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Equicontinuous geodesic flows

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Abstract. This article is about the interplay between topological dynamics and differential geometry. One could ask how much information about the geometry is carried in the dynamics of the geodesic flow. It was proved in Paternain [Expansive geodesic flows on surfaces. *Ergod. Th. & Dynam. Sys.* **13** (1993), 153–165] that an expansive geodesic flow on a surface implies that there exist no conjugate points. Instead of considering concepts that relate to chaotic behavior (such as expansiveness), we focus on notions for describing the stability of orbits in dynamical systems, specifically, equicontinuity and distality. In this paper we give a new sufficient and necessary condition for a compact Riemannian surface to have all geodesics closed; this is the idea of a P-manifold: (M, g) is a P-manifold if and only if the geodesic flow $SM \times \mathbb{R} \to SM$ is equicontinuous. We also prove a weaker theorem for flows on manifolds of dimension three. Finally, we discuss some properties of equicontinuous geodesic flows on non-compact surfaces and on higher-dimensional manifolds.

1. *Introduction*

Throughout this paper, all geodesics are parametrized by arc length and the geodesic flow is complete; the manifolds and Riemannian metrics are all assumed to be C^∞. We let $\pi : TM \to M$ denote the canonical projection. To begin with, we summarize some facts about recurrent maps and state Theorem 2.7 (due to Boris Kolev and Marie-Christine Pérouème) concerning the set of fixed points of recurrent maps on surfaces. In §3, we study the geodesic return map; then, in §4, we prove that equicontinuous geodesic flows must be periodic. In §5, we prove that if a flow without singularities on a three-dimensional manifold admits a global Poincaré section and has sufficiently many periodic orbits, then the flow is pointwise periodic. In the final section, we show that the existence of an equicontinuous geodesic flow on a compact manifold M implies that the fundamental group is finite. Moreover, we discuss equicontinuous geodesic flows on non-compact manifolds.

2. *Recurrent behavior*

Definition 2.1. A dynamical system (X, T) is called distal if $\inf\{d(xt, yt) \mid t \in T\} = 0$ implies $x = y$.

A system (X, T) is called equicontinuous (regular) if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for all x, y with $d(x, y) < \delta(\epsilon)$, we have $d(xt, yt) < \epsilon$ for all $t \in T$.

Here is a well-known fact about equicontinuous systems on compact metric spaces.

THEOREM 2.2. *An equicontinuous flow $\Phi : X \times \mathbb{R} \to X$ on a compact metric space is uniformly almost periodic, that is, for every $\epsilon > 0$ there exists a $\tau > 0$ such that in every interval I of length τ there is a $t \in I$ with $d(\Phi(t, x), x) < \epsilon$ for all x.*

Proof. See [**1**, Theorem 2.2]. □

A weaker form of uniform almost periodicity for maps is defined as follows.

Definition 2.3. A continuous map f on a metric space (X, d) is recurrent if there exists a sequence $n_k \to \infty$ such that $\sup_{x \in X} d(f^{n_k}(x), x) \to 0$ as $k \to \infty$.

Definition 2.4. A continuous map $f : X \to X$ on a metric space (X, d) is said to be *paracompact-recurrent* on $Y \subset X$ if there exists a sequence $n_k \to \infty$ such that $\sup_{x \in C} d(f^{n_k}(x), x) \to 0$ as $k \to \infty$, where $C \subset Y$ is any compact subset of X.

Note that in the definition of paracompact-recurrence, the sequence n_k is fixed and does not depend on $C \subset Y$; note also that paracompact-recurrence and recurrence are independent of the metric which defines the topology if the space X is compact.

LEMMA 2.5. *If f is recurrent, then f^m is recurrent.*

Proof. Set $s_k := \sup_{x \in X} d(f^{n_k}(x), x)$. Then

$$\begin{aligned} d(f^{n_k m}(x), x) &\leq \sum_{i=0}^{m-1} d(f^{(m-i)n_k}(x), f^{(m-i-1)n_k}(x)) \\ &\leq \sum_{i=0}^{m-1} d(f^{n_k}(f^{(m-i-1)n_k}(x)), f^{(m-i-1)n_k}(x)) \leq m s_k. \end{aligned}$$ □

LEMMA 2.6. *Let f be a continuous map on a compact surface S, and let F be a finite non-empty subset of* $\mathrm{Fix}(f)$ *(the fixed point set). If f is paracompact-recurrent on $S - F$, then f is recurrent.*

Proof. Without loss of generality we can suppose that our metric is induced by a Riemannian metric. Write $F = \{x_1, \ldots, x_m\}$; then choose a very small $\epsilon > 0$ and define

$$C := S - \left(\bigcup_i B(x_i, \epsilon) \right).$$

Each $c_i = \partial B(x_i, \epsilon)$ defines a simple closed curve. Choose a $K := K(\epsilon)$ such that for all $k > K$ we have

$$\sup_{x \in C} d(f^{n_k}(x), x) < 4\epsilon \quad \text{and} \quad f^{n_k}(c_i) \subset B\left(c_i, \frac{\epsilon}{2}\right).$$

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This is possible because there is a compact subset in $S - F$ that contains c_i and C.

Hence, $f^{n_k}(B(x_i, \epsilon)) \subset B(x_i, 2\epsilon)$, and therefore $d(f^{n_k}(x), x) < 4\epsilon$ for all $x \in C^c$ and $k > K$. Consequently, $d(f^{n_k}(x), x) < 4\epsilon$ for all $x \in X$ and $k > K$, i.e. f is recurrent. □

The following nice result is [**4**, Theorem 1.1].

