CAMBRIDGE
UNIVERSITY PRESS

PAPER

Cofree coalgebras and differential linear logic

James Clift and Daniel Murfet* ![ORCID]

Department of Mathematics, University of Melbourne
*Corresponding author. Email: d.murfet@unimelb.edu.au

Abstract
We prove that the semantics of intuitionistic linear logic in vector spaces which uses cofree coalgebras is also a model of differential linear logic, and that the Cartesian closed category of cofree coalgebras is a model of the simply typed differential λ-calculus.

> In the discrete world of computing, there is no meaningful metric in which 'small' changes and 'small' effects go hand in hand, and there never will be.
>
> E.W. Dijkstra, *On the cruelty of really teaching computer science*

1. Introduction

The idea of taking derivatives of programs is an old one (Paige and Koenig 1982, Section 2) with manifestations including automatic differentiation of algorithms computing real-valued functions (Corliss 1991) and incremental computation (Ramalingam and Reps 1993). However, these approaches are limited to restricted classes of computations, and it is only recently with the development of the differential λ-calculus by Ehrhard–Regnier (Ehrhard and Regnier 2003) and its refinement by differential linear logic (Blute et al. 2006; Ehrhard 2016) that derivatives have been defined for general higher-order programs. These theories assign to each program P another program ∂P, the derivative, which (in some sense) computes the infinitesimal change in the output of P resulting from an infinitesimal change to its input.

The connection between the Ehrhard–Regnier derivative and ordinary calculus is made explicit in the semantics of differential λ-calculus and differential linear logic, with the standard examples being the Köthe and finiteness space semantics of Ehrhard (2002; 2005) and the semantics of Blute–Ehrhard–Tasson (Blute et al. 2010) in convenient vector spaces. In this paper, we explain how the simplest semantics of intuitionistic linear logic in vector spaces (Hyland and Schalk 2003; Murfet 2014) is already a model of differential linear logic: tangent vectors and derivatives appear automatically when we use the cofree coalgebra to model the exponential.

Here is a sketch of the key point: let $[\![-]\!]$ denotes the natural semantics of linear logic in $\mathbb{C}$-vector spaces and suppose we are given a proof π in linear logic computing a function from inputs of type A to outputs of type B:

$$\pi$$
$$\vdots$$
$$!A \vdash B\,.$$

The space of inputs to $[\![\pi]\!]$ is $[\![A]\!]$, which for the sake of simplicity let us assume is finite-dimensional. A small change in the input starting from $P \in [\![A]\!]$ is a tangent vector v at the point P. We may identify v with a vector in $[\![A]\!]$, so that a representative infinitesimal curve segment of this tangent vector is

$$(-1, 1) \longrightarrow [\![A]\!], \quad t \longmapsto P + tv.$$

From the point of view on algebraic geometry given by Grothendieck's theory of schemes (Eisenbud and Harris 2000; Hartshorne 1977), tangent vectors are morphisms of (co)algebras involving the algebra $\mathbb{C}[\varepsilon]/\varepsilon^2$ and its dual coalgebra $(\mathbb{C}[\varepsilon]/\varepsilon^2)^*$. We review this connection in Section 2.3, but in short, a pair of vectors (P, v) in $[\![A]\!]$ is the same data as a linear map

$$\mathbb{C}1^* \oplus \mathbb{C}\varepsilon^* \longrightarrow [\![A]\!] \tag{1}$$

with $1^* \mapsto P, \varepsilon^* \mapsto v$, and so our tangent vector is associated to a linear map from the underlying vector space of the coalgebra $(\mathbb{C}[\varepsilon]/\varepsilon^2)^*$ to $[\![A]\!]$ (where the coalgebra structure on $(\mathbb{C}[\varepsilon]/\varepsilon^2)^*$ has comultiplication $1^* \mapsto 1^* \otimes 1^*$ and $\varepsilon^* \mapsto 1^* \otimes \varepsilon^* + \varepsilon^* \otimes 1^*$). However, there is a *universal* pair consisting of a cocommutative coalgebra $![\![A]\!]$ equipped with a linear map $![\![A]\!] \longrightarrow [\![A]\!]$. The universal property means that the linear map (1) encoding our tangent vector lifts uniquely to a morphism of coalgebras

$$(\mathbb{C}[\varepsilon]/\varepsilon^2)^* \longrightarrow ![\![A]\!]. \tag{2}$$

The function assigning to (P, v) this morphism of coalgebras is a bijection between pairs consisting of a point in $[\![A]\!]$ and a tangent vector, and morphisms of coalgebras (2).

Here is where π enters the story. In the semantics of linear logic in vector spaces, the denotation of $!A$ is $![\![A]\!]$, and the denotation of π is a linear map $[\![\pi]\!] : [\![!A]\!] \longrightarrow [\![B]\!]$ which lifts by the universal property to a morphism of coalgebras $![\![A]\!] \longrightarrow ![\![B]\!]$. This lifting may be composed with (2) to give a morphism of coalgebras

$$(\mathbb{C}[\varepsilon]/\varepsilon^2)^* \longrightarrow ![\![A]\!] \longrightarrow ![\![B]\!]. \tag{3}$$

Working the bijection between tangent vectors and coalgebra morphisms in reverse, we see that (3) uniquely specifies a point in $[\![B]\!]$ and a tangent vector. The point in $[\![B]\!]$ is just the output of the algorithm π on the given input, while the tangent vector gives the infinitesimal variation of the output, when the input is varied in the direction v.

The formal statement is that for any algebraically closed field k of characteristic zero, the semantics of intuitionistic linear logic in k-vector spaces defined using cofree coalgebras is a model of differential linear logic (Theorem 3.3). We refer to this as the *Sweedler semantics*, since the explicit description of this universal coalgebra is due to him (Murfet 2014; Sweedler 1969). The proof is elementary and we make no claim here to technical novelty; the link between the symmetric coalgebra and differential calculus is well known. Perhaps our main contribution is to give several detailed examples showing how to compute these derivatives (Section 4) and to check that this model of differential linear logic gives rise to a model of differential λ-calculus in the category of cofree coalgebras (Theorem 5.17).

We conclude this introduction with a sketch of one such example and a comparison of our work to other semantics of differential linear logic. To elaborate a little more on the notation: for any type A of linear logic (which for us has only connectives $\otimes, \multimap, !$) there is a vector space $[\![A]\!]$, and for any proof π of $A \vdash B$ there is a linear map $[\![\pi]\!] : [\![A]\!] \longrightarrow [\![B]\!]$. In particular, every proof ξ of type A has a denotation $[\![\xi]\!] \in [\![A]\!]$, and the promotion of ξ is a proof of type $!A$ which has for its denotation a vector in the cofree coalgebra $[\![!A]\!] = ![\![A]\!]$ which we denote $|\emptyset\rangle_{[\![\xi]\!]} \in [\![!A]\!]$, see Definition 2.14 below.

For any binary sequence $S \in \{0, 1\}^*$, there is an encoding of S as a proof $\underline{S}$ of type

$$\mathbf{bint}_A = !(A \multimap A) \multimap \big(!(A \multimap A) \multimap (A \multimap A)\big).$$

The idea is that such a proof encodes the algorithm which takes two endomorphisms α, β and returns the result of composing them according to the pattern contained in the binary sequence S (see Section 4.9). Repetition of sequences can be encoded as a proof

$$\underline{\text{repeat}}$$
$$\vdots$$
$$!\mathbf{bint}_A \vdash \mathbf{bint}_A \ .$$

The denotation is a linear map $[\![!\mathbf{bint}_A]\!] \longrightarrow [\![\mathbf{bint}_A]\!]$ sending $|\emptyset\rangle_{[\![S]\!]}$ to $[\![SS]\!]$. The derivative of repeat according to the theory of differential linear logic is another a proof

$$\partial\underline{\text{repeat}}$$
$$\vdots$$
$$!\mathbf{bint}_A, \mathbf{bint}_A \vdash \mathbf{bint}_A$$

which can be derived from repeat by new deduction rules called codereliction, cocontraction and coweakening (see Section 3.1). We prove in Section 4.2 that the denotation of this derivative in the Sweedler semantics is the linear map

$$[\![\partial\underline{\text{repeat}}]\!] : [\![!\mathbf{bint}_A]\!] \otimes [\![\mathbf{bint}_A]\!] \longrightarrow [\![\mathbf{bint}_A]\!] \ ,$$
$$|\emptyset\rangle_{[\![S]\!]} \otimes [\![T]\!] \longmapsto [\![ST]\!] + [\![TS]\!]$$

whose value on the tensor $|\emptyset\rangle_{[\![S]\!]} \otimes [\![T]\!]$ we interpret as the derivative of the repeat program at the sequence S in the direction T. This can be justified informally by the following calculation using an infinitesimal ε

$$(S + \varepsilon T)(S + \varepsilon T) = SS + \varepsilon(ST + TS) + \varepsilon^2 TT,$$

which says that varying the sequence infinitesimally from S in the direction T causes a variation of the repetition in the direction $ST + TS$.

The Sweedler semantics is far from the first semantics of differential linear logic: basic examples include the categories of sets and relations (Blute et al. 2006, Section 2.5.1) and suplattices (Blute et al. 2006, Section 2.5.2). The examples of Köthe and finiteness spaces (Ehrhard 2002; 2005) and convenient vector spaces (Blute et al. 2010; Frolicher and Kriegl 1988) have already been mentioned. These papers explain that the geometric 'avatar' of the exponential connective of linear logic is the functor sending a space X to the space of distributions on X (for a precise statement, see Remark 2.19). This remarkable analogy between logic and geometry deserves further study.

Conceptually, the Sweedler semantics is similar to these examples in that the exponential is modelled by a space of distributions (with finite support) but it is purely algebraic and there are simple explicit formulas for all the structure maps, which makes it suitable for concrete calculations with proof denotations. Moreover, in the algebraic approach, the differential structure emerges naturally from the exponential structure, rather than being 'baked in'. The downside is that the smoothness of proof denotations in our semantics is obscured; in particular, in the case $k = \mathbb{C}$, some extra work is required to see the relation between our differential structure and the derivatives in the usual sense.

2. Sweedler semantics

We review the Sweedler semantics of multiplicative exponential intuitionistic linear logic (henceforth, simply *linear logic*) in the category of vector spaces $\mathcal{V}$ over an algebraically closed field[1] k of

characteristic zero (e.g., $k = \mathbb{C}$) in Section 2.4. This was introduced in Hyland and Schalk (2003) and revisited in Murfet (2014) with a focus on explicit formulas for the involved structures (Murfet 2015). For background material on linear logic and its semantics, see Girard (1987), Girard et al. (1989), Melliès (2009). The multiplicative connectives $\otimes$ and $\multimap$ have the obvious interpretation; the only nontrivial ingredient in the Sweedler semantics is the cofree coalgebra which interprets the exponential. We begin this section with a review of cofree coalgebras (Section 2.1) and how to think about points (Section 2.2) and tangent vectors (Section 2.3) in coalgebraic language.

Let $\mathbf{Alg}_k$ denote the category of commutative unital k-algebras and $\mathbf{Coalg}_k$ the category of cocommutative counital k-coalgebras. Unless otherwise indicated, all algebras are commutative and unital, and all coalgebras are cocommutative and counital. Throughout, $\otimes = \otimes_k$ and Δ, ε denote, respectively, the comultiplication and counit of a coalgebra.

2.1 Cofree coalgebras

The following construction is from Sweedler (1969, Chapter VI):

Definition 2.1. *The* Hopf dual *or* continuous linear dual A° *of an algebra A is the subspace of $A^* = \mathrm{Hom}_k(A, k)$ consisting of linear maps $A \longrightarrow k$ which factor as a composite*

$$A \longrightarrow A/I \longrightarrow k$$

where $I \subseteq A$ is an ideal, the first map is the quotient and A/I is finite-dimensional.

The dual A° is sometimes denoted $\mathrm{Hom}^{\mathrm{cont}}(A, k)$, as for example in Murfet (2015). In Sweedler (1969, Lemma 6.0.1), it is proven that the canonical injective map $A^* \otimes A^* \longrightarrow (A \otimes A)^*$ identifies the subspace $A^\circ \otimes A^\circ$ with $(A \otimes A)^\circ$ and that the dual of the multiplication

$$M^* : A^* \longrightarrow (A \otimes A)^*$$

satisfies $M^*(A^\circ) \subseteq (A \otimes A)^\circ$. Identifying the codomain with $A^\circ \otimes A^\circ$ defines a linear map

$$\Delta : A^\circ \longrightarrow A^\circ \otimes A^\circ$$

and in this way $(A^\circ, \Delta, \varepsilon)$ is a cocommutative coalgebra (Sweedler 1969, Proposition 6.0.2) where the counit $\varepsilon : A^\circ \longrightarrow k$ is evaluation at the identity $1 \in A$. Clearly, if A is finite-dimensional then $A^\circ = A^*$. The fundamental theorem about the Hopf dual is

Theorem 2.2. (Sweedler). *Given an algebra A and coalgebra C, there is a natural bijection*

$$\mathbf{Alg}_k(A, C^*) \cong \mathbf{Coalg}_k(C, A^\circ).$$

Proof. See Sweedler (1969, Theorem 6.0.5). □

An important example is the Hopf dual of the symmetric algebra $A = \mathrm{Sym}(V^*)$ over a finite-dimensional vector space V. Suppose V has basis $e_1, \ldots, e_n$ with dual basis $x_i = e_i^*$ so that $\mathrm{Sym}(V^*)$ is isomorphic to $k[x_1, \ldots, x_n]$. In this case, the linear map

$$\eta : k \longrightarrow V \otimes V^* \longrightarrow V \otimes \mathrm{Sym}(V^*)$$

$$1 \mapsto \sum_{i=1}^{n} e_i \otimes x_i$$

which is independent of the choice of basis gives rise to a linear map

$$c_V : \mathrm{Sym}(V^*)^\circ \longrightarrow V$$

which sends $\theta \in \mathrm{Sym}(V^*)^\circ$ to the vector $c_V(\theta)$ computed by the composite

$$k \xrightarrow{\;\;\eta\;\;} V \otimes \mathrm{Sym}(V^*) \xrightarrow{\;\;1 \otimes \theta\;\;} V \otimes k \cong V .$$

In terms of our chosen basis

$$c_V(\theta) = \sum_{i=1}^{n} \theta(x_i) e_i .$$

The pair $(\mathrm{Sym}(V^*)^\circ, c_V)$ is the cofree coalgebra generated by V, more precisely:

Theorem 2.3. *For a finite-dimensional vector space V, the pair $(\mathrm{Sym}(V^*)^\circ, c_V)$ is universal among pairs consisting of a (cocommutative) coalgebra and a linear map from that coalgebra to V, in the sense that the map*

$$\mathbf{Coalg}_k(C, \mathrm{Sym}(V^*)^\circ) \longrightarrow \mathrm{Hom}_k(C, V) ,$$

$$\varphi \longmapsto c_V \circ \varphi$$

is a bijection for any coalgebra C.

Proof. See Sweedler (1969, Theorems 6.4.1, 6.4.3) or Murfet (2015, Theorem 2.20). Here is a sketch of the proof: since any coalgebra is a colimit of finite-dimensional subcoalgebras (Sweedler 1969, Theorem 2.2.1), we can reduce to the case of C finite-dimensional, where by Theorem 2.2

$$\mathbf{Coalg}_k(C, \mathrm{Sym}(V^*)^\circ) \cong \mathbf{Alg}_k(\mathrm{Sym}(V^*), C^*)$$
$$\cong \mathrm{Hom}_k(V^*, C^*)$$
$$\cong \mathrm{Hom}_k(C, V)$$

as claimed.   □

So much is immediate from Sweedler (1969). However, from the point of view of having a semantics of linear logic (or differential linear logic) in which one can actually do calculations, it is essential to have an *explicit* description of $\mathrm{Sym}(V^*)^\circ$ and c_V. Providing such a description was the purpose of Murfet (2015) and we now give a (partially new) exposition of the relevant facts.

In the following, let V be a finite-dimensional vector space.

Definition 2.4. *For $P \in V$, we define the linear map*

$$\Psi_P : \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(V^*)^\circ \qquad (4)$$

using a choice of basis $e_1, \ldots, e_n$ of V by

$$\Psi_P(e_1^{a_1} \cdots e_n^{a_n})(f) = \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}(f) \Big|_{x_1 = P_1, \ldots, x_n = P_n}$$

where $x_i = e_i^*$ is the dual basis, $P = \sum_{i=1}^{n} P_i e_i$. Writing $\mathrm{Sym}_P(V)$ for a copy of $\mathrm{Sym}(V)$

$$\Psi : \bigoplus_{P \in V} \mathrm{Sym}_P(V) \longrightarrow \mathrm{Sym}(V^*)^\circ$$

is the linear map with Ψ_P as its components.

There are two things that need to be checked, for this Ψ to be well defined:

Lemma 2.5. Ψ *is independent of the choice of basis used to define it.*

Proof. A change of coordinates affects $e_i^{a_i}$ in the same way as it affects $\frac{\partial^{a_i}}{\partial x_i^{a_i}}$. □

Lemma 2.6. *Given $P \in V$ and $a_1, \ldots, a_n \geq 0$, the functional*

$$\Psi_P(e_1^{a_1} \cdots e_n^{a_n}) \in \mathrm{Sym}(V^*)^*$$

belongs to the subspace $\mathrm{Sym}(V^*)^\circ$.

