*Combinatorics, Probability and Computing* (2014) 23, 1114–1147. © Cambridge University Press 2014
doi:10.1017/S0963548314000327

# Weighted Random Staircase Tableaux

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*Received 20 December 2012; revised 25 September 2013; first published online 15 July 2014*

This paper concerns a relatively new combinatorial structure called staircase tableaux. They were introduced in the context of the asymmetric exclusion process and Askey–Wilson polynomials; however, their purely combinatorial properties have gained considerable interest in the past few years.

In this paper we further study combinatorial properties of staircase tableaux. We consider a general model of random staircase tableaux in which symbols (Greek letters) that appear in staircase tableaux may have arbitrary positive weights. (We consider only the case with the parameters $u = q = 1$.) Under this general model we derive a number of results. Some of our results concern the limiting laws for the number of appearances of symbols in a random staircase tableaux. They generalize and subsume earlier results that were obtained for specific values of the weights.

One advantage of our generality is that we may let the weights approach extreme values of zero or infinity, which covers further special cases appearing earlier in the literature. Furthermore, our generality allows us to analyse the structure of random staircase tableaux, and we obtain several results in this direction.

One of the tools we use is the generating functions of the parameters of interest. This leads us to a two-parameter family of polynomials, generalizing the classical Eulerian polynomials.

We also briefly discuss the relation of staircase tableaux to the asymmetric exclusion process, to other recently introduced types of tableaux, and to an urn model studied by a number of researchers, including Philippe Flajolet.

2010 *Mathematics subject classification*: Primary 60C05
Secondary 05A15, 05E99, 60F05

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[†] Partially supported by the Simons Foundation (grant #208766 to Paweł Hitczenko)
[‡] Partially supported by the Knut and Alice Wallenberg Foundation.

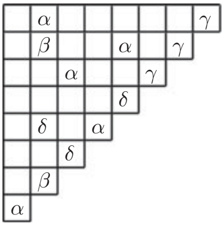


*Figure 1.* A staircase tableau of size 8; its weight is $\alpha^5\beta^2\delta^3\gamma^3$.

## 1. Introduction

This paper considers a combinatorial structure introduced recently by Corteel and Williams [17, 18] and called *staircase tableaux*. (The definition is given below.) The original motivations were in connection with the asymmetric exclusion process (ASEP) on a one-dimensional lattice with open boundaries, an important model in statistical mechanics (see below for a brief summary and [18] for the full story). The generating function for staircase tableaux was also used to give a combinatorial formula for the moments of the Askey–Wilson polynomials (see [18, 14] for details). Further work includes [11] where special situations in which the generating function of staircase tableaux took a particularly simple form were considered. Furthermore, [19] deals with the analysis of various parameters associated with appearances of the Greek letters $\alpha$, $\beta$, $\delta$, and $\gamma$ in a randomly chosen staircase tableau (see below, or [18, Section 2], for example, for the definitions and the meaning of these symbols).

The purpose of this paper is to continue the study of properties of staircase tableaux, regarding them as interesting combinatorial objects in themselves.

We recall the definition of a staircase tableau introduced in [17, 18].

**Definition 1.1.** A *staircase tableau of size* $n$ is a Young diagram of shape $(n, n-1, \dots, 2, 1)$ whose boxes are filled according to the following rules:

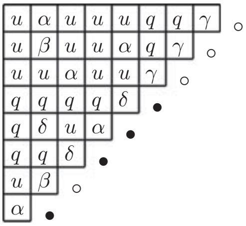
(Si) each box is either empty or contains one of the letters $\alpha$, $\beta$, $\delta$, or $\gamma$,
(Sii) no box on the diagonal is empty,
(Siii) all boxes in the same row and to the left of a $\beta$ or a $\delta$ are empty,
(Siv) all boxes in the same column and above an $\alpha$ or a $\gamma$ are empty.

An example of a staircase tableau is given in Figure 1.

The set of all staircase tableaux of size $n$ will be denoted by $\mathcal{S}_n$. There are several proofs of the fact that the number of staircase tableaux $|\mathcal{S}_n| = 4^n n!$; see, *e.g.*, [14, 11, 19] for some of them, or (1.5) below and its proof in Section 5.

Given a staircase tableau $S$, we let $N_\alpha, N_\beta, N_\gamma, N_\delta$ be the numbers of symbols $\alpha, \beta, \gamma, \delta$ in $S$ (we also use the notation $N_\alpha(S), \dots$). By (Siv), each column contains at most one $\alpha$ or $\gamma$, and thus $N_\alpha + N_\gamma \leqslant n$. Similarly, by (Siii), each row contains at most one $\beta$ or $\delta$ so $N_\beta + N_\delta \leqslant n$. Together with (Sii) this yields

$$n \leqslant N_\alpha + N_\beta + N_\gamma + N_\delta \leqslant 2n. \tag{1.1}$$



|  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|
| $u$ | $\alpha$ | $u$ | $u$ | $u$ | $q$ | $q$ | $\gamma$ | ○ |
| $u$ | $\beta$ | $u$ | $u$ | $\alpha$ | $q$ | $\gamma$ | ○ |  |
| $u$ | $u$ | $\alpha$ | $u$ | $u$ | $\gamma$ | ○ |  |  |
| $q$ | $q$ | $q$ | $q$ | $\delta$ | ● |  |  |  |
| $q$ | $\delta$ | $u$ | $\alpha$ | ● |  |  |  |  |
| $q$ | $q$ | $\delta$ | ● |  |  |  |  |  |
| $u$ | $\beta$ | ○ |  |  |  |  |  |  |
| $\alpha$ | ● |  |  |  |  |  |  |  |

*Figure 2.* The staircase tableau in Figure 1 filled with $u$s and $q$s and its type; the weight is $\alpha^5\beta^2\delta^3\gamma^3u^{13}q^{10}$ and the type is ○ ○ ○ ● ● ● ○ ●.

In fact, as is seen from (1.5) below, the maximum of $N_\alpha + N_\beta + N_\gamma + N_\delta$ is $2n-1$; see also Example 3.7 and Section 8. Note that there are $n(n+1)/2$ boxes in a staircase tableau in $\mathcal{S}_n$. Hence, in a large staircase tableau, only a small proportion of the boxes are filled.

As mentioned earlier, staircase tableaux were introduced in [17, 18] in connection with the *asymmetric exclusion process (ASEP)*; as a background, we give some details here. The ASEP is a Markov process describing a system of particles on a line with $n$ sites $1,\dots,n$; each site may contain at most one particle. Particles jump one step to the right with intensity $u$ and to the left with intensity $q$, provided the move is to a site that is empty; moreover, new particles enter site 1 with intensity $\alpha$ and site $n$ with intensity $\delta$, provided these sites are empty, and particles at site 1 and $n$ leave the system at rates $\gamma$ and $\beta$, respectively. (There is also a discrete-time version.) See further [18], which also contains references and information on applications and connections to other branches of science.

Explicit expressions for the steady-state probabilities of the ASEP were first given in [20]. Corteel and Williams [18] gave an expression using staircase tableaux, their weight $\mathrm{wt}(S)$ and their generating function. We first fill the tableau $S$ by labelling the empty boxes of $S$ with $u$s and $q$s as follows. First, we fill all the boxes to the left of a $\beta$ with $u$s, and all the boxes to the left of a $\delta$ with $q$s. Then, we fill the remaining boxes above an $\alpha$ or a $\delta$ with $u$s, and the remaining boxes above a $\beta$ or a $\gamma$ with $q$s. When the tableau is filled, its *weight*, $\mathrm{wt}(S)$, is defined as the product of labels of the boxes of $S$; this is thus a monomial of degree $n(n+1)/2$ in $\alpha$, $\beta$, $\gamma$, $\delta$, $u$ and $q$. For example, Figure 2 shows the tableau in Figure 1 filled with $u$s and $q$s; its weight is $\alpha^5\beta^2\delta^3\gamma^3u^{13}q^{10}$. We then let $Z_n(\alpha,\beta,\gamma,\delta,q,u)$ be the total weight of all filled staircase tableaux of size $n$, *i.e.*,

$$Z_n(\alpha,\beta,\gamma,\delta,q,u) = \sum_{S\in\mathcal{S}_n} \mathrm{wt}(S). \tag{1.2}$$

We define the *type* of a staircase tableau $S$ of size $n$ to be a word of the same size on the alphabet $\{\bullet,\circ\}$ (representing occupied and unoccupied sites, respectively) obtained by reading the diagonal boxes from northeast (NE) to southwest (SW) and writing $\bullet$ for each $\alpha$ or $\delta$, and $\circ$ for each $\beta$ or $\gamma$. (Thus a type of a tableau is a possible state for the ASEP.) Figure 2 shows the tableau of Figure 1 and its type: ○ ○ ○ ● ● ● ○ ●.

As Corteel and Williams [18, 17] have shown, the steady-state probability that the ASEP is in state $\sigma$ is

$$\frac{Z_\sigma(\alpha,\beta,\gamma,\delta,q,u)}{Z_n(\alpha,\beta,\gamma,\delta,q,u)},$$

where

$$Z_\sigma(\alpha,\beta,\gamma,\delta,q,u) = \sum_{S \text{ of type } \sigma} wt(S).$$

In the present paper, we will study staircase tableaux and the Greek letters in them, ignoring the $u$ and $q$ in the connection to ASEP above (*i.e.*, we consider Figure 1 rather than Figure 2). We will thus use the simplified versions of $\mathrm{wt}(S)$ and $Z_n$ obtained by putting $u=q=1$. That is, we let

$$\mathrm{wt}(S) := \alpha^{N_\alpha}\beta^{N_\beta}\gamma^{N_\gamma}\delta^{N_\delta}. \tag{1.3}$$

The generating function

$$Z_n(\alpha,\beta,\gamma,\delta) := \sum_{S\in\mathcal{S}_n} \mathrm{wt}(S) \tag{1.4}$$

has a particularly simple form, that is (see [14, 11]),

$$Z_n(\alpha,\beta,\delta,\gamma) = \prod_{i=0}^{n-1}\bigl(\alpha+\beta+\delta+\gamma+i(\alpha+\gamma)(\beta+\delta)\bigr). \tag{1.5}$$

(A proof is included in Section 5.) In particular, the number of staircase tableaux of size $n$ is

$$Z_n(1,1,1,1) = \prod_{i=0}^{n-1}(4+4i) = 4^n n!,$$

as stated above. (We use $\alpha,\beta,\gamma,\delta$ as fixed symbols in the tableaux, and in $N_\alpha,\dots,N_\delta$, but otherwise as variables or real-valued parameters. This should not cause any confusion.) Other special cases for which there is a simple form are discussed in [11] and [14].