THEOREM 2.7. *A non-trivial orientation-preserving and recurrent homeomorphism of the sphere S^2 has exactly two fixed points.*

3. *The geodesic return map*

A well-known tool for studying geodesic flows on a surface is the geodesic return map. Let A denote the open annulus.

Suppose that we have an equicontinuous geodesic flow Φ on the unit tangent bundle SM of an orientable Riemannian surface M. Let γ be a simple closed geodesic, and write $W = SM|\gamma - T\gamma$ (the set of all unit tangent vectors based on γ which are not elements of $T\gamma$). Note that W is homeomorphic to the union of two open annuli A_0 and A_1. We can identify $v \in A_0$ with (x, θ), where $\pi(v) = x$ and $\theta \in (0, 1)$ is the angle between v and $\dot{\gamma}(t)$ divided by π.

Since every orbit is recurrent, for our flow Φ we can define a map $F : A_0 \to A_0$ by

$$F(x, \theta) = (x_0, \theta_0),$$

where $x_0 = \pi(\Phi_{t_0}(x, \theta))$ is the next intersection point of $\{\pi(\Phi_t(x, \theta)) \mid t > 0\}$ with γ such that $\Phi_{t_0}(x, \theta) = (x_0, \theta_0) \in A_0$. For simplicity we will just write $F : A \to A$. The map F can be extended to a homeomorphism of S^2 by two-point compactification. In this case, F has two fixed points, $\{\infty\}$ and $\{-\infty\}$, as we shall see. If, for $v = (x, \theta)$, we have θ close to zero, then the geodesic $\Phi_t(v)$ stays near $\dot{\gamma}$ (by equicontinuity) and hence $F^n(x, \theta)$ is close to zero for all n; thus, $\{\infty\}$ and $\{-\infty\}$ are fixed points. In this paper, we call the extension of F the *geodesic return map* and denote it by $F : S^2 \to S^2$.

PROPOSITION 3.1. *If Φ is equicontinuous, then F is recurrent on S^2.*

We need only show that F is paracompact-recurrent on $S^2 - (\{\infty\} \cup \{-\infty\})$, since we can then apply Lemma 2.6.

For $0 < \theta_0 < \theta_1 < 1$, we set

$$K(\theta_0, \theta_1) = \{(x, v) \in A \mid \theta_0 \le v \le \theta_1\}.$$

LEMMA 3.2. *For $K := K(\theta_0, \theta_1)$ and large $N := N(\theta_0, \theta_1)$, the constant*

$$s(\theta_0, \theta_1, N) := \inf\{t_1 - t_0 \mid -N < t_0 < t_1 < N,\ \Phi_{t_0}(v) \in A,\ \Phi_{t_1}(v) \in A,\ v \in K\}$$

is strictly positive. If $v \in K$, then every geodesic γ_v intersects γ at least twice in the forward direction on the interval $[0, N]$ *and at least twice in the backward direction on the interval* $[-N, 0]$.

Proof. Use the compactness of K. □

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LEMMA 3.3. *For every $s(\theta_0, \theta_1, N)/2 > r > 0$, large N and $K(\theta_0, \theta_1)$, there exists a constant $\nu(\theta_0, \theta_1, N, r) > 0$ such that the following holds.*

If, for some $w \in SM$, we have a $v \in K(\theta_0, \theta_1)$ with $d(v, w) < \nu(\theta_0, \theta_1, N, r)$, then there is a unique t_w such that $|t_w| < r$ and $\Phi_{t_w}(w) \in A$.

Proof. Use the compactness of K. □

Given $v \in A$, for each integer n we define $t(n, v)$ as the unique element of $\mathbb{R}$ such that $\Phi_{t(n,v)}(v) = F^n(v)$ and $t(0, v) = 0$. Moreover, given $v \in A$ and $t \geq 0$, we define

$$P(v, t) := \max\{n \geq 0 \mid t(n, v) \leq t\}.$$

Note that we can find a neighbourhood U_0 of γ that looks like a strip (because M is orientable); therefore $U_0 - \gamma$ is the distinct union of two open strips S^* and S_*. In $U_0 - \gamma$, we can define what lies 'above' and what lies 'below' γ: the region where the points of A are directed inward is defined to be S^* ('above'), while the other points of $U_0 - \gamma$ are said to be lying in S_* ('below').

For $j \in \{0, 1\}$ choose sequences $(\theta_{j,i})_i$, with

$$0 < \theta_{0,i+1} < \theta_{0,i} < \theta_{1,i} < \theta_{1,i+1} < 1,$$

which converge strictly to j. Set $K_i = K(\theta_{0,i}, \theta_{1,i})$ and fix a $v_0 \in \bigcap_i K_i$.

LEMMA 3.4. *There exist sequences $T_i \to \infty$, $0 < \zeta_i \to 0$ and $0 < \beta_i \to 0$ such that:*

(1) $\pi(\Phi_{T_i+\zeta_i}(v)) \in S^*$ *for all* $v \in K_i$;

(2) $\pi(\Phi_{T_i-\zeta_i}(v)) \in S_*$ *for all* $v \in K_i$;

(3) $\Phi_{T_i}(v_0) \in A$;

(4) $d(\Phi_{T_i+s}(v), v) < \beta_i$ *for all* $v \in K_i$ *and* $|s| < 2\zeta_i$;

(5) *for each* $v \in K_i$ *there exists a unique intersection point of* γ *and* $\pi(\Phi_{T_i+s}(v))$, *where* s *ranges over* $|s| < 4\zeta_i$.

