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Powers of Principal Q-Borel ideals

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Abstract. Fix a poset Q on $\{x_1, \ldots, x_n\}$. A Q-Borel monomial ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a monomial ideal whose monomials are closed under the Borel-like moves induced by Q. A monomial ideal I is a principal Q-Borel ideal, denoted $I = Q(m)$, if there is a monomial m such that all the minimal generators of I can be obtained via Q-Borel moves from m. In this paper we study powers of principal Q-Borel ideals. Among our results, we show that all powers of $Q(m)$ agree with their symbolic powers, and that the ideal $Q(m)$ satisfies the persistence property for associated primes. We also compute the analytic spread of $Q(m)$ in terms of the poset Q.

1 Introduction

Throughout this paper, $S = \mathbb{K}[x_1, \ldots, x_n]$ denotes the polynomial ring over an arbitrary field $\mathbb{K}$. Francisco, Mermin, and Schweig [12] introduced the notion of a Q-Borel monomial ideal to generalize the properties of Borel monomial ideals, also called strongly stable monomial ideals (see [11, 15] and their references for more on Borel ideals and their importance). Specifically, we fix a poset Q on the set $\{x_1, \ldots, x_n\}$. Then a monomial ideal I is a *Q-Borel ideal* if for any monomial $m \in I$, if $x_i | m$ and $x_j \leq_Q x_i$, then $x_j \cdot \frac{m}{x_i} \in I$. We call $x_j \cdot \frac{m}{x_i}$ a *Q-Borel move* of m. A *Borel ideal* is then the special instance when Q is the chain $Q = C : x_1 < x_2 < \cdots < x_n$. A monomial ideal I is a *principal Q-Borel ideal*, denoted $Q(m)$, if there is a monomial m such that all the minimal generators of I can be obtained from m via Q-Borel moves. As shown in [12] and Bhat's thesis [1], many properties of $Q(m)$, e.g., projective dimension, primary decomposition, can be described in terms of the poset Q and order ideals of Q associated with the monomial m.

Our goal in this paper is to study the properties of powers of principal Q-Borel ideals. Understanding powers of ideals figures prominently in commutative algebra. Two examples of this theme are the ideal containment problem and the persistence of primes. The ideal containment problem compares the regular powers of an ideal with its symbolic powers. The persistence of primes asks whether $\operatorname{ass}(I^s) \subseteq \operatorname{ass}(I^{s+1})$ for all $s \geq 1$, where $\operatorname{ass}(J)$ denotes the set of associated primes of J. The references [2,

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3, 8, 10, 13, 16–18, 20] form a small subset of papers on these topics; see also [5, 9] for an introduction.

For principal Q-Borel ideals $Q(m)$, we consider these (and other) problems. Many of our results are expressed in terms of the combinatorics of the poset of Q, thus building upon [12, Question 1.3] which asked what other properties of Q-Borel ideals are determined by Q. One theme that becomes apparent is that principal Q-Borel ideals satisfy many of the same properties as principal monomial ideals (in fact, results about principal monomial ideals become special cases of our work when Q is the anti-chain).

We first compare the regular and symbolic powers (formal definitions postponed until later in the paper) of principal Q-Borel ideals. Our main result in this direction is the following theorem.

Theorem 1.1 (Theorem 3.8) *Let $I = Q(m)$ for some monomial m and poset Q. Then*

$$I^{(d)} = I^d \text{ for all } d \geq 1.$$

Our proof requires Francisco *et al.* [12] characterization of the associated primes of $Q(m)$, and Cooper *et al.* [8] description of the symbolic powers of monomial ideals. As a corollary, we obtain results on the Waldschmidt constant, the symbolic defect, and the resurgence (see Corollary 3.10).

The *analytic spread* of I, denoted $\ell(I)$, is the Krull dimension of the ring

$$\mathcal{F}(I) = \bigoplus_{i \geq 0} \frac{I^i}{\mathbf{m}I^i} \quad \text{where } I^0 = S \text{ and } \mathbf{m} = \langle x_1, \dots, x_n \rangle.$$

For principal Q-Borel ideals, we obtain the following formula for the analytic spread in terms of combinatorics of Q.

Theorem 1.2 (Theorem 5.4) *Let $I = Q(m)$ be a principal Q-Borel ideal, let $A(m)$ be the order ideal generated by the support of m. Then*

$$\ell(I) = |A(m)| - K(A(m)) + 1$$

where $K(A(m))$ is the number of connected components in the subposet induced by $A(m)$.

Our proof uses the fact that for ideals generated by monomials of the same degree, the analytic spread is the rank of the matrix of exponent vectors of the generators. The analytic spread of $Q(m)$ could also be computed using results of Herzog, Rauf, and Vladoiu [17], but our result highlights the connection to the poset of Q.

Herzog, Rauf, and Vladoiu's paper [17] is used to address the question of persistence of primes. Precisely, we show that $\mathrm{ass}(I) = \mathrm{ass}(I^s)$ for all $s \geq 1$ for any principal Q-Borel ideal (see Theorem 4.3). In fact, we give two different proofs for this result.

We also consider powers of *square-free principal Q-Borel ideals*, denoted $sfQ(m)$. These square-free monomial ideals are generated by the square-free monomial generators of $Q(m)$. For this class of ideals, we also compute their analytic spread (see Theorem 5.10) in terms of Q.

Our paper is structured as follows. Section 2 is the background on monomial ideals, posets, and (principal) Q-Borel ideals. In section 3 we prove Theorem 3.8. In

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Section 4 we examine the persistence of primes problem. Section 5 is devoted to the analytic spread of (square-free) principal Q-Borel ideals.

2 Background

In this section we recall the relevant background and definitions.

2.1 Basics of monomial ideals and posets

Given a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$ in S, we may write the monomial as $m = x^\alpha$ with $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The monomial m is a *square-free monomial* if $a_i = 0$ or 1 for all $i = 1, \ldots, n$. The *support* of $m = x_1^{a_1} \cdots x_n^{a_n}$ is the set $\mathrm{supp}(m) = \{j \mid a_j > 0\}$.

An ideal $I \subseteq S$ is a *(square-free) monomial ideal* if I is generated by (square-free) monomials. A monomial ideal has a unique set of minimal monomial generators denoted by $G(I)$.

Let Q be a poset on the ground set $\{x_1, \ldots, x_n\}$, where the partial order is denoted by $<_Q$. A poset Q' is an *induced poset* of Q if there exists an injective function $f : Q' \to Q$ such that $x \leq_{Q'} y$ if and only if $f(x) \leq_Q f(y)$.

Associated to any poset on a finite ground set is a *Hasse diagram*. In particular, the elements of Q are represented by vertices, and there exists a line segment from x to y in the "upwards" direction if $x <_Q y$ and if there is no other $z \in Q$ such that $x <_Q z <_Q y$. The Hasse diagram is an example of directed acyclic graph (a directed graph with no directed cycles). Given a poset Q, the number of *connected components* of Q, denoted $K(Q)$, is the number of connected components of the Hasse diagram, i.e., the connected components of the Hasse diagram when viewed as a undirected graph.

An *order ideal* of Q is a set $A \subseteq Q$ such that if $y \in A$ and if $x <_Q y$, then $x \in A$. Given any monomial $m = x_1^{a_1} \cdots x_n^{a_n} \in S$, we can associate with m the order ideal

$$A(m) = \left\{ x_j \;\middle|\; \text{there is an } x_i \text{ such that } x_j \leq_Q x_i \;\text{ and }\; x_i | m \right\}.$$

The order ideal $A(m)$ is an induced poset of Q via the inclusion map. We say an order ideal $A(m)$ is *connected* if the Hasse diagram of $A(m)$ is connected. The next lemma follows directly from the definitions.