Note that the symbols $\alpha$ and $\gamma$ have exactly the same role in the definition above of staircase tableaux, and so do $\beta$ and $\delta$. (This is no longer true in the connection to the ASEP, which is the reason for using four different symbols in the definition.) We say that a staircase tableau using only the symbols $\alpha$ and $\beta$ is an *$\alpha/\beta$-staircase tableau*, and we let $\bar{\mathcal{S}}_n \subset \mathcal{S}_n$ be the set of all $\alpha/\beta$-staircase tableaux of size $n$. We thus see that any staircase tableau can be obtained from an $\alpha/\beta$-staircase tableau by replacing some (or no) $\alpha$ by $\gamma$ and some (or no) $\beta$ by $\delta$; conversely, any staircase tableau can be reduced to an $\alpha/\beta$-staircase tableau by replacing every $\gamma$ by $\alpha$ and every $\delta$ by $\beta$.

There are also connections with other types of tableaux, namely *permutation tableaux* (see, *e.g.*, [48, 13, 15, 16, 12, 33]), *alternative tableaux* [39] and *tree-like tableaux* [1]. These tableaux are Young diagrams (of arbitrary shape) with some symbols added according to specific rules; for definitions see the references given above. The size of one of these is measured by its *length*, which is the sum of the number of rows and the number of columns. There are bijections between the $\alpha/\beta$-staircase tableaux of size $n$, the alternative tableaux of length $n$ and the permutation tableaux of length $n+1$ [18, Appendix], as

*Table 1.* Some correspondences between different types of tableaux

| Alternative tableaux | Permutation tableaux | Tree-like tableaux | Staircase tableaux |
|---|---|---|---|
| # rows | # rows −1 | # rows −1 | $A$ |
| # columns | # columns | # columns −1 | $B$ |
| # free rows | # unrestricted rows −1 | # left points | $n-N_\beta$ |
| # free columns | # top ones | # top points | $n-N_\alpha$ |
| # ← | # restricted rows −1 | # empty left cells | $N_\beta-B$ |
| # ↑ | # top zeros | # empty top cells | $N_\alpha-A$ |

well as between these and the tree-like tableaux of length $n+2$ [1]. (In particular, the numbers of tableaux of these four types are the same, *i.e.*, $(n+1)!$. In fact, there are also several bijections between these objects and permutations of size $n+1$ [48, 13, 39, 11, 1].) Some parameters are easily translated by these bijections: Table 1 gives some important examples from [18, 1]. (See these references for definitions; $A$ and $B$ denote, as below, the numbers of $\alpha$ and $\beta$ on the diagonal in a staircase tableaux.) Further, see Example 3.3 below.

We define the generating function of $\alpha/\beta$-staircase tableaux by

$$Z_n(\alpha,\beta) := \sum_{S\in\bar{\mathcal{S}}_n} \mathrm{wt}(S) = Z_n(\alpha,\beta,0,0), \tag{1.6}$$

and note that the relabelling argument just given implies

$$Z_n(\alpha,\beta,\gamma,\delta) = Z_n(\alpha+\gamma,\beta+\delta). \tag{1.7}$$

We let $x^{\overline{n}}$ denote the rising factorial defined by

$$x^{\overline{n}} := x(x+1)\cdots(x+n-1) = \Gamma(x+n)/\Gamma(x), \tag{1.8}$$

and note that by (1.5),

$$\begin{aligned} Z_n(\alpha,\beta) = Z_n(\alpha,\beta,0,0) &= \prod_{i=0}^{n-1}(\alpha+\beta+i\alpha\beta) = \alpha^n\beta^n(\alpha^{-1}+\beta^{-1})^{\overline{n}} \\ &= \alpha^n\beta^n\frac{\Gamma(n+\alpha^{-1}+\beta^{-1})}{\Gamma(\alpha^{-1}+\beta^{-1})}. \end{aligned} \tag{1.9}$$

In particular, as noted in [11] and [14], the number of $\alpha/\beta$-staircase tableaux is $Z_n(1,1) = 2^{\overline{n}} = (n+1)!$.

Dasse-Hartaut and Hitczenko [19] studied random staircase tableaux obtained by picking a staircase tableau in $\mathcal{S}_n$ uniformly at random. We can obtain the same result by picking an $\alpha/\beta$-staircase tableau in $\bar{\mathcal{S}}_n$ at random with probability proportional to $2^{N_\alpha+N_\beta}$ and then randomly replacing some symbols; each $\alpha$ is replaced by $\gamma$ with probability 1/2, and each $\beta$ by $\delta$ with probability 1/2, with all replacements independent. Note that the weight $2^{N_\alpha+N_\beta}$ is the weight (1.3) if we choose the parameters $\alpha=\beta=2$. The purpose of this paper, more generally, is to study random $\alpha/\beta$-staircase tableaux defined similarly with weights of this type for arbitrary parameters $\alpha,\beta\geqslant 0$. (As we will see in Section 3, this includes several cases considered earlier. It will also be useful in studying the structure of random staircase tableaux; see Section 6.) We generalize several results from [19].

**Definition 1.2.** Let $n \geqslant 1$ and let $\alpha, \beta \in [0, \infty)$ with $(\alpha, \beta) \neq (0, 0)$. Then $S_{n,\alpha,\beta}$ is the random $\alpha/\beta$-staircase tableau in $\bar{\mathcal{S}}_n$ with the distribution

$$\mathbb{P}_{\alpha,\beta}(S_{n,\alpha,\beta} = S) = \frac{\mathrm{wt}(S)}{Z_n(\alpha,\beta)} = \frac{\alpha^{N_\alpha(S)}\beta^{N_\beta(S)}}{Z_n(\alpha,\beta)}, \quad S \in \bar{\mathcal{S}}_n. \tag{1.10}$$

We also allow the parameters $\alpha = \infty$ or $\beta = \infty$; in this case (1.10) is interpreted as the limit when $\alpha \to \infty$ or $\beta \to \infty$, with the other parameters fixed. Similarly, we allow $\alpha = \beta = \infty$; in this case (1.10) is interpreted as the limit when $\alpha = \beta \to \infty$. See also Examples 3.5–3.7 and Section 8. (In the case $\alpha = \beta = \infty$, we tacitly assume $n \geqslant 2$ or sometimes even $n \geqslant 3$ to avoid trivial complications.)

**Remark 1.3.** There is a symmetry (involution) $S \mapsto S^\dagger$ of staircase tableaux defined by reflection in the NW–SE diagonal, thus interchanging rows and columns, together with an exchange of the symbols by $\alpha \leftrightarrow \beta$ and $\gamma \leftrightarrow \delta$; see also [11]. This maps $\bar{\mathcal{S}}_n$ onto itself, and maps the random $\alpha/\beta$-staircase tableau $S_{n,\alpha,\beta}$ to $S_{n,\beta,\alpha}$; the parameters $\alpha$ and $\beta$ thus play symmetric roles.

**Remark 1.4.** We can similarly define a random staircase tableaux $S_{n,\alpha,\beta,\gamma,\delta}$, with four parameters $\alpha, \beta, \gamma, \delta \geqslant 0$, by picking a staircase tableau $S \in \mathcal{S}_n$ with probability $\mathrm{wt}(S)/Z_n(\alpha,\beta,\gamma,\delta)$. By the argument above, this is the same as taking a random $S_{n,\alpha+\gamma,\beta+\delta}$ and randomly replacing each symbol $\alpha$ by $\gamma$ with probability $\gamma/(\alpha+\gamma)$, and each $\beta$ by $\delta$ with probability $\delta/(\beta+\delta)$. (The case $\alpha = \beta = \gamma = \delta = 1$ was mentioned above.) Our results can thus be translated to results for $S_{n,\alpha,\beta,\gamma,\delta}$, but we leave this to the reader.

**Remark 1.5.** For convenience (as a base case in inductions) we also allow $n = 0$; $\mathcal{S}_0 = \bar{\mathcal{S}}_0$ contains a single, empty staircase tableaux with $N_\alpha = N_\beta = N_\gamma = N_\delta = 0$ and thus weight $\mathrm{wt} = 1$, so $Z_0 = 1$. (In some places, *e.g.* in Section 8, we assume $n \geqslant 1$ to avoid trivial complications.)

**Remark 1.6.** It seems natural to use the parameters $\alpha$ and $\beta$ as above in Definition 1.2. However, in many of our results it is more convenient, and sometimes perhaps more natural, to use $\alpha^{-1}$ and $\beta^{-1}$ instead. We will generally use the notations $a := \alpha^{-1}$ and $b := \beta^{-1}$, and formulate results in terms of these parameters whenever convenient.

We state the main results in Section 2 and discuss some examples in Section 3. Section 4 contains further preliminaries, and the proofs of the theorems are given in Section 5. Sections 6 and 7 contain further results on subtableaux and on the positions of the symbols in a random staircase tableau. Finally, the limiting case $\alpha = \beta = \infty$ is studied in greater detail in Section 8.

## 2. Main results

We are interested in the distribution of various parameters of $S_{n,\alpha,\beta}$. In particular, we define $A(S)$ and $B(S)$ as the numbers of $\alpha$ and $\beta$, respectively, on the diagonal of

*Table 2.* The coefficients $v_{a,b}(n,k)$ of $P_{n,a,b}$ for small $n$

| $n\backslash k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | $a$ | $b$ | | |
| 2 | $a^2$ | $a+b+2ab$ | $b^2$ | |
| 3 | $a^3$ | $a+b+3a^2+3ab+3a^2b$ | $a+b+3ab+3b^2+3ab^2$ | $b^3$ |

an $\alpha/\beta$-staircase tableau $S$, and consider the random variables $A_{n,\alpha,\beta} := A(S_{n,\alpha,\beta})$ and $B_{n,\alpha,\beta} := B(S_{n,\alpha,\beta})$; note that $A_{n,\alpha,\beta} + B_{n,\alpha,\beta} = n$ by (Sii), so it suffices to consider one of these. Moreover, by Remark 1.3, $B_{n,\alpha,\beta} \overset{\mathrm{d}}{=} A_{n,\beta,\alpha}$.

In order to describe the distribution of $A_{n,\alpha,\beta}$, we need some further notation. Define numbers $v_{a,b}(n,k)$, for $a,b \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n = 0,1,\dots$, by the recursion

$$v_{a,b}(n,k) = (k+a)v_{a,b}(n-1,k) + (n-k+b)v_{a,b}(n-1,k-1), \quad n \geqslant 1, \tag{2.1}$$

with $v_{a,b}(0,0) = 1$ and $v_{a,b}(0,k) = 0$ for $k \neq 0$: see Table 2. (It is convenient to define $v_{a,b}(n,k)$ for all integers $k \in \mathbb{Z}$, but note that $v_{a,b}(n,k) = 0$ for $k < 0$ and $k > n$, for all $n \geqslant 0$, so it really suffices to consider $0 \leqslant k \leqslant n$.) These numbers were defined and studied by Carlitz and Scoville [9]. (Their notation is $A(n-k,k \mid a,b)$.) Furthermore, define polynomials

$$P_{n,a,b}(x) := \sum_{k=0}^{n} v_{a,b}(n,k)x^k = \sum_{k=-\infty}^{\infty} v_{a,b}(n,k)x^k. \tag{2.2}$$

Thus, $P_{0,a,b}(x) = 1$. Moreover, the recursion (2.1) is easily seen to be equivalent to the recursion

$$P_{n,a,b}(x) = \big((n-1+b)x + a\big)P_{n-1,a,b}(x) + x(1-x)P'_{n-1,a,b}(x), \quad n \geqslant 1. \tag{2.3}$$

In the cases $(a,b) = (1,0), (0,1)$ and $(1,1)$, the numbers $v_{a,b}(n,k)$ are the Eulerian numbers and $P_{n,a,b}(x)$ are the Eulerian polynomials (in different versions). We can thus see $v_{a,b}(n,k)$ and $P_{n,a,b}(x)$ as generalizations of Eulerian numbers and polynomials. Some properties of these numbers and polynomials are given in Section 4, where we also discuss some other special cases.