Proof. Choose sequences $0 < \delta_i \to 0$, $0 < s_i \to 0$ and $\alpha_i \to 0$ such that if $v \in K_i$ and $w \in SM$ with $d(v, w) < \delta_i$, then:

(i) $\pi(\Phi_{s_i}(w)) \in S^*$;

(ii) $\pi(\Phi_{-s_i}(w)) \in S_*$;

(iii) $d(\Phi_t(w), v) < \alpha_i$ for all $|t| \leq 8s_i$;

(iv) $\pi(\Phi_t(w))$ has a unique intersection point with γ, where t ranges over $|t| \leq 8s_i$;

(v) $\pi(\Phi_{[-20s_i,20s_i]}(w)) \subset U_0$.

This can be done easily. First, choose an increasing sequence N_i such that $s(\theta_{0,i+1}, \theta_{1,i+1}, N_i)$ (and hence also $s(\theta_{0,i+1}, \theta_{1,i+1}, N_{i+1})$) can be defined. Then, choose a decreasing sequence $s_i \to 0$ such that

$$32s_i < \min\left\{\frac{s(\theta_{0,i+1}, \theta_{1,i+1}, N_i)}{2}, \frac{s(\theta_{0,i}, \theta_{1,i}, N_i)}{2}\right\},$$

followed by another decreasing sequence $\delta_i > 0$ such that

$$\delta_i < \nu\left(\theta_{0,i}, \theta_{1,i}, N_i, \frac{s_i}{2}\right).$$

Furthermore, ensure that δ_i is chosen small enough so that $d(w, K_i) < \delta_i$ implies that $\Phi_{t_w}(w)$ lies in K_{i+1} (this is the same t_w as was given by Lemma 3.3). Since $s_i, \delta_i \to 0$ and our flow Φ is defined on a compact space, α_i is guaranteed to exist and, by considering only large i, we have $\pi(\Phi_{[-20s_i,20s_i]}(w)) \subset U_0$. If $d(v, w) < \delta_i$ and $v \in K_i$, then (i), (ii) and (v) hold. Next, we check that (iv) holds.

Note that for $d(v, w) < \delta_i$ and $v \in K_i$, we have $\Phi_{t_w}(w) \in K_{i+1}$ and

$$[-8s_i, 8s_i] \subset [-8s_i - t_w, 8s_i + t_w] \subset [-9s_i, 9s_i];$$

therefore $\Phi_{[-8s_i,8s_i]}(w)$ lies in $\Phi_{[-9s_i,9s_i]}(\Phi_{t_w}(w))$.

Hence we can conclude that (iv) holds, since $18s_i < s(\theta_{0,i+1}, \theta_{1,i+1}, N_i)$.

Now choose a sequence $S_i \to \infty$ such that $d(\Phi_{S_i}(x), x) < \delta_i$ for all x (apply Theorem 2.2). There exists an unique $|r_i| < s_i$ such that $\Phi_{S_i+r_i}(v_0) \in A$. Set $T_i := S_i + r_i$ and $\zeta_i := 2s_i$. Given $v \in K_i$, note that

$$T_i + \zeta_i = S_i + r_i + 2s_i > S_i + s_i$$

and $|T_i + \zeta_i - S_i| < 3s_i$. From this together with properties (i), (iv) and (v), we conclude that $\Phi_{S_i+s_i}(v) \in S^*$ and therefore $\Phi_{T_i+\zeta_i}(v) \in S^*$. Analogously, we can deduce from properties (ii), (iv) and (v) that $\Phi_{T_i-\zeta_i}(v) \in S_*$. Thus, statements (1), (2) and (3) of the lemma are proven.

If $|t| \leq 2\zeta_i = 4s_i$, then $|T_i + t - S_i| < 5s_i$; hence the orbit segment $\Phi_{[T_i-2\zeta_i,T_i+2\zeta_i]}(v)$ lies in the orbit segment $\Phi_{[S_i-8s_i,S_i+8s_i]}(v)$ and therefore, by (iv), has an unique intersection point with γ. This proves statement (5) of the lemma. Since $2\zeta_i \to 0$, $\Phi_{T_i}(x) \to x$ uniformly and M is compact, there exists a sequence β_i. □

Proof of Proposition 3.1. Set $p_i := P(v_0, T_i + \zeta_i)$. We will show that, for this sequence, F is paracompact-recurrent. Note that $P(v_0, T_i + \zeta_i) = P(v_0, T_i)$ by Lemma 3.4 and that $K_i \supset K_j$ if $j \leq i$. Given any compact set $C \subset A$, choose I_0 such that $C \subset K_{I_0}$. The functions

$$G_i : A \to \mathbb{N}$$

defined by $G_i(v) = P(v, T_i + \zeta_i)$ are constant on K_{I_0} if $i \geq I_0$.

Indeed, by construction, the functions G_i are locally constant on K_i and hence equal to the constant $p_i = P(v_0, T_i + \zeta_i)$ on the connected set K_i. Therefore, $G_i = p_i$ on K_{I_0} if $i \geq I_0$.

We have $|T_i - t(v, G_i(v))| < 2\zeta_i$ for $v \in K_i$ by construction, since we know from Lemma 3.4(1)–(2) that $P(v, T_i - \zeta_i) = p_i - 1$. Therefore, we conclude from Lemma 3.4(4) that for all $v \in K_{I_0}$ and $i \geq I_0$, we have $d(F^{p_i}(v), v) = d(\Phi_{t(v,p_i)}(v), v) < \beta_i$. □

4. *Equicontinuous geodesic flows on surfaces*

Definition 4.1. (M, g) is called a *P-manifold* if all geodesics are closed.

The following lemma is easy to prove.