Lemma 2.1 *Fix a poset Q on $\{x_1, \ldots, x_n\}$, and let $m_1, m_2 \in S$ be two monomials. If $supp(m_1) = supp(m_2)$, then $A(m_1) = A(m_2)$.*

The next lemma will be used in future calculations.

Lemma 2.2 *Fix a poset Q on $\{x_1, \ldots, x_n\}$ and let $m_1, m_2 \in S$ be two monomials. Then*

$$A(\mathrm{lcm}(m_1, m_2)) = A(m_1 m_2) = A(m_1) \cup A(m_2).$$

Proof Note that

$$\mathrm{supp}(\mathrm{lcm}(m_1, m_2)) = \mathrm{supp}(m_1 m_2) = \mathrm{supp}(m_1) \cup \mathrm{supp}(m_2).$$

Now apply Lemma 2.1. ∎

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The next lemma shows the relationships between the components of a monomial and the order subideals of its order ideal.

Lemma 2.3 *Fix a poset Q on $\{x_1, \dots, x_n\}$ and let $m \in S$ be a monomial. For any order ideal $O \subset Q$ such that $O = A(m')$ for some $m'|m$, there is a unique monomial m_O satisfying:*

- $m_O | m$.
- $O = A(m_O)$.
- *For any other monomial $m''|m$ such that $O = A(m'')$, we have $m''|m_O$.*

Proof Define

$$m_O = \operatorname{lcm}\{m'' \mid m'' \text{ a monomial, } m''|m, \text{ and } A(m'') = O\}.$$

The set on the right contains m' so m_O is well-defined and it is clear that $m_O|m$. From the last lemma, we have $A(m_O) = O$ and from the definition, if any other monomial $m''|m$ satisfies $A(m'') = O$, then we have $m''|m_O$. ∎

2.2 Q-Borel ideals

Q-Borel ideals were introduced by Francisco, Mermin, and Schweig [12] to generalize properties of Borel monomial ideals. We recall this definition.

Definition 2.4 Let $I \subseteq S$ be a monomial ideal and let Q be a poset on $\{x_1, \dots, x_n\}$. The ideal I is a *Q-Borel ideal* if whenever $x_j \leq_Q x_i$ and $x_i|m$ for some monomial $m \in I$, then $x_j \cdot (m/x_i) \in I$. We say that I is *Borel with respect to Q*.

Remark 2.5 Definition 2.4 generalizes the notion of a Borel monomial ideal. More precisely, a Q-Borel ideal is a *Borel ideal* if Q is the chain $Q = C : x_1 <_Q x_2 <_Q < \cdots <_Q x_n$. Note that any monomial ideal I is a Q-Borel ideal if we take Q to be the antichain.

If $x_i|m$ and $x_j \leq_Q x_i$, then we call $x_j \cdot (m/x_i)$ a *Q-Borel move* of the monomial m. It follows that a monomial ideal I is a Q-Borel ideal if I is closed under Q-Borel moves. Observe that if $m = x^\alpha$, then a Q-Borel move $x_j \cdot (m/x_i)$ corresponds to the existence of a vector $e_{(i,j)} \in \mathbb{N}^n$ whose k-th coordinate is given by

$$(e_{(i,j)})_k = \begin{cases} 1 & k = j \text{ and } x_j \leq_Q x_i \\ -1 & k = i \text{ and } x_j \leq_Q x_i \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

such that $x^{\alpha + e_{(i,j)}} = x_j \cdot (m/x_i)$. The following lemma shall be useful.

Lemma 2.6 *Fix a poset Q on $\{x_1, \dots, x_n\}$. Suppose that x^α and x^β are monomials of S such that x^β can be obtained via a series of Q-Borel moves on x^α. Then there exists $e_{(i_1,j_1)}, \dots, e_{(i_l,j_l)}$, not necessarily distinct, with $i_t \in supp(x^\alpha)$ for $t = 1, \dots, l$, such that*

$$\alpha + e_{(i_1,j_1)} + \cdots + e_{(i_l,j_l)} = \beta.$$

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Equivalently, expressed in terms of monomials, we have

$$x^{\beta} = x^{\alpha} \cdot \frac{x_{j_1} \cdots x_{j_l}}{x_{i_1} \cdots x_{i_l}}$$

where x_{i_t} divdes x^{α} for $t = 1, \dots, l$.

Proof Because x^{β} can be obtained from x^{α} by Q-Borel moves, there exists monomials $x^{\alpha} = x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_{r-1}}, x^{\alpha_r} = x^{\beta}$ such that $x^{\alpha_{t+1}}$ is obtained from x^{α_t} via a Q-Borel move for $t = 1, \dots, r-1$. In particular, there exists a vector of the form $e_{(a_t, b_t)}$ such that

$$\alpha_t + e_{(a_t, b_t)} = \alpha_{t+1} \text{ for each } t = 1, \dots, r-1$$

where $a_t \in \operatorname{supp}(x^{\alpha_t})$ and $x_{b_t} \leq_Q x_{a_t}$. Consequently,

$$\alpha + e_{(a_1, b_1)} + \cdots + e_{(a_{r-1}, b_{r-1})} = \beta.$$

If $a_t \in \operatorname{supp}(x^{\alpha})$ for all $t = 1, \dots, r-1$, then we are done.

On the other hand, suppose that there is some $e_{(a_t, b_t)}$ such that $a_t \notin \operatorname{supp}(x^{\alpha})$. Let t be the smallest index such that $a_t \notin \operatorname{supp}(x^{\alpha})$. That is, t is the smallest index such that α_{t+1} has not been expressed in the form $\alpha + e_{(i_1, j_1)} + \cdots + e_{(i_t, j_t)}$ with all $i_k \in \operatorname{supp}(x^{\alpha})$. Note that $t \geq 2$ because $a_1 \in \operatorname{supp}(x^{\alpha})$. Now

$$\alpha_{t+1} = \alpha_t + e_{(a_t, b_t)} = \left(\alpha + e_{(a_1, b_1)} + \cdots + e_{(a_{t-1}, b_{t-1})}\right) + e_{(a_t, b_t)}.$$

Because a_t is not in the support of x^{α}, but in the support of x^{α_t}, this means that $a_t \in \{b_1, \dots, b_{t-1}\}$ since the b_k's correspond to the supports of the new variables by which we multiply after dividing by a_k. Say $a_t = b_s$ with $s \in \{1, \dots, t-1\}$. But then by equation (2.1)

$$e_{(a_s, b_s)} + e_{(a_t, b_t)} = e_{(a_s, b_t)},$$

that is, the coordinate which is 1 in the first vector cancels out with -1 in the second vector. Furthermore, $b_t \leq_Q a_s$, because $b_t \leq_Q a_t = b_s \leq_Q a_s$. So, we can rewrite α_{t+1} as

$$\begin{aligned} \alpha_{t+1} &= \alpha + e_{(a_1, b_1)} + \cdots + \left(e_{(a_s, b_s)} + e_{(a_t, b_t)}\right) + \cdots + e_{(a_{t-1}, b_{t-1})} \\ &= \alpha + e_{(a_1, b_1)} + \cdots + e_{(a_s, b_t)} + \cdots + e_{(a_{t-1}, b_{t-1})} \end{aligned}$$

where all the a_k's are in the $\operatorname{supp}(x^{\alpha})$. So α_{t+1} has the desired form.