In the case $a = b = 0$, we trivially have $v_{0,0}(n,k) = 0$ and $P_{n,0,0} = 0$ for all $n \geqslant 1$; in this case we define the substitutes, for $n \geqslant 2$,

$$\tilde{v}_{0,0}(n,k) := v_{1,1}(n-2,k-1) \tag{2.4}$$

and

$$\tilde{P}_{n,0,0}(x) := \sum_{k=0}^{n} \tilde{v}_{0,0}(n,k)x^k = xP_{n-2,1,1}(x). \tag{2.5}$$

See also Lemmas 4.10 and 4.11.

Our main results are as follows. Proofs are given in Section 5.

**Theorem 2.1.** *Let* $\alpha,\beta \in (0,\infty]$ *and let* $a := \alpha^{-1}$, $b := \beta^{-1}$. *If* $(\alpha,\beta) \neq (\infty,\infty)$, *then the probability generating function* $g_A(x)$ *of the random variable* $A_{n,\alpha,\beta}$ *is given by*

$$\begin{aligned} g_A(x) &:= \mathbb{E}\, x^{A_{n,\alpha,\beta}} = \sum_{k=0}^{n} \mathbb{P}(A_{n,\alpha,\beta} = k)x^k \\ &= \frac{P_{n,a,b}(x)}{P_{n,a,b}(1)} = \frac{P_{n,a,b}(x)}{(a+b)^{\overline{n}}} = \frac{\Gamma(a+b)}{\Gamma(n+a+b)} P_{n,a,b}(x). \end{aligned} \tag{2.6}$$

*Equivalently,*

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \frac{v_{a,b}(n,k)}{P_{n,a,b}(1)} = \frac{v_{a,b}(n,k)}{(a+b)^{\overline{n}}} = \frac{\Gamma(a+b)}{\Gamma(n+a+b)} v_{a,b}(n,k). \tag{2.7}$$

*In the case* $\alpha = \beta = \infty$, *and* $n \geqslant 2$, *we instead have*

$$g_A(x) := \sum_{k=0}^{n} \mathbb{P}(A_{n,\alpha,\beta} = k)x^k = \frac{\tilde{P}_{n,0,0}(x)}{\tilde{P}_{n,0,0}(1)} = \frac{\tilde{P}_{n,0,0}(x)}{(n-1)!}, \tag{2.8}$$

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \frac{\tilde{v}_{0,0}(n,k)}{\tilde{P}_{n,0,0}(1)} = \frac{\tilde{v}_{0,0}(n,k)}{(n-1)!}. \tag{2.9}$$

**Remark 2.2.** Consider the following generalized Pólya urn model (an instance of the so-called Friedman's urn [28, 27], which was studied by Bernstein [4, 5]; see also Flajolet, Dumas and Puyhaubert [25]). An urn contains white and black balls. There are initially $a$ white and $b$ black balls. At times $1, 2, \ldots$, one ball is drawn at random from the urn and then replaced, together with a new ball of the opposite colour.

Let $A_n$ $[B_n]$ be the number of white [black] balls added in the $n$ first draws; we thus have $A_n + B_n = n$. Furthermore, after $n$ draws there are $A_n + a$ white and $B_n + b$ black balls in the urn, and thus

$$\mathbb{P}(A_{n+1} = k) = \frac{a+k}{n+a+b}\mathbb{P}(A_n = k) + \frac{n-(k-1)+b}{n+a+b}\mathbb{P}(A_n = k-1). \tag{2.10}$$

Comparing (2.10) to (2.1), we find by induction

$$\mathbb{P}(A_n = k) = \frac{v_{a,b}(n,k)}{(a+b)^{\overline{n}}}. \tag{2.11}$$

In the description of the urn model, it is natural to assume that $a$ and $b$ are integers. However, urn models of this type can easily be extended to allow fractional balls and thus non-integer 'numbers' of balls; see, *e.g.*, [34]. (It is then perhaps better to talk about weights instead of numbers, allowing balls of different weights.) We may thus allow the initial numbers $a$ and $b$ to be any non-negative real numbers with $a + b > 0$; we still add one (whole) ball each time. Equation (2.11) still holds, which by Theorem 2.1 shows the following.

**Theorem 2.3.** *Let* $\alpha,\beta \in (0,\infty]$, *with* $(\alpha,\beta) \neq (\infty,\infty)$. *Then* $(A_{n,\alpha,\beta}, B_{n,\alpha,\beta})$ *has the same distribution as* $(A_n, B_n)$ *in the urn model above, for every* $n \geqslant 0$, *starting with* $a := \alpha^{-1}$ *white and* $b := \beta^{-1}$ *black balls.* □

We now state further consequences of Theorem 2.1.

**Theorem 2.4.** *Let* $\alpha,\beta \in (0,\infty]$ *and let* $a := \alpha^{-1}$ *and* $b := \beta^{-1}$. *Then*

$$\mathbb{E}(A_{n,\alpha,\beta}) = \frac{n(n+2b-1)}{2(n+a+b-1)}$$

*and*

$$\operatorname{Var}(A_{n,\alpha,\beta}) = n\frac{(n-1)(n-2)(n+4a+4b-1)+6(n-1)(a+b)^2+12ab(a+b-1)}{12(n+a+b-1)^2(n+a+b-2)}.$$

**Remark 2.5.** In the symmetric case $\alpha=\beta$ we thus obtain $\mathbb{E}(A_{n,\alpha,\alpha}) = n/2$; this is also obvious by symmetry, since $A_{n,\alpha,\alpha} \overset{\mathrm{d}}{=} B_{n,\alpha,\alpha}$ by Remark 1.3.

**Theorem 2.6.** *The probability generating function* $g_A(x)$ *of the random variable* $A_{n,\alpha,\beta}$ *has all its roots simple and on the negative half-line* $(-\infty,0]$. *As a consequence, for any given* $n,\alpha,\beta$ *there exist* $p_1,\dots,p_n \in [0,1]$ *such that*

$$A_{n,\alpha,\beta} \overset{\mathrm{d}}{=} \sum_{i=1}^{n} \operatorname{Be}(p_i), \tag{2.12}$$

*where* $\operatorname{Be}(p_i)$ *is a Bernoulli random variable with parameter* $p_i$ *and the summands are independent. It follows that the distribution of* $A_{n,\alpha,\beta}$ *and the sequence* $v_{a,b}(n,k)$, $k\in\mathbb{Z}$, *are unimodal and log-concave.*

These results lead to a central limit theorem.

**Theorem 2.7.** *Let* $\alpha,\beta\in(0,\infty]$ *be fixed and let* $n\to\infty$. *Then* $A_{n,\alpha,\beta}$ *is asymptotically normal,*

$$\frac{A_{n,\alpha,\beta} - \mathbb{E}\,A_{n,\alpha,\beta}}{(\operatorname{Var} A_{n,\alpha,\beta})^{1/2}} \xrightarrow{\mathrm{d}} N(0,1), \tag{2.13}$$

*or, more explicitly,*

$$\frac{A_{n,\alpha,\beta} - n/2}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0,1/12). \tag{2.14}$$

*Moreover, a corresponding local limit theorem holds:*

$$\mathbb{P}(A_{n,\alpha,\beta}=k) = \bigl(2\pi \operatorname{Var} A_{n,\alpha,\beta}\bigr)^{-1/2}\biggl(\exp\biggl(-\frac{(k-\mathbb{E}\,A_{n,\alpha,\beta})^2}{2\operatorname{Var} A_{n,\alpha,\beta}}\biggr)+o(1)\biggr), \tag{2.15}$$

*as* $n\to\infty$, *uniformly in* $k\in\mathbb{Z}$, *or, more explicitly,*

$$\mathbb{P}(A_{n,\alpha,\beta}=k) = \sqrt{\frac{6}{\pi n}}\bigl(e^{-6(k-n/2)^2/n}+o(1)\bigr), \tag{2.16}$$

*as* $n\to\infty$, *uniformly in* $k\in\mathbb{Z}$.

**Remark 2.8.** The proof shows that the central limit theorem in the forms (2.13) and (2.15) also holds if $\alpha$ and $\beta$ are allowed to depend on $n$, provided only that $\operatorname{Var}(A_{n,\alpha,\beta})\to\infty$, which by Theorem 2.4 holds as soon as $n^2/(a+b) \to \infty$ or $nab/(a+b)^2 \to \infty$; hence this holds except when $a$ or $b$ is $\infty$ or tends to $\infty$ rapidly, *i.e.*, unless $\alpha$ or $\beta$ is 0 or tends to 0 rapidly. Example 3.8 illustrates that asymptotic normality may fail in extreme cases.

**Remark 2.9.** Asymptotic normality (2.13) is well known for many generalized Pólya urn models, including the one discussed above [4, 5, 27, 34]. We do not know any general local limit theorems for such urn models.

We can also study the total numbers $N_\alpha$ and $N_\beta$ of symbols $\alpha$ and $\beta$ in a random $S_{n,\alpha,\beta}$. This is simpler, and follows directly from (1.9), as we show in Section 5. (Recall that in $N_\alpha$ and $N_\beta$, $\alpha$ and $\beta$ are symbols and not parameter values.)