Proof. We prove $\theta = \Psi_P(e_1^{a_1} \cdots e_n^{a_n})$ vanishes on $(x_1 - P_1, \ldots, x_n - P_n)^{\sum_i a_i + 1}$. Suppose given a monomial $f = (x_1 - P_1)^{b_1} \cdots (x_n - P_n)^{b_n}$ with $\sum_i b_i > \sum_i a_i$. Then the derivative of f involved in $\theta(f)$ will be divisible by some $x_i - P_i$ and so vanishes at P. □

Theorem 2.7. Ψ *is an isomorphism of vector spaces.*

Proof. Set $R = \mathrm{Sym}(V^*)$ and given $P \in V$ let $\mathfrak{m}_P = (x_1 - P_1, \ldots, x_n - P_n)$ denote the maximal ideal of R consisting of all polynomials $f(x_1, \ldots, x_n)$ with $f(P_1, \ldots, P_n) = 0$. By the Chinese remainder theorem (see the proof of Murfet 2015, Lemma A.1) for any functional $\theta \in R^\circ$, there is a unique $P \in V$ such that for some $j > 0$ the map θ factors as

$$R \longrightarrow R/\mathfrak{m}_P^j \longrightarrow k \,.$$

Since $R_{\mathfrak{m}_P}/\mathfrak{m}_P^j R_{\mathfrak{m}_P} \cong R/\mathfrak{m}_P^j$, we have

$$(R_{\mathfrak{m}_P})^\circ = \varinjlim_{j>0} \mathrm{Hom}_k(R/\mathfrak{m}_P^j, k) \,.$$

One way to restate the consequence of the Chinese remainder theorem is that every $\theta \in R^\circ$ belongs to $(R_{\mathfrak{m}_P})^\circ$ for a unique P, that is, there is an isomorphism

$$\bigoplus_{P \in V} (R_{\mathfrak{m}_P})^\circ \xrightarrow{\ \cong\ } R^\circ \,.$$

The proof of Lemma 2.6 shows that the image of Ψ_P lies in the subspace $(R_{\mathfrak{m}_P})^\circ \subseteq R^\circ$, so it suffices to show that for every $P \in V$ the map

$$\Psi_P : \mathrm{Sym}_P(V) \longrightarrow (R_{\mathfrak{m}})^\circ$$

is a bijection. But the left-hand side is a direct limit of subspaces $\mathrm{Sym}_P(V)_{\leq j}$ spanned by monomials of degree $\leq j$ and the right-hand side is a direct limit of subspaces $\mathrm{Hom}_k(R/\mathfrak{m}^{j+1}, k)$. Moreover, there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sym}_P(V) & \xrightarrow{\ \ \Psi_P\ \ } & (R_{\mathfrak{m}})^\circ \\
\uparrow & & \uparrow \\
\mathrm{Sym}_P(V)_{\leq j} & \xrightarrow[\ \ \Psi_P^j\ \ ]{} & \mathrm{Hom}_k(R/\mathfrak{m}^{j+1}, k)
\end{array}
$$

where the vertical maps are inclusions. So it suffices to prove Ψ_P^j is an isomorphism for each j. But this is clearly true for $j = 0$ and for $j > 0$ we proceed by induction. We have an exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \longrightarrow R/\mathfrak{m}^{j+1} \longrightarrow R/\mathfrak{m}^j \longrightarrow 0$$

and hence a commutative diagram with exact rows

$$0 \longrightarrow \mathrm{Hom}_k(R/\mathfrak{m}^j, k) \longrightarrow \mathrm{Hom}_k(R/\mathfrak{m}^{j+1}, k) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}^j/\mathfrak{m}^{j+1}, k) \longrightarrow 0$$

$$\cong \Big\uparrow \qquad \Psi_P^j \Big\uparrow \qquad \bar{\Psi}_P^j \Big\uparrow$$

$$0 \longrightarrow \mathrm{Sym}_P(V)_{\leq j-1} \longrightarrow \mathrm{Sym}_P(V)_{\leq j} \longrightarrow \mathrm{Sym}_P(V)_{\leq j}/\mathrm{Sym}_P(V)_{\leq j-1} \longrightarrow 0$$

where $\bar{\Psi}_P^j$ is the induced map on the quotients, and the leftmost vertical map is an isomorphism by the inductive hypothesis. So it suffices to prove that $\bar{\Psi}_P^j$ is an isomorphism (by the Five Lemma). But the domain and codomain both pick out 'monomials' of degree j, in one case by removing from monomials of degree $\leq j$ all those of degree $\leq j-1$ and in the other case by removing from monomials of degree $\geq j$ all those of degree $\geq j+1$.

More formally, we may directly calculate that

$$\bar{\Psi}_P^j(e_1^{a_1} \cdots e_n^{a_n}) = \sum_{b_1 + \cdots + b_n = j} \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}(\omega_{\mathbf{b}})\Big|_P \cdot \omega_{\mathbf{b}}^*$$

$$= a_1! \cdots a_n! \cdot \omega_{\mathbf{a}}^*$$

where for $\mathbf{b} = (b_1, \ldots, b_n)$ we write $\omega_{\mathbf{b}} = \prod_{j=1}^n (x_j - P_j)^{b_j}$ which under the restriction $|\mathbf{b}| \leq j$ give a k-basis for $\mathfrak{m}^j$ with dual basis $\omega_{\mathbf{b}}^*$. $\qquad \square$

Remark 2.8. In the notation of Murfet (2015), the map Ψ_P is the composite of the isomorphisms in Murfet (2015, Lemma 2.12) and Murfet (2015, Theorem 2.6)

$$\mathrm{Sym}_P(V) \xrightarrow{\ \cong\ } \mathrm{LC}(V, P) \xrightarrow{\ \cong\ } \mathrm{Sym}(V^*)_{\mathfrak{m}_P}^\circ$$

defined by (set $z_i = x_i - P_i$)

$$e_1^{a_1} \cdots e_n^{a_n} \longmapsto a_1! \cdots a_n! \left[\frac{f}{z_1^{a_1}, \ldots, z_n^{a_n}} \frac{d\underline{z}}{\underline{z}} \right] \longmapsto \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}(-)\Big|_P$$

where $\mathrm{LC}(V, P) = H_P^n(R, \Omega_{R/k}^n)$ is a local cohomology module, see Section 2.5.

For any vector space V (not necessarily finite-dimensional), the underlying vector space of the symmetric algebra $\mathrm{Sym}(V)$ is naturally equipped with the structure of a coalgebra (the *symmetric coalgebra*) with comultiplication Δ defined by

$$\Delta(v_1 \cdots v_n) = \sum_{I \subseteq \{1, \ldots, n\}} v_I \otimes v_{I^c}$$

where $v_i \in V$ for $1 \leq i \leq n$ and for $I \subseteq \{1, \ldots, n\}$ we denote by v_I the tensor which is the product in $\mathrm{Sym}(V)$ of the set $\{v_i \,|\, i \in I\}$. By convention if $I = \emptyset$ then $v_I = 1$. The counit $\varepsilon : \mathrm{Sym}(V) \longrightarrow k$ satisfies $\varepsilon(1) = 1$ and vanishes on monomials of positive degree; for the details see Bourbaki (Bourbaki 1989, III Section 11) (our coalgebras are their coassociative counital *cogebras*).

Proposition 2.9. Ψ *is an isomorphism of coalgebras.*

Proof. It suffices to show that

$$\Psi_P^j : \mathrm{Sym}_P(V)_{\leq j} \longrightarrow \mathrm{Hom}_k(R/\mathfrak{m}_P^{j+1}, k)$$

is a morphism of coalgebras, where $\mathrm{Sym}_P(V)_{\leq j}$ is a subcoalgebra of the symmetric coalgebra $\mathrm{Sym}_P(V)$ and $\mathrm{Hom}_k(R/\mathfrak{m}_P^{j+1}, k)$ is given the coalgebra structure as the dual of the

finite-dimensional algebra $R/\mathfrak{m}_P^{j+1}$. Given $\theta : R/\mathfrak{m}_P^{j+1} \longrightarrow k$, we have

$$\Delta(\theta) = \sum_{|\mathbf{b}| \leq j, |\mathbf{b'}| \leq j} \theta(\omega_{\mathbf{b}} \cdot \omega_{\mathbf{b'}}) \cdot \omega_{\mathbf{b}}^* \otimes \omega_{\mathbf{b'}}^*$$

where $\omega_{\mathbf{b}} = \prod_{i=1}^{n} (x_i - P_i)^{b_j}$ and $|\mathbf{b}| = \sum_i b_i$. Hence,

$$\Delta(\omega_{\mathbf{c}}^*) = \sum_{\mathbf{b}+\mathbf{b'}=\mathbf{c}} \omega_{\mathbf{b}}^* \otimes \omega_{\mathbf{b'}}^* .$$

Now, we have already calculated that

$$\Psi_P(e_1^{a_1} \cdots e_n^{a_n}) = a_1! \cdots a_n! \cdot \omega_{\mathbf{a}}^*$$

so we have

$$(\Psi_P \otimes \Psi_P)\Delta(e_1^{a_1} \cdots e_n^{a_n}) = (\Psi_P \otimes \Psi_P) \sum_{\mathbf{b}+\mathbf{b'}=\mathbf{a}} \binom{a_1}{b_1} \cdots \binom{a_n}{b_n} e_1^{b_1} \cdots e_n^{b_n} \otimes e_1^{b'_1} \cdots e_n^{b'_n}$$

$$= \sum_{\mathbf{b}+\mathbf{b'}=\mathbf{a}} \binom{a_1}{b_1} \cdots \binom{a_n}{b_n} b_1! \cdots b_n!(b'_1)! \cdots (b'_n)! \omega_{\mathbf{b}}^* \otimes \omega_{\mathbf{b'}}^*$$

$$= a_1! \cdots a_n! \sum_{\mathbf{b}+\mathbf{b'}=\mathbf{a}} \omega_{\mathbf{b}}^* \otimes \omega_{\mathbf{b'}}^*$$

$$= \Delta \Psi_P(e_1^{a_1} \cdots e_n^{a_n})$$

proving the claim. The compatibility of the counits is clear. $\square$

The vector space $\mathrm{Sym}_P(V) = \bigoplus_{i \geq 0} \mathrm{Sym}_P^i(V)$ is graded, where $\mathrm{Sym}_P^i(V)$ is the image in the symmetric algebra of $V^{\otimes i}$. The projection from this graded vector space to its components $k \oplus V$ of degree ≤ 1 followed by the map $(\lambda, v) \mapsto \lambda P + v$ defines

$$\mathrm{Sym}_P(V) \longrightarrow\!\!\!\!\!\rightarrow k \oplus V \xrightarrow{(P, 1_V)} V$$

and as P varies these maps give the components of the linear map

$$d_V : \bigoplus_{P \in V} \mathrm{Sym}_P(V) \longrightarrow V, \qquad d_V|_{\mathrm{Sym}_P(V)}(v_0, v_1, v_2, \dots) = v_0 P + v_1 .$$

Recall the linear map c_V, which we may compute in our basis to be

$$c_V : \mathrm{Sym}(V^*)^{\circ} \longrightarrow V, \qquad c_V(\theta) = \sum_{i=1}^{n} \theta(x_i)e_i .$$

Lemma 2.10. *For a finite-dimensional vector space V, the diagram*

$$\bigoplus_P \mathrm{Sym}_P(V) \xrightarrow[\cong]{\Psi} \mathrm{Sym}(V^*)^{\circ}$$

with d_V and c_V mapping to V

commutes, and hence the pair $\left(\bigoplus_P \mathrm{Sym}_P(V), d_V\right)$ is also universal among pairs consisting of a (cocommutative) coalgebra and a linear map from that coalgebra to V.

Proof. We have $c_V \Psi_P(1) = \sum_i P_i e_i = P$ and

$$c_V \Psi_P(e_j) = \sum_i \Psi_P(e_j)(x_i) e_i = \sum_i \frac{\partial}{\partial x_j}(x_i) e_i = e_j$$

whereas for a monomial $m \in \mathrm{Sym}_P(V)$ of degree > 1, both legs of the diagram vanish. $\quad\square$

In summary, there is a linear map d_V from the coproduct of copies of the symmetric coalgebra indexed by the points of V. By the universal property of the Hopf dual, there is a unique morphism of coalgebras Ψ making the diagram of Lemma 2.10 commute, and what we have done in the above is compute explicitly this unique morphism of coalgebras in terms of differential operators, and prove that it is an isomorphism.

In the above, we have focused on finite-dimensional vector spaces V because in this case the role of differential operators is most transparent. But the symmetric coalgebra $\mathrm{Sym}(V)$ is defined for any vector space V, with the same formulas for Δ, ε as those given above, and the coproduct $\bigoplus_{P \in V} \mathrm{Sym}_P(V)$ is therefore still a coalgebra. The linear map d_V defined above remains well defined, even if V is infinite-dimensional.

Proposition 2.11. *For any vector space V (not necessarily finite-dimensional), the pair*

$$!V := \Big(\bigoplus_{P \in V} \mathrm{Sym}_P(V), d_V \Big)$$

is universal among pairs consisting of a cocommutative coalgebra and a linear map to V.

Proof. This is already implicit in Sweedler, as explained in Murfet (2015, Appendix B), but we give here another argument. Let C be a coalgebra and $\varphi : C \longrightarrow V$ a linear map. We write C as a direct limit of its finite-dimensional subcoalgebras $\{C_i\}_{i \in I}$ (Sweedler 1969, Theorem 2.2.1). For each $i \in I$, let V_i be the finite-dimensional subspace $\varphi(C_i)$ of V and $\varphi_i : C_i \longrightarrow V_i$ be the restriction of φ. By Lemma 2.10, there is a unique morphism of coalgebras Φ_i making

$$
\begin{array}{ccc}
C_i & \xrightarrow{\ \Phi_i\ } & \bigoplus_{P \in V_i} \mathrm{Sym}_P(V_i) \\
& {\varphi_i}\searrow & \big\downarrow{d_{V_i}} \\
& & V_i
\end{array}
$$

commute. The inclusions $V_i \subseteq V$ induce morphisms of algebras $\mathrm{Sym}_P(V_i) \longrightarrow \mathrm{Sym}_P(V)$ which are easily checked to be injective morphisms of coalgebras. We let Φ_i' denote the composite with the direct sum of these inclusions:

$$C_i \xrightarrow{\ \Phi_i\ } \bigoplus_{P \in V_i} \mathrm{Sym}_P(V_i) \longrightarrow \bigoplus_{P \in V_i} \mathrm{Sym}_P(V) \subseteq \bigoplus_{P \in V} \mathrm{Sym}_P(V).$$

These morphisms are compatible with inclusions $C_i \subseteq C_j$ and so induce a morphism of coalgebras $\Phi : C \longrightarrow \bigoplus_{P \in V} \mathrm{Sym}_P(V)$ which satisfies $d_V \circ \Phi = \varphi$. To show that Φ is unique with this property, let Φ' be some other morphism of coalgebras satisfying $d_V \circ \Phi' = \varphi$. For each $i \in I$, the restriction $\Phi'|_{C_i}$ factors as a morphism of coalgebras through

$$\mathrm{Sym}_{P_1}(V) \oplus \cdots \oplus \mathrm{Sym}_{P_n}(V)$$

for some finite set of points $P_1, \ldots, P_n \in V$ (depending on i) since C_i is finite-dimensional. Moreover, $\Phi'|_{C_i}$ must factor further through a finite-dimensional subspace of this sum, so there exists a finite-dimensional subspace $W_i \subseteq V$ such that $P_1, \ldots, P_n \in W_i$ and $\Phi'|_{C_i}$ factors as

follows:

We may without loss of generality assume $V_i \subseteq W_i$ so that this factorisation Ψ_i and the composite of Φ_i with the inclusion into $\bigoplus_{P \in W_i} \mathrm{Sym}_P (W_i)$ are two morphisms of coalgebras which agree when post-composed with d_{W_i} (since $d_V \circ \Phi' = \varphi$). By the universal property, we conclude that they agree, and hence $\Phi'|_{C_i} = \Phi|_{C_i}$. It follows that $\Phi' = \Phi$. $\qquad\square$

Definition 2.12. *The pair* $(!V, d_V)$ *is referred to as the* cofree coalgebra generated by V.

The cofree coalgebra is used to model the exponential connective of linear logic in the Sweedler semantics, as explained in Section 2.4 below. For this reason, we refer to d_V as the *dereliction map*. If there is no chance of confusion, we will often write d for d_V. Note that the comultiplication Δ on $!V$ and the counit ε are derived from the comultiplication and counit on the summands $\mathrm{Sym}_P (V)$ in the obvious fashion; see Sweedler (1969, p. 50).

In the remainder of this section, V is an arbitrary vector space.

Definition 2.13. *For an integer $n > 0$, set $[n] = \{1, \ldots, n\}$.*

Definition 2.14. **(Ket notation).** *For $P, v_1, \ldots, v_s \in V$, we write*

$$|v_1, \ldots, v_s\rangle_P \in \mathrm{Sym}_P (V) \subseteq !V \tag{5}$$

for the image in $\mathrm{Sym}_P (V)$ of $v_1 \otimes \cdots \otimes v_s \in V^{\otimes s}$. Since such tensors span $\mathrm{Sym}_P (V)$, the kets in (5) span the vector space $!V$ as P ranges over all points of V. The identity $1 \in \mathrm{Sym}_P (V)$ is denoted as $|\emptyset\rangle_P$. Given $I = \{i_1, \ldots, i_t\} \subseteq [s]$, we write

$$|v_I\rangle_P := |v_{i_1}, \ldots, v_{i_t}\rangle_P . \tag{6}$$

In this notation, the formulas for the comultiplication, dereliction and counit are

$$\Delta : !V \longrightarrow !V \otimes !V \qquad \Delta|v_1, \ldots, v_s\rangle_P = \sum_{I \subseteq [s]} |v_I\rangle_P \otimes |v_{I^c}\rangle_P , \tag{7}$$

$$d : !V \longrightarrow V \qquad d|v_1, \ldots, v_s\rangle_P = \delta_{s=0} \cdot P + \delta_{s=1} v_1 , \tag{8}$$

$$\varepsilon : !V \longrightarrow k \qquad \varepsilon|v_1, \ldots, v_s\rangle_P = \delta_{s=0} \cdot 1 \tag{9}$$

where in the formula for Δ, I^c denotes the complement in $[s]$ and I ranges over all subsets, including the empty set. In particular, $\Delta|\emptyset\rangle_P = |\emptyset\rangle_P \otimes |\emptyset\rangle_P$, $d|v\rangle_P = v$ and $\varepsilon|\emptyset\rangle_P = 1$.

