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REAL ORDER-AUTOMORPHISM GROUPS

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Abstract

By using the concept of tame embeddings of chains, a characterization is given of the subobjects of the lattice-ordered groups of order-automorphisms of the chains of rational and real numbers.

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1. Introduction

For a chain (totally ordered set) T, let $\underline{A}(T)$ denote the l-group (lattice-ordered group) of o-automorphisms (order-automorphisms) of T. Each l-group is embeddable in $\underline{A}(T)$ for some chain T (Holland (1963)).

In Weinberg (1978) a characterization is given of the l-groups which are embeddable in $\underline{A}(H_\alpha)$, where H_α is a minimal η_α-set. Since $H_0 = Q$, the chain of rational numbers, this describes the sub-l-groups of $\underline{A}(Q)$. The purpose of this note is to characterize the sub-l-groups of $\underline{A}(R)$, where R is the chain of real numbers, and to contrast them with the sub-l-groups of $\underline{A}(Q)$. The characterization of the latter will be repeated without the unnecessary details concerning η_α-sets.

The notion of tame embeddings of chains plays a key role in the discussion. A subset S of a chain T is said to be *tamely embedded* in T if every o-automorphism of S extends to an o-automorphism of T. For example, every chain is tamely embedded in its Dedekind completion and in its chain of ideals. The *lexicographic product* of chains is defined as the cartesian product ordered by the principle of last differences. If S and T are nonempty chains, then each is tamely embeddable in the lexicographic product $S \times T$. Thus any (nonempty) countable chain T is tamely embeddable in Q, the latter being o-isomorphic to $Q \times T$. The reals have subchains which are not tamely embedded, although, as we will see, any chain that can be embedded in the reals can be reembedded tamely.

1.1 Lemma. *Let S be a subchain of T, and let G be a sub-l-group of $\underline{A}(S)$. If every element of G can be extended to an o-automorphism of T, then the extensions can be chosen so as to embed G as a sub-l-group of $\underline{A}(T)$.*

Proof. Consider the family $\underline{I}$ of nonempty convex subsets of T which are maximal with respect to the property of missing S. Order-isomorphism is an equivalence relation on this family. For each equivalence class choose a transitive family of o-isomorphisms $\varphi_{J,I}\colon I \to J$, one for each ordered pair of members of the class. Define an l-monomorphism $\psi\colon G \to \underline{A}(T)$ as follows. Let $g \in G$. Since g extends to an o-automorphism of T, it induces an equivalence-preserving permutation of the family $\underline{I}$ of intervals. Let $\psi(g)$ be the unique extension of g to T which agrees with $\varphi_{g(I),I}$ for each interval I. That ψ is a group monomorphism follows from the transitivity of the families $\{\varphi_{J,I}\}$. To see that ψ is an l-monomorphism it suffices to observe that it preserves disjointness.

1.2. Theorem (Weinberg (1967), Lemma 1). *If S is tamely embedded in the chain T, then the l-group $\underline{A}(S)$ can be embedded in $\underline{A}(T)$ by extending o-automorphisms.*

2. The characterizations

A family of prime subgroups of an l-group is called *dense* if its intersection is the identity; the family is called *representing* if the intersection contains no nontrivial l-ideals. Any representing family $\underline{P}$ of prime subgroups of G induces an embedding of G in the product of the l-groups $\underline{A}(G/P)$, $P \in \underline{P}$. Let $\lessdot$ be any total order of $\underline{P}$, and let $T = \bigcup_{P \in \underline{P}} G/P$ be the lexicographic union: $gP_1 \leqslant hP_2$ exactly when $P_1 \lessdot P_2$, or $P_1 = P_2$ and $gP_1 \leqslant hP_1$. This induces the natural Holland embedding of G in $\underline{A}(T)$: $g \to \bar{g}$ where $\bar{g}(hP) = ghP$ (see Holland (1963)).

We say that a prime subgroup P of G has an order-theoretic property *residually* if the chain G/P of residue classes has the property.

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2.1. THEOREM. *Let G be an l-group. The following are equivalent.*

(a) *G is embeddable in $\underline{A}(Q)$.*

(b) *G has a countable dense family of prime subgroups which are residually countable.*

(c) *G has a countable representing family of prime subgroups which are residually countable.*

PROOF. Suppose that G is a sub-l-group of $\underline{A}(Q)$. For each $t \in Q$, consider the stabilizer subgroup G_t of elements of G which fix t. There are at most countably many such subgroups, each is prime, their intersection is the identity, and each chain G/G_t, being isomorphic to the orbit G_t of t in Q, is countable. Thus (b) holds. Clearly, (b) implies (c).