LEMMA 4.2. *If (M, g) is a P-manifold, then the geodesic flow (SM, Φ) is equicontinuous.*

Proof. It is a well-known fact that if M is a P-manifold then the flow is periodic (see, for instance, [**8**]) and M is compact. Let L denote the smallest period. For $\epsilon > 0$, choose a $\delta > 0$ such that $d(v, w) < \delta$ implies

$$d(\Phi_t(v), \Phi_t(w)) < \epsilon$$

for $|t| < 2L$; then the above holds for all t. □

To prove our first theorem, we need the following result.

THEOREM 4.3. (Ballmann) *Every compact Riemannian manifold (M, g) of dimension two has at least three simple closed geodesics.*

Proof. See [**2**]. □

THEOREM 4.4. *Given a compact Riemannian manifold (M, g) of dimension two, the following conditions are equivalent.*
(1) *M is a P-manifold.*
(2) *(SM, Φ) is equicontinuous.*

(1) implies (2) by Lemma 4.2. We will show that (2) implies (1). Take a simple closed geodesic γ.

LEMMA 4.5. *If Z is a compact set in $M - \gamma$, then there are $0 < \theta_{0,Z} < \theta_{1,Z} < 1$ such that for every geodesic α intersecting Z, $\Phi(Z \times \mathbb{R}) \cap A$ is a subset of $K(\theta_{0,Z}, \theta_{1,Z})$.*

Proof. Choose an open set V around γ such that $V \cap Z = \emptyset$. Choose $\delta > 0$ such that $d(w, T\gamma) < \delta$ implies $\gamma_w \subset V$. If there were no $\theta_{0,Z}$ or $\theta_{1,Z}$, we would be able to find a geodesic $\gamma_w \subset V$ but starting in Z. □

LEMMA 4.6. *Every geodesic intersects γ.*

Proof. Given a point $x \in M - \gamma$ and $w \in S_x M$, choose a path-connected compact set C in SM such that $S_x M \subset C$ and $\pi(C) \cap \gamma = \emptyset$. Apply Lemma 4.5 to $Z = \pi(C)$. Choose a curve β in C from $w \in S_x M$ to $q \in S_x M$ where γ_q intersects γ. Cover β with finitely many (say n) balls B_i such that for any two vectors $p, u \in B_i$ we have, for a large N and all t,

$$d(\Phi_t(u), \Phi_t(p)) < \nu(\theta_{0,Z} - \epsilon, \theta_{1,Z} + \epsilon, N, b),$$

where $b = s(\theta_{0,Z} - \epsilon, \theta_{1,Z} + \epsilon, N)/4$ and ϵ is small. By induction, we conclude that if $\gamma_q(T_0) \in \gamma$, then $\gamma_w([T_0 - (n+1)b, T_0 + (n+1)b])$ intersects γ. □

Proof of Theorem 4.4. Since the lifted geodesic flow on the orientable double cover $\tilde{M}$ is equicontinuous, it suffices to show that the theorem holds for orientable surfaces, otherwise we can consider the orientable double cover $\tilde{M}$ and conclude that $\tilde{M}$ is a P-manifold. Apply the construction of F in §3 to (M, g) and γ. By Theorem 4.3 and Lemma 4.6, we have a periodic point in $y \in A$; thus, for some m, F^{2m} is an orientation-preserving

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homeomorphism with three fixed points (namely $\{\infty\}$,$\{-\infty\}$ and y). From Proposition 3.1, we know that F^{2m} is recurrent and hence trivial by Theorem 2.7. Lemma 4.6 then implies that every geodesic is closed. □

Note that distality of the geodesic flow does not imply that the manifold is a P-manifold. Take the torus T^n with the standard flat metric. The flow is distal. Indeed, for an unit vector $v \in \mathbb{R}^n$ define the vector field $X_v(x) = v$. Note that the solutions of these vector fields are our lifted geodesics.

If $\inf\{d(xt, yt) \mid t \in T\} = 0$ for $x, y \in ST^n$, then the lifted geodesics xt, yt are solutions of the same vector field; however, the projected flows on T^n of these vector fields are equicontinuous and, since equicontinuity implies distality (see [**1**]), it follows that $y = x$. Note also that on surfaces of higher genus the geodesic flow has positive entropy, but distal flows on compact metric spaces always have zero entropy (see [**5**]); thus, the only compact surfaces M that can possibly admit distal geodesic flows are S^2, $\mathbb{R}P^2$, T^2 and the Klein bottle—and they do.

5. *Flows on manifolds of dimension three*

Definition 5.1. A *global surface of section* Σ for a C^∞ flow Φ without singularities on a three-dimensional manifold is a compact submanifold with the following properties.

(1) If Σ has a boundary, then its boundary components are periodic orbits.

(2) The interior of the surface (denoted by $\overset{\circ}{\Sigma}$) is transversal to Φ.

(3) The orbit through a point not lying on the boundary of Σ hits the interior in forward and backward time.

(4) Every orbit intersects Σ.

There is a natural compactification of $\overset{\circ}{\Sigma}$ to a closed surface, obtained by collapsing the boundary components to a point. We call this unique compactification *the compactification of* $\overset{\circ}{\Sigma}$ *to a closed surface*. If the flow is equicontinuous, we can do more.

The return map $F : \overset{\circ}{\Sigma} \to \overset{\circ}{\Sigma}$ can be extended to the compactification of $\overset{\circ}{\Sigma}$ to a closed surface by defining the collapsed boundary components to be fixed points. Such an extension is well-defined, and this can be proven as in the beginning of §3; we call this map the *extended Poincaré section map*.