Repeating this process allows β to be expressed in the desired form. ∎

Because Q-Borel ideals are closed under Q-Borel moves, the generators of Q-Borel ideals can be described as subsets of monomials of S from which other monomial generators in the ideal can be obtained via Q-Borel moves. The following terminology shall be helpful.

Definition 2.7 Let X be a subset of monomials of S. The smallest Q-Borel ideal I that contains X is denoted $Q(X)$, and we say X is a *Q-Borel generating set* of $I = Q(X)$. A square-free monomial ideal J is a *square-free Q-Borel ideal* if it is generated by the

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square-free monomials of a Q-Borel ideal. Given a set Y of square-free monomials, we let $sfQ(Y)$ denote the smallest square-free Q-Borel ideal containing Y.

The following fact follows directly from the definitions.

Lemma 2.8 [12, Proposition 2.6] *If all the monomials of X have the same degree, then all the minimal generators of the Q-Borel ideal $I = Q(X)$ have the same degree.*

2.3 Q-Borel principal ideals

We are primarily interested in the following ideals.

Definition 2.9 If $X = \{m\}$ contains a single monomial, then we call $I = Q(X)$ a *Q-Borel principal ideal*, and we abuse notation and write $I = Q(m)$. Similarly, if $Y = \{m\}$ contains a single square-free monomial, then we call $I = sfQ(Y)$ a *square-free Q-Borel principal ideal* and write $I = sfQ(m)$.

Principal Q-Borel ideals are preserved under ideal multiplication.

Lemma 2.10 *Fix a poset Q on $\{x_1,\dots,x_n\}$, and let $m_1, m_2 \in S$ be two monomials. Then*

$$Q(m_1)Q(m_2) = Q(m_1m_2).$$

Proof Let $p_1 \in Q(m_1)$, respectively $p_2 \in Q(m_2)$, be any monomial generator of $Q(m_1)$, respectively $Q(m_2)$. So p_1 is a Q-Borel move of m_1, and similarly for p_2 and m_2. Thus

$$p_1 = m_1\frac{x_{j_1}x_{j_2}\cdots x_{j_r}}{x_{i_1}x_{i_2}\cdots x_{i_r}} \text{ with } x_{j_\ell} <_Q x_{i_\ell} \text{ for } \ell = 1,\dots,r$$

and

$$p_2 = m_2\frac{x_{b_1}x_{b_2}\cdots x_{b_s}}{x_{a_1}x_{a_2}\cdots x_{a_s}} \text{ with } x_{b_\ell} <_Q x_{a_\ell} \text{ for } \ell = 1,\dots,s.$$

But this means that

$$p_1p_2 = m_1m_2\frac{x_{j_1}x_{j_2}\cdots x_{j_r}}{x_{i_1}x_{i_2}\cdots x_{i_r}}\frac{x_{b_1}x_{b_2}\cdots x_{b_s}}{x_{a_1}x_{a_2}\cdots x_{a_s}}$$

is a Q-Borel move of m_1m_2, so $p_1p_2 \in Q(m_1m_2)$, thus showing $Q(m_1)Q(m_2) \subseteq Q(m_1m_2)$.

For the reverse containment, if $p \in Q(m_1m_2)$ is a generator of $Q(m_1m_2)$ obtained via a series of Q-Borel moves on m_1m_2. So, by Lemma 2.6 and Lemma 2.2, we have

$$p = m_1m_2\frac{x_{j_1}x_{j_2}\cdots x_{j_r}}{x_{i_1}x_{i_2}\cdots x_{i_r}}\frac{x_{b_1}x_{b_2}\cdots x_{b_s}}{x_{a_1}x_{a_2}\cdots x_{a_s}}\frac{x_{c_1}x_{c_2}\cdots x_{c_t}}{x_{d_1}x_{d_2}\cdots x_{d_t}}$$

where $x_{j_\ell} <_Q x_{i_\ell}$, $x_{b_\ell} <_Q x_{a_\ell}$, $x_{c_\ell} <_Q x_{d_\ell}$ and $i_\ell \in \operatorname{supp}(m_1)\setminus\operatorname{supp}(m_2)$, $a_\ell \in \operatorname{supp}(m_2)\setminus\operatorname{supp}(m_1)$ and $d_\ell \in \operatorname{supp}(m_1)\cap\operatorname{supp}(m_2)$ for all relevant ℓ. Since

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$x_{d_1}\cdots x_{d_t}|(\gcd(m_1,m_2))^2$, we can re-index, if necessary, so that for some $1 \le t' \le t-1$ we have

$$x_{d_1}\cdots x_{d_{t'}}|\gcd(m_1,m_2) \text{ and } x_{d_{t'+1}}\cdots x_{d_t}|\gcd(m_1,m_2).$$

We then have

$$\left(m_1\frac{x_{j_1}\cdots x_{j_r}}{x_{i_1}\cdots x_{i_r}}\frac{x_{c_1}\cdots x_{c_{t'}}}{x_{d_1}\cdots x_{d_{t'}}}\right) \in Q(m_1), \left(m_2\frac{x_{b_1}\cdots x_{b_s}}{x_{a_1}\cdots x_{a_s}}\frac{x_{c_{t'+1}}\cdots x_{c_t}}{x_{d_{t'+1}}\cdots x_{d_t}}\right) \in Q(m_2)$$

implying that $p \in Q(m_1)Q(m_2)$. Therefore, $Q(m_1m_2) \subseteq Q(m_1)Q(m_2)$ and we have the conclusion. ∎

We also have the following property of ideal intersections.

Lemma 2.11 *Fix a poset Q on $\{x_1,\dots,x_n\}$, and let $m_1, m_2 \in S$ be two monomials. If $A(m_1) \cap A(m_2) = \varnothing$, then $Q(m_1) \cap Q(m_2) = Q(m_1)Q(m_2)$.*

Proof It suffices to show that $Q(m_1) \cap Q(m_2) \subseteq Q(m_1)Q(m_2)$.

Note that for any monomial m, if $p \in G(Q(m))$ is a minimal generator of $Q(m)$, then $\{x_j \mid j \in \operatorname{supp}(p)\} \subseteq A(m)$. In fact, we have

$$A(m) = \bigcup_{p\in G(Q(m))} \{x_j \mid j \in \operatorname{supp}(p)\}.$$

That is, $A(m)$ is precisely the set of variables that divide at least one minimal generator of $Q(m)$.

Because $A(m_1)$ and $A(m_2)$ are disjoint, this implies that for any Q-Borel movement m' of m_1 and any Q-Borel movement m'' of m_2, $\gcd(m',m'') = 1$, and thus $\operatorname{lcm}(m',m'') = m'm''$. It then follows that

$$\begin{aligned} Q(m_1) \cap Q(m_2) &= \langle \operatorname{lcm}(m',m'') = m'm'' \mid m' \in G(Q(m_1)) \text{ and } m'' \in G(Q(m_2))\rangle \\ &= Q(m_1)Q(m_2), \end{aligned}$$

as desired.

As first shown by Francisco, *et al.* [12], the associated primes of principal Q-Borel ideals are related to order ideals of Q. Recall that for any ideal $I \subseteq S$, a prime ideal P is an *associated prime* of I if there exists an element $f \in S$ such that

$$I : \langle f\rangle = \{g \in S \mid gf \in I\} = P.$$

We denote the set of all associated primes of I by $\operatorname{ass}(I)$. We then have the following theorem.