2.2 Group-like elements are points

In algebraic geometry, a k-point of an algebra A is by definition a morphism of algebras $A \longrightarrow k$, or equivalently a morphism of schemes $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A)$. In the case where $A = \mathrm{Sym}(V^*)$ for V finite-dimensional, such points are canonically identified with V itself:

$$\mathbf{Alg}_k(\mathrm{Sym}(V^*), k) \cong \mathrm{Hom}_k (V^*, k) \cong V^{**} \cong V .$$

This bijection identifies a point $P = (P_1, \ldots, P_n)$ in V with coordinates $P_i \in k$ in a chosen basis $e_1, \ldots, e_n$ for V with the maximal ideal

$$\mathfrak{m}_P = (x_1 - P_1, \ldots, x_n - P_n) \subseteq \mathrm{Sym}(V^*)$$

where as above we write $x_i = e_i^*$ for the dual basis. From the coalgebraic point of view, a k-point of a coalgebra C is a morphism of coalgebras $k \longrightarrow C$. Such morphisms are in canonical bijection with the *group-like elements* (Sweedler 1969, p. 57) of C

$$G(C) = \{x \in C \,|\, \Delta(x) = x \otimes x \text{ and } \varepsilon(x) = 1\}$$

via the bijection

$$\mathbf{Coalg}_k(k, C) \longrightarrow G(C)$$
$$\varphi \longmapsto \varphi(1) \,.$$

The points of a k-algebra A are related to the points of A° since by Theorem 2.2

$$\mathbf{Alg}_k(A, k) \cong \mathbf{Coalg}_k(k, A^\circ)$$

and so in particular

$$V \cong \mathbf{Alg}_k(\,\mathrm{Sym}(V^*), k) \cong \mathbf{Coalg}_k(k, \mathrm{Sym}(V^*)^\circ) \cong \mathbf{Coalg}_k(k, !V) \cong G(!V) \,.$$

2.3 Primitive elements are tangent vectors

From the point of view of algebraic geometry, a tangent vector at a k-point of an algebra A is a morphism of algebras $\varphi : A \longrightarrow k[\varepsilon]/(\varepsilon^2)$ (Hartshorne 1977, Example II.2.8), (Eisenbud and Harris 2000, II.3.1). The point of A at which such a tangent vector is 'attached' is given by the composite

$$A \xrightarrow{\quad \varphi \quad} k[\varepsilon]/(\varepsilon^2) \xrightarrow{\qquad\qquad} k \qquad\qquad (10)$$

where in the second map, $\varepsilon \mapsto 0$. The appropriateness of this definition can be seen readily in the case $A = \mathrm{Sym}(V^*)$ for V finite-dimensional where, using the coordinates x_i of the previous section to identify A with $k[x_1, \ldots, x_n]$, tangent vectors

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \qquad v_i \in k$$

at a point $P = (P_1, \ldots, P_n)$ are in bijective correspondence with k-algebra morphisms

$$\varphi_v : k[x_1, \ldots, x_n] \longrightarrow k[\varepsilon]/(\varepsilon^2)$$
$$f(x_1, \ldots, x_n) \longmapsto f(P) \cdot 1 + \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\Big|_P \cdot \varepsilon \,.$$

From the coalgebraic point of view, a tangent vector at a k-point of a coalgebra C is a morphism of coalgebras $\varphi : (k[\varepsilon]/(\varepsilon^2))^* \longrightarrow C$. We set $\mathcal{T} = (k[\varepsilon]/(\varepsilon^2))^*$. The k-point at which the tangent vector is attached is given by the composite

$$k \xrightarrow{\qquad\qquad} \mathcal{T} \xrightarrow{\quad \varphi \quad} C$$

where the first map is the dual of the second map in (10). Such morphisms are in canonical bijection with the *primitive elements* of C (Sweedler 1969, p. 199)

$$\mathrm{Prim}(C) = \{x \in C \,|\, \Delta(x) = x \otimes g + g \otimes x \text{ for some } g \in G(C)\}$$

via the bijection

$$\mathbf{Coalg}_k(\mathcal{T}, C) \longrightarrow \mathrm{Prim}(C)$$
$$\varphi \longmapsto \varphi(\varepsilon^*).$$

The tangent vectors at points of a k-algebra A are related to tangent vectors at points of the coalgebra A° via Theorem 2.2

$$\mathbf{Alg}_k(A, k[\varepsilon]/(\varepsilon^2)) \cong \mathbf{Alg}_k(A, (k[\varepsilon]/(\varepsilon^2))^{**}) \cong \mathbf{Coalg}_k(\mathcal{T}, A^\circ)$$

and in particular there is a canonical bijection

$$\mathrm{Prim}(!V) \cong \mathbf{Coalg}_k(\mathcal{T}, !V)$$
$$\cong \mathbf{Coalg}_k(\mathcal{T}, \mathrm{Sym}(V^*)^\circ)$$
$$\cong \mathbf{Alg}_k(\mathrm{Sym}(V^*), k[\varepsilon]/(\varepsilon^2))$$
$$\cong \mathrm{Hom}_k(V^*, k[\varepsilon]/(\varepsilon^2))$$
$$\cong V \otimes (k \oplus k\varepsilon)$$
$$\cong V \oplus V\varepsilon.$$

Given $P, v \in V$, the morphism of coalgebras $\mathcal{T} \longrightarrow !V$ corresponding to the pair $(P, v\varepsilon)$ under this bijection is precisely the morphism (2) alluded to in the introduction.

2.4 Definition of the Sweedler semantics

With the notation for the cofree coalgebra $!V$ and universal map d_V as introduced above (see Definition 2.12), we now recall the definition of the Sweedler semantics $[\![-]\!]$ in the category $\mathcal{V}$ of k-vector spaces from Hyland and Schalk (2003) and Murfet (2014, Sections 5.1 and 5.3). This is a Lafont model of intuitionistic linear logic in the sense of Melliès (2009, Section 7.2). For each atomic formula x of the logic, we choose a vector space $[\![x]\!]$. For formulas A, B, define

- $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$,
- $[\![A \multimap B]\!] = \mathrm{Hom}_k([\![A]\!], [\![B]\!])$,
- $[\![!A]\!] = ![\![A]\!]$.

If Γ is $A_1, ..., A_n$, then we define $[\![\Gamma]\!] = [\![A_1]\!] \otimes ... \otimes [\![A_n]\!]$. Given a linear map $\pi : C \longrightarrow V$ with C a coalgebra, we write $\mathrm{prom}(\pi) : C \longrightarrow !V$ for the unique morphism of coalgebras with $d_V \circ \mathrm{prom}(\pi) = \pi$ and we use the same notation in the syntax for the proof obtained from a proof π of $!\Gamma \vdash B$ by applying the promotion rule to obtain a proof of $!\Gamma \vdash !B$.

Definition 2.15. *The denotation $[\![\pi]\!]$ of a proof $\pi : \Gamma \vdash B$ is a linear map $[\![\pi]\!] : [\![\Gamma]\!] \to [\![B]\!]$ defined recursively on the structure of proofs. The proof π must match one of the proofs in the first column of Table 1, and the second column gives its denotation. Here, d denotes the dereliction map, Δ the comultiplication and ε the counit:*

Definition 2.16. *For a vector space V, we denote by*

$$\delta_V : !V \longrightarrow !!V$$

the unique morphism of coalgebras satisfying $d_{!V} \circ \delta_V = 1_{!V}$. Where there is no possibility of confusion, we write δ for δ_V.

Table 1. Denotations of proofs in the Sweedler semantics

$\dfrac{}{A \vdash A}\ {}^{\text{axiom}}$	$[\![\pi]\!](a) = a$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ {}^{\text{exch}} \end{array}$	$[\![\pi]\!](\gamma \otimes b \otimes a \otimes \delta) = [\![\pi_1]\!](\gamma \otimes a \otimes b \otimes \delta)$
$\begin{array}{cc} \pi_1 & \pi_2 \\ \vdots & \vdots \\ \multicolumn{2}{c}{\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ {}^{\text{cut}}} \end{array}$	$[\![\pi]\!](\gamma \otimes \delta) = [\![\pi_2]\!](\delta \otimes [\![\pi_1]\!](\gamma))$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ {}^{\otimes L} \end{array}$	$[\![\pi]\!](\gamma \otimes (a \otimes b)) = [\![\pi_1]\!](\gamma \otimes a \otimes b)$
$\begin{array}{cc} \pi_1 & \pi_2 \\ \vdots & \vdots \\ \multicolumn{2}{c}{\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ {}^{\otimes R}} \end{array}$	$[\![\pi]\!](\gamma \otimes \delta) = [\![\pi_1]\!](\gamma) \otimes [\![\pi_2]\!](\delta)$
$\begin{array}{cc} \pi_1 & \pi_2 \\ \vdots & \vdots \\ \multicolumn{2}{c}{\dfrac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ {}^{\multimap L}} \end{array}$	$[\![\pi]\!](\gamma \otimes \delta \otimes \varphi) = [\![\pi_2]\!](\delta \otimes \varphi \circ [\![\pi_1]\!](\gamma))$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ {}^{\multimap R} \end{array}$	$[\![\pi]\!](\gamma) = \{a \mapsto [\![\pi_1]\!](\gamma \otimes a)\}$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ {}^{\text{der}} \end{array}$	$[\![\pi]\!](\gamma \otimes \bar{a}) = [\![\pi_1]\!](\gamma \otimes d(\bar{a}))$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ {}^{\text{prom}} \end{array}$	$[\![\pi]\!] = \text{prom}[\![\pi_1]\!]$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ {}^{\text{weak}} \end{array}$	$[\![\pi]\!](\gamma \otimes \bar{a}) = \varepsilon(\bar{a})[\![\pi_1]\!](\gamma)$
$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ {}^{\text{ctr}} \end{array}$	$[\![\pi]\!](\gamma \otimes \bar{a}) = [\![\pi_1]\!](\gamma \otimes \Delta(\bar{a}))$

Given $P, v_1, \ldots, v_s \in V$, we have by Murfet (2015, Theorem 2.22)

$$\delta |v_1, \ldots, v_s\rangle_P = \sum_{\{C_1, \ldots, C_l\} \in \mathcal{P}_{[s]}} \Big| |v_{C_1}\rangle_P, \ldots, |v_{C_l}\rangle_P \Big\rangle_Q \tag{11}$$

where $\mathcal{P}_{[s]}$ denotes the set of partitions of $[s] = \{1, \ldots, s\}$ and $Q = |\emptyset\rangle_P$. Our partitions do not contain the empty set. As a special case $\delta |\emptyset\rangle_P = |\emptyset\rangle_Q$.

Given a linear map $f : V \longrightarrow W$, there is a unique morphism of coalgebras $!f : !V \longrightarrow !W$ with the property that $d_W \circ !f = f \circ d_V$ and this makes $!$ into a functor $! : \mathcal{V} \longrightarrow \mathcal{V}$. In fact, this functor is a comonad on $\mathcal{V}$, when equipped with the natural transformations $\delta : ! \longrightarrow !!$ and $d : ! \longrightarrow \mathrm{id}_{\mathcal{V}}$ which are component-wise the morphisms δ_V and d_V defined above. If we let Δ, ε denote the assignment of comultiplication and counit maps to all the vector spaces $!V$ then, by construction, we have in the sense of Blute et al. (2006, Definition 2.1):

Lemma 2.17. *The tuple $(!, \delta, d, \Delta, \varepsilon)$ is a coalgebra modality on $\mathcal{V}$.*

2.5 Local cohomology and distributions

In this section, we explain how the cofree coalgebra arises in algebraic geometry, since this connection gives a useful context for the differential structure of the Sweedler semantics; the contents of this section will however not be used in the sequel. For a finite-dimensional vector space V of dimension n with $R = \mathrm{Sym}(V^*)$, one proves using local duality (Murfet 2015, Theorem 2.6) that there is an isomorphism

$$\bigoplus_{P \in V} H_P^n(R, \Omega_{R/k}^n) \cong \mathrm{Sym}(V^*)^\circ \tag{12}$$

where H_P^n denotes local cohomology at P (Hartshorne 1966). This isomorphism is defined by sending a class τ in the local cohomology at P to the functional $f \mapsto \mathrm{Res}_P(f\tau)$ where Res_P denotes the generalised residue and $f\tau$ the action by R on local cohomology. The isomorphism (12) arises from isomorphisms $H_P^n(R, \Omega_{R/k}^n) \cong \mathrm{Sym}_P(V)$ identifying the identity $|\emptyset\rangle_P$ in $\mathrm{Sym}_P(V)$ with the class of the meromorphic differential form (Murfet 2015, Definition 2.9)

$$\left[\frac{dx_1 \wedge \cdots \wedge dx_n}{(x_1 - P_1), \ldots, (x_n - P_n)} \right] \in H_P^n(R, \Omega_{R/k}^n). \tag{13}$$

It is easy to see that

$$\mathrm{Res}_P\Big(f|v\rangle_P\Big) = \partial_v(f)|_{x=P}, \tag{14}$$

and more generally that (Murfet 2015, Lemma 2.13)

$$\mathrm{Res}_P\Big(f|v_1, \ldots, v_s\rangle_P\Big) = \partial_{v_1} \cdots \partial_{v_s}(f)|_{x=P}. \tag{15}$$

Thus, we may identify elements of $!V$ with functionals on the space of polynomial functions, given by evaluating derivatives at points of V.

Remark 2.18. When $k = \mathbb{C}$ with $V = \mathbb{C}v$ and $z = v^*$ the generator of $R = \mathbb{C}[z]$, this is nothing but the Cauchy integral formula since under the isomorphisms of Remark 2.8 we have the following identifications (of vectors in $\mathrm{Sym}_P(V)$ written in ket notation, with classes in local cohomology)

$$|\emptyset\rangle_P = \left[\frac{dz}{z - P} \right], \qquad |v\rangle_P = \left[\frac{dz}{(z - P)^2} \right] \tag{16}$$

and the Cauchy formula says

$$f'(P) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-P)^2} \, \mathrm{d}z \, .$$

Remark 2.19. When $k = \mathbb{R}$ this agrees with the analytic theory of distributions, since by Friedlander and Joshi (1998, Theorem 3.2.1) the $\mathbb{C}$-vector space of distributions on the real manifold V supported at a point P is spanned by the functions

$$f \longmapsto \partial_{v_1} \cdots \partial_{v_s}(f)|_{x=P}$$

as $s \geq 0$ and $v_1, \ldots, v_s$ varies over all sequences in V. So, in this case, we can identify the coalgebra $!V \otimes_{\mathbb{R}} \mathbb{C}$ with the space of distributions on V with finite support.

In the semantics of differential linear logic defined using finiteness spaces (Ehrhard 2005) and convenient vector spaces (Blute et al. 2010), the space $!V$ is a closure of the linear span of Dirac distributions (in our notation, $|\emptyset\rangle_P$) on V. More precisely, if V is a finite-dimensional convenient vector space, then $!V$ consists of distributions of compact support. For example, see Blute et al. (2006, Theorem 5.7) for the limit defining the distribution $|v\rangle_0$ in our notation. There is a similar role for Dirac distributions in the Coherent Banach space semantics of linear logic in Girard (1999, Section 3.2).

It is interesting to note that functional programs extended with Dirac distributions have already been considered in the literature on automatic differentiation; see Nilsson (2003). For an abstract categorical theory of distributions via monads, see Kock (2012).

Remark 2.20. Any cocommutative coalgebra is the direct limit of finite-dimensional coalgebras, and the category of finite-dimensional cocommutative coalgebras is isomorphic to the category of zero-dimensional schemes over k. This is taken as the starting point of one approach to noncommutative geometry which has been influential in the study of A_∞-algebras, where one posits that an arbitrary coalgebra is the coalgebra of distributions on a 'noncommutative space' (Kontsevich and Soibelman 2008, p. 15; Kontsevich and Soibelman 2002; Le Bruyn 2008).

3. Differential Linear Logic

Let k be an algebraically closed field of characteristic zero and $\mathcal{V}$ the category of k-vector spaces. This is a model of linear logic (see Section 2.4) when equipped with the comonad $!$ arising from the cofree coalgebra. We now explain how to equip this category with the structure necessary to make it a model of differential linear logic, following Blute et al. (2006).

Given vector spaces V, W, we write $\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V$ for the linear *swap map* defined on tensors by $\sigma_{V,W}(x \otimes y) = y \otimes x$. By Blute et al. (2006, Proposition 2.6) to equip $\mathcal{V}$ with the coalgebra modality $(!, \delta, d, \Delta, \varepsilon)$ of Lemma 2.17 as a differential category, we need to define a deriving transformation in the sense of Blute et al. (2006, Definition 2.5).

Definition 3.1. *A deriving transformation for* $(\mathcal{V}, !, \delta, d, \Delta, \varepsilon)$ *is a family of linear maps*

$$D_V : !V \otimes V \longrightarrow !V$$

defined for all $V \in \mathcal{V}$ *and natural in* V, *satisfying the following properties:*

(D.1) $\varepsilon \circ D = 0$, *that is,*

$$!V \otimes V \xrightarrow{\ D_V\ } !V \xrightarrow{\ \varepsilon\ } k = 0 \, . \tag{17}$$

(D.2) $\Delta \circ D = (1 \otimes D) \circ (\Delta \otimes 1) + (D \otimes 1) \circ (1 \otimes \sigma) \circ (\Delta \otimes 1)$, *that is,*

$$!V \otimes V \xrightarrow{\;\;D_V\;\;} !V \xrightarrow{\;\;\Delta\;\;} !V \otimes !V \tag{18}$$

is equal to the sum

$$!V \otimes V \xrightarrow{\;\;\Delta \otimes 1\;\;} !V \otimes !V \otimes V \xrightarrow{\;\;1 \otimes D_V\;\;} !V \otimes !V \quad +$$

$$!V \otimes V \xrightarrow{\;\;\Delta \otimes 1\;\;} !V \otimes !V \otimes V \xrightarrow[\cong]{\;\;1 \otimes \sigma_{!V,V}\;\;} !V \otimes V \otimes !V \xrightarrow{\;\;D_V \otimes 1\;\;} !V \otimes !V$$

(D.3) $d \circ D = a \circ (\varepsilon \otimes 1)$, *that is,*

$$!V \otimes V \xrightarrow{\;\;D_V\;\;} !V \xrightarrow{\;\;d\;\;} V \quad = \quad !V \otimes V \xrightarrow{\;\;\varepsilon \otimes 1\;\;} k \otimes V \xrightarrow[\cong]{\;\;a\;\;} V \tag{19}$$

where $a(\lambda \otimes x) = \lambda x$.

(D.4) $\delta \circ D = D \circ (\delta \otimes D) \circ (\Delta \otimes 1)$, *that is,*

$$!V \otimes V \xrightarrow{\;\;D_V\;\;} !V \xrightarrow{\;\;\delta\;\;} !!V \tag{20}$$

is equal to

$$!V \otimes V \xrightarrow{\;\;\Delta \otimes 1\;\;} !V \otimes !V \otimes V \xrightarrow{\;\;\delta \otimes D_V\;\;} !!V \otimes !V \xrightarrow{\;\;D_{!V}\;\;} !!V\,. \tag{21}$$

We refer to Blute et al. (2006, Section 2.2) for an explanation of these axioms. Briefly, (D.1) says the derivative of constant maps is zero, (D.2) is the product rule, (D.3) says the derivative of a linear map is constant and (D.4) is the chain rule. Clearly, the rules specify how to commute D past the structural maps $\delta, d, \Delta, \varepsilon$. Here d stands for the dereliction rule in linear logic, Δ for contraction and ε for weakening. The map δ stands for promotion, since for a linear map $\phi : !V \longrightarrow W$ the unique lifting to a morphism of coalgebras $\Phi : !V \longrightarrow !W$ can be obtained as the composite

$$!V \xrightarrow{\;\;\delta\;\;} !!V \xrightarrow{\;\;!\phi\;\;} !W\,. \tag{22}$$

Definition 3.2. *We define the k-linear map* $D_V : !V \otimes V \longrightarrow !V$ *by*

$$D_V\big(|v_1, \ldots, v_s\rangle_P \otimes v\big) = |v, v_1, \ldots, v_s\rangle_P\,. \tag{23}$$

Theorem 3.3. D_V *is a deriving transformation for any vector space* V.

We split the proof of the theorem into lemmas. We prefer to give the proofs without first choosing a basis of V, but if one is willing to do so, then the connection between these identities and the usual rules of calculus follows from writing the formula for the comultiplication Δ as a kind of Taylor expansion; see for example Seiler (2009, (B.65)). Throughout, we use the ket notation of Definition 2.14.

Lemma 3.4. *(D.1) holds for* V.

Proof. This is clear, since the counit $\varepsilon : !V \longrightarrow k$ vanishes on $|v_1, \ldots, v_s\rangle_P$ if $s > 0$. $\qquad\square$

Lemma 3.5. *(D.2) holds for* V.