THEOREM 5.2. *Let Φ be a C^∞ equicontinuous flow without singularities on a three-dimensional manifold that admits a global surface of section Σ. Let X be the compactification of $\overset{\circ}{\Sigma}$ to a closed surface; then the following properties hold.*

(1) *If X is homeomorphic to the torus or the Klein bottle and the extended Poincaré section map on X has at least one periodic point, then the flow is pointwise periodic.*

(2) *If X is homeomorphic to the sphere and the extended Poincaré section map on X has at least three periodic points, then the flow is pointwise periodic.*

(3) *If X is homeomorphic to the projective plane and the extended Poincaré section map on X has at least two periodic points, then the flow is pointwise periodic.*

(4) *If X is homeomorphic to a surface of negative Euler characteristic, then the flow is pointwise periodic.*

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The proof is quite similar to that of Theorem 4.4, so we shall only repeat some of the ideas. Indeed, the proof of Theorem 4.4 was of a topological nature. The following result will be important.

THEOREM 5.3. *A recurrent homeomorphism of a compact surface with negative Euler characteristic is periodic. If a recurrent homeomorphism on the torus, the annulus, the Möbius strip or the Klein bottle has a periodic point, then the homeomorphism is periodic.*

Proof. See [**4**, Corollary 4.2 and Remark 4.3]. □

Proof of Theorem 5.2. Let $F : \mathring{\Sigma} \to \mathring{\Sigma}$ denote the return map. If we can show that our extension is recurrent, then it follows from our assumptions and Theorem 5.3 that F is periodic and every orbit of the flow is periodic; thus we only need to show that F is recurrent.

If the section Σ is a surface with boundary, then $\mathring{\Sigma}$ denotes the interior. Choose a finite number (say N) of connected, compact, orientable surfaces Σ_j $(j = 1, \ldots, Q)$ which are diffeomorphic to discs and are such that $\bigcup_j \Sigma_j = \Sigma$. For each j, choose a sequence of connected, compact, orientable surfaces $K_{j,i}$ such that $K_{j,i} \subset \mathring{K}_{j,i+1}$ and $\bigcup_i K_{j,i}$ is an open set. We construct $K_{j,i}$ in such a way that given any compact set $C \subset \mathring{\Sigma}$, we have that $C \subset \bigcup_j K_{j,I}$ for some large I. Moreover, we suppose that $\bigcup_{i,j} K_{j,i} = \mathring{\Sigma}$ and that $\bigcup_j K_{j,i}$ is connected.

For every $K_{j,i}$, we can find a tubular neighbourhood $U_{j,i}$ which is diffeomorphic to $K_{j,i} \times (-1, 1)$ via a diffeomorphism $\tau_{i,j}$. We say that a point in $\tau_{i,j}^{-1}(K_{j,i} \times (-1, 0))$ lies 'above' $K_{j,i}$ and that a point in $\tau_{i,j}^{-1}(K_{j,i} \times (0, 1))$ lies 'below' $K_{j,i}$. Set $U_{*,i,j} := \tau_{i,j}(K_{j,i} \times (-1, 0))$ and $U^*_{i,j} := \tau_{i,j}(K_{j,i} \times (0, 1))$. We can suppose that $U_{*,i,j} \subset U_{*,i+1,j}$ and that $U^*_{i,j} \subset U^*_{i+1,j}$. Since Φ is transversal to the section and does not have singularities, we conclude that $\bigcup_j K_{j,i}$ is an orientable surface. Therefore, we can define in a tubular neighbourhood of $\bigcup_j K_{j,i}$ what lies 'below' and and what lies 'above'.

Similarly, we can again define for $K_{j,i}$ and large N the constant

$$s(j, i, N) := \inf\{|t_0 - t_1| : -N < t_0 < t_1 < N,\ \Phi_{t_0}(v) \in \Sigma,\ \Phi_{t_1}(v) \in \Sigma,\ v \in K_{j,i}\}.$$

We can also define, in an analogous manner, the constant $\nu(j, i, N, r) > 0$. Given $v \in \mathring{\Sigma}$, for each integer n we define $t(n, v)$ as the unique element of $\mathbb{R}$ such that $\Phi_{t(n,v)}(v) = F^n(v)$ and $t(0, v) = 0$. Moreover, given $v \in \mathring{\Sigma}$ and $t \geq 0$, we define

$$P(v, t) := \max\{n \geq 0 \mid t(n, v) \leq t\}.$$

The counterpart of Lemma 3.4 is the following result.

LEMMA 5.4. *Fix $v_j \in \bigcap_i K_{j,i}$. There exist sequences $T_i \to \infty$, $0 < \zeta_i \to 0$ and $0 < \beta_i \to 0$ such that $\Phi_{T_i}(v_1) \in \mathring{\Sigma}$ and, for all $j \in \{1, \ldots, Q\}$:*

(1) *$\Phi_{T_i+\zeta_i}(v) \in U^*_{i+1,j}$ for all $v \in K_{j,i}$;*
(2) *$\pi(\Phi_{T_i-\zeta_i}(v)) \in U_{*,i+1,j}$ for all $v \in K_{j,i}$;*
(3) *$d(\Phi_{T_i+s}(v), v) < \beta_i$ for all $v \in K_{j,i}$ and $|s| < 2\zeta_i$;*
(4) *for every $v \in K_{j,i}$ there exists an unique intersection point of $\mathring{\Sigma}$ and $\Phi_{T_i+s}(v)$, where s ranges over $|s| < 4\zeta_i$.*

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Proof. This is analogous to the proof of Lemma 3.4. □

It is now easy to complete the proof of the theorem. Set $p_{j,i} := P(v_j, T_i + \zeta_i)$. The functions

$$G_i : A \to \mathbb{N}$$

defined by $G_i(v) = P(v, T_i + \zeta_i)$ are constant on K_{j,I_0} if $i \geq I_0$; however, since $\bigcup_i K_{j,I_0}$ is connected, we know that $p_{j,i}$ is independent from j, and so $p_{j,i} = p_i$. We deduce that $d(F^{p_i}(v), v) = d(\Phi_{t(v,p_i)}(v), v) < \beta_i$, and thus F is paracompact-recurrent on $\overset{\circ}{\Sigma}$. Since the collapsed boundary components are fixed points, we conclude from Lemma 2.6 that F is recurrent on X. □

COROLLARY 5.5. *If Φ is a C^∞ equicontinuous flow without singularities on a three-dimensional manifold that admits a global surface of section Σ and has at least three distinct periodic orbits, then the flow is pointwise periodic.*

6. *Non-compact manifolds*

In this section we consider non-compact manifolds; therefore, two metrics that generate the same topology may be non-equivalent. We can prove some facts about pointwise equicontinuous geodesic flows on non-compact surfaces if we restrict to a canonical metric for geodesic flows; in our case, this would be the Sasaki metric.