**Theorem 2.10.** *Let* $\alpha, \beta \in (0,\infty]$, *and let* $a := \alpha^{-1}$, $b := \beta^{-1}$. *The joint probability generating function of* $N_\alpha$ *and* $N_\beta$ *for the random staircase tableau* $S_{n,\alpha,\beta}$ *is*

$$\mathbb{E}_{\alpha,\beta}\left(x^{N_\alpha} y^{N_\beta}\right) = \prod_{i=0}^{n-1} \frac{\alpha x + \beta y + i\alpha\beta xy}{\alpha + \beta + i\alpha\beta} = \prod_{i=0}^{n-1} \frac{bx + ay + ixy}{a+b+i}. \tag{2.17}$$

*In other words,*

$$\left(N_\alpha, N_\beta\right) \overset{\mathrm{d}}{=} \left(\sum_{i=0}^{n-1} I_i, \sum_{i=0}^{n-1} J_i\right), \tag{2.18}$$

*where* $(I_i, J_i)$ *are independent pairs of* $0/1$*-variables with the distributions*

$$\mathbb{P}(I_i = \imath, J_i = \imath') = \begin{cases} 0 & (\imath,\imath') = (0,0), \\ \dfrac{b}{(a+b+i)} & (\imath,\imath') = (1,0), \\ \dfrac{a}{(a+b+i)} & (\imath,\imath') = (0,1), \\ \dfrac{i}{(a+b+i)} & (\imath,\imath') = (1,1). \end{cases} \tag{2.19}$$

*In particular, the marginal distributions are*

$$I_i \sim \mathrm{Be}\left(1 - \frac{a}{a+b+i}\right), \quad J_i \sim \mathrm{Be}\left(1 - \frac{b}{a+b+i}\right). \tag{2.20}$$

*Hence,*

$$\mathbb{E}\, N_\alpha = \sum_{i=0}^{n-1}\left(1 - \frac{a}{a+b+i}\right) = n - \sum_{i=0}^{n-1} \frac{a}{a+b+i}, \tag{2.21}$$

$$\mathrm{Var}\, N_\alpha = \sum_{i=0}^{n-1} \frac{a}{a+b+i}\left(1 - \frac{a}{a+b+i}\right), \tag{2.22}$$

$$\mathrm{Cov}(N_\alpha, N_\beta) = -\sum_{i=0}^{n-1} \frac{ab}{(a+b+i)^2}. \tag{2.23}$$

*In the case* $\alpha = \beta = \infty$ $(a = b = 0)$ *and* $i = 0$, *we interpret*

$$\frac{a}{a+b+i} = \frac{b}{a+b+i} = \frac{1}{2} \quad \textit{and} \quad \frac{i}{a+b+i} = 0$$

*in* (2.19)–(2.23), *and the factor in* (2.17) *as* $(x+y)/2$.

**Theorem 2.11.** *Let* $\alpha, \beta \in (0,\infty]$ *be fixed and let* $n \to \infty$. *Then, with* $a := \alpha^{-1}$ *and* $b := \beta^{-1}$,

$$\mathbb{E}\, N_\alpha = n - a \log n + O(1), \tag{2.24}$$

$$\operatorname{Var} N_\alpha = a \log n + O(1), \tag{2.25}$$

$$\operatorname{Cov}(N_\alpha, N_\beta) = O(1). \tag{2.26}$$

*Furthermore,*

$$\frac{N_\alpha - \mathbb{E}\, N_\alpha}{\sqrt{\log n}} \xrightarrow{\mathrm{d}} N(0,a), \tag{2.27}$$

$$\frac{N_\beta - \mathbb{E}\, N_\beta}{\sqrt{\log n}} \xrightarrow{\mathrm{d}} N(0,b), \tag{2.28}$$

*jointly, with independent limits.*

**Remark 2.12.** A local limit theorem holds too. Moreover, Theorem 2.10 implies that $n - N_\alpha$ can be approximated in the total variation sense by a Poisson distribution $\mathbb{P}(n - \mathbb{E}\, N_\alpha)$; see, *e.g.*, [2, Theorem 2.M]. We omit the details.

**Remark 2.13.** We can similarly also study the joint distribution of $N_\alpha$ and $A$, for example (the total number of $\alpha$s and the number on the diagonal), but we leave this to the reader.

The results above show that the effects of changing the parameters $\alpha$ and $\beta$ are surprisingly small. Typically, probability weights of the type (1.3) (which are common in statistical physics) shift the distributions of the random variables considerably, but here the effects in Theorems 2.4 and 2.11, for example, are only second-order. The reason seems to be that the variables are so constrained; we have $N_\alpha, N_\beta \leqslant n$ and by Theorem 2.10, both are close to their maximum and thus the weights do not differ as much between different random staircase tableaux as might be expected.

**Remark 2.14.** In order to get stronger effects, we may let the weights tend to 0 as $n \to \infty$. For example, taking $\alpha = 1/(sn)$ and $\beta = 1/(tn)$ for some fixed $s, t > 0$, and thus $a = sn$, $b = tn$, we obtain by Theorem 2.4

$$\mathbb{E}(A_{n,\alpha,\beta}) = \frac{2t+1}{2(s+t+1)} n + O(1), \tag{2.29}$$

$$\operatorname{Var}(A_{n,\alpha,\beta}) = \frac{1 + 4s + 4t + 6(s+t)^2 + 12st(s+t)}{12(s+t+1)^3} n + O(1). \tag{2.30}$$

A central limit theorem holds by Remark 2.8. Similarly, one easily shows joint asymptotic normality for $N_\alpha, N_\beta$ in this case too; unlike the case of fixed $\alpha$ and $\beta$ in Theorem 2.11, the limits are now dependent normal variables. We omit the details. Note that by Theorem 6.1, the central part of a uniformly random $\alpha/\beta$-staircase tableau, say the part comprising the middle third of the rows and columns, is an example of this type.

Finally, we note a correspondence with permutations; it would be interesting to find a bijective proof. Given a permutation $\sigma$ of $[n]$, we say that $i$ is an ascent (descent) if $1 \leqslant i \leqslant n-1$ and $\sigma(i) < \sigma(i+1)$ ($\sigma(i) > \sigma(i+1)$), and that $i$ is a left (right) record (or maximum) if $1 \leqslant i \leqslant n$ and $\sigma(i) > \sigma(j)$ for every $j < i$ ($j > i$).

**Theorem 2.15.** *The number of $\alpha/\beta$-staircase tableau of size $n$ with parameters $A = k$, $B = n-k$, $N_\alpha = r$ and $N_\beta = s$ equals the number of permutations of $[n+1]$ with $k$ ascents, $n-k$ descents, $n+1-s$ left records and $n+1-r$ right records.*

## 3. Special cases

**Example 3.1 ($\alpha = \beta = 2$).** This yields the uniformly random staircase tableaux studied by Dasse-Hartaut and Hitczenko [19], as stated above. More precisely, in the notation of Remark 1.4, the uniformly random staircase tableaux is $S_{n,1,1,1,1}$, which is obtained from $S_{n,2,2}$ by a simple random replacement of symbols.

The main results of [19] can be recovered as special cases of the theorems above, with $a = b = 1/2$. Note that in this case, the formulas in Theorem 2.4 simplify to $\mathbb{E}(A_{n,2,2}) = n/2$ (see Remark 2.5) and $\operatorname{Var}(A_{n,2,2}) = (n+1)/12$.

**Example 3.2 ($\alpha = \beta = 1$).** This yields the uniformly random $\alpha/\beta$-staircase tableau $S_{n,1,1}$. As stated above, the number of $\alpha/\beta$-staircase tableaux of size $n$ is $Z_n(1,1) = (n+1)!$. Indeed, Corteel and Williams [18] gave a bijection between $\alpha/\beta$-staircase tableaux of size $n$ and permutation tableaux of size (length) $n+1$, and there are several bijections between the latter and permutations of size $n+1$ [48, 13]; $\alpha/\beta$-staircase tableaux are further studied in [14, 11].

The theorems above, with $a = b = 1$, yield results on uniformly random $\alpha/\beta$-staircase tableaux. For example, Theorem 2.1 shows, using (4.5), that the distribution of $A_{n,1,1}$ is given by the Eulerian numbers

$$\mathbb{P}(A_{n,1,1} = k) = \frac{v_{1,1}(n,k)}{(n+1)!} = \frac{\left\langle {n+1 \atop k} \right\rangle}{(n+1)!}. \tag{3.1}$$

In other words, the number of $\alpha/\beta$-staircase tableaux of size $n$ with $k$ $\alpha$s on the diagonal is $\left\langle {n+1 \atop k} \right\rangle$. (This also follows by the bijections mentioned above between $\alpha/\beta$-staircase tableaux and permutation tableaux [18] and between the latter and permutations [13].) Theorems 2.6 and 2.7 give in this case well-known results for Eulerian numbers; see [29] and [8], respectively.

Furthermore, the formulas in Theorem 2.4 simplify and yield $\mathbb{E}\,A_{n,1,1} = n/2$ (see Remark 2.5) and $\operatorname{Var} A_{n,1,1} = (n+2)/12$. As another example, Theorem 2.10 shows that

$$n - N_\alpha \overset{\mathrm{d}}{=} \sum_{i=0}^{n-1} (1 - I_i) \sim \sum_{i=0}^{n-1} \operatorname{Be}\left(\frac{1}{i+2}\right) = \sum_{i=2}^{n+1} \operatorname{Be}\left(\frac{1}{i}\right), \tag{3.2}$$

with the summands independent; note that this has the same distribution as $C_{n+1}-1$, where $C_{n+1}$ is the number of cycles in a random permutation of size $n+1$, or, equivalently, the number of maxima (records) in such a random permutation. (Again, a bijective proof can be given using the bijections with permutation tableaux and permutations in [18] and [13].)

Similarly, several of the results for permutation tableaux in [33] can be recovered from our results.

Furthermore, deleting the top row of a staircase tableau corresponds for the alternative tableau to deleting the first step on its SE boundary; this means deleting its last column if it is empty, and otherwise deleting the first row. (And similarly for deleting the first column.) Hence, for example, our Theorem 6.1 translates to a result on subtableaux of random alternative tableaux.

**Example 3.3 ($\alpha = 1$).** As stated above, there are several bijections between various types of tableaux and permutations of size $n+1$. In particular (see [13] and [24, Section 6]), one of these bijections yields a correspondence between the number of cycles in the permutation, and the number of unrestricted rows in the permutation tableaux; by Table 2, this equals $n+1-N_\beta$ in the staircase tableaux. Hence, our random staircase tableau $S_{n,1,\beta}$ corresponds to a random permutation of size $n+1$ with probability proportional to $\theta^{\#\text{cycles}}$, where $\theta = 1/\beta$; this is a much-studied distribution of permutations, in particular in connection with the *Ewens sampling formula* [23]. Moreover, Féray [24] shows that under this bijection, the sequence of weak exceedances in the permutation corresponds to the shape of the permutation tableau; translated to the staircase tableau, the diagonal element in row $i$ in the staircase tableau is $\alpha$ if and only if $i+1$ is a weak exceedance (*i.e.*, $\sigma(i+1) \geqslant i+1$) in the permutation.

**Example 3.4 ($\alpha = 2$, $\beta = 1$).** This corresponds to staircase tableaux without $\delta$s briefly studied in [14]. The number of such staircase tableaux is, by (1.9),

$$Z_n(2,1) = 2^n(3/2)^{\overline{n}} = \prod_{i=0}^{n-1}(3+2i) = (2n+1)!! \tag{3.3}$$

(see [14, 11]). Our theorems yield results on random $\delta$-free staircase tableaux.