Theorem 2.12 [12, Theorem 4.3] *Let $I = Q(m)$ for some monomial m and poset Q. Then $P \in \operatorname{ass}(I)$ if and only if*

$$P = \langle x_i \mid x_i \in A(m')\rangle$$

for some $m'|m$ with the property that $A(m')$ is connected.

Remark 2.13 As we will see in Section 4, principal Q-Borel ideals are products of prime monomial ideals, that is, all principal Q-Borel ideals are examples of ideals that

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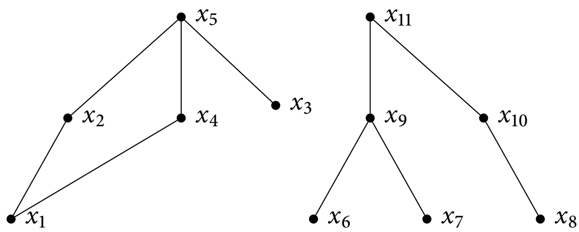
are products of ideals generated by linear forms. There are a number of papers on this topic, for example [6, 7].

In particular, the primary decomposition of principal Q-Borel ideals can also be deduced from the work of [6]. We use the statement of [12] since it relates the associated primes directly to the Hasse diagram of Q,

Example 2.14 We illustrate some of the above ideas with the following example. Let $S = \mathbb{K}[x_1, \ldots, x_{11}]$ and let Q be the poset on $\{x_1, \ldots, x_{11}\}$ with Hasse diagram:



In the above drawing, $x_i <_Q x_j$ if there is a path from x_i to x_j such that the path from x_i to x_j only moves "upward". For example $x_1 <_Q x_5$, but x_1 and x_3 are not comparable.

If we consider the monomial $m = x_4x_9^2$, then because $x_1 <_Q x_4$ and $x_4|m$, the monomial $x_1 \cdot (m/x_4) = x_1x_9^2$ is a Q-Borel move of m. The Q-Borel principal ideal $I = Q(x_4x_9^2)$ is the monomial ideal generated by all the Q-Borel moves one can obtain from $x_4x_9^2$. In particular,

$$Q(x_4x_9^2) = \langle x_1x_6^2, x_1x_6x_7, x_1x_7^2, x_1x_6x_9, x_1x_7x_9, x_1x_9^2, x_4x_6^2, x_4x_6x_7, x_4x_7^2, x_4x_6x_9, x_4x_7x_9, x_4x_9^2\rangle.$$

Observe that all the generators of $Q(x_4x_9^2)$ have degree three, as expected by Lemma 2.8.

We apply Theorem 2.12 to compute $\operatorname{ass}(Q(x_4x_9^2))$. The monomials that divide $x_4x_9^2$ are x_4, x_9, x_9^2, x_4x_9 and $x_4x_9^2$. Now $A(x_4x_9) = A(x_4x_9^2) = \{x_1, x_4, x_6, x_7, x_9\}$ is not connected, but the order ideals $A(x_4) = \{x_1, x_4\}$ and $A(x_9) = A(x_9^2) = \{x_9, x_6, x_7\}$ are. So

$$\operatorname{ass}(Q(x_4x_9^2)) = \{\langle x_1, x_4\rangle, \langle x_6, x_7, x_9\rangle\}.$$

3 The ideal containment problem for $Q(m)$

The *d-th symbolic power* of an ideal $I \subseteq S$, denoted $I^{(d)}$, is the ideal

$$I^{(d)} = \bigcap_{P \in \operatorname{ass}(I)} (I^d S_P \cap S)$$

where S_P is the ring S localized at the ideal P, and the intersection is over the set of all the associated primes of I. (The definition of symbolic powers is not uniform in

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the literature, where in some references, the indexing set is only over the minimal associated primes, as in [21, Definition 4.3.22].)

The regular d-th power of I, that is I^d, always satisfies $I^d \subseteq I^{(d)}$. Ein-Lazersfeld-Smith [10] and Hochster-Huneke [18] showed that, for every positive integer d, there is an integer $r \geq d$ such that $I^{(r)} \subseteq I^d$. The "ideal containment problem" pertains to the problem of determining, for each positive integer d, the smallest integer r such that $I^{(r)} \subseteq I^d$. In this section, we show that for any principal Q-Borel ideal, we can take $r = d$.

The following results of Cooper *et al.* [8] about symbolic powers of monomial ideals will be useful. If $I = Q_1 \cap \cdots \cap Q_s$ is a primary decomposition of the monomial ideal I, and if $P \in \operatorname{ass}(I)$, then we define

$$Q_{\subseteq P} = \bigcap_{\sqrt{Q_i} \subseteq P} Q_i.$$

That is, $Q_{\subseteq P}$ is the intersection of all the primary ideals in the primary decomposition of I such that $\sqrt{Q_i}$ is contained in P. Then, we have the following theorem.

Theorem 3.1 [8, Theorem 3.7] *The d-th symbolic power of a monomial ideal I is*

$$I^{(d)} = \bigcap_{P \in maxass(I)} Q_{\subseteq P}^d$$

where maxass(I) denotes the maximal associated primes of I, ordered by inclusion.

Thus, to compute the symbolic powers of principal Q-Borel ideals, we need to determine maxass(I). We introduce the following terminology.

Definition 3.2 Let $S = \mathbb{K}[x_1, \ldots, x_n]$ and let Q be a poset over its variables. Fix a monomial $m \in S$ and suppose that $m'|m$. We say that m' is a *maximal connected component* of m if

- $A(m')$ is connected,
- $A(m')$ is maximal with respect to inclusion, i.e., there is no other m'' that divides m such that $A(m'')$ is connected and $A(m') \subsetneq A(m'')$, and
- $m' = m_O$ with $O = A(m')$, i.e., m' is the unique monomial of Lemma 2.3.

Note that by Lemma 2.3, the maximal connected components of a monomial exist and are unique.

Remark 3.3 Using Lemma 2.3, we can give an equivalent definition of a maximal connected component in terms of the poset Q. Specifically, let m be a monomial and Q a poset as before. Let L be the lattice of divisors of m and Λ the subposet of L consisting of $\{\mu \mid A(\mu) \text{ is connected.}\}$. Then m' is a maximal connected component if and only if m' is a maximal element of Λ. This alternative viewpoint may be helpful.

Lemma 3.4 *Let $I = Q(m)$ for some monomial m and poset Q. Then $P \in$ maxass(I) if and only if $P = \langle x \mid x \in A(m') \rangle$ with m' a maximal connected component of m.*

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Proof ($\Rightarrow$) Suppose that $P \in \mathrm{maxass}(I)$. By Theorem 2.12, there exists a monomial m' such that $m'|m$, $A(m')$ is connected, and $P = \langle x_i \mid x_i \in A(m')\rangle$. We can assume that $m' = m_O$ with $O = A(m')$. If m' is not a maximal connected component of m, then there is some m'' that divides m such that the connected component $A(m'')$ properly contains $A(m')$. But since $A(m'')$ is connected, $P' = \langle x_i \mid x_i \in A(m'')\rangle$ is an associated prime of I that properly contains P, contradicting the maximality of P. We now have the desired contradiction.