Proof. Setting $v_0 = v$ we have (recall the notational conventions of Definition 2.14)

$$\Delta D\big(|v_1, \ldots, v_s\rangle_P \otimes v\big) = \Delta |v, v_1, \ldots, v_s\rangle_P$$

$$= \sum_{I \subseteq \{0,1,\ldots,s\}} |v_I\rangle_P \otimes |v_{I^c}\rangle_P$$

$$= \sum_{0 \in I} |v_I\rangle_P \otimes |v_{I^c}\rangle_P + \sum_{0 \notin I} |v_I\rangle_P \otimes |v_{I^c}\rangle_P$$

$$= \sum_{J \subseteq \{1,\ldots,s\}} |v, v_J\rangle_P \otimes |v_{J^c}\rangle_P + \sum_{J \subseteq \{1,\ldots,s\}} |v_J\rangle_P \otimes |v, v_{J^c}\rangle_P$$

$$= \sum_{J \subseteq \{1,\ldots,s\}} \Big\{ D\big(|v_J\rangle_P \otimes v\big) \otimes |v_{J^c}\rangle_P + |v_J\rangle_P \otimes D\big(|v_{J^c}\rangle_P \otimes v\big) \Big\}$$

as claimed, where for $I \subseteq \{0, 1, \ldots, s\}$ we write I^c for $\{0, \ldots, s\} \setminus I$ and for $J \subseteq \{1, \ldots, s\}$, we write J^c for $\{1, \ldots, s\} \setminus J$. $\qquad \square$

Lemma 3.6. *(D.3) holds for V.*

Proof. We have

$$dD\big(|v_1, \ldots, v_s\rangle_P \otimes v\big) = d|v, v_1, \ldots, v_s\rangle_P = \delta_{s=0} v$$

while

$$a(\varepsilon \otimes 1)\big(|v_1, \ldots, v_s\rangle_P \otimes v\big) = \varepsilon |v_1, \ldots, v_s\rangle_P \cdot v = \delta_{s=0} v.$$

$\qquad \square$

Lemma 3.7. *(D.4) holds for V.*

Proof. The trivial case is, with $Q = |\emptyset\rangle_P$,

$$\delta D(|\emptyset\rangle_P \otimes v) = \delta |v\rangle_P = \big| |v\rangle_P \big\rangle_Q$$

and on the other side

$$D_{!V}(\delta \otimes D)(\Delta \otimes 1)(|\emptyset\rangle_P \otimes v) = D_{!V}(\delta \otimes D)(|\emptyset\rangle_P \otimes |\emptyset\rangle_P \otimes v)$$

$$= D_{!V}(|\emptyset\rangle_Q \otimes |v\rangle_P)$$

$$= \big| |v\rangle_P \big\rangle_Q.$$

Now, we consider the case $s > 0$. Putting $v_0 = v$, we have

$$\delta D\big(|v_1, \ldots, v_s\rangle_P \otimes v\big) = \sum_{X \in \mathcal{P}_{\{0,1,\ldots,s\}}} \Big| \prod_{x \in X} |v_x\rangle_P \Big\rangle_Q$$

where for a partition $X = \{x_1, \ldots, x_t\}$ the notation means multiplication in the symmetric algebra, that is,

$$\Big| \prod_{x \in X} |v_x\rangle_P \Big\rangle_Q = \Big| |v_{x_1}\rangle_P, \ldots, |v_{x_t}\rangle_P \Big\rangle_Q.$$

There is a surjective function

$$\theta : \mathcal{P}_{\{0,1,\ldots,s\}} \longrightarrow \mathcal{P}_{\{1,\ldots,s\}}$$

$$\theta(X) = \big\{ x \setminus \{0\} \,|\, x \in X \text{ and } x \neq \{0\} \big\}$$

and given a partition $X = \{x_1, \ldots, x_t\}$ of $\{1, \ldots, s\}$,

$$\theta^{-1}(X) = \Big\{\{x_1 \cup \{0\}, x_2, \ldots, x_t\},$$
$$\{x_1, x_2 \cup \{0\}, \ldots, x_t\},$$
$$\ldots,$$
$$\{x_1, x_2, \ldots, x_{t-1}, x_t \cup \{0\}\}$$
$$\{x_1, x_2, \ldots, x_t, \{0\}\}\Big\}.$$

With this in mind we have, writing $\prod_{x' \neq x} |v_{x'}\rangle_P$ for the product in the symmetric algebra of $|v_{x'}\rangle_P$ as x' ranges over elements of $X \setminus \{x\}$, that $\delta D(|v_1, \ldots, v_s\rangle_P \otimes v)$ is equal to

$$\sum_{X \in \mathcal{P}_{\{1,\ldots,s\}}} \left\{ \sum_{x \in X} \Big| |v, v_x\rangle_P, \prod_{x' \neq x} |v_{x'}\rangle_P \Big\rangle_Q + \Big| |v\rangle_P, \prod_{x \in X} |v_x\rangle_P \Big\rangle_Q \right\}. \tag{24}$$

Note that when $X = \{\{1, \ldots, s\}\}$, the summand is

$$\Big| |v, v_1, \ldots, v_s\rangle_P \Big\rangle_Q + \Big| |v\rangle_P, |v_1, \ldots, v_s\rangle_P \Big\rangle_Q.$$

On the other hand, the right-hand side (21) of the (D.4) identity is

$$D_{!V}(\delta \otimes D)(\Delta \otimes 1)\big(|v_1, \ldots, v_s\rangle_P \otimes v\big)$$
$$= \sum_{I \subseteq \{1,\ldots,s\}} D_{!V}(\delta \otimes D)\big(|v_I\rangle_P \otimes |v_{I^c}\rangle_P \otimes v\big)$$
$$= \sum_{I \subseteq \{1,\ldots,s\}} D_{!V}\big(\delta |v_I\rangle_P \otimes |v, v_{I^c}\rangle_P\big)$$
$$= D_{!V}\big(\delta |\emptyset\rangle_P \otimes |v, v_1, \ldots, v_s\rangle_P\big)$$
$$\quad + \sum_{\emptyset \subset I} \sum_{Y \in \mathcal{P}_I} D_{!V}\Big(\big| \prod_{y \in Y} |v_y\rangle_P \big\rangle_Q \otimes |v, v_{I^c}\rangle_P\Big)$$
$$= \Big| |v, v_1, \ldots, v_s\rangle_P \Big\rangle_Q + \sum_{\emptyset \subset I} \sum_{Y \in \mathcal{P}_I} \Big| |v, v_{I^c}\rangle_P, \prod_{y \in Y} |v_y\rangle_P \Big\rangle_Q$$
$$= \Big| |v, v_1, \ldots, v_s\rangle_P \Big\rangle_Q + \sum_{Y \in \mathcal{P}_{\{1,\ldots,s\}}} \Big| |v\rangle_P, \prod_{y \in Y} |v_y\rangle_P \Big\rangle_Q$$
$$\quad + \sum_{\emptyset \subset I \subset \{1,\ldots,s\}} \sum_{Y \in \mathcal{P}_I} \Big| |v, v_{I^c}\rangle_P, \prod_{y \in Y} |v_y\rangle_P \Big\rangle_Q$$

which matches (24) since the last sum can be rewritten as

$$\sum_{\substack{X \in \mathcal{P}_{\{1,\ldots,s\}} \\ \text{with } |X| > 1}} \sum_{x \in X} \Big| |v, v_x\rangle_P, \prod_{x' \neq x} |v_{x'}\rangle_P \Big\rangle_Q.$$

$\square$

Together the previous lemmas complete the proof of Theorem 3.3.

Corollary 3.8. $\mathcal{V}$ *is a differential category.*

Proof. This follows from Blute et al. (2006, Proposition 2.6). $\square$

We are grateful to the referee who pointed out to us (see Cockett and Lemay 2018, Section 4.1) that an additional axiom is now routinely added to deriving transformations:

Lemma 3.9. *For any vector space V, the diagram*

$$\begin{array}{ccc}
!V \otimes V \otimes V & \xrightarrow{1 \otimes \sigma_{V,V}} !V \otimes V \otimes V \xrightarrow{D_V \otimes 1} & !V \otimes V \\
\downarrow{\scriptstyle D_V \otimes 1} & & \downarrow{\scriptstyle D_V} \\
!V \otimes V & \xrightarrow{\hspace{4cm} D_V \hspace{4cm}} & !V
\end{array}$$

commutes.

Proof. We compute

$$
\begin{aligned}
D_V(D_V \otimes 1)(1 \otimes \sigma)\left(|v_1, ..., v_s\rangle_P \otimes v \otimes v'\right) &= D_V(D_V \otimes 1)\left(|v_1, ..., v_s\rangle_P \otimes v' \otimes v\right) \\
&= D_V\left(|v', v_1, ..., v_s\rangle_P \otimes v\right) \\
&= |v, v', v_1, ..., v_s\rangle_P \\
&= |v', v, v_1, ..., v_s\rangle_P \\
&= |v, v_1, ..., v_s\rangle_P \otimes v' \\
&= D_V\left(|v, v_1, ..., v_s\rangle_P \otimes v'\right) \\
&= D_V(D_V \otimes 1)\left(|v_1, ..., v_s\rangle_P \otimes v \otimes v'\right).
\end{aligned}
$$

$\square$

Remark 3.10. From the geometric point of view (see Section 2.5), the definition of D_V is justified as follows: adding v to a ket contributes, inside a residue, a partial derivative in the direction v by (15). To state this more formally, let $\mathcal{D}(R)$ denote the algebra of k-linear differential operators on $R = \mathrm{Sym}(V^*)$ and observe that there is a canonical map $\iota : V \longrightarrow \mathcal{D}(R)$ sending $v \in V$ to the differential operator ∂_v and we have a map

$$H_P^n(R, \Omega_{R/k}^n) \otimes V \xrightarrow{\quad 1 \otimes \iota \quad} H_P^n(R, \Omega_{R/k}^n) \otimes \mathcal{D}(R) \xrightarrow{\quad a \quad} H_P^n(R, \Omega_{R/k}^n) \qquad (25)$$

where a denotes the action of the ring $\mathcal{D}(R)$ on local cohomology (Murfet 2015, Lemma 2.7). These maps assemble in the colimit (12) to give (23).

3.1 Codereliction, cocontraction and coweakening

An alternative formulation of the differential structure in differential linear logic is in terms of *codereliction, cocontraction* and *coweakening* maps; see Fiore (2007) and Blute et al. (2010, Section 5.1). This has the advantage of providing an appealing symmetry to the formulation of the syntax. In this section, we briefly sketch the definition of these maps in the Sweedler semantics.

First, we recall the canonical commutative Hopf structure on $!V$ of Sweedler (1969, Section 6.4).

Lemma 3.11. *Given vector spaces V_1, V_2, then there is an isomorphism of coalgebras*

$$\Theta : !V_1 \otimes !V_2 \longrightarrow !(V_1 \oplus V_2),$$

$$|v_1, \ldots, v_s\rangle_P \otimes |w_1, \ldots, w_t\rangle_Q \longmapsto |v_1, \ldots, v_s, w_1, \ldots, w_t\rangle_{(P,Q)}.$$

Proof. The existence of this isomorphism is due to Sweedler, for the explicit calculation of the map see Sweedler (1969, Remark 2.19). □

Using this and the definitions in Sweedler (1969), it is easy to check that the product ∇ is

$$\nabla : !V \otimes !V \longrightarrow !V,$$
$$|v_1, \ldots, v_s\rangle_P \otimes |w_1, \ldots, w_t\rangle_Q \longmapsto |v_1, \ldots, v_s, w_1, \ldots, w_t\rangle_{P+Q},$$

while the antipode S is

$$S : !V \longrightarrow !V,$$
$$|v_1, \ldots, v_s\rangle_P \longmapsto |-v_1, \ldots, -v_s\rangle_{-P}$$

and the unit $u : k \longrightarrow !V$ is $u(1) = |\emptyset\rangle_0$. By Sweedler (1969, Theorem 6.4.8), these maps make $!V$ into a commutative (and cocommutative) Hopf algebra. In the terminology of Ehrhard (2016), the map ∇ is the *cocontraction* map and u is the *coweakening* map (the antipode seems not to have a formal role in differential linear logic). Finally,

Definition 3.12. *The* coderelicition $\bar{d}$ *is the composite*

$$V \cong k \otimes V \xrightarrow{\;u \otimes 1\;} !V \otimes V \xrightarrow{\quad D \quad} !V$$

which is given by $v \mapsto |v\rangle_0$.

Note that we can recover D as

$$!V \otimes V \xrightarrow{\;1 \otimes \bar{d}\;} !V \otimes !V \xrightarrow{\quad \nabla \quad} !V$$
$$|v_1, \ldots, v_s\rangle_P \otimes v \mapsto |v_1, \ldots, v_s\rangle_P \otimes |v\rangle_0 \mapsto |v, v_1, \ldots, v_s\rangle_P.$$

It seems more convenient to model differentiation syntactically using the coderelicition, cocontraction and coweakening maps, rather than the deriving transformation D itself. We briefly sketch how this works, following Ehrhard (2016). In the sequent calculus for linear logic, one introduces three new deduction rules 'dual' to derelicition, contraction and weakening:

$$(\text{Coderelicition}): \frac{\Gamma, !A, \Delta \vdash B}{\Gamma, A, \Delta \vdash B} \; \text{coder}$$

$$(\text{Cocontraction}): \frac{\Gamma, !A, \Delta \vdash B}{\Gamma, !A, !A, \Delta \vdash B} \; \text{coctr}$$

$$(\text{Coweakening}): \frac{\Gamma, !A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \; \text{coweak}$$

together with new cut-elimination rules (Ehrhard 2016, Section 1.4.3).

Definition 3.13. *Given a proof π of $!A \vdash B$ in linear logic, the* derivative $\partial\pi$ *is the proof*

$$
\begin{array}{c}
\pi \\
\vdots \\
\cline{1-1}
!A \vdash B \\[-2pt]
\hline
\rule{0pt}{2ex}!A, !A \vdash B \\[-2pt]
\hline
\rule{0pt}{2ex}!A, A \vdash B
\end{array}
\;\; \text{coctr} \;\; \text{coder}
$$

(26)

whose denotation is, by our earlier remark, the composite

$$![\![A]\!] \otimes [\![A]\!] \xrightarrow{\quad D \quad} ![\![A]\!] \xrightarrow{\quad [\![\pi]\!] \quad} [\![B]\!] \,. \tag{27}$$

Remark 3.14. Given π as above, we have the function (Murfet 2014, Definition 5.10)

$$[\![\pi]\!]_{nl} : [\![A]\!] \longrightarrow [\![B]\!] \,, \qquad P \longmapsto [\![\pi]\!]|\emptyset\rangle_P \,, \tag{28}$$

and for $P, v \in [\![A]\!]$ we interpret the vector

$$[\![\pi]\!]D(|\emptyset\rangle_P \otimes v) = [\![\pi]\!]|v\rangle_P \in [\![B]\!] \tag{29}$$

as the derivative of $[\![\pi]\!]_{nl}$ at the point P in the direction v. Here, we implicitly identify $[\![A]\!]$ with the tangent space $T_P[\![A]\!]$ and $[\![B]\!]$ with the tangent space $T_{[\![\pi]\!]_{nl}(P)}[\![B]\!]$. This interpretation is justified by the following elaboration of the remarks in the Introduction.

Let $\mathrm{prom}\,(\pi)$ denote the promotion of π, which has for its denotation the unique morphism of coalgebras $[\![\mathrm{prom}\,(\pi)]\!] : ![\![A]\!] \longrightarrow ![\![B]\!]$ with $d \circ [\![\mathrm{prom}\,(\pi)]\!] = [\![\pi]\!]$. Let $\gamma : (k[\varepsilon]/\varepsilon^2)^* \longrightarrow ![\![A]\!]$ be the morphism of coalgebras as in Section 2.3 corresponding to the tangent vector v at a point $P \in [\![A]\!]$. Then the morphism of coalgebras

$$[\![\mathrm{prom}\,(\pi)]\!] \circ \gamma : (k[\varepsilon]/\varepsilon^2)^* \longrightarrow ![\![B]\!] \tag{30}$$

has the following values, writing $Q = [\![\pi]\!]_{nl}(P)$, we have by Murfet (2015, Theorem 2.22)

$$[\![\mathrm{prom}\,(\pi)]\!]\gamma(1) = [\![\mathrm{prom}\,(\pi)]\!]|\emptyset\rangle_P = |\emptyset\rangle_Q \,,$$

$$[\![\mathrm{prom}\,(\pi)]\!]\gamma(\varepsilon^*) = [\![\mathrm{prom}\,(\pi)]\!]|v\rangle_P = \Big| \, [\![\pi]\!]|v\rangle_P \Big\rangle_Q \,.$$

Under the bijection of Section 2.3, the morphism of coalgebras (30) therefore corresponds to the tangent vector $[\![\pi]\!]|v\rangle_P \in [\![B]\!]$ at Q.

It is easy using the formulas for ∇, D to check that the ∇-rule of Blute et al. (2006, Section 4.3) is satisfied:

Lemma 3.15. *The diagram*

$$\begin{array}{ccccc}
V \otimes !V \otimes !V & \xrightarrow{\ 1 \otimes \nabla\ } & V \otimes !V & \xrightarrow{\ \sigma_{V,!V}\ } & !V \otimes V \\[2pt]
{\scriptstyle \sigma_{V,!V} \otimes 1}\Big\downarrow & & & & \Big\downarrow{\scriptstyle D} \\[2pt]
!V \otimes V \otimes !V & \xrightarrow[\ D \otimes 1\ ]{} & !V \otimes !V & \xrightarrow[\ \nabla\ ]{} & !V
\end{array} \tag{31}$$

commutes.

This, together with Blute et al. (2006, Theorem 4.12), shows that $\mathcal{V}$ with the comonad ! and deriving transformation D is a model of the differential calculus in the sense of Blute et al. (2006, Definition 4.11).

Corollary 3.16. $\mathcal{V}$ *is a categorical model of the differential calculus.*

4. Examples

In this section, we give various examples of proofs π and the derivatives $[\![\pi]\!] \circ D$ of their denotations, according to Definition 3.13. The encoding of integers and binary sequences in linear logic is based on the following encoding of the composition rule.

Definition 4.1. *For any formula A, let C_A^1 denote the proof*

$$\frac{\overline{A \vdash A} \quad \overline{A \vdash A}}{A, A \multimap A \vdash A} \; {\multimap} L$$

We define recursively for $n > 1$ a proof C_A^n of $A, (A \multimap A)^n \vdash A$, where $(A \multimap A)^n$ denotes a sequence of n copies of $A \multimap A$, to be

$$
\begin{array}{c}
C_A^{n-1} \\
\vdots \\
\dfrac{\overline{A \vdash A} \quad A, (A \multimap A)^{n-1} \vdash A}{A, (A \multimap A)^n \vdash A} \; {\multimap} L
\end{array}
$$

Definition 4.2. *For $n \geq 1$, let $\underline{\mathrm{comp}}_A^n$ denote the proof*

$$
\begin{array}{c}
C_A^n \\
\vdots \\
\dfrac{A, (A \multimap A)^n \vdash A}{(A \multimap A)^n \vdash A \multimap A} \; {\multimap} R
\end{array}
$$

We define $\underline{\mathrm{comp}}_A^0$ to be the proof

$$\frac{\overline{A \vdash A}}{\vdash A \multimap A} \; {\multimap} R$$

Remark 4.3. If $V = [\![A]\!]$ and $\alpha_i \in [\![A \multimap A]\!] = \mathrm{End}_k(V)$ for $1 \leq i \leq n$, then

$$[\![\underline{\mathrm{comp}}_A^n]\!](\alpha_1 \otimes \cdots \otimes \alpha_n) = \alpha_n \circ \cdots \circ \alpha_1 \,, \tag{32}$$

while $[\![\underline{\mathrm{comp}}_A^0]\!] = 1_V$. Note the reversed ordering on the right-hand side!