Let M be a surface (which may be non-compact); d will always denote the induced metric on M of the Riemannian metric and $\tilde{d}$ the induced metric on SM of the Sasaki metric. Note that, by construction of the Sasaki metric, we have $\tilde{d}(v, w) \geq d(\pi(v), \pi(w))$.

Definition 6.1. A system (X, T) is called *pointwise equicontinuous* (or pointwise regular) if for any $\epsilon > 0$ and any $x \in X$ there exists an $\delta(\epsilon, x) > 0$ such that for all y with $d(x, y) < \delta(\epsilon, x)$ we have $d(xt, yt) < \epsilon$ for all $t \in T$.

On compact metric spaces, pointwise equicontinuity implies equicontinuity; but on non-compact metric spaces this is not true, and pointwise equicontinuity is not independent of the metric that generates the topology. Moreover, given any compact set K of X, we can find an $\delta(\epsilon, x) > 0$ in the definition above that is independent of $x \in K$.

LEMMA 6.2. *Let $(\Phi, SM, \tilde{d})$ be a geodesic flow that is pointwise equicontinuous. For any compact set K, there exists a number $C(K)$ such that for all $v, w \in K$ and $t \in \mathbb{R}$ we have $d(\gamma_{v_0}(t), \gamma_w(t)) \leq C(K)$.*

Proof. First, choose an open and bounded set O that contains $SM|K$. For $\overline{O}$, choose a constant $\delta(\epsilon) > 0$ such that if $v, w \in SM|K$ satisfy $d(v, w) < \delta(\epsilon)$, then $d(\Phi_t(v), \Phi_t(w)) < \epsilon$. Cover $SM|K$ with $N = N(\epsilon, K)$ balls of radius smaller than $\delta(\epsilon)$ such that the union of these balls is connected and lies in O. We conclude that

$$d(\gamma_v(t), \gamma_w(t)) \leq N\epsilon := C(K)$$

for all $v, w \in SM|K$ and $t \in \mathbb{R}$. □

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PROPOSITION 6.3. *If M is compact and the geodesic flow (Φ, SM) is equicontinuous, then $\pi_1(M)$ is finite.*

Proof. (Φ, SM) is equicontinuous with respect to the Sasaki metric; therefore the lifted geodesic flow on the universal covering $\tilde{M}$ is equicontinuous with respect to the lifted Sasaki metric (denoted by $\tilde{d}$). Since M is compact, we conclude that for each $r > 0$, there exists a number $Z(r)$ such that for any point x of M the sphere $S_x M$ can be covered by $Z(r)$ balls of radius r (with respect to the metric d). Given any point x, and assuming that M is not compact, choose $v \in S_x M$ such that $\gamma_v : \mathbb{R}^+ \to M$ is a ray and choose a sequence $t_i \to \infty$. Set $v_i = -\dot{\gamma}_{v_i}(t_i)$. By the proof of the preceding lemma and the existence of $Z(r)$, we have that $d(\gamma_{v_i}(t_i), \gamma_{-v_i}(t_i) = \gamma_v(2t_i)) \leq Z(\delta(\epsilon))\epsilon$ for an $\epsilon > 0$; hence $2t_i = d(x = \gamma_{v_i}(t_i), \gamma_v(2t_i)) \leq Z(\delta(\epsilon))\epsilon$, but t_i grows. □

COROLLARY 6.4. *If M is non-compact and the geodesic flow (Φ, SM) is pointwise equicontinuous with respect to the Sasaki metric, then the following hold.*

(1) *There is no minimal geodesic (also called a 'line') in M.*

(2) $\operatorname{diam}(\partial B(x, r)) := \sup\{d(x, y) \mid x, y \in \partial B(x, r)\}$ *is bounded by a constant $C(x)$ for all x; the constant $C(x)$ can be chosen uniformly on compact sets.*

Proof. Suppose γ is minimal; then

$$2t = d(\gamma(t), \gamma(-t)) \leq C,$$

for some constant C, and therefore γ cannot be minimal.

If $\operatorname{diam}(\partial B(x, r))$ is not bounded, choose sequences a_i and b_i such that $a_i, b_i \in \partial B(x, r_i)$ and $d(a_i, b_i) \to \infty$ as $r_i \to \infty$. For each a_i, choose a $v_i \in S_x M$ such that γ_{v_i} : $[0, d(x, a_i)]$ is a minimal geodesic segement that starts in x and ends in a_i. Without loss of generality we can suppose that $v_i \to v$; therefore γ_v is a ray and, from equicontinuity, we conclude that $d(\gamma_v(r_i), a_i) \to 0$. We repeat the construction for b_i to get a ray γ_w such that $d(\gamma_w(r_i), b_i) \to 0$. The flow is equicontinuous and therefore we have, by Lemma 6.2, that $d(\gamma_v(r_i), \gamma_w(r_i))$ is bounded and hence that $d(a_i, b_i)$ is bounded. It follows from the proof that $C(x)$ can be chosen uniformly on compact sets. □

One might conjecture that Theorem 4.4 holds in higher-dimensional cases, but it seems that there are no tools to prove such a conjecture. Proposition 6.3 holds for P-manifolds (see [**3**, Theorem 7.37]) and, using the Morse index theorem, one can see that non-compact manifolds with strictly positive sectional curvature have no line; thus, there are some reasons to conjecture this.