**Example 3.5 ($\alpha = \infty$).** This means that we take the limit as $\alpha \to \infty$ in (1.10), which means that we have a non-zero probability only for staircase tableaux with the maximum number of symbols $\alpha$, *i.e.*, with $N_\alpha = n$. For such $\alpha/\beta$-staircase tableaux, the probability is proportional to $\beta^{N_\beta}$.

We let $\mathcal{S}_n^* \subset \bar{\mathcal{S}}_n$ be the set of such $\alpha/\beta$-staircase tableaux of size $n$; by (Siv), these are the $\alpha/\beta$-staircase tableaux of size $n$ with exactly one $\alpha$ in each column. (Such staircase tableaux were studied in [11].) We define the corresponding generating function

$$Z_n^*(\beta) := \sum_{S \in \mathcal{S}_n^*} \beta^{N_\beta} = \lim_{\alpha\to\infty} \alpha^{-n} Z_n(\alpha,\beta) = \prod_{i=0}^{n-1}(1+i\beta), \tag{3.4}$$

where the final equality follows from (1.9). Thus, $S_{n,\infty,\beta}$ is the random element of $\mathcal{S}_n^*$ with the distribution $\mathbb{P}(S_{n,\infty,\beta} = S) = \beta^{N_\beta(S)}/Z_n^*(\beta)$.

**Example 3.6 ($\boldsymbol{\alpha = \infty}$, $\boldsymbol{\beta = 1}$).** As a special case of the preceding example, $S_{n,\infty,1}$ is a uniformly random element of $\mathcal{S}_n^*$. By (3.4), the number of $\alpha/\beta$-staircase tableaux of size $n$ with $n$ $\alpha$s is

$$Z_n^*(1) = n!. \tag{3.5}$$

Hence, the probability that a uniformly random $\alpha/\beta$-staircase tableau has the maximum number $n$ of $\alpha$s is $Z_n^*(1)/Z_n(1,1) = n!/(n+1)! = 1/(n+1)$. (See also Theorem 2.10 and (3.2).)

The theorems above, with $a = 0$ and $b = 1$, yield results on uniformly random $\alpha/\beta$-staircase tableaux with $n$ $\alpha$s (*i.e.*, one in each column). For example, Theorem 2.1 shows, using (4.4), that the distribution of $A_{n,\infty,1}$ is given by the Eulerian numbers

$$\mathbb{P}(A_{n,\infty,1} = k) = \frac{v_{0,1}(n,k)}{n!} = \frac{\left\langle {n \atop k-1} \right\rangle}{n!}. \tag{3.6}$$

In other words, the number of $\alpha/\beta$-staircase tableaux of size $n$ with $n$ $\alpha$s, of which $k$ are on the diagonal, is $\left\langle {n \atop k-1} \right\rangle$. (A bijective proof is given in [11].) By symmetry, counting instead the number of $\beta$s on the diagonal, by (4.2),

$$\mathbb{P}(B_{n,\infty,1} = k) = \mathbb{P}(A_{n,1,\infty} = k) = \frac{v_{1,0}(n,k)}{n!} = \frac{\left\langle {n \atop k} \right\rangle}{n!}. \tag{3.7}$$

Compare with Example 3.2, where the distributions of $A$ and $B \overset{\mathrm{d}}{=} A$ are also given by Eulerian numbers. By (3.6)–(3.7) and (3.1), we see that $A_{n,\infty,1} \overset{\mathrm{d}}{=} A_{n-1,1,1} + 1$ and $B_{n,\infty,1} \overset{\mathrm{d}}{=} A_{n,\infty,1} - 1 \overset{\mathrm{d}}{=} A_{n-1,1,1} \overset{\mathrm{d}}{=} B_{n-1,1,1}$.

The formulas in Theorem 2.4 simplify and yield $\mathbb{E}\, A_{n,\infty,1} = (n+1)/2$ and $\operatorname{Var} A_{n,\infty,1} = (n+1)/12$. As another example, Theorem 2.10 shows that

$$n - N_\beta \overset{\mathrm{d}}{=} \sum_{i=0}^{n-1} (1 - J_i) \sim \sum_{i=0}^{n-1} \mathrm{Be}\left(\frac{1}{i+1}\right) = \sum_{i=1}^{n} \mathrm{Be}\left(\frac{1}{i}\right), \tag{3.8}$$

with the summands independent; this has the same distribution as $C_n$, with $C_n$ as in the corresponding result in Example 3.2. (A bijective proof is given in [11].)

**Example 3.7 ($\boldsymbol{\alpha = \beta = \infty}$).** This means that we take the limit as $\alpha = \beta \to \infty$ in (1.10), which means that we have a non-zero probability only for $\alpha/\beta$-staircase tableau with the maximum number of symbols. These tableaux correspond to the terms with maximal total degree in $Z_n(\alpha,\beta)$, and it follows from (1.9) that they have $2n-1$ symbols. (We assume $n \geqslant 1$.)

We let $\mathcal{S}_n^{**} \subset \bar{\mathcal{S}}_n$ be the set of $\alpha/\beta$-staircase tableaux with $N_\alpha + N_\beta = 2n-1$; thus $S_{n,\infty,\infty}$ is a uniformly random element of $\mathcal{S}_n^{**}$.

We further define the corresponding generating function

$$Z_n^{**}(\alpha,\beta) := \sum_{S \in \mathcal{S}_n^{**}} \alpha^{N_\alpha} \beta^{N_\beta}. \tag{3.9}$$

This can be obtained by extracting the terms with largest degrees in (1.9), and thus

$$Z_n^{**}(\alpha,\beta)=(\alpha+\beta)\prod_{i=1}^{n-1}(i\alpha\beta)=(n-1)!\,\big(\alpha^n\beta^{n-1}+\alpha^{n-1}\beta^n\big). \tag{3.10}$$

Hence there are $2(n-1)!$ tableaux in $\mathcal{S}_n^{**}$; $(n-1)!$ with $n$ $\alpha$s and $n-1$ $\beta$s, and $(n-1)!$ with $n-1$ $\alpha$s and $n$ $\beta$s. See also Section 8. (It follows that the corresponding number of staircase tableaux with $2n-1$ symbols $\alpha,\beta,\gamma,\delta$ is $2^{2n}(n-1)!$: see [11].)

By Theorem 2.1 and (4.7) below, assuming $n\geqslant 2$,

$$\mathbb{P}\big(A_{n,\infty,\infty}=k\big)=\frac{\tilde{v}_{0,0}(n,k)}{(n-1)!}=\frac{\left\langle {n-1 \atop k-1}\right\rangle}{(n-1)!}, \tag{3.11}$$

and thus by (3.1), $A_{n,\infty,\infty}\overset{\mathrm{d}}{=}A_{n-2,1,1}+1$.

**Example 3.8 ($\boldsymbol{\beta=0}$).** This gives weight 0 to any staircase tableaux with a symbol $\beta$, so only tableaux with just the symbol $\alpha$ may occur. By (Sii) and (Siv) in the definition, the only such tableau is the one with $\alpha$ in every diagonal box, and no other symbols. This limiting case is thus trivial, with $S_{n,\alpha,0}$ deterministic (and independent of the parameter $\alpha$), and $N_\alpha=A_{n,\alpha,\beta}=n$, $N_\beta=B_{n,\alpha,\beta}=0$, and $Z_n(\alpha,0)=\alpha^n$.

This case (and the symmetric $\alpha=0$) is excluded from most of our results, but since it is trivial, the reader can easily supplement corresponding, trivial, results for it. Note that this case occurs as a natural limiting case when $\beta\to 0$.

**Example 3.9 ($\boldsymbol{\alpha=\beta=0}$).** This case is really excluded, since it would give weight 0 to every $\alpha/\beta$-staircase tableau. However, we can define it as the limit as $\alpha=\beta\to 0$. This gives a non-zero probability only to $\alpha/\beta$-staircase tableaux with a minimum number of symbols, *i.e.*, with $n$ symbols on the diagonal and no others. There are $2^n$ such $\alpha/\beta$-staircase tableaux, and all get the same probability, so $S_{n,0,0}$ is obtained by putting a random symbol in each diagonal box, uniformly and independently. This leads to a classical case and we will not discuss it any further.

More generally, taking the limit as $\alpha,\beta\to 0$ with $\alpha/(\alpha+\beta)\to\rho\in[0,1]$ yields an $\alpha/\beta$-staircase tableau with symbols only on the diagonal and each diagonal box having symbol $\alpha$ with probability $\rho$, independently of the other boxes (*cf.* Theorem 8.4).

## 4. The polynomials $P_{n,a,b}$

The numbers $v_{a,b}(n,k)$ and polynomials $P_{n,a,b}(x)$ are defined by (2.1)–(2.3) for all real (or complex) $a$ and $b$, but for our purposes we are only interested in $a,b\geqslant 0$. We regard $a$ and $b$ as fixed parameters, but we note that the numbers $v_{a,b}(n,k)$ are polynomials in $a$ and $b$ (of degree exactly $n$ in the non-trivial case $0\leqslant k\leqslant n$).

The case $a=b=0$ is trivial: by (2.1) or (2.3) and induction,

$$v_{0,0}(n,k)=0 \quad\text{and}\quad P_{n,0,0}(x)=0 \quad\text{for all } n\geqslant 1. \tag{4.1}$$

For $a = 1$, $b = 0$, the recursion (2.1) is the standard recursion for *Eulerian numbers* $\left\langle {n \atop k} \right\rangle$ (see, *e.g.*, [31, Section 6.2], [40, §26.14], [41, A008292]). Thus

$$v_{1,0}(n,k) = \left\langle {n \atop k} \right\rangle. \tag{4.2}$$

(These are often defined as the number of permutations of $n$ elements with $k$ descents (or ascents). See, *e.g.*, [47, Section 1.3], where other relations to permutations are given.) The corresponding polynomials

$$P_{n,1,0}(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k \tag{4.3}$$

are known as *Eulerian polynomials*: see Euler [21, 22].

Furthermore, the cases $(a,b) = (0,1)$ and $(1,1)$ also lead to Eulerian numbers, with different indexing. By (2.1) and induction, or by (4.16) below,

$$v_{0,1}(n,k) = v_{1,0}(n, n-k) = \left\langle {n \atop n-k} \right\rangle = \left\langle {n \atop k-1} \right\rangle, \quad n \geqslant 1 \tag{4.4}$$

(which is non-zero for $1 \leqslant k \leqslant n$). Similarly, by (2.1) and induction,

$$v_{1,1}(n,k) = v_{1,0}(n+1,k) = \left\langle {n+1 \atop k} \right\rangle, \quad n \geqslant 0. \tag{4.5}$$

Equivalently,

$$P_{n,0,1}(x) = xP_{n,1,0}(x), \quad P_{n,1,1}(x) = P_{n+1,1,0}(x). \tag{4.6}$$

Similarly, by the definition (2.4) and (4.5),

$$\tilde{v}_{0,0}(n,k) = \left\langle {n-1 \atop k-1} \right\rangle, \quad n \geqslant 2, \tag{4.7}$$

and by (2.5) and (4.6),

$$\tilde{P}_{n,0,0}(x) = P_{n-1,0,1}(x) = xP_{n-1,1,0}(x). \tag{4.8}$$

Returning to the general case, when $a = 0$ or $b = 0$ we have the following simple relations, generalizing the results for Eulerian numbers and polynomials in (4.4)–(4.6).