4.1 Church numerals

Definition 4.4. *The type of* integers on A *(Girard 1987, Section 5.3.2) is*

$$\boldsymbol{int}_A = !(A \multimap A) \multimap (A \multimap A) \,.$$

For $n \geq 0$, we define the Church numeral $\underline{n}_A$ to be the proof

$$\mathrm{comp}^n_A$$

$$\vdots$$

$$\frac{\dfrac{(A \multimap A)^n \vdash A \multimap A}{\dfrac{!(A \multimap A)^n \vdash A \multimap A}{\dfrac{!(A \multimap A) \vdash A \multimap A}{\vdash \boldsymbol{int}_A} \,{\multimap R}}\, {n \times \, ctr}}\, {n \times \, der}}{}$$

Generally, we omit the final step, since it is irrelevant semantically. In the case $n = 0$, the formula $!(A \multimap A)$ is introduced on the left by a weakening rule.

Example 4.5. The proof $\underline{2}_A$ (see e.g., Murfet 2014, Example 5.9) is

$$\frac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{A \vdash A}}{A, A \multimap A \vdash A} \,{\multimap L}}{\dfrac{A, A \multimap A, A \multimap A \vdash A}{\dfrac{A \multimap A, A \multimap A \vdash A \multimap A}{\dfrac{!(A \multimap A), A \multimap A \vdash A \multimap A}{\dfrac{!(A \multimap A), !(A \multimap A) \vdash A \multimap A}{!(A \multimap A) \vdash A \multimap A} \,{ctr}} \,{der}} \,{der}} \,{\multimap R}} \,{\multimap L}}}{}$$

From now on, A is fixed and we write $\underline{n}$ for $\underline{n}_A$. Let $V = [\![A]\!]$ so $[\![A \multimap A]\!] = \mathrm{End}_k(V)$. In the notation of Remark 3.14, there is a function

$$[\![\underline{n}]\!]_{nl} : \mathrm{End}_k(V) \longrightarrow \mathrm{End}_k(V). \tag{33}$$

Lemma 4.6. *For $n \geq 0$ and $\alpha \in \mathrm{End}_k(V)$, we have $[\![\underline{n}]\!]|\emptyset\rangle_\alpha = \alpha^n$ so $[\![\underline{n}]\!]_{nl}(\alpha) = \alpha^n$.*

Proof. This is an easy exercise, see Murfet (2014) for the case $n = 2$. $\qquad\square$

The derivative $\partial \underline{n}$ of Definition 3.13 is a proof of $!(A \multimap A), A \multimap A \vdash A \multimap A$ and for $\alpha, v \in \mathrm{End}_k(V)$ the value of its denotation $[\![\partial \underline{n}]\!] = [\![\underline{n}]\!] \circ D$ on $|\emptyset\rangle_\alpha \otimes v$, that is, the derivative of $\underline{n}$ at α in the direction v, is $[\![\underline{n}]\!]|v\rangle_\alpha$.

Lemma 4.7. $[\![\underline{n}]\!]|v\rangle_\alpha = \sum_{i=1}^n \alpha^{i-1} v \alpha^{n-i}$.

Proof. This may be easily computed using the formulas (7), (8) for the comultiplication and dereliction. For example, in the case $n = 2$, the image of $|v\rangle_\alpha$ under $[\![\underline{n}]\!]$ is given by

$$|v\rangle_\alpha \xmapsto{\;\;ctr\;\;} |v\rangle_\alpha \otimes |\emptyset\rangle_\alpha + |\emptyset\rangle_\alpha \otimes |v\rangle_\alpha$$
$$\xmapsto{\;\;2 \times der\;\;} v \otimes \alpha + \alpha \otimes v$$
$$\xmapsto{\;\;-\circ-\;\;} \alpha \circ v + v \circ \alpha,$$

as claimed. $\qquad\square$

Remark 4.8. When $k = \mathbb{C}$, V is r-dimensional and $\varphi = [\![\underline{n}]\!]_{nl}$, the vector $[\![\underline{n}]\!]|v\rangle_\alpha$ agrees with the image of v under the usual tangent map of the smooth map φ

$$M_r(\mathbb{C}) \cong T_\alpha \, \mathrm{End}_k \, (V) \xrightarrow{\;\;T_\alpha\varphi\;\;} T_{\alpha^n} \, \mathrm{End}_k \, (V) \cong M_r(\mathbb{C}) \, .$$

This justifies in this case the interpretation of $[\![\underline{n}]\!] |v\rangle_\alpha$ as the derivative.

4.2 Binary integers

Definition 4.9. *The type of* binary integers on A *(Girard* 1995, *Section 2.5.3) is*

$$\textbf{bint}_A = \,!(A \multimap A) \multimap (!(A \multimap A) \multimap (A \multimap A)).$$

Given a sequence $S \in \{0,1\}^*$, we define a proof $\underline{S}_A$ of $\textbf{bint}_A$ as follows. Let $l \geq 0$ be the length of S. The proof tree for $\underline{S}_A$ matches that of the Church numeral $\underline{l}$ up to the step where we perform contractions, that is,

$$
\begin{array}{c}
\underline{\mathrm{comp}}^l_A \\[4pt]
\vdots \\[4pt]
\dfrac{(A \multimap A)^l \vdash A \multimap A}{!(A \multimap A)^l \vdash A \multimap A} \; {\scriptstyle n\times \, der}
\end{array}
\tag{34}
$$

We match each copy of $!(A \multimap A)$ on the left with the corresponding position in S, and using a series of contractions we identify all copies corresponding to a position in which 0 appears in S, and likewise all copies corresponding to positions with a 1. After these contractions, there will be two copies of $!(A \multimap A)$ on the left (the first being by convention the remnant of all the 0-associated copies) unless S contains only 0's or only 1's. In this case, we use further a weakening rule to introduce the 'missing' $!(A \multimap A)$, giving finally the desired proof $\underline{S}_A$:

$$
\begin{array}{c}
\underline{\mathrm{comp}}^l_A \\[4pt]
\vdots \\[4pt]
\dfrac{(A \multimap A)^l \vdash A \multimap A}{!(A \multimap A)^l \vdash A \multimap A} \; {\scriptstyle n\times \, der} \\[4pt]
\dfrac{}{!(A \multimap A), !(A \multimap A) \vdash A \multimap A} \; {\scriptstyle ctr \; and \; possibly \; weak} \\[4pt]
\dfrac{}{\vdash \textbf{bint}_A} \; {\scriptstyle 2\times \, \multimap R}
\end{array}
$$

In the final right $\multimap R$ introduction rules, the second copy of $!(A \multimap A)$ (associated with the 1's in S) is moved across the turnstile first. If S is the empty sequence, then $l = 0$ and the proof is a pair of weakenings on the left followed by the $\multimap R$ introduction rules.

For the rest of this section, A is fixed and we write $\underline{S}$ for $\underline{S}_A$.

Example 4.10. The proof $\underline{001}$ is

$$
\dfrac{
  \dfrac{
    A \vdash A \qquad
    \dfrac{
      \dfrac{\overline{A \vdash A} \qquad \overline{A \vdash A}}{A, A \multimap A \vdash A} \; {\scriptstyle \multimap L}
    }{A, A \multimap A, A \multimap A \vdash A} \; {\scriptstyle \multimap L}
  }{
    \dfrac{A \vdash A \qquad A, A \multimap A, A \multimap A \vdash A}{A, A \multimap A, A \multimap A, A \multimap A \vdash A} \; {\scriptstyle \multimap L}
  }
}{
  \dfrac{
    \dfrac{A \multimap A, A \multimap A, A \multimap A \vdash A \multimap A}{\color{red}!(A \multimap A), !(A \multimap A), !(A \multimap A) \vdash A \multimap A} \; {\scriptstyle 3\times \, der}
  }{
    \dfrac{\color{red}!(A \multimap A), !(A \multimap A) \vdash A \multimap A}{\vdash \textbf{bint}_A} \; {\scriptstyle 2\times \, \multimap R}
  } \; {\scriptstyle ctr}
} \; {\scriptstyle \multimap R}
$$

where the colouring indicates which copies of $!(A \multimap A)$ are contracted. Using (32),

$$[\![\underline{001}]\!]\big(|\emptyset\rangle_\gamma \otimes |\emptyset\rangle_\delta\big) = [\![\underline{\mathrm{comp}^3_A}]\!]\big(|\emptyset\rangle_\gamma \otimes |\emptyset\rangle_\gamma \otimes |\emptyset\rangle_\delta\big) = \delta \circ \gamma \circ \gamma \,. \tag{35}$$

Generalising the calculation of Section 4.1, we now describe the derivatives of binary integers. The general formula computes, for $S \in \{0, 1\}^*$, the linear operator

$$[\![\underline{S}]\!]\big(|\alpha_1, \ldots, \alpha_r\rangle_\gamma \otimes |\beta_1, \ldots, \beta_s\rangle_\delta\big) \in \mathrm{End}_k (V)\,.$$

Informally, this operator is described by inserting γ for 0 and δ for 1 in (the reversal of) S, and then summing over all ways of replacing r of the γ's in this composite with α_i's, and t of the δ's with β_j's. Let Inj (P, Q) denote the set of injective functions $P \longrightarrow Q$, and write $[s] = \{1, \ldots, s\}$.

Lemma 4.11. *Let* $S = a_l a_{l-1} \cdots a_1$ *with* $a_i \in \{0, 1\}$ *be a binary sequence, and set*

$$N_0 = \{j \,|\, a_j = 0\}\,, \qquad N_1 = \{j \,|\, a_j = 1\}\,.$$

Then we have

$$[\![\underline{S}]\!]\big(|\alpha_1, \ldots, \alpha_s\rangle_\gamma \otimes |\beta_1, \ldots, \beta_r\rangle_\delta\big) = \sum_{f \in \mathrm{Inj}\,([s], N_0)} \sum_{g \in \mathrm{Inj}\,([r], N_1)} \Gamma^{f,g}_1 \circ \cdots \circ \Gamma^{f,g}_l \,, \tag{36}$$

where

$$\Gamma^{f,g}_i = \begin{cases} \gamma & i \in N_0 \setminus \mathrm{Im}\,(f)\,, \\ \delta & i \in N_1 \setminus \mathrm{Im}\,(g)\,, \\ \alpha_j & \text{if } i \in \mathrm{Im}\,(f) \text{ and } f(j) = i\,, \\ \beta_j & \text{if } i \in \mathrm{Im}\,(g) \text{ and } g(j) = i\,. \end{cases}$$

In particular, this vanishes if $s > |N_0|$ *or* $r > |N_1|$.

Proof. Since $[\![\underline{S}]\!]$ applies $n = |N_0|$ coproducts to $|\alpha_1, \ldots, \alpha_s\rangle_\gamma$ yielding

$$\sum_{\substack{J_1, \ldots, J_n \\ \text{pairwise disjoint, s.t.} \\ J_1 \cup \cdots \cup J_n = \{1, \ldots, s\}}} |\alpha_{J_1}\rangle_\gamma \otimes \cdots \otimes |\alpha_{J_n}\rangle_\gamma \,,$$

to which the dereliction operator $d^{\otimes n}$ is applied, which annihilates those tuples $(J_1, \ldots, J_n)$ where any J_i contains more than one element. The resulting sum is over $f \in \mathrm{Int}\,([s], N_0)$ and each summand is $\gamma \otimes \cdots \otimes \alpha_{\sigma(1)} \otimes \cdots \otimes \gamma \otimes \cdots \alpha_{\sigma(s)} \otimes \cdots \otimes \gamma$ for a permutation σ. The same is true of $|\beta_1, \ldots, \beta_r\rangle_\delta$, and after the two resulting tensors are intertwined the final step is compose all the operators, yielding (36).   □

Example 4.12. For $S = 001$, we have

$$[\![\underline{001}]\!]\big(|\alpha\rangle_\gamma \otimes |\emptyset\rangle_\delta\big) = \delta \circ \alpha \circ \gamma + \delta \circ \gamma \circ \alpha\,,$$

$$[\![\underline{001}]\!]\big(|\alpha_1, \alpha_2\rangle_\gamma \otimes |\emptyset\rangle_\delta\big) = \delta \circ \alpha_1 \circ \alpha_2 + \delta \circ \alpha_2 \circ \alpha_1\,,$$

$$[\![\underline{001}]\!]\big(|\emptyset\rangle_\gamma \otimes |\beta\rangle_\delta\big) = \beta \circ \gamma \circ \gamma\,,$$

$$[\![\underline{001}]\!]\big(|\alpha\rangle_\gamma \otimes |\beta\rangle_\delta\big) = \beta \circ \alpha \circ \gamma + \beta \circ \gamma \circ \alpha\,,$$

$$[\![\underline{001}]\!]\big(|\alpha_1, \alpha_2\rangle_\gamma \otimes |\beta\rangle_\delta\big) = \beta \circ \alpha_1 \circ \alpha_2 + \beta \circ \alpha_2 \circ \alpha_1\,.$$

and zero for all other inputs.

More interestingly, we can also compute the derivatives of proofs of $!\mathbf{bint}_A \vdash \mathbf{bint}_A$. In what follows, A is fixed and $E = A \multimap A$.

Definition 4.13. *The proof* repeat *is*

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{!E \vdash !E \quad \cfrac{\cfrac{!E \vdash !E \quad \cfrac{\cfrac{!E \vdash !E \quad E, E \vdash E}{!E, !E \multimap E, E \vdash E}{\scriptstyle \multimap L}}{!E, !E, \boldsymbol{bint}_A, E \vdash E}{\scriptstyle \multimap L}}{!E, !E, !E, \boldsymbol{bint}_A, !E \multimap E \vdash E}{\scriptstyle \multimap L}}{!E, !E, !E, !E, \boldsymbol{bint}_A, \boldsymbol{bint}_A \vdash E}{\scriptstyle \multimap L}}}{!E, !E, !E, \boldsymbol{bint}_A, \boldsymbol{bint}_A \vdash E}{\scriptstyle ctr}}{!E, !E, \boldsymbol{bint}_A, \boldsymbol{bint}_A \vdash E}{\scriptstyle ctr}}{\boldsymbol{bint}_A, \boldsymbol{bint}_A \vdash \boldsymbol{bint}_A}{\scriptstyle 2\times \multimap R}}{!\boldsymbol{bint}_A, !\boldsymbol{bint}_A \vdash \boldsymbol{bint}_A}{\scriptstyle 2\times der}}{!\boldsymbol{bint}_A \vdash \boldsymbol{bint}_A}{\scriptstyle ctr}
$$

with comp^2_A above.

which repeats a binary sequence in the sense that the cutting it against the promotion of $\underline{S}$ is equivalent under cut-elimination to $\underline{SS}$. In particular, $[\![\text{repeat}]\!]|\emptyset\rangle_{[\![S]\!]} = [\![SS]\!]$.

Given $S, T \in \{0, 1\}^*$, the derivative of repeat at S in the direction T is

$$
[\![\text{repeat}]\!]|[\![T]\!]\rangle_{[\![S]\!]} \in [\![\boldsymbol{bint}_A]\!] = \mathrm{Hom}_k \left(!\, \mathrm{End}_k \left(V \right) \otimes !\, \mathrm{End}_k \left(V \right), \mathrm{End}_k \left(V \right) \right), \tag{37}
$$

and as promised in the Introduction:

Lemma 4.14. $[\![\text{repeat}]\!]|[\![T]\!]\rangle_{[\![S]\!]} = [\![ST]\!] + [\![TS]\!]$.

Proof. The value of the left-hand side on a tensor $|\alpha_1, \ldots, \alpha_s\rangle_\gamma \otimes |\beta_1, \ldots, \beta_r\rangle_\delta$ is computed by reading the proof tree for repeat from bottom to top:

$$
\begin{aligned}
|[\![T]\!]\rangle_{[\![S]\!]} &\xmapsto{\;ctr\;} |[\![T]\!]\rangle_{[\![S]\!]} \otimes |\emptyset\rangle_{[\![S]\!]} + |\emptyset\rangle_{[\![S]\!]} \otimes |[\![T]\!]\rangle_{[\![S]\!]} \\
&\xmapsto{\;2\times der\;} [\![T]\!] \otimes [\![S]\!] + [\![S]\!] \otimes [\![T]\!] \\
&\xmapsto{\;2\times R \multimap\;} |\alpha_1, \ldots, \alpha_s\rangle_\gamma \otimes |\beta_1, \ldots, \beta_r\rangle_\delta \otimes \left([\![T]\!] \otimes [\![S]\!] + [\![S]\!] \otimes [\![T]\!] \right) \\
&\xmapsto{\;2\times ctr\;} \sum_{I,J} |\alpha_I\rangle_\gamma \otimes |\beta_J\rangle_\delta \otimes |\alpha_{I^c}\rangle_\gamma \otimes |\beta_{J^c}\rangle_\delta \otimes \left([\![T]\!] \otimes [\![S]\!] + [\![S]\!] \otimes [\![T]\!] \right) \\
&\xmapsto{\hspace{1.5cm}} \sum_{I,J} [\![S]\!]\left(|\alpha_I\rangle_\gamma \otimes |\beta_J\rangle_\delta \right) \circ [\![T]\!]\left(|\alpha_{I^c}\rangle_\gamma \otimes |\beta_{J^c}\rangle_\delta \right) \\
&\qquad + \sum_{I,J} [\![T]\!]\left(|\alpha_I\rangle_\gamma \otimes |\beta_J\rangle_\delta \right) \circ [\![S]\!]\left(|\alpha_{I^c}\rangle_\gamma \otimes |\beta_{J^c}\rangle_\delta \right)
\end{aligned}
$$

which agrees with $[\![ST]\!] + [\![TS]\!]$ on $|\alpha_1, \ldots, \alpha_s\rangle_\gamma \otimes |\beta_1, \ldots, \beta_r\rangle_\delta$ by Lemma 4.11. □

4.3 Multiplication

The multiplication of Church numerals is encoded by a proof $\underline{\text{mult}}_A$ of $!\boldsymbol{int}_A, \boldsymbol{int}_A \vdash \boldsymbol{int}_A$, see for example Girard (1995, Section 2.5.2). To construct the proof tree, it will be convenient to introduce the following intermediate proof γ, writing $E = A \multimap A$ as above:

$$\frac{\dfrac{\overline{!E \vdash !E} \quad \overline{E \vdash E}}{!E, \mathbf{int}_A \vdash E}{}_{\multimap L}}{\dfrac{!E, !\mathbf{int}_A \vdash E}{!E, !\mathbf{int}_A \vdash !E}{}_{\text{prom}}}{}_{\text{der}}$$

