adjustment dynamics predicted by the model performs quite well, at least with respect to the long-run economic development of the Western world.

To inspect adjustment dynamics, we set the balanced growth rate to 2 percent annually and the savings rate along the balanced growth path to 30 percent. This implies that $A = 0.0667$. A real rate of return on capital around 7 percent accords well with the average real return on the stock market for the last century and has been used in other calibration studies [e.g., Jones and Williams (2000)]. As long as we are interested in the evolution of rates, parameters of the utility function can be kept unspecified. Pairs that support the balanced growth path are, for example, $(\theta = 0.5, \rho = 0.0567)$ and $(\theta = 2, \rho = 0.0267)$.

To extract the path from stagnation towards balanced growth, I start the economy close to the steady state of stagnation $x = A$ and solve (4). Figure 1 shows adjustment dynamics for x and the implied savings rate and growth rate. For better

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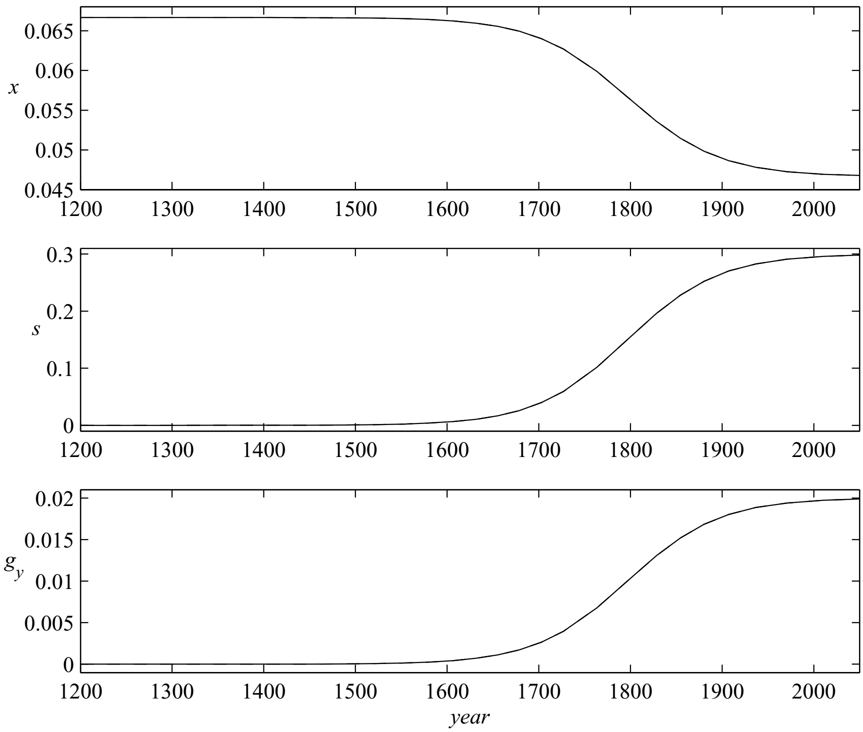

FIGURE 1. Subsistence consumption: Transitional dynamics.

comparison with the real data I have normalized time such that $t = 1800$ when $s = 0.14$ [as observed for England according to Feinstein (1981)]. Against the data provided in the Introduction, the model performs reasonably well. The economy gets the most momentum between 1700 and 1900, that is, during the phase that includes the industrial revolution. The take-off to modern growth is quite gradual and if anything it is somewhat “too slow”; that is, savings rates and growth rates predicted between 1600 and 1700 are somewhat too small.

As explained, the model predicts the same adjustment paths also for countries other than model England, at least, if they are assumed to arrive at the same balanced growth path. The only (ad hoc) possibility of introducing cross-country differences is to assume that some countries initiated the transition earlier than others.[3]

Although the subsistence-augmented Ak model works as a crude approximation for the economic development of the Western world, there are many details of the transition that the model fails to predict—for example, England’s overtaking of France and Italy. But then, it is only a very small and crude model from which one cannot expect everything.

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If we are interested in how the model performs with respect to levels, we have to specify subsistence needs $\bar{c}$. For that purpose I take $\bar{c} = 400$, the annual GDP per capita in England in year 1200 [according to Maddison (2001)]. The model then predicts a GDP in 1900 of 2800, while it was actually 4492 (according to Maddison). Note that the model allows no way to manipulate this result through alternative choices of ρ or θ (as long as we keep the steady-state specification) because the path of income is completely determined by its initial value and its growth rate, which are independent from the values of the preference parameter (as long as they support the steady-state specification). The underestimation of GDP per capita is not a real surprise because the crude model neglects other important drivers of economic development such as growth of factor productivity, structural change, and the demographic transition [Galor (2005)].

The same observation, however, can also be formulated positively. Compared with its peer group, the Solow model, the Ramsey model, and the standard Ak model, the subsistence-augmented Ak model performs astonishingly well. With respect to savings rates and growth rates, this good performance is obtained irrespective of the specification of the size of subsistence needs.

With respect to the (non-)growth experience of today's less developed countries, the subsistence-augmented Ak model is less appropriate. A general impression there is that many of these countries seem to grow at very low rates at very different levels of income. An explanation within the present model would require to assume unreasonably large cross-country differences of subsistence needs. Thus the original critique of the poverty trap model and the derived conclusion [Kraay and Raddatz (2007)] remains valid. The incidence of subsistence consumption cannot explain slow economic growth far off subsistence level. In that case one has to search for other explanations.

NOTES

1. For the general neoclassical growth model with decreasing returns (and $\theta > 1$), the savings rate is decreasing during transition toward the steady state. If subsistence consumption is introduced in this framework, we have two counteracting forces. As the economy develops and the capital stock rises, decreasing returns on capital lead to a lower propensity to save, whereas increasing distance of income from subsistence needs leads per se to a higher propensity to save. It is not clear a priori which effect dominates. For the Ak model investigated in the present paper, however, the effect is unambiguous. The return on capital is constant and thus there are no transitional dynamics without subsistence needs. Consequently, introducing subsistence needs leads to a savings rate increasing with economic development. An alternative yet more involved way to get the adjustment dynamics of the Ak growth model right is the introduction of habit formation [Carroll et al. (2000)].

2. See Kraay and Raddatz (2007) for a critique of subsistence needs as a driver of poverty traps. See Sharif (1986) on conceptualization and measurement of subsistence.

3. An economy starting further above $\bar{c}$ than England, that is, an economy starting at a higher $k(0)$ and thus a lower $x(0)$, would industrialize earlier. It would follow the same trajectory toward the steady state but would arrive at any given x earlier than England. Employing the intuition developed in Section 2, one sees that we can alternatively state that an economy starting at the same $k(0)$ but a lower level of $\bar{c}$ would start at a lower $x(0)$ than England. Again, following the same trajectory $x(t)$ toward the steady state, it would industrialize earlier. In other words, the invariance result refers to

how an economy adjusts towards the steady state and not to *when*, that is, at what particular time, a particular level of income is attained.

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