**Lemma 4.1.** *For all $n \geqslant 1$,*

$$v_{a,0}(n,k) = av_{a,1}(n-1,k), \tag{4.9}$$

$$v_{0,b}(n,k) = bv_{1,b}(n-1,k-1), \tag{4.10}$$

*and, equivalently,*

$$P_{n,a,0}(x) = aP_{n-1,a,1}(x), \tag{4.11}$$

$$P_{n,0,b}(x) = bxP_{n-1,1,b}(x). \tag{4.12}$$

**Proof.** We use induction, with (2.1) or (2.3). □

We collect some further properties in the following theorems.

**Theorem 4.2.** *For all* $a, b$ *and* $n \geqslant 0$,

$$P_{n,a,b}(0) = v_{a,b}(n,0) = a^n, \tag{4.13}$$

$$v_{a,b}(n,n) = b^n, \tag{4.14}$$

$$P_{n,a,b}(1) = \sum_{k=0}^{n} v_{a,b}(n,k) = (a+b)^{\overline{n}} = \frac{\Gamma(n+a+b)}{\Gamma(a+b)}. \tag{4.15}$$

*Furthermore, we have the symmetry*

$$v_{a,b}(n,k) = v_{b,a}(n,n-k) \tag{4.16}$$

*and thus*

$$P_{n,a,b}(x) = x^n P_{n,b,a}(1/x). \tag{4.17}$$

**Proof.** We use induction, with (2.1) or (2.3). □

**Remark 4.3.** The symmetries (4.16)–(4.17) between $a$ and $b$ are more evident if we define the homogeneous two-variable polynomials

$$\widehat{P}_{n,a,b}(x,y) := \sum_{k=0}^{n} v_{a,b}(n,k) x^k y^{n-k}, \tag{4.18}$$

which satisfy the recursion

$$\widehat{P}_{n,a,b}(x,y) = \left(bx + ay + xy\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right)\widehat{P}_{n-1,a,b}(x,y), \quad n \geqslant 1, \tag{4.19}$$

and the symmetry

$$\widehat{P}_{n,a,b}(x,y) = \widehat{P}_{n,b,a}(y,x).$$

(Note that $\widehat{P}_{n,a,b}(x,y) = y^n P_{n,a,b}(x/y)$ and $P_{n,a,b}(x) = \widehat{P}_{n,a,b}(x,1)$.)

Then (2.6) can be written in the symmetric form

$$\mathbb{E}\, x^{A_{n,\alpha,\beta}} y^{B_{n,\alpha,\beta}} = \sum_{k=0}^{n} \mathbb{P}(A_{n,\alpha,\beta} = k) x^k y^{n-k} = \frac{\Gamma(a+b)}{\Gamma(n+a+b)} \widehat{P}_{n,a,b}(x,y). \tag{4.20}$$

However, we find it more convenient to work with polynomials in one variable.

**Theorem 4.4.** *For all* $a, b$ *and* $n \geqslant 0$,

$$P'_{n,a,b}(1) = \sum_{k=0}^{n} k v_{a,b}(n,k) = \frac{n(n+2b-1)}{2}(a+b)^{\overline{n-1}} \tag{4.21}$$

*and*

$$P''_{n,a,b}(1) = \sum_{k=0}^{n} k(k-1)v_{a,b}(n,k) \\ = \frac{n(n-1)(3n^2+(12b-11)n+12b^2-24b+10)}{12}(a+b)^{\overline{n-2}}. \tag{4.22}$$

**Proof.** This can be shown by induction, differentiating (2.3) once or twice and then taking $x = 1$. We omit the details, and instead give another proof in Section 5. □

**Theorem 4.5.**

(i) *If $a, b > 0$, then $v_{a,b}(n,k) > 0$ for $0 \leqslant k \leqslant n$, and $P_{n,a,b}(x)$ is a polynomial of degree $n$ with $n$ simple negative roots.*

(ii) *If $a > b = 0$, then $v_{a,b}(n,k) > 0$ for $0 \leqslant k < n$, and $P_{n,a,b}(x)$ is a polynomial of degree $n-1$ with $n-1$ simple negative roots.*

(iii) *If $a = 0 < b$, then $v_{a,b}(n,k) > 0$ for $1 \leqslant k \leqslant n$, and $P_{n,a,b}(x)$ is a polynomial of degree $n$ with $n$ simple roots in $(-\infty, 0]$. One of the roots is $0$, provided $n > 0$.*

(iv) *If $a = b = 0$, then $\tilde{v}_{0,0}(n,k) > 0$ for $1 \leqslant k \leqslant n-1$, and $\tilde{P}_{n,0,0}(x)$ is a polynomial of degree $n-1$ with $n-1$ simple roots in $(-\infty, 0]$. One of the roots is $0$, provided $n \geqslant 2$.*

**Proof.** **(i)** Induction shows that $v_{a,b}(n,k) > 0$ for $0 \leqslant k \leqslant n$, so $P_{n,a,b}$ has degree exactly $n$. The fact that all roots are negative and simple follows from (2.3), as noted already by Frobenius [29] for the Eulerian polynomials; this can be seen by the following standard argument. Suppose, by induction, that $P_{n-1,a,b}$ has $n-1$ simple roots $-\infty < x_{n-1} < \cdots < x_1 < 0$. Then $P_{n-1,a,b}$ changes sign at each root, with a non-zero derivative, and since $P_{n-1,a,b}(0) > 0$ by (4.13), we have $\operatorname{sign}(P'_{n-1,a,b}(x_i)) = (-1)^{i-1}$, $i = 1, \ldots, n-1$. Since (2.3) yields $P_{n,a,b}(x_i) = x_i(1-x_i)P'_{n-1,a,b}(x_i)$ and $x_i < 0$, this implies $\operatorname{sign}(P_{n,a,b}(x_i)) = (-1)^i$, $i = 1, \ldots, n-1$. Moreover, $\operatorname{sign}(P_{n,a,b}(0)) = +1$ and $\lim_{x\to-\infty} \operatorname{sign}(P_{n,a,b}(x)) = (-1)^n \operatorname{sign}(v_{a,b}(n,n)) = (-1)^n$ by (4.13) and (4.14). Hence $P_{n,a,b}$ changes sign at least $n$ times in $(-\infty, 0)$, and thus has at least $n$ roots there. Since $P_{n,a,b}$ has degree $n$, these are all the roots, and they are all simple.

**(ii, iii)** These follow from (i) and Lemma 4.1. (Alternatively, the proof above works with minor modifications.)

**(iv)** This follows from (i) and the definitions (2.4)–(2.5). □

The proof also shows that the roots of $P_{n-1,a,b}$ and $P_{n,a,b}$ are interlaced (except that 0 is a common root when $a = 0$). For more general results of this kind, see, *e.g.*, [50] and [36, Proposition 3.5].

**Remark 4.6.** As stated in the Introduction, $v_{a,b}(n,k)$ equals the number $A(n-k, k \mid a, b)$ defined by Carlitz and Scoville [9], who used the generating function (in our notation)

$$\sum_{n,k\geqslant 0} v_{a,b}(n,k)\frac{x^{n-k}y^k}{n!} = \sum_{n=0}^{\infty} \frac{\widehat{P}_{n,a,b}(y,x)}{n!} = \bigl(1+xF(x,y)\bigr)^a \bigl(1+yF(x,y)\bigr)^b, \tag{4.23}$$

where

$$F(x,y) := \frac{e^x - e^y}{xe^y - ye^x} \tag{4.24}$$

(interpreted as its limit $1/(1-x)$ when $x=y$); this can also be written as

$$\sum_{n,k\geqslant 0} v_{a,b}(n,k)\frac{x^{n-k}y^k t^n}{n!} = \sum_{n=0}^{\infty} \frac{\widehat{P}_{n,a,b}(y,x)t^n}{n!} = e^{atx+bty}\left(\frac{x-y}{xe^{ty}-ye^{tx}}\right)^{a+b}. \tag{4.25}$$

**Example 4.7.** The case $a=b=1/2$ appeared in [19]; see Example 3.1. In this case, it is more convenient to study the numbers $B(n,k) := 2^n v_{1/2,1/2}(n,k)$ which are integers and satisfy the recursion

$$B(n,k) = (2k+1)B(n-1,k) + (2n-2k+1)B(n-1,k-1), \quad n \geqslant 1. \tag{4.26}$$

These are called *Eulerian numbers of type B* [41, A060187]. The numbers $B(n,k)$ seem to have been introduced by MacMahon [37, p. 331] in number theory. They also have combinatorial interpretations, for example as the numbers of descents in signed permutations, *i.e.*, in the hyperoctahedral group [6, 10, 45].

Note that this case is a special case of both of the following examples.

**Example 4.8.** Franssens [26] studied numbers and polynomials equivalent to the case $a=b$ of ours; more precisely, his $B_{n,k}(c) = 2^n v_{c/2,c/2}(n,k)$, as is seen by comparing his recursion formula to (2.1), and thus his $B_n(x,y;c) = 2^n\widehat{P}_{n,c/2,c/2}(x,y)$, using the notation (4.18). The generating function in [26, Proposition 3.1] thus yields (for small $|t|$)

$$\sum_{n=0}^{\infty} \widehat{P}_{n,a,a}(x,y)\frac{t^n}{n!} = B(x,y,t)^{2a}, \tag{4.27}$$

with

$$B(x,y,t) := \begin{cases} \dfrac{x-y}{xe^{-(x-y)t/2} - ye^{(x-y)t/2}} & x \neq y, \\ \dfrac{1}{1-xt} & x = y, \end{cases} \tag{4.28}$$

which is a special case of (4.25).

**Example 4.9.** The case $a+b=1$ yields polynomials $P_{n,a,1-a}(x)$ generalizing the Eulerian polynomials (the case $a=1$, or $a=0$); they satisfy (extending results by Euler [22, Chapter II.7])

$$\sum_{k=0}^{\infty} (k+a)^n x^k = \frac{P_{n,a,1-a}(x)}{(1-x)^{n+1}} \tag{4.29}$$

and (as follows from (4.25))

$$\sum_{n=0}^{\infty} P_{n,a,1-a}(x)\frac{z^n}{n!} = \frac{(1-x)e^{az(1-x)}}{1-xe^{z(1-x)}}. \tag{4.30}$$

These polynomials are sometimes called (generalized) *Euler–Frobenius polynomials* and appear, for example, in spline theory; see [38, 49, 43, 44, 46, 30, 35]. The function $P_{n,1-a,a}(x)/(x-1)^n$ was studied by Carlitz [7] (there denoted $H_n(a \mid x)$).