Then $\llbracket \gamma \rrbracket : \,! \operatorname{End}_k (V) \otimes \,! \llbracket \mathbf{int}_A \rrbracket \to \,! \operatorname{End}_k (V)$ is a morphism of coalgebras such that

$$\llbracket \gamma \rrbracket (t \otimes |\emptyset\rangle_\alpha) = |\emptyset\rangle_{\alpha(t)} \qquad \text{and} \qquad \llbracket \gamma \rrbracket (t \otimes |v\rangle_\alpha) = |v(t)\rangle_{\alpha(t)},$$

for $\alpha, v \in \llbracket \mathbf{int}_A \rrbracket = \operatorname{Hom}_k (\,! \operatorname{End}_k (V), \operatorname{End}_k (V))$ and $t \in \,! \operatorname{End}_k (V)$. The proof $\underline{\operatorname{mult}}_A$ is

$$
\gamma
$$
$$
\vdots
$$
$$\frac{\dfrac{!E, !\mathbf{int}_A \vdash !E \quad \overline{E \vdash E}}{!E, !\mathbf{int}_A, \mathbf{int}_A \vdash E}{}_{\multimap L}}{!\mathbf{int}_A, \mathbf{int}_A \vdash \mathbf{int}_A}{}_{\multimap R}$$

Let $l, m, n \geq 0$ be integers. We write $\underline{\operatorname{mult}}_A(-, n)$ for the proof of $!\mathbf{int}_A \vdash \mathbf{int}_A$ obtained from the above by cutting against the proof $\underline{n}$ of $\vdash \mathbf{int}_A$. The derivative of this proof at $\alpha = \llbracket l \rrbracket$ in the direction $v = \llbracket m \rrbracket$ is the element of $\llbracket \mathbf{int}_A \rrbracket$ given on $t \in \,! \operatorname{End}_k (V)$ by

$$\llbracket \underline{\operatorname{mult}}_A \rrbracket \big(|v\rangle_\alpha \otimes \llbracket \underline{n} \rrbracket\big)(t) = \llbracket \underline{n} \rrbracket \big(|v(t)\rangle_{\alpha(t)}\big) = \sum_{i=1}^{n} \alpha(t)^{i-1} v(t) \alpha(t)^{n-i}$$

using Lemma 4.7 in the last step. When $t = |\emptyset\rangle_x$ for $x \in \operatorname{End}_k (V)$, this evaluates to

$$\sum_{i=1}^{n} \alpha(t)^{i-1} v(t) \alpha(t)^{n-i} = \sum_{i=1}^{n} x^{l(i-1)} x^m x^{l(n-i)} = n x^{l(n-1)+m}.$$

This result agrees with a more traditional calculus approach using limits:

$$\lim_{h \to 0} \frac{\llbracket \underline{\operatorname{mult}}_A \rrbracket (|\emptyset\rangle_{\llbracket l \rrbracket + h \llbracket m \rrbracket} \otimes \llbracket \underline{n} \rrbracket) |\emptyset\rangle_x - \llbracket \underline{\operatorname{mult}}_A \rrbracket (|\emptyset\rangle_{\llbracket l \rrbracket} \otimes \llbracket \underline{n} \rrbracket) |\emptyset\rangle_x}{h}$$

$$= \lim_{h \to 0} \frac{\llbracket \underline{n} \rrbracket |\emptyset\rangle_{x^l + hx^m} - \llbracket \underline{n} \rrbracket |\emptyset\rangle_{x^l}}{h}$$

$$= \lim_{h \to 0} \frac{(x^l + hx^m)^n - x^{ln}}{h}$$

$$= n x^{l(n-1)+m}.$$

5. Differential Lambda Calculus

Categorically speaking λ-calculi are modelled by Cartesian closed categories (Lambek and Scott 1986) and any model of linear logic gives rise to a Cartesian closed category by taking the Kleisli category of the comonad interpreting the exponential connective; see Melliès (2009, Section 7). There is a parallel relationship between differential λ-calculus and differential linear logic, which has been worked out in the language of Cartesian differential categories (Blute et al. 2009; Bucciarelli et al. 2010).

As above k is an algebraically closed field of characteristic zero, and $\mathcal{V}$ is the category of k-vector spaces. In this section, we explain how the category of cofree coalgebras over k gives a model of differential λ-calculus, by

- checking that $\mathcal{V}$ is a differential storage category with biproducts, so that
- by Blute et al. (2009, Proposition 3.2.1), the Kleisli category $\mathcal{V}_!$ of $(\mathcal{V}, !)$ is a Cartesian differential category,
- and we check that $\mathcal{V}_!$ is a differential λ-category in the sense of Bucciarelli et al. (2010, Definition 4.4).

We have already checked that the comonad $!$ arising from the cofree coalgebra construction defines a coalgebra modality in the sense of Blute et al. (2006, Definition 2.1), and that we have deriving transformations $D_V : !V \otimes V \longrightarrow !V$. By Corollary 3.8, the tuple $(\mathcal{V}, !, \delta, d, \Delta, \varepsilon, D)$ is a differential category in the sense of Blute et al. (2006, Definition 2.4). We now prove further that this is an example of a *differential storage category*.

For this, we need to understand $! : \mathcal{V} \longrightarrow \mathcal{V}$ as a monoidal functor.

Definition 5.1. *Given vector spaces V, W, let $m_{V,W}$ be the unique morphism of coalgebras which makes the following diagram commute:*

$$
\begin{array}{ccc}
!V \otimes !W & \xrightarrow{\;m_{V,W}\;} & !(V \otimes W) \\
& \searrow{\scriptstyle d_V \otimes d_W} & \downarrow{\scriptstyle d_{V \otimes W}} \\
& & V \otimes W
\end{array}
$$

Lemma 5.2. *The morphism of coalgebras $m_{V,W}$ is natural in V, W.*

Proof. If $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$ are linear, then the outside diagram in

$$
\begin{array}{ccccc}
!V \otimes !W & \xrightarrow{\;!\varphi \otimes !\psi\;} & & & !V' \otimes !W' \\
\downarrow{\scriptstyle m_{V,W}} & \searrow{\scriptstyle d_V \otimes d_W} & & \swarrow{\scriptstyle d_{V'} \otimes d_{W'}} & \downarrow{\scriptstyle m_{V',W'}} \\
& V \otimes W & \xrightarrow{\;\varphi \otimes \psi\;} & V' \otimes W' & \\
& \searrow{\scriptstyle d_{V \otimes W}} & & \swarrow{\scriptstyle d_{V \otimes W}} & \\
!(V \otimes W) & & \xrightarrow{\;!(\varphi \otimes \psi)\;} & & !(V' \otimes W')
\end{array}
$$

commutes, by virtue of each of the sub-diagrams commuting. □

Definition 5.3. *Let $u : k \longrightarrow !k$ be the unique morphism of coalgebras with the property that $d_k \circ u = 1_k$. Explicitly, $u(1) = |\emptyset\rangle_1$.*

Lemma 5.4. *The data $\{m_{V,W}\}_{V,W \in \mathcal{V}}$ and u make $! : \mathcal{V} \longrightarrow \mathcal{V}$ a lax monoidal functor.*

Proof. We have to check commutativity of the outer square in

$$(!A \otimes !B) \otimes !C \xrightarrow{\;\;\alpha\;\;} !A \otimes (!B \otimes !C)$$

with morphisms $m_{A,B} \otimes 1$, $d^{\otimes 3}$, $d^{\otimes 3}$, $1 \otimes m_{B,C}$,
$!(A \otimes B) \otimes !C \xrightarrow{d^{\otimes 2}} (A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C) \xleftarrow{d^{\otimes 2}} !A \otimes !(B \otimes C)$,
$m_{A \otimes B, C}$, d, d, $m_{A, B \otimes C}$,
$!((A \otimes B) \otimes C) \xrightarrow{\;\;!\alpha\;\;} !(A \otimes (B \otimes C))$
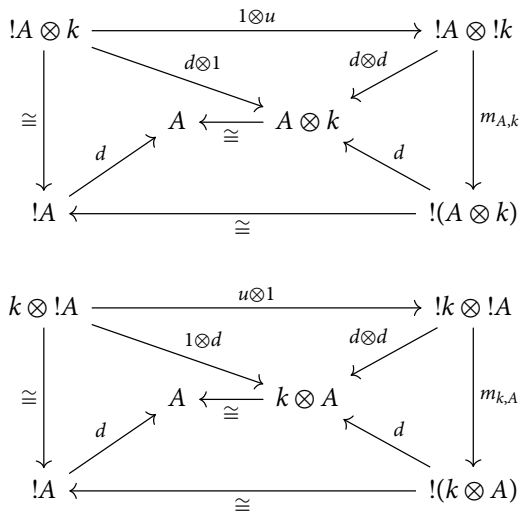
and the outer squares in

$$!A \otimes k \xrightarrow{\;\;1 \otimes u\;\;} !A \otimes !k$$

with $d \otimes 1$, $d \otimes d$, $A \xleftarrow{\cong} A \otimes k$, $m_{A,k}$, d, d,
$!A \xleftarrow{\;\;\cong\;\;} !(A \otimes k)$

$$k \otimes !A \xrightarrow{\;\;u \otimes 1\;\;} !k \otimes !A$$

with $1 \otimes d$, $d \otimes d$, $A \xleftarrow{\cong} k \otimes A$, $m_{k,A}$, d, d,
$!A \xleftarrow{\;\;\cong\;\;} !(k \otimes A)$
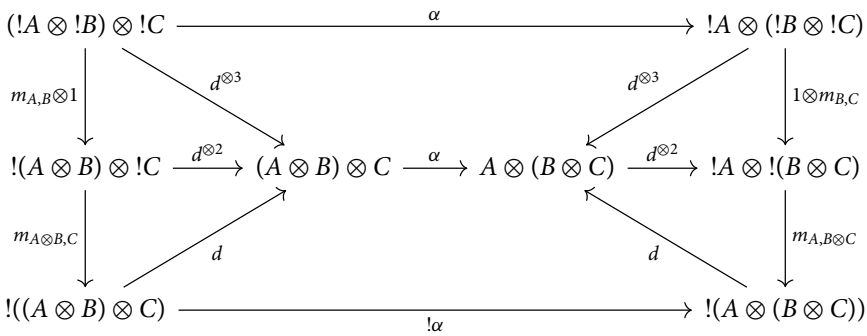
but from the given decompositions of these squares, this is clear. □

Lemma 5.5. *The tuple* $(!, m, u)$ *is a symmetric monoidal functor on* $(\mathcal{V}, \otimes, k)$.

Proof. This means that the following diagram commutes

$$!V \otimes !W \xrightarrow{\;\;m\;\;} !(V \otimes W)$$

with $d \otimes d$, $V \otimes W$, d, σ, σ, $!\sigma$,
$W \otimes V$, $d \otimes d$, d,
$!W \otimes !V \xrightarrow{\;\;m\;\;} !(W \otimes V)$
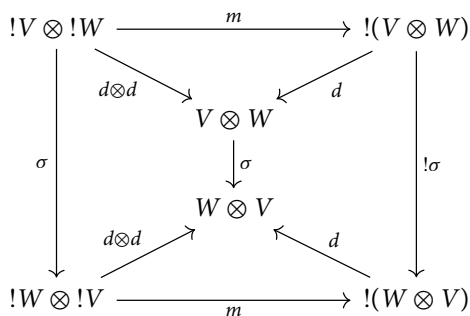
which is clear from the given decomposition. □

Lemma 5.6. *The natural transformations* $\delta : ! \longrightarrow !!$ *and* $d : ! \longrightarrow \mathrm{id}_\mathcal{V}$ *are lax monoidal.*

Proof. We need to check commutativity of

$$
\begin{array}{ccc}
!A \otimes !B \xrightarrow{\ \delta_A \otimes \delta_B\ } !!A \otimes !!B \xrightarrow{\ m_{!A,!B}\ } !(!A \otimes !B)
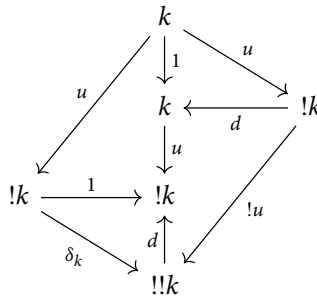\end{array}
$$
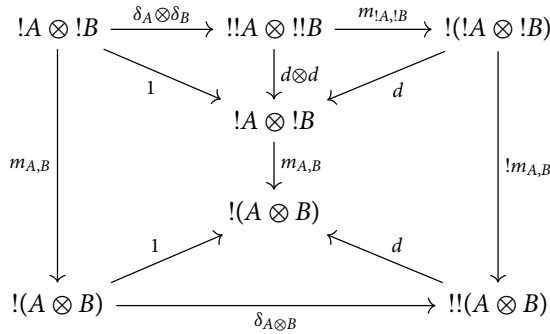
and also

and
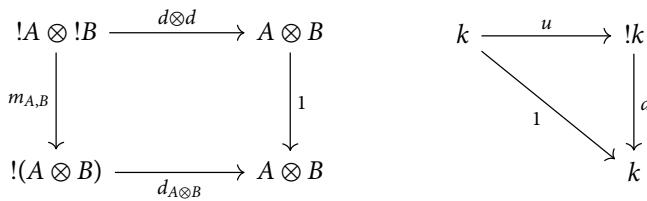
which are all clear from the given decompositions. □

The above results together prove that:

Lemma 5.7. $(!, \delta, d, m, u)$ *is a symmetric monoidal comonad.*

Now each $!A$ is by construction a cocommutative comonoid in the category of vector spaces, and the comultiplication $\Delta : !A \longrightarrow !A \otimes !A$ and counit $\varepsilon : !A \longrightarrow k$ are morphisms of !-coalgebras in the following sense. Recall that a coalgebra (Borceux 1994, Section 4.1) for the comonad $(!, \delta, d)$ (henceforth called a !-*coalgebra*) is a vector space V and linear map $\varphi : V \longrightarrow !V$ with the property that the diagrams

(38)
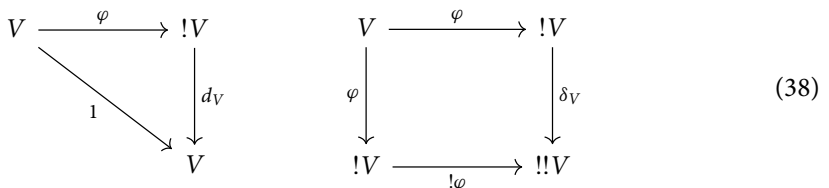
commute. We make $!A \otimes !A$ into a !-coalgebra by

$$!A \otimes !A \xrightarrow{\delta_A \otimes \delta_A} !!A \otimes !!A \xrightarrow{m_{!A,!A}} !(!A \otimes !A)$$

and k is a coalgebra via $u : k \longrightarrow !k$. Then (see Blute et al. 2006, Definition 4.1):

Lemma 5.8. *The tuple* $(!, \delta, d)$ *is a storage modality on* $(\mathcal{V}, \otimes, k)$ *when we take the canonical comonoid structures* $(!A, \Delta, \varepsilon)$.

Proof. We need only check Δ, ε are morphisms of !-coalgebras. In the first case, consider the commutative diagram

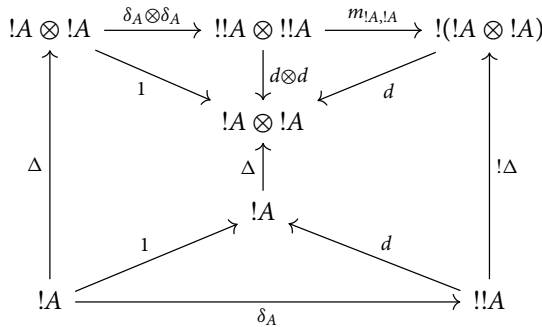Now since $!A$ is cocommutative Δ is a morphism of coalgebras, so we may use the universal property of $!(!A \otimes !A)$ as indicated. The fact that $\varepsilon : !A \longrightarrow k$ is a morphism of !-coalgebras is exhibited by the diagram
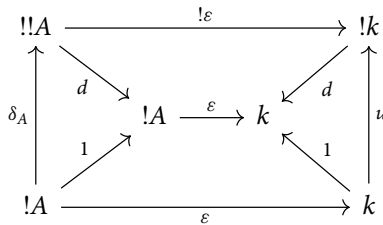
where we use that ε is a morphism of coalgebras. $\square$

We conclude that $\mathcal{V}$ has the structure of a differential storage category, in the sense of Blute et al. (2006, Definition 4.10).

Theorem 5.9. *With the above structure* $\mathcal{V}$ *is a differential storage category.*

Proof. We have shown $\mathcal{V}$ is an additive storage category (Blute et al. 2006, Definition 4.4), with a deriving transformation satisfying the ∇-rule (Lemma 3.15) so we are done. $\square$

5.1 Cartesian differential categories

The Kleisli category of the comonad ! on $\mathcal{V}$ is equivalent to the category of cofree coalgebras; for the reader's convenience, we recall the proof in Appendix A. Under this equivalence, the tensor product of coalgebras, which gives the categorical product in **Coalg**$_k$ by Sweedler (1969, p. 49, 65), corresponds to the product $(X, Y) \mapsto X \oplus Y$ on the Kleisli category.

By Blute et al. (2009, Proposition 3.2.1), the Kleisli category $\mathcal{V}_!$ is a Cartesian differential category. In order to describe the Cartesian differential structure, recall that there is a bijection

$$\mathbf{Coalg}_k(!X, !Y) \xrightarrow{\;\cong\;} \mathrm{Hom}_k\,(!X, Y) = \mathcal{V}_!(X, Y)$$
$$F \longmapsto d_Y \circ F\,.$$

The Cartesian differential operator on $\mathcal{V}_!$ prescribed by Blute et al. (2009) is a function

$$\mathbb{D}_{X,Y} : \mathcal{V}_!(X, Y) \longrightarrow \mathcal{V}_!(X \times X, Y)$$

which may be viewed as a function

$$\mathrm{Hom}_k\,(!X, Y) \longrightarrow \mathrm{Hom}_k\,(!(X \oplus X), Y)$$

or equivalently as a function

$$\mathbf{Coalg}_k(!X, !Y) \longrightarrow \mathbf{Coalg}_k(!X \otimes !X, !Y)\,.$$

In the following, we write D_X for both the deriving transformation $!X \otimes X \longrightarrow !X$ and the map obtained from D_X by precomposing with the swap $\sigma_{!X,X}$. By definition $\mathbb{D}_{X,Y}$ assigns to a linear map $f : !X \longrightarrow Y$ the composite

where D_X is the deriving transformation, Θ is the canonical isomorphism of Lemma 3.11 and $\pi_i : X \oplus X \longrightarrow X$ denote the projections. In our case, we can simplify this definition. Recall that a Cartesian differential category (Blute et al. 2009, Definition 2.1.4) is a Cartesian left additive category equipped with a Cartesian differential operator, denoted here by $\mathbb{D}$. The left additive structure plus $(-, -)$ on $\mathcal{V}_!$ is given by Blute et al. (2009, Proposition 1.3.3) from the Cartesian and additive structure on $\mathcal{V}$ (the usual $\oplus$ and $+$) as follows: given $F, G \in \mathbf{Coalg}_k(!X, !Y)$ we define plus $(F, G) \in \mathbf{Coalg}_k(!X, !Y)$ to be the unique morphism of coalgebras with

$$d_Y \circ \mathrm{plus}\,(F, G) = d_Y \circ (F + G)\,.$$

Lemma 5.10. *The composite*

$$!X \otimes !X \xrightarrow[\cong]{\;\Theta\;} !(X \oplus X) \xrightarrow{\;\Delta\;} !(X \oplus X) \otimes !(X \oplus X) \xrightarrow{\;!\pi_0 \otimes !\pi_1\;} !X \otimes !X$$

is the identity map.