($\Leftarrow$) We reverse the above argument. Let m' be a maximal connected component of m. By Theorem 2.12, there is a prime ideal $P \in \mathrm{ass}(I)$ such that $P = \langle x_i \mid x_i \in A(m')\rangle$ since $A(m')$ is connected. If P is not a maximal associated prime, then there is a prime ideal P' with $P \subsetneq P'$. But then $P' = \langle x_i \mid x_i \in A(m'')\rangle$ for some m'' such that $m''|m$ and $A(m'')$ is connected. But then $A(m') \subsetneq A(m'')$ contradicting the fact that m' is a maximal connected component of m. ■

The following lemma on distinct maximal connected components is required.

Lemma 3.5 *Let $m \in S$ be a monomial, and let m_1 an m_2 be two distinct maximal connected components of m. Then $A(m_1) \cap A(m_2) = \varnothing$.*

Proof Suppose that $y \in A(m_1) \cap A(m_2)$. Then y is path connected to every element in $A(m_1)$, and similarly, to every element in $A(m_2)$ since both $A(m_1)$ and $A(m_2)$ are connected. But then $A(\mathrm{lcm}(m_1, m_2))$ is a connected component of $A(m)$ that properly contains $A(m_1)$ and $A(m_2)$. But this contradicts the fact that $A(m_1)$ and $A(m_2)$ are maximal.

Lemma 3.6 *Let $m \in S$ be a monomial and let $m_1, \ldots, m_r$ be all the maximal connected components of m. Then $m = m_1 \cdots m_r$.*

Proof Note that by Lemma 3.5, it follows that all the supports of $m_1, \ldots, m_r$ are pairwise disjoint, so $m_1 \cdots m_r$ divides m. If $m_1 \cdots m_r$ strictly divides m, that means that there is either: (1) a variable x_j that divides m that does not divide any of $m_1, \ldots, m_r$, or (2) a variable x_j such that $x_j^d|m$ and x_j^a divides some m_i, but $a < d$. We show that neither case can happen.

If $x_j|m$, then $A(x_j) \subseteq A(m)$ and $A(x_j)$ is connected. Consider all m' such that $m'|m$, $A(x_j) \subseteq A(m')$, and $A(m')$ is connected. In addition, suppose m' is picked to be maximal with the property with respect to both inclusion and the degree of m'. But then m' would be a maximal connected component, which is a contradiction.

For case (2), suppose that $x_j^d|m$. Since the $m_1, \ldots, m_r$ have distinct support, x_j can only divide one of these monomials. After relabeling, suppose $x_j|m_1$. Suppose x_j^a with $a \geq 1$ is the largest power of x_j that divides m_1. We claim that $a = d$. Since $m_1|m$ we know $a \leq d$. If $1 \leq a < d$, then $A(m_1 x_j) = A(m_1)$ since m_1 and $m_1 x_j$ have the same support. But then m_1 is not a maximal connected component since $\deg m_1 x_j > \deg m_1$ and $m_1 x_j|m$. So case (2) cannot happen. ■

We relate the primary decomposition of $Q(m)$ with its maximal connected components.

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Lemma 3.7 *Let $m \in S$ be a monomial and let $m_1, \ldots, m_r$ be all the maximal connected components of m. Then*

$$Q(m) = Q(m_1) \cap \cdots \cap Q(m_r).$$

Furthermore, if $Q(m) = Q_1 \cap \cdots \cap Q_s$ is a primary decomposition of $Q(m)$, then

$$Q(m_i) = Q_{\subseteq \langle A(m_i) \rangle} \text{ for } i = 1, \ldots, r$$

where $\langle A(m_i) \rangle = \langle x \mid x \in A(m_i) \rangle$.

Proof By Lemma 3.6, we have $m = m_1 \cdots m_r$. By Lemmas 2.2 and 3.5, we have that $A(m_1 \cdots m_{j-1}) \cap A(m_j) = \left(\bigcup_{i=1}^{j-1} A(m_i) \right) \cap A(m_j) = \emptyset$, for $j = 2, \ldots, r$. So by repeatedly applying Lemma 2.11, we have

$$Q(m) = \prod_{i=1}^{r} Q(m_i) = \bigcap_{i=1}^{r} Q(m_i).$$

For the second claim, observe that any associated prime of $Q(m)$ is an associated prime of $Q(m_j)$ for just one j (due to Theorem 2.12 and the definition of a maximal connected component); for the same reason, any associated prime of $Q(m_i)$ is an associated prime of $Q(m)$. Since $Q(m_i)$ has just one maximal associated prime, namely, $\langle A(m_i) \rangle$, we then have $Q_{\subseteq \langle A(m_i) \rangle} = Q(m_i)$, as desired. ∎

We arrive at the main result of this section.

Theorem 3.8 *Let $I = Q(m)$ for some monomial m and poset Q. Then*

$$I^{(d)} = I^d \text{ for all } d \geq 1.$$

Proof Let $m_1, \ldots, m_r$ be the maximal connected components of m. By Lemma 3.4, $\operatorname{maxass}(I) = \{\langle A(m_i) \rangle \mid i = 1, \ldots, r\}$. By Theorem 3.1 and Lemma 3.7 we have

$$I^{(d)} = \bigcap_{i=1}^{r} Q^d_{\subseteq \langle A(m_i) \rangle} = \bigcap_{i=1}^{r} (Q(m_i))^d .$$

But, by Lemma 2.10, we have

$$\bigcap_{i=1}^{r} (Q(m_i))^d = \bigcap_{i=1}^{r} \left(Q(m_i^d) \right)$$

Because $A(m_i) = A(m_i^d)$, it follows from Lemma 3.5 that all the generators of $Q(m_i^d)$ are relatively prime with the all generators of $Q(m_j^d)$ for any $i \neq j$. Thus

$$I^{(d)} = \bigcap_{i=1}^{r} \left(Q(m_i^d) \right) = \prod_{i=1}^{r} Q(m_i^d) = Q(m^d) = Q(m)^d = I^d$$

The third and fourth equality follow from Lemma 2.10 and the fact that $m = m_1 \cdots m_r$. ∎

Theorem 3.8 allows us to compute some invariants related to the ideal containment problem. We recall these definitions (see [5] for more on the properties of these

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invariants). For a homogeneous ideal I, $\alpha(I)$ denotes the smallest degree of an element in a minimal set of homogeneous generators for I. For a graded R-module M, $\mu(M)$ denotes its minimal number of generators.

Definition 3.9 Let I be a homogeneous ideal of S.

(1) (see [3]) The *Waldschmidt constant* of I, denoted by $\widehat{\alpha}(I)$, is

$$\hat{\alpha}(I) := \lim_{s\to\infty} \frac{\alpha(I^{(s)})}{s}.$$

(2) (see [13]) The *d -th symbolic defect* of I, denoted by sdefect(I,d), as

$$\text{sdefect}(I,d) = \mu\left(I^{(d)}/I^d\right).$$

(3) (see [3]) The *resurgence* of I, denoted by $\rho(I)$, is

$$\rho(I) = \sup\left\{\frac{s}{r} \mid I^{(s)} \nsubseteq I^r\right\}.$$

Corollary 3.10 *Let $I = Q(m)$ for some monomial m and poset Q. Then*

(1) $\widehat{\alpha}(I) = \deg(m)$,
(2) *sdefect*$(I,d) = 0$ *for all* $d \geq 1$, *and*
(3) $\rho(I) = 1$.

Proof These results follow directly from the fact that $I^d = I^{(d)}$ for all $d \geq 1$. ∎

Remark 3.11 Observe that Corollary 3.10 holds for principal ideals in the regular sense, thus illustrating the theme that principal Q-Borel ideals behave like principal ideals.