We now discuss whether there exists an equicontinuous (or even a pointwise equicontinuous) geodesic flow with respect to the Sasaki metric on a non-compact surface.

Here is a sub-result.

PROPOSITION 6.5. *Let (M, g) be a Riemannian manifold of dimension two, and suppose that the geodesic flow (Φ, SM) is pointwise equicontinuous with respect to the Sasaki metric; then M is homeomorphic to the plane.*

Our proof of this proposition is based on the following theorem.

THEOREM 6.6. *Every surface is homeomorphic to a surface formed from a sphere S by first removing a closed totally disconnected set X from S, then removing the interiors of a finite or infinite sequence D_i of non-overlapping closed discs in $S - X$, and finally identifying in a suitable way the boundaries of these discs in pairs. It may be necessary to identify the boundary of one disc with itself to produce a 'cross cap'. The sequence D_i approaches X in the sense that, for any open set U in S containing X, all but a finite number of the D_i are contained in U.*

Proof. See [**7**]. □

Proof of Proposition 6.5. Recall that d is the metric induced by the Riemannian metric. We first prove that M has genus zero. Suppose that M is not diffeomorphic to $S^2 - X$. Choose a curve β that is not contractible and lies on a 'handle' or 'crosscap' such that we can find a compact set O which contains β and is diffeomorphic to a closed disc where a cylinder or a crosscap is glued in. Let $[\beta]$ denote the free homotopy class. If there is a sequence β_i such that $\beta_i \in [\beta]$ and $L(\beta_i) \to 0$, then, since β_i always intersects O, we conclude that β_i is contractible for large i. Now, let β_i be a sequence such that $\beta_i \in [\beta]$ and $L(\beta_i) \to \inf_{c \in [\beta]} L(c) > 0$. Again, we know that β_i must always intersect O. If β_i lies in a compact subset of M, then we know that a subsequence converges to a closed geodesic; however, it is clear that β_i lies in a compact subset K of M, since otherwise there would be $x_i \in \beta_i$ that tends to infinity, and hence $L(\beta_i)$ would not be bounded. Therefore, we have found a closed geodesic β. Since there exists a ray starting at a point of β, we conclude from Lemma 6.2 that the ray does not tend to infinity, and thus we get a contradiction.

We prove now that X is just a simple point. We endow S^2 with a metric d_0 that generates the standard topology of S^2. Let $-\infty$ and ∞ be two different points of X. Choose sequences a_i and b_i such that $a_i \to -\infty$ and $b_i \to \infty$ with respect to d_0. We want to show that $d(a_i, b_i)$ tends to infinity for a subsequence. Suppose that $d(a_i, b_i)$ is bounded. Let $\delta_i : [0, d(a_i, b_i)] \to M$ be a minimal geodesic segment from a_i to b_i. Let K_i denote the image of δ_i. Define

$$\lim(K_i) := \{y \in S^2 \mid \exists x_i \in K_i \text{ such that } d_0(x_i, y) \to 0\}.$$

It is easy to see that $K := \lim(K_i)$ is closed and connected.

Assume that a point of M lies in K. Then, without loss of generality, a tangent vector v_i of δ_i converges to a vector v of SM. Since we are supposing that $d(a_i, b_i)$ is bounded by a constant C, γ_v will be a geodesic that meets $-\infty$ and ∞ on the interval $[-2C, 2C]$, since $\delta_i[0, C]$ converges to a segment of $\gamma_v[-2C, 2C]$ with respect to d. Hence K is a subset of X, but this means that K is reduced to a point; so $d_0(a_i, b_i) \to 0$, and therefore we have obtained a contradiction to the fact that $d(a_i, b_i)$ is bounded. Thus, given any sequence $a_i \to -\infty$ and $b_i \to \infty$, $d_1(a_i, b_i)$ tends to infinity for a subsequence, as claimed.

We now construct sequences $n_i \to -\infty$ and $m_i \to \infty$ such that $d(n_i, m_i)$ is bounded, and thus derive a contradiction to the fact that X contains more than one point. Define

$$\omega(v) := \{y \in S^2 \mid \exists t_i \to \infty \text{ such that } d_0(\gamma_v(t_i), y) \to 0\}.$$

It is easy to see that $\omega(v)$ is closed and connected in S^2. If, for a ray γ_v, the set $\omega(v)$ is not a subset of X, then there is a sequence $t_i \to \infty$ such that $\dot{\gamma}_v(t_i)$ converges to a vector

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v^* of SM and hence γ_{v^*} will be a minimal geodesic, since γ_v is a ray. This contradicts Corollary 6.4, and therefore $\omega(v)$ is a subset of X. Hence, for a ray γ_v, the set $\omega(v)$ is reduced to a point of X (X is totally disconnected).

Take sequences $a_i \to -\infty$ and $b_i \to \infty$. For a_i, choose $v_i \in S_x M$ such that γ_{v_i} : $[0, d(x, a_i)]$ is a minimal geodesic segment that starts in x and ends in a_i. The sequence v_i tends to a vector v. The curve γ_v will be a ray and therefore $\omega(v)$ will be a point. We show that $-\infty = \omega(v)$. Note that $d(a_i, \gamma_v(t_i)) \to 0$, since our flow is equicontinuous in x. Suppose that $d_0(\gamma_v(t_i), q) \to 0$ for a point $q \in S^2$. If $q \neq -\infty$, then choose a minimal geodesic segment $\delta_i : [0, d(a_i, \gamma_v(t_i))] \to M$ that starts in a_i and ends in $\gamma_v(t_i)$. We denote the image of δ_i by K_i. Again we conclude, as above, that $\lim(K_i)$ will be a connected subset of X, and therefore $q = -\infty$.