Proof. We can see this by direct calculation (see Definition 2.14 for notation):

$$(!\pi_0 \otimes !\pi_1) \circ \Delta \circ \Theta \left(|v_1, ..., v_s\rangle_P \otimes |w_1, ..., w_t\rangle_Q \right)$$

$$= (!\pi_0 \otimes !\pi_1) \circ \Delta \left(|(v_1, 0), ..., (v_s, 0), (0, w_1), ..., (0, w_t)\rangle_{(P,Q)} \right)$$

$$= (!\pi_0 \otimes !\pi_1) \sum_{I \subseteq [s]} \sum_{J \subseteq [t]} |v_I, w_J\rangle_{(P,Q)} \otimes |v_{I^c}, w_{J^c}\rangle_{(P,Q)}$$

$$= \sum_{I \subseteq [s]} \sum_{J \subseteq [t]} \delta_{J=\emptyset} |v_I\rangle_P \otimes \delta_{I^c=\emptyset} |w_{J^c}\rangle_Q$$

$$= \sum_{I \subseteq [s]} \sum_{J \subseteq [t]} \delta_{J^c=[t]} \delta_{I=[s]} |v_I\rangle_P \otimes |w_{J^c}\rangle_Q$$

$$= |v_1, ..., v_s\rangle_P \otimes |w_1, ..., w_t\rangle_Q$$

which proves the claim. □

The upshot is that (Blute et al. 2009, Proposition 3.2.1) implies the following Cartesian differential operator makes $\mathcal{V}_!$ into a Cartesian differential category:

Definition 5.11. *The Cartesian differential operator*

$$\mathbb{D}_{X,Y} : \mathcal{V}_!(X, Y) \longrightarrow \mathcal{V}_!(X \times X, Y)$$

sends a linear map $f : !X \longrightarrow Y$ to the linear map

$$!(X \oplus X) \xrightarrow[\cong]{\Theta^{-1}} !X \otimes !X \xrightarrow{d_X \otimes 1} X \otimes !X \xrightarrow{D_X} !X \xrightarrow{f} Y$$

Direct computation shows that

$$\mathbb{D}_{X,Y}(f) \left(|v_1, ..., v_s\rangle_P \otimes |w_1, ..., w_t\rangle_Q \right)$$

$$= fD(d \otimes 1) \left(|v_1, ..., v_s\rangle_P \otimes |w_1, ..., w_t\rangle_Q \right)$$

$$= fD(\delta_{s=0} P \otimes |w_1, ..., w_t\rangle_Q + \delta_{s=1} v_1 \otimes |w_1, ..., w_t\rangle_Q)$$

$$= \delta_{s=0} f|P, w_1, ..., w_t\rangle_Q + \delta_{s=1} f|v_1, w_1, ..., w_t\rangle_Q.$$

and the lifting of $\mathbb{D}_{X,Y}(f)$ to a morphism of coalgebras $!(X \oplus X) \longrightarrow !Y$ may be described explicitly using Murfet (2015, Theorem 2.22).

Next, we show that $\mathcal{V}_!$ together with the maps $\mathbb{D}_{X,Y}$ is a Cartesian *closed* differential category in the sense of Blute et al. (2009, Section 1.4). First, we recall the closed structure on $\mathcal{V}_!$. The closed structure on the Kleisli category is well known, see for example, the general treatment in Blute et al. (2015). Throughout, A, B denote arbitrary vector spaces.

Definition 5.12. *Let $\Gamma_{A,B}$ denote the unique morphism of coalgebras making*

$$
\begin{array}{ccc}
!A \otimes ! \operatorname{Hom}_k(!A, B) & \xrightarrow{1 \otimes d} & !A \otimes \operatorname{Hom}_k(!A, B) \\
\Big\downarrow{\scriptstyle \Gamma_{A,B}} & & \Big\downarrow{\scriptstyle \mathrm{ev}} \\
!B & \xrightarrow{d} & B
\end{array}
$$

commute, where we take the usual coalgebra structure on the tensor product.

Definition 5.13. *We define*

$$\operatorname{HOM}(A, B) = \operatorname{Hom}_k(!A, B).$$

Given a morphism $f \in \mathcal{V}_!(B, B')$, that is a linear map $f : !B \longrightarrow B'$, we define

$$\mathrm{HOM}(A, f) \in \mathcal{V}_!(\mathrm{HOM}(A, B), \mathrm{HOM}(A, B')) = \mathrm{Hom}_k \left(! \, \mathrm{Hom}_k \, (!A, B), \mathrm{Hom}_k \, (!A, B')\right)$$

to be the linear map corresponding under the Hom-tensor adjunction to $f \circ \Gamma_{A,B}$.

Again, this may be computed explicitly using Murfet (2015, Theorem 2.22), but we will not need the formulas here.

Lemma 5.14. $\mathrm{HOM}(A, -)$ *is a functor* $\mathcal{V}_! \longrightarrow \mathcal{V}_!$.

Proof. To show $\mathrm{HOM}(A, 1_B) = 1_{\mathrm{HOM}(A,B)}$, we have to show that $\mathrm{ev} \circ (1 \otimes d)$ corresponds under the Hom-tensor adjunction to the dereliction $d : ! \, \mathrm{Hom}_k(!A, B) \longrightarrow \mathrm{Hom}_k(!A, B)$. This we can do by direct calculation: the map $\mathrm{ev} \circ (1 \otimes d)$ is

$$|v_1, ..., v_s\rangle_P \otimes |\zeta_1, ..., \zeta_t\rangle_\alpha \mapsto \delta_{t=0}\alpha|v_1, ..., v_s\rangle_P + \delta_{t=1}\zeta_1|v_1, ..., v_s\rangle_P, \tag{39}$$

which agrees with the dereliction on $! \, \mathrm{Hom}_k(!A, B)$. Now suppose given linear maps $g : !B' \longrightarrow B''$ and $f : !B \longrightarrow B'$, and let $\bullet$ denote the Kleisli composition. To show that

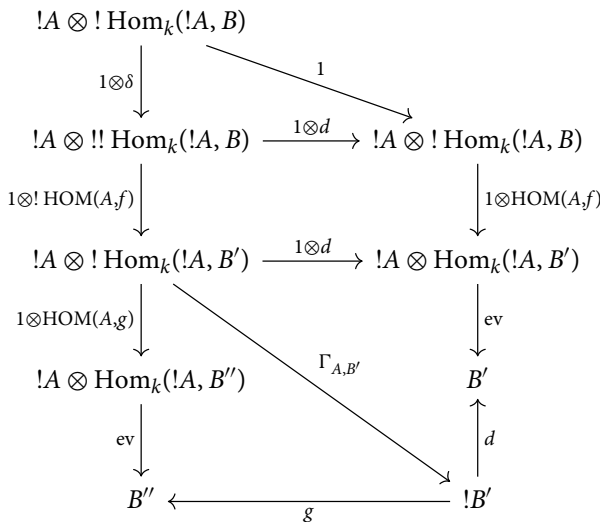$$\mathrm{HOM}(A, g) \bullet \mathrm{HOM}(A, f) = \mathrm{HOM}(A, g \bullet f), \tag{40}$$

we first observe $g \bullet f$ is the linear map

$$!B \xrightarrow{\ \delta\ } !!B \xrightarrow{\ !f\ } !B' \xrightarrow{\ g\ } B''$$

and the left-hand side of (40) is the composite

$$! \, \mathrm{Hom}_k(!A, B) \xrightarrow{\ \delta\ } !! \, \mathrm{Hom}_k(!A, B) \xrightarrow{\ ! \, \mathrm{HOM}(A,f)\ } ! \, \mathrm{Hom}_k(!A, B') \xrightarrow{\ \mathrm{HOM}(A,g)\ } \mathrm{Hom}_k(!A, B'')$$

which corresponds under the Hom-tensor adjunction to the left-hand vertical composite in the following commutative diagram

So to prove (40), it suffices to show

$$\Gamma_{A,B'} \circ (1 \otimes ! \, \mathrm{HOM}(A, f)) \circ (1 \otimes \delta) = !f \circ \delta \circ \Gamma_{A,B}.$$

But both sides are morphisms of coalgebras, so we may compare them after post-composition with d, and this reduces to $\mathrm{ev} \circ (1 \otimes \mathrm{HOM}(A, f)) = f \circ \Gamma_{A,B}$ which is true by definition. $\qquad\square$

Lemma 5.15. *The functor* $\mathrm{HOM}(A, -) : \mathcal{V}_! \longrightarrow \mathcal{V}_!$ *is right adjoint to* $- \times A$.

Proof. Recall that $- \times A$ as a functor on $\mathcal{V}_!$ sends $f : !B \longrightarrow B'$ to the linear map

$$f \times A : !(B \oplus A) \cong !B \otimes !A \xrightarrow{F \otimes 1} !B' \otimes !A \cong !(B' \oplus A) \xrightarrow{d} B' \oplus A$$

where F is the morphism of coalgebras lifting f, given explicitly by Murfet (2015, Theorem 2.22). Given $v_1, ..., v_s, P \in A$ and $\omega_1, ..., \omega_t, Q \in B$ and writing $\mathcal{P}_{[t]}$ for the set of partitions of $[t] = \{1, \ldots, t\}$ we may calculate $f \times A$ as the map

$$|w_1, ..., w_t\rangle_Q \otimes |v_1, ..., v_s\rangle_P \xmapsto{F \otimes 1} \sum_{C \in \mathcal{P}_{[t]}} \left| \prod_{c \in C} f|w_c\rangle_Q \right\rangle_{f|\emptyset\rangle_Q} \otimes |v_1, ..., v_s\rangle_P$$

$$\xmapsto{\Theta} \sum_{C \in \mathcal{P}_{[t]}} \left| \prod_{c \in C} f|w_c\rangle_Q, v_1, ..., v_s \right\rangle_{(f|\emptyset\rangle_Q, P)}$$

$$\xmapsto{d} \delta_{s=t=0}(f|\emptyset\rangle_Q, P) + \delta_{s=0,t>0}(f|w_1, ..., w_t\rangle_Q, 0)$$
$$+ \delta_{s=1,t=0}(0, v_1).$$

So in summary:

$$(f \times A)(|w_1, ..., w_t\rangle_Q \otimes |v_1, ..., v_s\rangle_P)$$
$$= \left(\delta_{s=t=0} f|\emptyset\rangle_Q + \delta_{s=0,t>0} f|w_1, ..., w_t\rangle_Q, \delta_{s=t=0} P + \delta_{s=1,t=0} v_1 \right).$$

We have bijections for vector spaces A, B, C

$$\mathbf{Coalg}_k(!C \otimes !A, !B) \cong \mathrm{Hom}_k(!C \otimes !A, B)$$
$$\cong \mathrm{Hom}_k(!C, \mathrm{Hom}_k(!A, B))$$
$$\cong \mathbf{Coalg}_k(!C, ! \mathrm{Hom}_k(!A, B))$$
$$= \mathbf{Coalg}_k(!C, ! \mathrm{HOM}(A, B)),$$

and hence a bijection

$$\mathcal{V}_!(C \times A, B) \cong \mathcal{V}_!(C, \mathrm{HOM}(A, B)). \tag{41}$$

The question that remains is whether these bijections are natural in C, B. Clearly, they are natural in C. To prove naturality in B, we have to show that for a linear map $f : !B \longrightarrow B'$

$$
\begin{array}{ccc}
\mathcal{V}_!(C \times A, B) & \xrightarrow{\cong} & \mathcal{V}_!(C, \mathrm{HOM}(A, B)) \\
{\scriptstyle \mathcal{V}_!(1,f)} \downarrow & & \downarrow {\scriptstyle \mathcal{V}_!(1, \mathrm{HOM}(A,f))} \\
\mathcal{V}_!(C \times A, B') & \xrightarrow{\cong} & \mathcal{V}_!(C, \mathrm{HOM}(A, B'))
\end{array}
$$

commutes. That is, given a morphism of coalgebras $\gamma : !C \otimes !A \longrightarrow !B$, we have to show

$$\mathrm{HOM}(A,f) \circ \mathrm{prom}\, \widetilde{(d \circ \gamma)} = \widetilde{f \circ \gamma} : !C \longrightarrow \mathrm{Hom}_k(!A, B') \tag{42}$$

where $\widetilde{z}$ denotes the morphism corresponding to z under the Hom-tensor adjunction. For this, consider the diagram
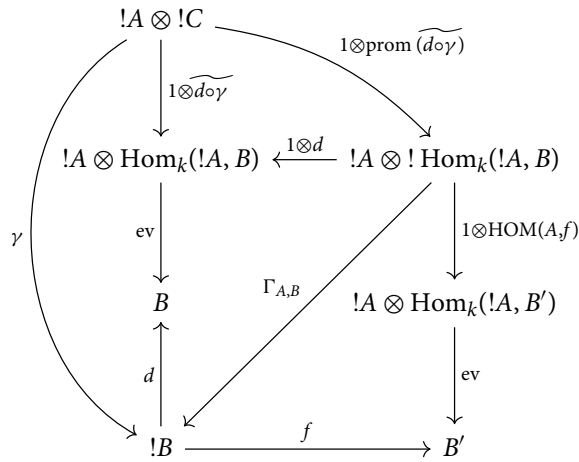
From the calculation

$$d \circ \Gamma_{A,B} \circ (1 \otimes \mathrm{prom}\, \widetilde{(d \circ \gamma)}) = \mathrm{ev} \circ (1 \otimes d) \circ \mathrm{prom}\, \widetilde{(d \circ \gamma)}$$
$$= \mathrm{ev} \circ (1 \otimes \widetilde{d \circ \gamma})$$
$$= d \circ \gamma$$

we deduce that $\Gamma_{A,B} \circ (1 \otimes \mathrm{prom}\, \widetilde{(d \circ \gamma)}) = \gamma$ since both sides are morphisms of coalgebras. From this and the above diagram, we easily deduce (42). □

Lemma 5.16. *With the above structure $\mathcal{V}_!$ is a Cartesian closed left additive category.*

Proof. We need to show that

$$\mathcal{V}_!(A \times B, C) \xrightarrow{\;\cong\;} \mathcal{V}_!(A, \mathrm{HOM}(B, C))$$

is an isomorphism of monoids. But this map is the Hom-tensor adjunction

$$\mathrm{Hom}_k(!A \otimes !B, C) \xrightarrow{\;\cong\;} \mathrm{Hom}_k(!A, \mathrm{Hom}_k(!B, C))$$

which is linear, so this is clear. □

Theorem 5.17. *$\mathcal{V}_!$ is a differential λ-category (Bucciarelli et al. 2010, Definition 4.4) and thus a model of the simply typed differential λ-calculus (Bucciarelli et al. 2010, Section 4.3).*

Proof. First, we observe that $\mathcal{V}_!$ is a Cartesian closed differential category in the sense of Bucciarelli et al. (2010, Definition 4.2). It is a Cartesian closed left additive category, and we have already observed in Section 5.1 it has an operator $\mathbb{D}_{X,Y}(-)$ satisfying the axioms of a Cartesian differential category. It remains to check the axiom (D-Curry) which says given $f : C \times A \longrightarrow B$ in $\mathcal{V}_!$ and denoting currying by Λ, that

$$\mathbb{D}(\Lambda f) = \Lambda(\mathbb{D}(f) \circ \langle \pi_1 \times 0_A, \pi_2 \times 1_A \rangle). \tag{43}$$

Here $\Lambda f : C \longrightarrow \mathrm{HOM}(A, B)$ and so $\mathbb{D}(\Lambda f) : C \times C \longrightarrow \mathrm{HOM}(A, B)$, whereas the right-hand side corresponds under adjunction to

$$(C \times C) \times A \xrightarrow{\langle \pi_1 \times 0_A, \pi_2 \times 1_A \rangle} (C \times A) \times (C \times A) \xrightarrow{\mathbb{D}(f)} B. \tag{44}$$

In $\mathcal{V}$, this map is the composition of (where 0_A denotes the lift of $0 : !A \longrightarrow A$)

$$
\begin{array}{ccc}
!C \otimes !C \otimes !A & & !C \otimes !A \otimes !C \otimes !A \\
\cong \downarrow & & \uparrow !\pi_1 \otimes !0_A \otimes !\pi_2 \otimes 1_{!A} \\
!(C \oplus C) \otimes !A & & \\
\Delta \otimes \Delta \downarrow & & \\
!(C \oplus C) \otimes !(C \oplus C) \otimes !A \otimes !A \xrightarrow{\cong} !(C \oplus C) \otimes !A \otimes !(C \oplus C) \otimes !A
\end{array}
$$

with $\mathbb{D}(f)$ which is

$$
\begin{array}{ccc}
!C \otimes !A \otimes !C \otimes !A & & B \\
\cong \downarrow & & \uparrow f \\
!(C \oplus A) \otimes !(C \oplus A) & & \\
d_{C \oplus A} \otimes 1 \downarrow & & \\
(C \oplus A) \otimes !(C \oplus A) \xrightarrow{D_{C \oplus A}} !(C \oplus A)
\end{array}
$$

As above we write $[n] = \{1, \dots, n\}$. This composite is the linear map $!C \otimes !C \otimes !A \longrightarrow B$ given by the formula

$$
\begin{aligned}
|\alpha_1, &\dots, \alpha_r\rangle_P \otimes |\beta_1, \dots, \beta_s\rangle_Q \otimes |\gamma_1, \dots, \gamma_t\rangle_R \\
&\longmapsto |\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\rangle_{(P,Q)} \otimes |\gamma_1, \dots, \gamma_t\rangle_R \\
&\xrightarrow{\Delta \otimes \Delta} \sum_{A \subseteq [r]} \sum_{B \subseteq [s]} \sum_{C \subseteq [t]} |\alpha_A, \beta_B\rangle_{(P,Q)} \otimes |\alpha_{A^c}, \beta_{B^c}\rangle_{(P,Q)} \otimes |\gamma_C\rangle_R \otimes |\gamma_{C^c}\rangle_R \\
&\longmapsto \sum_{A,B,C} |\alpha_A, \beta_B\rangle_{(P,Q)} \otimes |\gamma_C\rangle_R \otimes |\alpha_{A^c}, \beta_{B^c}\rangle_{(P,Q)} \otimes |\gamma_C^c\rangle_R \\
\\
&\longmapsto \sum_{A,B,C} \delta_{B=\emptyset} |\alpha_A\rangle_P \otimes \delta_{C=\emptyset} |\emptyset\rangle_0 \otimes \delta_{A^c=\emptyset} |\beta_{B^c}\rangle_Q \otimes |\gamma_C^c\rangle_R \\
&= |\alpha_1, \dots, \alpha_r\rangle_P \otimes |\emptyset\rangle_0 \otimes |\beta_1, \dots, \beta_s\rangle_Q \otimes |\gamma_1, \dots, \gamma_t\rangle_R \\
&\longmapsto |\alpha_1, \dots, \alpha_r\rangle_{(P,0)} \otimes |\beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)} \\
&\xrightarrow{d \otimes 1} \delta_{r=0}(P, 0) \otimes |\beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)} \\
&\qquad + \delta_{r=1}\alpha \otimes |\beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)} \\
&\xrightarrow{D} \delta_{r=0}|P, \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)} \\
&\qquad + \delta_{r=1}|\alpha, \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)} \\
&\xrightarrow{f} \delta_{r=0} f|P, \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)} \\
&\qquad + \delta_{r=1} f|\alpha, \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\rangle_{(Q,R)}.
\end{aligned}
$$

On the other hand, $\mathbb{D}(\Lambda f)$ is the linear map

$$
!C \otimes !C \xrightarrow{d \otimes 1} C \otimes !C \xrightarrow{D} !C \xrightarrow{\Lambda f} \mathrm{Hom}_k(!A, B)
$$

which sends $|\alpha_1, ..., \alpha_r\rangle_P \otimes |\beta_1, ..., \beta_s\rangle_Q$ to the linear map

$$|\gamma_1, \ldots, \gamma_t\rangle_R \longmapsto f\Big(\big(\delta_{r=0}|P, \beta_1, \ldots, \beta_s\rangle_Q + \delta_{r=1}|\alpha, \beta_1, \ldots, \beta_s\rangle_Q\big) \otimes |\gamma_1, \ldots, \gamma_t\rangle_R\Big)$$

This corresponds under adjunction to the linear map $!C \otimes !C \otimes !A \longrightarrow B$ computed above, completing the proof. $\qquad\qquad\square$

Notes

1 The formal theory of coalgebras is simpler over algebraically closed fields, which explains why we use $k = \mathbb{C}$ in our examples, but this is not really important: one could work over $k = \mathbb{R}$ by taking $\mathbb{C}$-points into account in the explicit description of the cofree coalgebra.