Remark 3.12 For principal Q-Borel ideals $I = Q(m)$, Corollary 3.10 shows that the Waldschmidt constant is very easy to obtain from m. If we consider square-free Q-Borel ideals, it becomes much harder to determine this invariant. In a follow up paper [4], we look at the Waldschmidt constant of square-free Q-Borel ideals in the special case that Q is the chain $C : x_1 < \cdots < x_n$, or in other words, square-free Borel ideals.

4 Associated primes of powers of principal Q-Borel ideals

As noted in the introduction, studying the set of the associated primes of a power of an ideal has been of recent interest. One property that has been studied is the persistence property. Formally, an ideal I is said to have the *persistence property* if $\text{ass}(I^i) \subseteq \text{ass}(I^{i+1})$ for all $i \geq 1$. Given this interest, it makes sense to determine if principal Q-Borel ideals have this property. This short section gives two different proofs that principal Q-Borel ideals have this property.

Our first proof relies on the work of Herzog, Rauf, and Vladoiu [17]; we recall a key definition from [17].

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Definition 4.1 A monomial ideal I is a *transversal polymatroidal* ideal if

$$I = P_1 P_2 \cdots P_t$$

for prime monomial ideals $P_1, \dots, P_t$.

Lemma 4.2 *Let $I = Q(m)$ for some monomial m and poset Q. Then, I is a transversal polymatroidal ideal.*

Proof This result follows from [12, Proposition 2.7], which states that a principal Q-Borel ideal is a product of prime monomial ideals. ∎

We then have following result, which implies that principal Q-Borel ideals have the persistence property. Our first proof makes use of a property of polymatroidal ideals, while our second proof uses Lemma 2.10, and is self-contained.

Theorem 4.3 *Let $I = Q(m)$ for some monomial m and poset Q. Then we have*

$$ass(I) = ass(I^s) \text{ for all } s \geq 1.$$

First Proof By [17, Corollary 3.6], every transversal polymatroidal ideal J satisfies $\text{ass}(J) = \text{ass}(J^s)$ for all $s \geq 1$. Now apply Lemma 4.2. ∎

Second Proof By repeatedly applying Lemma 2.10, $I^s = Q(m)^s = Q(m^s)$. If $P \in \text{ass}(I)$, then by Theorem 2.12, there is a m' such $m'|m$ and $A(m')$ is connected and $P = \langle x_i \mid x_i \in A(m')\rangle$. But then $m'|m^s$ and $A(m')$ is connected, so P is also an associated prime of $I^s = Q(m^s)$.

Conversely, suppose that $P \in \text{ass}(I^s) = \text{ass}(Q(m^s))$. By Theorem 2.12, there is a monomial m' that divides m^s such that $A(m')$ is connected and $P = \langle x_i \mid x_i \in A(m')\rangle$. If $m' = x_{i_1}^{b_{i_1}} \cdots x_{i_r}^{b_{i_r}}$ with $b_{i_j} > 0$, let $m'' = x_{i_1} \cdots x_{i_r}$. Since $m'|m^s$, we have $m''|m$. Furthermore, because m' and m'' share the same support, $A(m') = A(m'')$ by Lemma 2.1. So, we have m'' divides m and $A(m'')$ is connected. So by Theorem 2.12, $P = \langle x_i \mid x_i \in A(m') = A(m'')\rangle$ is an associated prime of I, as desired. ∎

5 The analytic spread of principal Q-Borel ideals

In this section, we compute the analytic spread of principal Q-Borel ideals $Q(m)$ and square-free principal Q-Borel ideals $sfQ(m)$. In particular, this invariant is expressed in terms of the properties of the order ideal $A(m)$ viewed as an induced subposet of Q. We recall the definition of analytic spread.

Definition 5.1 Let $I \subseteq S = \mathbb{K}[x_1, \dots, x_n]$ be a homogeneous ideal, and let $\mathbf{m} = \langle x_1, \dots, x_n\rangle$. The *analytic spread* of I, denoted $\ell(I)$, is the Krull dimension of the ring

$$\mathcal{F}(I) = \bigoplus_{i \geq 0} \frac{I^i}{\mathbf{m} I^i} \quad \text{where } I^0 = S.$$

Remark 5.2 The ring $\mathcal{F}(I)$ is usually referred to as the *special fiber ring*. The special fiber ring is also isomorphic to $\mathcal{R}(I)/\mathbf{m}\mathcal{R}(I)$ where $\mathcal{R}(I) = R[It] = \bigoplus_{i \geq 0} I^i t^i \subseteq R[t]$

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is the Rees algebra of I. Roughly speaking, the analytic spread is the minimum number of generators of an ideal J that is a reduction of I (e.g., see [19, Corollary 8.2.5]).

The next lemma gives us a tool to compute $\ell(I)$ when I is generated by monomials all of the same degree.

Lemma 5.3 [20, Lemma 3.2] *Let $I = \langle x^{\alpha_1}, \ldots, x^{\alpha_r} \rangle$ be a monomial ideal and let A be the matrix with columns α_i. If $\deg x^{\alpha_i} = d$ for all i, then the analytic spread of I is*

$$\ell(I) = \operatorname{rank} A.$$

Since $I = Q(m)$ is generated by monomials of the same degree (see Lemma 2.8), to compute $\ell(Q(m))$ it is enough compute the rank of the matrix corresponding to the degrees of the generators. The rank of this matrix is encoded in $A(m)$, as we now show.

Theorem 5.4 *Let $I = Q(m)$ for some monomial m and poset Q. Then*

$$\ell(I) = |A(m)| - K(A(m)) + 1$$

where $A(m)$ is the order ideal of m and $K(A(m))$ is the number of connected components of $A(m)$ as an induced subposet of Q.

Proof We can write $I = Q(m)$ as $I = \langle x^{\alpha_1}, \ldots, x^{\alpha_r} \rangle$ where $\{x^{\alpha_1}, \ldots, x^{\alpha_r}\}$ are the minimal generators, and $m = x^{\alpha_r}$. By Lemma 2.8, the generators all have the same degree.

Let $A = \begin{bmatrix} \alpha_1 & \cdots & \alpha_r \end{bmatrix}$ be the $n \times r$ matrix where the i-th column is given by α_i. By Lemma 5.3 we need to compute $\operatorname{rank}(A)$, or equivalently, the rank of the matrix

$$A' = \begin{bmatrix} \alpha_1 - \alpha_r & \alpha_2 - \alpha_r & \cdots & \alpha_{r-1} - \alpha_r & \alpha_r \end{bmatrix}$$

because the column space of A and A' is the same.

For all $x_j \leq_Q x_i$, let $e_{(i,j)} \in \mathbb{N}^n$ denote the vector defined in (2.1). Note that x^{α_k} is the monomial obtained from $m = x^{\alpha_r}$ via a series of Q-Borel moves. In particular by Lemma 2.6 there exists vectors $e_{(i_1,j_1)}, e_{(i_2,j_2)}, \ldots, e_{(i_l,j_l)}$ with $i_t \in \operatorname{supp}(x^{\alpha_r})$ for $t = 1, \ldots, l$ such that

$$\alpha_r + e_{(i_1,j_1)} + \cdots + e_{(i_l,j_l)} = \alpha_k.$$

Thus $\alpha_k - \alpha_r \in \operatorname{Span}\{e_{(i,j)} \mid i \in \operatorname{supp}(x^{\alpha_r}) \text{ and } x_j \leq_Q x_i\}$ for any $1 \leq k \leq r-1$. Because $\alpha_r + e_{(i,j)}$ is a column of A for any $i \in \operatorname{supp}(x^{\alpha_r})$ and $x_j \leq_Q x_i$, the vectors $e_{(i,j)}$ appear as columns of A'. This implies that

$$\operatorname{rank} A' = 1 + \dim_{\mathbb{K}}(\operatorname{Span}\{e_{(i,j)} \mid i \in \operatorname{supp}(x^{\alpha_r}) \text{ and } x_j \leq_Q x_i\}).$$

where the 1 corresponds to the column corresponding to α_r.