Thus, we get a ray γ_v that starts in x and converges to $-\infty$. By repeating this construction, we can generate a ray γ_w that starts in x and converges to ∞. From Lemma 6.2, we then know that $d(n_i := \gamma_v(i), m_i := \gamma_w(i))$ is bounded. □

We end this paper by showing that a pointwise equicontinuous geodesic flow with respect to a metric d_1 exists. This metric d_1 is not equivalent to the induced metric of the Sasaki metric of h (h will be defined later). Let g_0 be the standard metric on $\mathbb{R}^2$, let d_0 be the metric induced by g_0, and identify $\mathbb{R}^2$ with $\mathbb{C}$. Choose a diffeomorphism $f : [0, \infty) \to [0, \infty)$ such that $f|_{[0,1/2)} = \mathrm{Id}$ and $f(t) = \exp(t)$ for $r \geq 1$. Let F denote a diffeomorphism from $\mathbb{R}^{>0} \times S^1$ to $\mathbb{R}^2 - \{0\}$, defined by $F(r, \phi) = (f(r), \phi)$ where (r, ϕ) are the standard polar coordinates. The pull-back of the metric g_0 under F defines a new metric h on $\mathbb{R}^2$ whose geodesic flow is complete. Consider the metric d_1 on $S\mathbb{R}^2$ defined by $d_1((x, v), (y, w)) = \|x - y\| + \|v - w\|$. We show that the geodesic flow of $(\mathbb{R}^2, h)$ will be pointwise equicontinuous with respect to d_1.

Consider the coordinate system $F : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$. Given a vector v at a point x, we consider the geodesic $g_{x,v}(t) = x + tv$ of the metric g_0. Let us fix a point $x = a_0 + ib_0$ and a vector $v = a_1 + ib_1$. Since our metric is invariant with respect to revolutions, we can assume that $a_1 \neq 0$. For a small $\epsilon > 0$, choose a $M > 0$ and a $\delta(\epsilon) > 0$ such that $d_0(\dot{g}_{x,v}(t), \dot{h}_{y,w}(t)) < \epsilon$ whenever $t \in [-M, M]$ and $d((x, v), (y, w)) < \delta(\epsilon)$. If M is large and $\delta(\epsilon) > 0$ is small enough, then the geodesic $h_{y,w}(t)$ lies outside the compact set $F([0, 2], \mathbb{R})$ for $t \notin [-M, M]$.

Note that $|g(t)| = g(t) \cos \phi(t) + ig(t) \sin \phi(t)$, where

$$\phi(t) = \arctan\left(\frac{b_0 + tb_1}{a_0 + ta_1}\right), \quad |g(t)|^2 = (a_0 + ta_1)^2 + (b_0 + tb_1)^2.$$

Note that

$$F^{-1} = g^*_{x,v}(t) := \left(f^{-1}(|g_{x,v}(t)|), \arctan\left(\frac{b_0 + tb_1}{a_0 + ta_1}\right)\right)$$

will be a geodesic of our manifold $(\mathbb{R}^2, h)$.

We show that if $\epsilon > 0$ is small enough, the distance $d_1(\dot{g}^*_{x,v}(t), \dot{h}^*_{y,w}(t))$ remains arbitrarily small for all t. A calculation shows that for $t \notin [-M, M]$, we have

$$\dot{g}^*_{x,v}(t) = \left(\frac{a_1(a_0 + ta_1) + b_1(b_0 + tb_1)}{|g(t)|^2}, \frac{b_1(a_0 + ta_1) + a_1(b_0 + tb_1)}{(1 + ((b_0 + tb_1)/(a_0 + ta_1))^2)(a_0 + ta_1)^2}\right).$$

Hence, for a small $\epsilon > 0$ and large M, we have $\|\dot{g}^*_{x,v}(t) - \dot{h}^*_{y,w}(t)\|$ small for $t \notin [-M, M]$. Let $h_{y,w}(t) = y + tw = n_0 + im_0 + t(n_1 + im_1)$; then one can compute that

$$\|h_{y,w}(t) - g_{x,y}(t)\| \leq \|f^{-1}(|g(t)|) - f^{-1}(|h(t)|)\| + \left\|\arctan\left(\frac{b_0 + tb_1}{a_0 + ta_1}\right) - \arctan\left(\frac{m_0 + tm_1}{n_0 + tn_1}\right)\right\|.$$

In the same way, we conclude that for a small $\epsilon > 0$ and large M, we will have the second term small for $t \notin [-M, M]$. Another computation shows that

$$\|f^{-1}(|g(t)|) - f^{-1}(|h(t)|)\| = \left\|\frac{1}{2}\ln\left(\frac{(a_0^2 + b_0^2)/t^2 + (2(a_0a_1 + b_0b_1))/t + a_1^2 + b_1^2}{(n_0^2 + m_0^2)/t^2 + (2(n_0n_1 + m_0m_1))/t + n_1^2 + m_1^2}\right)\right\|,$$

but this term will be small for small $\epsilon > 0$, large M and $t \notin [-M, M]$. Therefore, we know that the geodesic flow is equicontinuous with respect to the metric d_1. The metric is not equivalent to the induced metric of h, since the Riemannian distance between two geodesics of different directions grows.

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