To see that (c) implies (a), totally order $\underline{P}$, the given family of prime subgroups. Let $T = \bigcup_{P \in P} G/P$ be the lexicographic union. Then G is contained in $\underline{A}(T)$. Since T is countable, T is tamely embeddable in Q, so $\underline{A}(T)$ and G are embeddable in $\underline{A}(Q)$.

As a simple consequence we can identify many l-groups which are not embeddable in $\underline{A}(Q)$. For example the o-group R of real numbers, as well as any of its uncountable subgroups, and hence $\underline{A}(R)$, is not embeddable in $A(Q)$. In particular $\underline{A}(R)$ and $\underline{A}(Q)$ are not l-isomorphic. The latter fact was first obtained by Holland (1963), p. 408, by proving that $\underline{A}(R)$ is o-complete but $\underline{A}(Q)$ is not, while Glass (1976), 1.9.11, obtained the result as a corollary to the study of automorphisms of $\underline{A}(R)$. Gurevich and Holland have proved that if $\underline{A}(T)$ is transitive and satisfies the same group sentences as $\underline{A}(R)$, then T is isomorphic as a chain to R. Hence $\underline{A}(Q)$ and $\underline{A}(R)$ are not isomorphic as groups. Michele Jambu (unpublished) has recently obtained the analogous result for lattices.

The family of sub-l-groups of $\underline{A}(Q)$ is however quite large, containing all countable l-groups and being closed under countable products and sub-l-groups. Since R is contained in an l-homomorpc image of a countable product of copies of Q, the family is not closed under l-homomorphic images. However, R is embeddable in $\underline{A}(Q)$ as a subgroup and (observation by Gurevich) is also embeddable as a sublattice. Indeed, $\underline{A}(Q)$ contains as a sub-l-group the product of denumerably many copies of the rationals. As a group the product is isomorphic to R, while, as a lattice, it contains a copy of R.

A subset S of a chain T is called *order-dense* if each element of T is the supremum of the set of its lower bounds in S, and dually. Call T *order-separable* if it has a countable order-dense subset. For example, of the lexicographic products $R \times Q$ and $Q \times R$, the first is order-separable, but the second is not.

It is well-known that order-separability characterizes the subchains of R (Sierpinski, p. 223, Ex. 5). The fact that it characterizes tamely embeddable subchains is needed.

2.2. LEMMA. *Let T be a chain. The following are equivalent.*

(a) *T is tamely embeddable in R.*

(b) *T is embeddable in R.*

(c) *T is order-separable.*

PROOF. Any tamely embeddable chain is embeddable. Suppose that T is embeddable in R. Then T is separable with respect to the subspace topology and hence with respect to the open interval topology which is weaker. Adjoin to the consequent countable dense set those points which are isolated on the right or the left. The resulting subset is countable and order-dense in T.

Finally, suppose that S is a countable order-dense subset of T. Any point of T which is isolated on the right or the left belongs to S. By an inductive construction, alternately putting new points to the right of points which are isolated on the right, and, dually, putting new points to the left of points which are isolated on the left, we can tamely embed T in an order-separable chain W which has a countable order-dense subset V with no one-sided isolated points. Then V is o-isomorphic to Q, so the Dedekind completion of W is R, in which it is tamely embedded. Thus, T is tamely embeddable in R.

2.3. THEOREM. *Let G be an l-group. The following are equivalent.*

(a) *G is embeddable in $\underline{A}(R)$.*

(b) *G has a countable dense family of prime subgroups which are residually order-separable.*

(c) *G has a countable representing family of prime subgroups which are residually order-separable.*

PROOF. We mention only (c) implies (a). Let $\{P_n\}$ be the hypothesized family of prime subgroups. Since, for each n, the chain G/P_n is order-separable, the lexicographic union $T = \bigcup_n G/P_n$ is order-separable. Hence $\underline{A}(T)$ and G are embeddable in $\underline{A}(R)$.

As a consequence we can see, for example, that the lexicographic product $R \times R$, and hence also $\underline{A}(R \times R)$ is not a sub-l-group of $\underline{A}(R)$. Indeed, in any dense family of prime subgroups of $R \times R$, one is (0), but the underlying ordered set of the lexicographic product is not order-separable. Thus 2.3(c) is not satisfied by $R \times R$. By induction it can be shown that the lexicographic product R^{n+1}, and hence also $\underline{A}(R^{n+1})$, cannot be embedded in $\underline{A}(R^n)$. Arguing as above, it suffices to show that the chain R^{n+1} is not embeddable in the chain R^n. The latter is a consequence of a theorem of Richard Arens (1951).

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