Consider the order ideal $A(x^{\alpha_r})$ and view it as an induced poset of of Q. Let B denote the incidence matrix of the Hasse diagram associated to $A(x^{\alpha_r})$. That is, B is the matrix whose rows are indexed by the elements of $A(x^{\alpha_r})$ and whose columns are indexed by the directed edges in $A(x^{\alpha_r})$. Furthermore, in the column indexed by the edge of $A(x^{\alpha_r})$ between x_j and x_i with $x_j <_Q x_i$, we put a -1 in the row indexed by x_i and a 1 in the row indexed by j.

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It follows from the proof of [14, Theorem 8.3.1] that the kernel of B is generated by the vectors $v_C = \sum_{x_i \in C} e_i$ where C is a connected component of $A(m)$. Given that the columns of B belong to $\mathrm{Span}\{e_{(i,j)} \mid i \in \mathrm{supp}(x^{\alpha_r}) \text{ and } x_j \leq_Q x_i\}$ and the generators of this space are orthogonal to the elements in $\{v_C \mid C \text{ is a connected component of } A(m)\}$, then

$$\mathrm{Col}(B) = \mathrm{Span}\{e_{(i,j)} \mid i \in \mathrm{supp}(x^{\alpha_r}) \text{ and } x_j \leq_Q x_i\}.$$

Thus,

$$\ell(Q(m)) = \mathrm{rank}\, A = \mathrm{rank}\, A' = 1 + \mathrm{rank}\, B$$

and from [14, Theorem 8.3.1] we know $\mathrm{rank}\, B = |A(m)| - K(A(m))$ from where we obtain the desired conclusion. ■

Before considering square-free principal Q-Borel ideals, we make a brief aside to differentiate our work from that of Herzog and Qureshi [16]. As shown in [16], the analytic spread of a polymatroidal ideal [16, Definition 2.3] can be computed via the linear relation graph of the ideal.

Definition 5.5 Let $G(I) = \{m_1, \ldots, m_s\}$ be the minimal generators of a monomial ideal I. The *linear relation graph* Γ of I is the graph with edge set

$$E = \{\{i, j\} \mid \text{there exists } m_k, m_l \in G(I) \text{ such } x_i m_k = x_j m_l\}$$

and vertex set $V = \bigcup_{\{i,j\} \in E} \{i, j\}$.

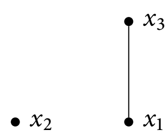
The analytic spread of a polymatroidal ideal is related to its linear relation graph.

Lemma 5.6 [16, Lemma 4.2] *Let I be a polymatroidal ideal with linear relation graph Γ. If r is the number of vertices of Γ and s is the number of connected components of Γ, then*

$$\ell(I) = r - s + 1.$$

As shown in [12, Proposition 2.9], a principal Q-Borel ideal $I = Q(m)$ is a polymatroidal ideal. Consequently, one can compute $\ell(Q(m))$ via Lemma 5.6. However, our Theorem 5.4 has the advantage of expressing the analytic spread in terms of the poset Q and order ideal $A(m)$. As the next example shows, we do not necessarily have $|A(m)| = r$ and $K(A(m)) = s$, with r and s as in Lemma 5.6.

Example 5.7 Consider $S = \mathbb{K}[x_1, x_2, x_3]$, and let our poset Q on $\{x_1, x_2, x_3\}$ have Hasse diagram



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Consider $I = Q(x_2x_3) = \langle x_1x_2, x_2x_3\rangle$. Then $|A(x_2x_3)| = 3$ and $K(A(x_2x_3)) = 2$. However, the linear relation graph Γ of I contains the single edge $\{1,3\}$ since $x_3(x_1x_2) = x_1(x_2x_3)$ is the only linear relation among the generators of I. So $r = 2$ and $s = 1$.

In light of the above example, it is natural to ask if there is any connection between $A(m)$ and the linear relation graph Γ of the principal Q-Borel ideal $I = Q(m)$. This relationship is explained in the following theorem.

Theorem 5.8 *Fix a poset Q on $X = \{x_1,\dots,x_n\}$ and take $m \in S$ a monomial. Let $I = Q(m)$ and let Γ be its linear relation graph. Consider H, the Hasse diagram of $A(m)$, but as an undirected graph; that is, the vertex set is $V(H) = A(m)$ and $\{x_i,x_j\} \in E(H)$ is an edge if $x_i <_Q x_j$ or $x_j <_Q x_i$ and there is no element $y \in X$ with $x_i <_Q y <_Q x_j$ or $x_j <_Q y < x_i$.*

Then Γ is the transitive closure of H after removing the isolated vertices of H.

Proof First, it is clear that $E(H) \subseteq E(\Gamma)$. Also $V(H) \setminus V(\Gamma)$ is precisely the set of isolated vertices of H. Now, take $\{i,j\} \in E(\Gamma)$. Then there exists $x^\alpha, x^\beta \in G(I)$ such that

$$e_i + \alpha = e_j + \beta.$$

But x^α, x^β are also Q-Borel movements of $m = x^\nu$. Then, by Lemma 2.6, there exists $i_1,\dots,i_t \in \mathrm{supp}(m)$, $j_1,\dots,j_t$ with $x_{j_k} <_Q x_{i_k}$ for $1 \le k \le t$ and $\{i'_1,\dots,i'_s\} \in \mathrm{supp}(m)$, $j'_1,\dots,j'_s$ with $x_{j'_k} <_Q x_{i'_k}$, $1 \le k \le s$, such that:

$$\nu = \alpha + \sum_{k=1}^{t} e_{(i_k,j_k)} = \beta + \sum_{k=1}^{s} e_{(i'_k,j'_k)}$$

and then

$$e_j - e_i = \sum_{k=1}^{s} e_{(i'_k,j'_k)} - \sum_{k=1}^{t} e_{(i_k,j_k)}.$$

But this means that there is a path from i to j along the vertices of H and then $\{i,j\}$ is in the transitive closure of H. ∎

Remark 5.9 The previous theorem implies that if c is the number of isolated vertices of $A(m)$, then $|A(m)| = r + c$ and $K(A(m)) = s + c$ where r and s are as in Lemma 5.6. Using the fact that a principal Q-Borel ideal is a polymatroidal ideal, we could then use Lemma 5.6 and Theorem 5.8 to give a different proof of Theorem 5.4. In particular, if Γ is the linear relation graph of $Q(m)$, we have

$$\ell(Q(m)) = r - s + 1 = (r+c) - (s+c) + 1 = |A(m)| - K(A(m)) + 1.$$

Our proof of Theorem 5.4 avoids using the polymatroidal property.

Our analysis of the square-free principal Q-Borel case is similar to the principal Q-Borel case. We require the following notation. Suppose that Q is a poset on $X =$

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$\{x_1, \ldots, x_n\}$. If $Y = \{x_{j_1}, \ldots, x_{j_s}\}$ is a subset of X, then Q induces a poset Q' on Y if we define $x_j <_{Q'} x_i$ if $x_j <_Q x_i$.