We defined in (2.4)–(2.5) $\tilde{v}_{0,0}(n,k)$ and $\tilde{P}_{n,0,0}(x)$ as substitutes for the vanishing $v_{0,0}(n,k)$ and $P_{n,0,0}(x)$. To justify this, we first note that these numbers and polynomials satisfy the recursions obtained by putting $a = b = 0$ in (2.1) and (2.3).

**Lemma 4.10.** *We have*

$$\tilde{v}_{0,0}(n,k) = k\tilde{v}_{0,0}(n-1,k) + (n-k)\tilde{v}_{0,0}(n-1,k-1), \quad n \geqslant 3, \tag{4.31}$$

*with* $\tilde{v}_{0,0}(2,1) = 1$ *and* $\tilde{v}_{0,0}(2,k) = 0$ *for* $k \neq 1$. *Similarly,*

$$\tilde{P}_{n,0,0}(x) = (n-1)x\tilde{P}_{n-1,0,0}(x) + x(1-x)\tilde{P}'_{n-1,0,0}(x), \quad n \geqslant 3. \tag{4.32}$$

*with* $\tilde{P}_{2,0,0}(x) = x$.

**Proof.** This follows easily by substituting the definitions (2.4) and (2.5) in (2.1) and (2.3). □

Moreover, these numbers and polynomials appear as limits as $a,b \to 0$ if we renormalize.

**Lemma 4.11.** *For any* $n \geqslant 2$ *and* $k \in \mathbb{Z}$ *or* $x \in \mathbb{R}$, *as* $a,b \searrow 0$,

$$\frac{v_{a,b}(n,k)}{a+b} \to \tilde{v}_{0,0}(n,k), \tag{4.33}$$

$$\frac{P_{n,a,b}(x)}{a+b} \to \tilde{P}_{n,0,0}(x). \tag{4.34}$$

**Proof.** We first verify (4.33) for $n = 2$ by inspection: see Table 2. For $n > 2$ we divide (2.1) by $a+b$, let $a,b \searrow 0$ and use induction together with (4.31).

Finally, (4.34) follows from (4.33) by (2.5) and (2.2). □

**Remark 4.12.** More general numbers, defined by a more general version of the recursion formula (2.1), are studied in [50].

## 5. Proofs of Theorems 2.1–2.15

To prove Theorem 2.1 we use induction on the size $n$, where we extend a staircase tableau of size $n-1$ by adding a column of length $n$ to the left and consider all possible ways of filling it out with the symbols. This method was used, in a probabilistic context, in [19] and its origins seem to go back to [14, Remark 3.14]; see also [11]. For permutation tableaux an analogous technique was used in [12] and [33].

In order to do the necessary recursive analysis, we introduce a suitable generating function with an additional 'catalytic' parameter, which we now define.

We say that a row of a staircase tableau is *indexed by* $\alpha$ if its leftmost entry is $\alpha$. Thus, for example, in the tableau depicted in Figure 1, the first, third and eighth rows are indexed by $\alpha$. The number of rows indexed by $\alpha$ in a staircase tableau $S$ will be denoted by $r = r(S)$.

We introduce the generating function for the pair of parameters $(A, r)$:

$$D_n(x,z) := \sum_{S\in\bar{\mathcal{S}}_n} \mathrm{wt}(S)x^{A(S)}z^{r(S)} = \sum_{S\in\bar{\mathcal{S}}_n} \alpha^{N_\alpha}\beta^{N_\beta}x^A z^r. \tag{5.1}$$

We regard $\alpha$ and $\beta$ as fixed in this section, and for simplicity we omit them from the notation $D_n(x,z)$. We assume that $0 < \alpha, \beta < \infty$.

**Remark 5.1.** In an $\alpha/\beta$-staircase tableau, a row containing a $\beta$ must by (Siii) have the $\beta$ as its leftmost entry; hence it is not indexed by $\alpha$. Conversely, a row without $\beta$ is necessarily indexed by $\alpha$. Since no row contains more than one $\beta$, it follows that $r = n - N_\beta$ [19]. We thus have $D_n(x,z) = z^n\tilde{D}_n(x,\alpha,\beta/z)$, where

$$\tilde{D}_n(x,\alpha,\beta) := \sum_{S\in\bar{\mathcal{S}}_n} \alpha^{N_\alpha}\beta^{N_\beta}x^A = D_n(x,1). \tag{5.2}$$

Hence it is possible to avoid $r$ and instead argue with the simpler $\tilde{D}_n(x,\alpha,\beta)$ and a varying $\beta$. However, we find it more convenient to keep $\alpha$ and $\beta$ fixed and to use $r$ in the argument below.

Trivially, $D_0(x,z) = 1$ (see Remark 1.5).

**Lemma 5.2.** *$D_n$ satisfies the recursion, for $n \geqslant 1$,*

$$D_n(x,z) = \alpha z(x-1)D_{n-1}(x,z) + (\alpha z + \beta)D_{n-1}(x, z+\beta). \tag{5.3}$$

**Proof.** Fix an $\alpha/\beta$-staircase tableau $S$ of size $n-1$ with parameters $N_\alpha$, $N_\beta$, $A$, $r$, and consider all ways to extend it to a tableau of size $n$ by adding a column of length $n$ on the left and filling some boxes in it. There are three cases (see [11, 19]).

(i) We put $\alpha$ in the bottom box of the added column. By (Siv), no other boxes in the new column can be filled, so this gives a single staircase tableau of size $n$; this tableau has parameters $N_\alpha + 1$, $N_\beta$, $A+1$ and $r+1$, so its contribution to $D_n(x,z)$ is

$$\alpha^{N_\alpha+1}\beta^{N_\beta}x^{A+1}z^{r+1} = \alpha x z\, \alpha^{N_\alpha}\beta^{N_\beta}x^A z^r. \tag{5.4}$$

(ii) We put $\beta$ in the bottom box of the added column; we may also put $\alpha$ or $\beta$ in some other boxes in the new column, and we consider first the case when we put no $\alpha$, so only $\beta$s are added. By (Siii), we may put a $\beta$ only in the rows indexed by $\alpha$ (apart from the bottom box). For $0 \leqslant k \leqslant r$, there are thus $\binom{r}{k}$ possibilities to add $k$ further $\beta$; each choice yields a staircase tableau with parameters $N_\alpha$, $N_\beta + 1 + k$, $A$, $r-k$, and their total contribution to $D_n(x,z)$ is

$$\sum_{k=0}^{r}\binom{r}{k}\alpha^{N_\alpha}\beta^{N_\beta+1+k}x^A z^{r-k} = \alpha^{N_\alpha}\beta^{N_\beta+1}x^A(z+\beta)^r. \tag{5.5}$$

(iii) We put $\beta$ in the bottom box of the added column and $\alpha$ or $\beta$ in some other boxes in the new column, including an $\alpha$. By (Siv), we may add only one $\alpha$, and it has to be the top one of the added symbols. Again, the new symbols may (apart from the bottom box) only be added in rows indexed by $\alpha$. For $1 \leqslant k \leqslant r$, there are thus $\binom{r}{k}$ possibilities to add $k-1$ further $\beta$ and one $\alpha$; each choice yields a staircase tableau with parameters $N_\alpha + 1$, $N_\beta + k$, $A$, $r-k+1$, and their total contribution to $D_n(x,z)$ is

$$\sum_{k=1}^{r} \binom{r}{k} \alpha^{N_\alpha+1}\beta^{N_\beta+k}x^A z^{r-k+1} = \alpha^{N_\alpha+1}\beta^{N_\beta}x^A z\big((z+\beta)^r - z^r\big). \tag{5.6}$$

Combining (5.4)–(5.6), we obtain the total contribution from extensions of $S$ to be

$$\alpha xz\,\alpha^{N_\alpha}\beta^{N_\beta}x^A z^r + (\beta+\alpha z)\alpha^{N_\alpha}\beta^{N_\beta}x^A(z+\beta)^r - \alpha z\,\alpha^{N_\alpha}\beta^{N_\beta}x^A z^r, \tag{5.7}$$

and summing over all $S \in \bar{\mathcal{S}}_{n-1}$ yields (5.3). ☐

Iterating (5.3), we obtain the following, recalling that $x^{\overline{\ell}}$ denotes the rising factorial and that $a = \alpha^{-1}$ and $b = \beta^{-1}$.

**Lemma 5.3.** *Assume* $0 < \alpha, \beta < \infty$. *For* $0 \leqslant m \leqslant n$,

$$D_n(x,z) = (\alpha\beta)^m \sum_{\ell=0}^{m} c_{m,\ell}(z)(a+bz)^{\overline{\ell}}(x-1)^{m-\ell}D_{n-m}(x, z+\ell\beta), \tag{5.8}$$

*where* $c_{0,0}(z) = 1$ *and, for* $m \geqslant 0$, *with* $c_{m,-1}(z) = c_{m,m+1}(z) = 0$,

$$c_{m+1,\ell}(z) = (\ell + bz)c_{m,\ell}(z) + c_{m,\ell-1}(z), \quad 0 \leqslant \ell \leqslant m+1. \tag{5.9}$$

**Proof.** The case $m = 0$ is trivial. Suppose that (5.8) holds for some $m \geqslant 0$ and all $n \geqslant m$. If $n > m$, we use Lemma 5.2 on the right-hand side of (5.8) and obtain

$$\begin{aligned}
&(\alpha\beta)^{-m}D_n(x,z)\\
&\quad= \sum_{\ell=0}^{m} c_{m,\ell}(z)(a+bz)^{\overline{\ell}}(x-1)^{m-\ell}\big(\alpha(z+\ell\beta)(x-1)D_{n-m-1}(x,z+\ell\beta)\\
&\qquad\quad + (\alpha z + \alpha\ell\beta + \beta)D_{n-m-1}(x, z+\ell\beta+\beta)\big)\\
&\quad= \alpha\beta\sum_{\ell=0}^{m} c_{m,\ell}(z)(a+bz)^{\overline{\ell}}(x-1)^{m+1-\ell}(bz+\ell)D_{n-m-1}(x,z+\ell\beta)\\
&\qquad\quad + \alpha\beta\sum_{\ell=0}^{m} c_{m,\ell}(z)(a+bz)^{\overline{\ell}}(x-1)^{m-\ell}(bz+\ell+a)D_{n-m-1}(x,z+\ell\beta+\beta)\\
&\quad= \alpha\beta\sum_{\ell=0}^{m} (\ell+bz)c_{m,\ell}(z)(a+bz)^{\overline{\ell}}(x-1)^{m+1-\ell}D_{n-m-1}(x,z+\ell\beta)\\
&\qquad\quad + \alpha\beta\sum_{j=1}^{m+1} c_{m,j-1}(z)(a+bz)^{\overline{j}}(x-1)^{m+1-j}D_{n-m-1}(x,z+j\beta).
\end{aligned}$$

The result for $m+1$ follows, and the lemma follows by induction. ☐

We now take $z = 1$, thus forgetting $r$. (We will not use $r$ further. If desired, $r$ can be recovered by Remark 5.1.) This yields the following formula for the generating function $D_n(x, 1)$ for $A$. We write $c_{n,\ell} = c_{n,\ell}(1)$.