References

Blute, R., Ehrhard, T. and Tasson, C. (2010). A convenient differential category, arXiv preprint [arXiv:1006.3140].

Blute, R., Cockett, J. and Seely, R. (2006). Differential categories. *Mathematical Structures in Computer Science* **16** (06) 1049–1083.

Blute, R., Cockett, J. and Seely, R. (2009). Cartesian differential categories. *Theory and Applications of Categories* **22** (23) 622–672.

Blute, R., Cockett, J. and Seely, R. (2015). Cartesian differential storage categories. *Theory and Applications of Categories* **30** 620–686.

Borceux, F. (1994). *Handbook of Categorical Algebra 2: Categories and Structures*, Encyclopedia of Mathematics and Its Applications, Cambridge, Cambridge University Press.

Bourbaki, N. (1989). *Algebra I*, Springer-Verlag Berlin Heidelberg.

Bucciarelli, A., Ehrhard, T. and Manzonetto, G. (2010). Categorical models for simply typed resource calculi. *Electronic Notes in Theoretical Computer Science* **265** 213–230.

Cockett, J. and Lemay, J.-S. (2019). Integral categories and calculus categories. *Mathematical Structures in Computer Science* **29**(2) 1–66.

Corliss, G. (1991). Automatic differentiation bibliography. In: *Automatic Differentiation of Algorithms: Theory, Implementation, and Application*, Philadelphia, Pennsylvania, USA, SIAM, 331–353. updated on-line version.

Ehrhard, T. (2002). On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science* **12** 579–623.

Ehrhard, T. (2005). Finiteness spaces. *Mathematical Structures in Computer Science* **15** 615–646.

Ehrhard, T. (2016). An introduction to Differential Linear Logic: proof-nets, models and antiderivatives, arXiv preprint [arXiv:1606.01642].

Ehrhard, T. and Laurent, O. (2007). Interpreting a finitary π-calculus in differential interaction nets. In: *International Conference on Concurrency Theory*, Berlin Heidelberg, Springer.

Ehrhard, T. and Regnier, L. (2003). The differential λ-calculus. *Theoretical Computer Science* **309** 1–41.

Ehrhard, T. and Regnier, L. (2006). Differential interaction nets. *Theoretical Computer Science* **364** 166–195.

Eisenbud, D. and Harris, J. (2000). *The Geometry of Schemes*, Springer Graduate Texts in Mathematics, vol. 197, New York, Springer-Verlag.

Fiore, M. (2007). Differential structure in models of multiplicative biadditive intuitionistic linear logic. In: *Proceedings of TLCA*, 163–177.

Friedlander, F. and Joshi, M. (1998). *Introduction to the Theory of Distributions*, Cambridge, Cambridge University Press.

Frolicher, A. and Kriegl, A. (1988). *Linear Spaces and Differentiation Theory*, Great Britain, Wiley.

Girard, J.-Y. (1987). Linear logic. *Theoretical Computer Science* **50** (1) 1–102.

Girard, J.-Y. (1988). Normal functors, power series and the λ-calculus. *Annals of Pure and Applied Logic* **37** 129–177.

Girard, J.-Y. (1995). Light linear logic. *Information and Computation* **143**(2) 175–204.

Girard, J.-Y. (1999). Coherent Banach spaces: a continuous denotational semantics. *Theoretical Computer Science* **227** (1) 275–297.

Girard, J.-Y., Lafont, Y. and Taylor, P. (1989). *Proofs and Types*, Cambridge Tracts in Theoretical Computer Science, vol. 7, Cambridge University Press.

Hartshorne, R. (1966). *Residues and Duality*, Lecture Notes in Mathematics, vol. 20, Berlin, Springer-Verlag.

Hartshorne, R. (1977). *Algebraic Geometry*, Springer Graduate Texts in Mathematics, vol. 52, New York, Springer-Verlag.

Hyland, M. and Schalk, A. (2003). Glueing and orthogonality for models of linear logic. *Theoretical Computer Science* **294** 183–231.

Kock, A. (2012). Commutative monads as a theory of distributions. *Theory and Applications of Categories* **26** (4) 97–131.

Kontsevich, M. and Soibelman, Y. (2008). Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. In: *Homological Mirror Symmetry*, Berlin Heidelberg, Springer, 1–67.

Kontsevich, M. and Soibelman, Y. (2002). Deformation theory, in preparation, accessible [online].

Lambek, J. and Scott, P. J. (1986). *Introduction to Higher Order Categorical Logic*, Cambridge Studies in Advanced Mathematics, vol. 7, Cambridge, Cambridge University Press.

Le Bruyn, L. (2008). Noncommutative geometry and dual coalgebras, arXiv preprint, [arXiv:0805.2377].

Melliès, P.-A. (2009). Categorical semantics of linear logic. In: *Interactive Models of Computation and Program Behaviour*, Panoramas et Synthèses vol. 27, Société Mathématique de France.

Murfet, D. (2015). On Sweedler's cofree cocommutative coalgebra. *Journal of Pure and Applied Algebra* **219** 5289–5304. [arXiv:1406.5749].

Murfet, D. (2014). Logic and linear algebra: an introduction, arXiv preprint, [arXiv:1407.2650].

Nilsson, H. (2003). Functional automatic differentiation with Dirac impulses. *ACM SIGPLAN Notices* **38** (9) 153–164.

Paige, R. and Koenig, S. (1982). Finite differencing of computable expressions. *ACM Transactions on Programming Languages and Systems (TOPLAS)* **4** (3) 402–454.

Ramalingam, G. and Reps, T. (1993). A categorized bibliography on incremental computation. In: *POPL*, ACM, 502–510.

Seiler, W. (2009). *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*, vol. 24, Springer-Verlag Berlin Heidelberg.

Sweedler, M. (1969). *Hopf Algebras*, New York, W. A. Benjamin.

Appendix A. Kleisli Categories and Coalgebras

For background on (co)monads and their (co)algebras, see Borceux (1994, Section 4.1). Let $! : \mathcal{V} \longrightarrow \mathcal{V}$ be the comonad discussed above, and $\delta : {!} \longrightarrow {!!}$ the natural transformation of Definition 2.16.

Lemma A.1. *Given a vector space V, there is a bijection between cocommutative counital coalgebra structures on V and $!$-coalgebra structures on V.*

Proof. Suppose (V, Δ, ε) is such a coalgebra. There is a unique morphism of coalgebras $\varphi_{\Delta,\varepsilon} : V \longrightarrow {!}V$ such that $d_V \circ \varphi_{\Delta,\varepsilon} = 1_V$, and moreover,

$$
\begin{array}{ccc}
V & \xrightarrow{\ \varphi_{\Delta,\varepsilon}\ } & {!}V \\
{\scriptstyle \varphi_{\Delta,\varepsilon}}\downarrow & & \downarrow{\scriptstyle \delta_V} \\
{!}V & \xrightarrow[\ {!}\varphi_{\Delta,\varepsilon}\ ]{} & {!!}V
\end{array}
$$

is easily seen to commute, so $(V, \varphi_{\Delta,\varepsilon})$ is a $!$-coalgebra. Then $(\Delta, \varepsilon) \mapsto \varphi_{\Delta,\varepsilon}$ is our claimed bijection. Given a $!$-coalgebra (V, φ) define Δ as the composite

$$
V \xrightarrow{\ \varphi\ } {!}V \xrightarrow{\ \Delta\ } {!}V \otimes {!}V \xrightarrow{\ d \otimes d\ } V \otimes V,
$$

and ε as

$$
V \xrightarrow{\ \varphi\ } {!}V \xrightarrow{\ \varepsilon\ } k.
$$

From commutativity of

$$
\begin{array}{ccc}
V & \xrightarrow{\ \varphi\ } & {!}V \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle d} \\
& & V
\end{array}
\qquad\qquad
\begin{array}{ccc}
V & \xrightarrow{\ \varphi\ } & {!}V \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \delta_V} \\
{!}V & \xrightarrow[\ {!}\varphi\ ]{} & {!!}V
\end{array}
$$

we deduce commutativity of
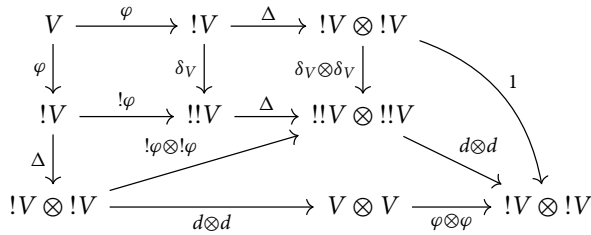
That is,

$$\Delta \circ \varphi = (\varphi \otimes \varphi) \circ (d \otimes d) \circ \Delta \circ \varphi \tag{A1}$$

We use this to prove coassociativity of (V, Δ, ε) as follows. We must show that
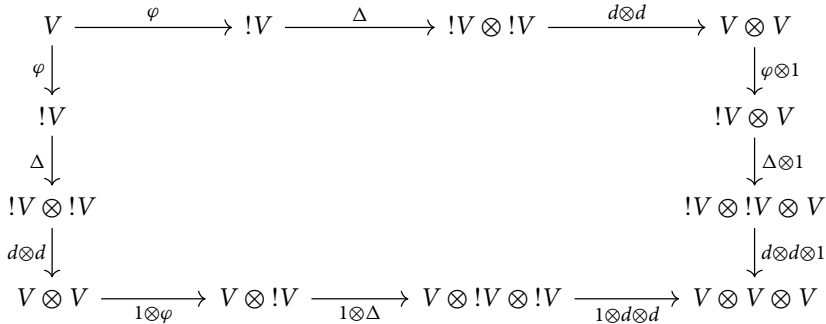
commutes, or stated differently, that the two ways around the outside of the following diagram agree when precomposed with $(d \otimes d) \circ \Delta \circ \varphi$:
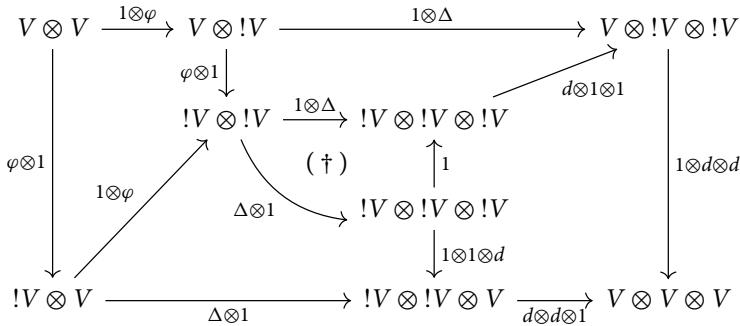
In this diagram every square but the one marked ($\dagger$) commutes. Now precomposing both ways around this diagram with $(d \otimes d) \circ \Delta \circ \varphi$ amounts to precomposing the two ways around the triangle ($\dagger$) with the right hand side of (A1) and therefore to precomposing with $\Delta \circ \varphi$. But by coassociativity of Δ, the two ways around ($\dagger$) agree when precomposed with Δ. The usphot is that the above diagram commutes, when precomposed with $(d \otimes d) \circ \Delta \circ \varphi$.

For counitality we need

to commute, which is clear from the given decomposition. For cocommutativity of Δ we need commutativity of

$$
\begin{array}{ccccccc}
V & \xrightarrow{\varphi} & !V & \xrightarrow{\Delta} & !V \otimes !V & \xrightarrow{d \otimes d} & V \otimes V \\
& \searrow{\varphi} & \downarrow{1} & & \downarrow{\sigma} & & \downarrow{\sigma} \\
& & !V & \xrightarrow{\Delta} & !V \otimes !V & \xrightarrow{d \otimes d} & V \otimes V
\end{array}
$$

which is again clear. We have now assigned to any !-coalgebra structure φ on V a coalgebra structure $\Delta_\varphi, \varepsilon_\varphi$. Next we observe that the assignments

$$(\Delta, \varepsilon) \longmapsto \varphi_{\Delta, \varepsilon}, \qquad \varphi \longmapsto (\Delta_\varphi, \varepsilon_\varphi)$$

are mutually inverse. Clearly given (Δ, ε) the diagrams

$$
\begin{array}{ccccccc}
V & \xrightarrow{\varphi_{\Delta, \varepsilon}} & !V & \xrightarrow{\Delta} & !V \otimes !V & \xrightarrow{d \otimes d} & V \otimes V \\
\downarrow{\Delta} & & & & & & \\
V \otimes V & & & & & &
\end{array}
\qquad
\begin{array}{ccc}
V & \xrightarrow{\varphi_{\Delta, \varepsilon}} & !V & \xrightarrow{\varepsilon} & k
\end{array}
$$

show that $\Delta_{\varphi_{\Delta, \varepsilon}} = \Delta$ and $\varepsilon_{\varphi_{\Delta, \varepsilon}} = \varepsilon$. Given φ we observe that $\varphi : (V, \Delta_\varphi, \varepsilon_\varphi) \longrightarrow !V$ is a morphism of coalgebras and satisfies $d \circ \varphi = 1$, which completes the proof. □

Lemma A.2. *Let $(V, \varphi), (W, \psi)$ be !-coalgebras and $(V, \Delta_\varphi, \varepsilon_\varphi), (W, \Delta_\psi, \varepsilon_\psi)$ the associated coalgebras. A linear map $f : V \longrightarrow W$ is a morphism of !-coalgebras if and only if it is a morphism of coalgebras.*

Proof. To say f is a morphism of !-coalgebras is to say

$$
\begin{array}{ccc}
V & \xrightarrow{\varphi} & !V \\
\downarrow{f} & & \downarrow{!f} \\
W & \xrightarrow{\psi} & !W
\end{array}
\tag{$*$}
$$

commutes, whereas to say f is a morphism of coalgebras is to say

$$
\begin{array}{ccc}
V & \xrightarrow{\Delta_\varphi} & V \otimes V \\
\downarrow{f} & & \downarrow{f \otimes f} \\
W & \xrightarrow{\Delta_\psi} & W \otimes W
\end{array}
\qquad
\begin{array}{ccc}
V & \xrightarrow{\varepsilon_\varphi} & k \\
\downarrow{f} & \nearrow{\varepsilon_\psi} & \\
W & &
\end{array}
\tag{$\dagger$}
$$

commute. If $(*)$ commutes it is clear that the diagrams in $(\dagger)$ commute. Conversely suppose the diagrams in $(\dagger)$ commute. We know then that φ, ψ are morphisms of coalgebras, so it suffices to check $(*)$ commutes after post composition by d, which is trivial. □

Recall that for a comonad T on a category $\mathscr{C}$ we have the Kleisli category $\mathscr{C}_T$ and the Eilenberg-Moore category $\mathscr{C}^T$ (Borceux 1994, Section 4.1). Note that a comonad on $\mathcal{C}$ is the same thing as a monad on $\mathcal{C}^{\mathrm{op}}$. Explicitly, $\mathscr{C}_T$ is the category with

- ob $\mathscr{C}_T = $ ob $\mathscr{C}$,
- $\mathscr{C}_T(x, y) = \mathscr{C}(Tx, y)$,

- $\mathrm{id}_x^{\mathscr{C}_T} \in \mathscr{C}(Tx, x)$ is the counit ε,
- $\mathscr{C}_T(y, z) \times \mathscr{C}_T(x, y) \longrightarrow \mathscr{C}_T(x, z)$ is $(g, f) \mapsto g \circ T(f) \circ \delta_X$, where $\delta : T \longrightarrow TT$.

while $\mathscr{C}^T$ is category of coalgebras for T as defined in (38). There is a fully faithful functor $\mathscr{C}_T \longrightarrow \mathscr{C}^T$ defined by $X \mapsto (TX, \delta_X)$.

Proposition A.3. $\mathcal{V}^!$ *is isomorphic to* $\mathbf{Coalg}_k$.

Proof. We define $F : \mathcal{V}^! \longrightarrow \mathbf{Coalg}_k$ by $F(V, \varphi) = (V, \Delta_\varphi, \varepsilon_\varphi)$ and $G : \mathbf{Coalg}_k \longrightarrow \mathcal{V}^!$ by $G(V, \Delta, \varepsilon) = (V, \varphi_{\Delta, \varepsilon})$ in the above notation. On morphism sets both of F and G are the identity (Lemma A.2). Clearly $F \circ G = 1$ and $G \circ F = 1$. $\square$

Corollary A.4. $\mathcal{V}_!$ *is equivalent to the full subcategory of cofree coalgebras in* $\mathbf{Coalg}_k$.

Proof. This is immediate from the above but can also be seen directly using

$$\mathbf{Coalg}_k(!A, !B) \cong \mathrm{Hom}_k\,(!A, B) \cong \mathcal{V}_!(A, B).$$

$\square$