If m is a monomial only in the variables of Y, then we write $A_Q(m)$ or $A_{Q'}(m)$ if wish to view the order ideal in Q on the set X or in Q' on the set Y. Similarly, we write $sfQ(m)$ or $sfQ'(m)$, and $Q(m)$ and $Q'(m)$ if we wish to denote which partial order and ground set we are using.

Theorem 5.10 *Fix a poset Q on $X = \{x_1, \ldots, x_n\}$, and suppose that $m \in S$ is a square-free monomial. Let $m' = gcd(G(sfQ(m)))$ be the greatest common divisor of all the generators of the square-free principal Q-Borel ideal $I = sfQ(m)$. Then,*

$$\ell(I) = \ell(Q'(m/m')) = |A_{Q'}(m/m')| - K(A_{Q'}(m/m')) + 1,$$

where Q' is the induced poset on $Y = X \setminus \{x_j \mid j \in supp(m')\}$.

Proof Let $m = x^\alpha = x_{i_1}x_{i_2}\cdots x_{i_s}$ and $m' = x^\delta = x_{j_1}\cdots x_{j_t}$. Since m' is the greatest common divisor of all the generators, $m'|m$. Furthermore, suppose $x_j|m'$, and thus $x_j|m$. If $x_k <_Q x_j$, then $\frac{x_k}{x_j}m \notin I$ because otherwise we would have a generator of I not divisible by x_j. If $x_j <_Q x_i$ and $x_i|m$, then $\frac{x_j}{x_i}m$ is a Q-Borel move of m, but it is not in I since this monomial is not square-free. Thus, $x \in A(m')$ implies that $A(x) = \{x\}$ or for any $y \in Q \setminus \{x\}$ comparable to x, the corresponding Q-Borel movement is not in $sfQ(m)$.

We first consider the case that $m' = 1$. Note that this means that every x_i that divides m is not a minimal element of $A(m)$. Indeed, if x_i is a minimal element, then x_i would appear in every generator of I, contradicting the fact $m' = 1$.

Set $I = sfQ(m)$ and $J = Q(m)$. Let A be the matrix whose column entries have the form β where x^β is a generator of I, and similarly, let B be the matrix whose columns have the form γ where x^γ is a generator of J. By Lemma 5.3 and Theorem 5.4 we have

$$\ell(I) = \text{rank}(A) \le \text{rank}(B) = \ell(J) = |A(m)| - K(A(m)) + 1.$$

The inequality follows from the fact that all of the columns of A are in B.

Fix any $i \in \text{supp}(m)$ (and thus, $x_i|m$), and suppose $x_j <_Q x_i$. If $x_j \nmid m$, then $\frac{x_j}{x_i}m \in G(I)$, and therefore $\alpha + e_{(i,j)}$ is a column of A. Since α is a column of A, we have $e_{(i,j)} \in \text{Col}(A)$. If x_j also divides m, then there exists a minimal element $x_k <_Q x_j$. Since x_k is minimal, our hypotheses imply that $x_k \nmid m$. But then

$$\alpha + e_{(i,k)} \text{ and } \alpha + e_{(j,k)}$$

are columns of A, and then $e_{(i,k)} - e_{(j,k)} = e_{(i,j)}$ is in $\text{Col}(A)$. But from Theorem 5.4 we have

$$\text{Col}(B) = \text{Span}(\{\alpha\} \cup \{e_{(i,j)} \mid i \in \text{supp}(m), x_j \le_Q x_i\}) \subset \text{Col}(A).$$

Consequently, $\text{rank}(B) \le \text{rank}(A)$, giving the desired result.

Now suppose that $m' = x^\delta = x_{j_1}\cdots x_{j_k} > 1$. Since every generator of $sfQ(m)$ is divisible by m', we have

$$I = sfQ'(m) = m' \cdot sfQ'(m/m').$$

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If A, respectively B, is the matrix whose columns have the form γ with x^γ a generator of $sfQ'(m)$, respectively, $sfQ'(m/m')$, we can use Lemma 5.3 and the proof of Theorem 5.4 to show that $\operatorname{rank}(A) = \operatorname{rank}(B)$; in particular, one needs to verify

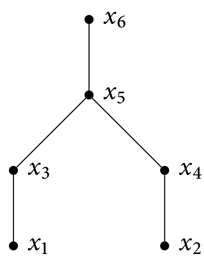
$$\begin{aligned}\dim \operatorname{Col}(A) &= \dim \operatorname{Span}(\{\delta+\beta\} \cup \{e_{(i,j)} \in \mathbb{N}^{|Q|} \mid i \in \operatorname{supp}(m/m') \text{ and } x_j \leq_{Q'} x_i\}) \\ &= \dim \operatorname{Span}(\{\beta\} \cup \{e_{(i,j)} \in \mathbb{N}^{|Q'|} \mid i \in \operatorname{supp}(m/m') \text{ and } x_j \leq_{Q'} x_i\}) \\ &= \dim \operatorname{Col}(B)\end{aligned}$$

where $m = x^{\delta+\beta}$ and $m/m' = x^\beta$. Consequently

$$\ell(I) = \operatorname{rank}(A) = \operatorname{rank}(B) = \ell(sfQ'(m/m')) = \ell(Q'(m/m'))$$

where the last equality follows from the first part of the proof. ∎

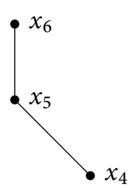
Example 5.11 We illustrate the above result. Let Q be the poset with Hasse diagram



Let $m = x_1x_2x_3x_6$ and $I = sfQ(m)$, and thus

$$I = \langle x_1x_2x_3x_6, x_1x_2x_3x_5, x_1x_2x_3x_4\rangle.$$

We have $\gcd(G(I)) = x_1x_2x_3$. Therefore Q' is the poset on $\{x_4, x_5, x_6\}$ with Hasse diagram:



Hence, $\ell(I) = \ell(Q'(x_6)) = 3$.

Remark 5.12 It can be shown that $m' = \gcd(G(I))$ in Theorem 5.10 is the largest monomial (by degree) that divides m such that

$$\{x_j \mid j \in \operatorname{supp}(m')\} = A(m').$$

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That is, the variables that divide m' form an order ideal. Returning to the above example, note that $\{x_j \mid j \in \text{supp}(x_1x_2x_3)\} = \{x_1, x_2, x_3\} = A(x_1x_2x_3)$ in the poset Q. Note that if no such monomial exists, we use the convention that $A(1) = \varnothing$.

Using the above interpretation of m', we have the following corollary, which uses the following terminology. Given a poset Q on $\{x_1, \ldots, x_n\}$, the *minimal elements* of $\{x_1, \ldots, x_n\}$ are those x_i that are minimal with respect to the partial order on Q.

Corollary 5.13 *Fix a poset Q and suppose that $m \in S$ is a square-free monomial. Suppose $\{x_j \mid j \in supp(m)\}$ contains no minimal elements of Q. If $I = sfQ(m)$, then*

$$\ell(I) = \ell(Q(m)) = |A(m)| - K(A(m)) + 1.$$

Proof No subset of $\{x_j \mid j \in \text{supp}(m)\}$ is an order ideal in Q. So $m' = \text{gcd}(G(I)) = 1$. Now, apply Theorem 5.10. ∎

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