**Lemma 5.4.** *Assume* $0 < \alpha, \beta < \infty$. *For* $n \geqslant 0$,

$$D_n(x,1) = (\alpha\beta)^n \sum_{\ell=0}^{n} c_{n,\ell}(a+b)^{\overline{\ell}}(x-1)^{n-\ell}, \tag{5.10}$$

*where* $c_{0,0} = 1$ *and, for* $n \geqslant 0$, *with* $c_{n,-1} = c_{n,n+1} = 0$,

$$c_{n+1,\ell} = (\ell + b)c_{n,\ell} + c_{n,\ell-1}, \quad 0 \leqslant \ell \leqslant n+1. \tag{5.11}$$

**Proof.** Take $z = 1$ and $m = n$ in Lemma 5.3, recalling that $D_0 = 1$, so the factor $D_{n-m}(x, z + \ell\beta)$ on the right-hand side of (5.8) disappears. $\square$

We have found a formula for $D_n(x, 1)$ as a polynomial in $x - 1$. We can identify it as $P_{n,a,b}(x)$ (up to a constant factor).

**Lemma 5.5.** *Assume* $0 < \alpha, \beta < \infty$. *For* $n \geqslant 0$,

$$D_n(x,1) = (\alpha\beta)^n P_{n,a,b}(x). \tag{5.12}$$

**Proof.** Define $\widehat{D}_n(x) := (\alpha\beta)^{-n} D_n(x,1)$. Clearly, $\widehat{D}_0(x) = 1 = P_{0,a,b}(x)$. We show that $\widehat{D}_n$ satisfies the recursion (2.3), which implies that $\widehat{D}_n = P_{n,a,b}$ for all $n \geqslant 0$ and thus completes the proof. By Lemma 5.4,

$$\begin{aligned}
&((n+b)x + a)\widehat{D}_n(x) + x(1-x)\widehat{D}_n'(x) \\
&\quad = \sum_{\ell=0}^{n} \big(nx + bx + a - (n-\ell)x\big) c_{n,\ell}(a+b)^{\overline{\ell}}(x-1)^{n-\ell} \\
&\quad = \sum_{\ell=0}^{n} \big((\ell+b)(x-1) + \ell + b + a\big) c_{n,\ell}(a+b)^{\overline{\ell}}(x-1)^{n-\ell} \\
&\quad = \sum_{\ell=0}^{n} (\ell+b) c_{n,\ell}(a+b)^{\overline{\ell}}(x-1)^{n+1-\ell} + \sum_{\ell=0}^{n} (a+b+\ell) c_{n,\ell}(a+b)^{\overline{\ell}}(x-1)^{n-\ell} \\
&\quad = \sum_{\ell=0}^{n} (\ell+b) c_{n,\ell}(a+b)^{\overline{\ell}}(x-1)^{n+1-\ell} + \sum_{j=1}^{n+1} c_{n,j-1}(a+b)^{\overline{j}}(x-1)^{n+1-j} \\
&\quad = \sum_{j=0}^{n+1} c_{n+1,\ell}(a+b)^{\overline{\ell}}(x-1)^{n+1-\ell} = \widehat{D}_{n+1}(x),
\end{aligned}$$

where we used (5.11) and (5.10) in the last line. $\square$

**Proof of Theorem 2.1.** Assume $\alpha, \beta \in (0, \infty)$. We have $D_n(1,1) = Z_n(\alpha, \beta)$ by (5.1) and (1.6). Moreover, it follows immediately from $A_{n,\alpha,\beta} = A(S_{n,\alpha,\beta})$ and the definitions

(2.6) and (1.10) that

$$g_A(x) = \sum_{S \in \bar{\mathcal{S}}_n} x^{A(S)}\,\mathbb{P}(S_{n,\alpha,\beta} = S) = \sum_{S \in \bar{\mathcal{S}}_n} x^{A(S)} \frac{\mathrm{wt}(S)}{Z_n(\alpha,\beta)} = \frac{D_n(x,1)}{D_n(1,1)}. \tag{5.13}$$

Hence, Lemma 5.5 yields

$$g_A(x) = \frac{P_{n,a,b}(x)}{P_{n,a,b}(1)}, \tag{5.14}$$

which shows (2.6), using (4.15). Extracting coefficients yields (2.7).

The case $\alpha = \infty$ or $\beta = \infty$ follows by taking limits as $\alpha \to \infty$ ($\beta \to \infty$).

The case $\alpha = \beta = \infty$ follows similarly by taking limits as $\alpha = \beta \to \infty$, using Lemma 4.11. ☐

The proof above contains (as a simpler special case) the calculation of $Z_n$ in [11]; we record this for completeness.

**Proof of (1.5) and (1.9).** Taking $x = 1$ in Lemma 5.4, we obtain

$$Z_n(\alpha,\beta) = D_n(1,1) = (\alpha\beta)^n c_{n,n}(a+b)^{\overline{n}} = (\alpha\beta)^n c_{n,n}(a+b)^{\overline{n}}, \tag{5.15}$$

since $c_{n,n} = 1$ by (5.11) and induction. (Alternatively, we may use Lemma 5.5 and (4.15).) This yields (1.9), and (1.5) follows by (1.7). ☐

**Proof of Theorem 4.4.** We assume $a, b > 0$; the general case then follows since all quantities are polynomials in $a$ and $b$. By Lemmas 5.5 and 5.4, for any $k \geqslant 0$,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} P_{n,a,b}(1) = k!\, c_{n,n-k}(a+b)^{\overline{n-k}} \tag{5.16}$$

(with $c_{n,\ell} = 0$ for $\ell < 0$). In particular, for $k = 1$ we have by (5.11)

$$c_{n+1,n} = (n+b)c_{n,n} + c_{n,n-1} = n + b + c_{n,n-1}, \tag{5.17}$$

and a simple induction yields

$$c_{n,n-1} = \sum_{m=0}^{n-1}(m+b) = \frac{n(n+2b-1)}{2}, \tag{5.18}$$

which by (5.16) yields (4.21).

Similarly,

$$\begin{aligned} c_{n,n-2} &= \sum_{m=1}^{n}(m+b-2)c_{m-1,m-2} \\ &= \frac{n(n-1)(3n^2 + (12b-11)n + 12b^2 - 24b + 10)}{24}, \end{aligned} \tag{5.19}$$

which by (5.16) yields (4.22). ☐

**Proof of Theorem 2.4.** Assume first $(a,b) \neq (0,0)$. Then (2.6) yields

$$\mathbb{E}\, A_{n,\alpha,\beta} = g_A'(1) = \frac{P'_{n,a,b}(1)}{P_{n,a,b}(1)}$$

and

$$\operatorname{Var} A_{n,\alpha,\beta} = g_A''(1) + g_A'(1) - \bigl(g_A'(1)\bigr)^2 = \frac{P''_{n,a,b}(1) + P'_{n,a,b}(1)}{P_{n,a,b}(1)} - \frac{P'_{n,a,b}(1)^2}{P_{n,a,b}(1)^2}$$

and the result follows from Theorem 4.4 and (4.15) (after some calculations).

The case $a = b = 0$ follows by continuity. □

**Proof of Theorem 2.6.** The first claim is immediate by Theorems 2.1 and 4.5. This implies (2.12) and the following claims by standard arguments: If $g_A(x)$ has roots $-\xi_1, \dots, -\xi_n \leqslant 0$, then, using $g_A(1) = 1$,

$$g_A(x) = \frac{\prod_{i=1}^n (x + \xi_i)}{\prod_{i=1}^n (1 + \xi_i)} = \prod_{i=1}^n \left( \frac{\xi_i}{1 + \xi_i} + \frac{1}{1 + \xi_i} x \right), \tag{5.20}$$

which equals the probability generating function of $\sum_{i=1}^n \mathrm{Be}(p_i)$ for independent $\mathrm{Be}(p_i)$ with $p_i = 1/(1 + \xi_i)$; this verifies (2.12). If $b = 0$ so $g_A(x)$ has only $n - 1$ roots, the same holds with $p_n = 0$. (We may then formally set $\xi_n = \infty$.)

The fact that the distribution of $A_{n,a,b}$ is log-concave and thus unimodal follows easily from (2.12) by induction; the same holds for the sequence $v_{a,b}(n,k)$, $k \in \mathbb{Z}$, by (2.7). □

**Proof of Theorem 2.7.** By Theorem 2.6,

$$A_{n,\alpha,\beta} \overset{\mathrm{d}}{=} \sum_{i=1}^n I_i, \tag{5.21}$$

with $I_i \sim \mathrm{Be}(p_i)$ independent. Note that then

$$\mathbb{E}\, A_{n,\alpha,\beta} = \sum_{i=1}^n p_i \quad \text{and} \quad \operatorname{Var} A_{n,\alpha,\beta} = \sum_{i=1}^n p_i(1 - p_i).$$

Moreover,

$$\sum_{i=1}^n \mathbb{E}\, |I_i - p_i|^3 \leqslant \sum_{i=1}^n \mathbb{E}\, |I_i - p_i|^2 = \operatorname{Var} A_{n,\alpha,\beta}. \tag{5.22}$$

The central limit theorem with Lyapunov's condition (see, *e.g.*, [32, Theorem 7.2.2]) shows that any sequence of sums of this type is asymptotically normal, provided the variance tends to infinity, which holds in our case by Theorem 2.4. Theorem 2.4 further shows

$$\mathbb{E}\, A_{n,\alpha,\beta} = n/2 + O(1), \tag{5.23}$$

$$\operatorname{Var} A_{n,\alpha,\beta} = n/12 + O(1), \tag{5.24}$$

which implies that the versions (2.13) and (2.14) are equivalent.

Finally, [42, Theorem VII.3] shows that a local limit theorem (2.15) also holds for any sum of the type (5.21); again we use (5.23)–(5.24) to simplify the result and obtain (2.16). □

**Proof of Theorem 2.10.** Assume first $\alpha, \beta < \infty$. The joint probability generating function of $(N_\alpha, N_\beta)$ is by definition

$$\frac{\sum_{S\in\bar{\mathcal{S}}_n} \mathrm{wt}(S)x^{N_\alpha}y^{N_\beta}}{Z_n(\alpha,\beta)} = \frac{\sum_{S\in\bar{\mathcal{S}}_n} \alpha^{N_\alpha}\beta^{N_\beta}x^{N_\alpha}y^{N_\beta}}{Z_n(\alpha,\beta)} = \frac{Z_n(\alpha x,\beta y)}{Z_n(\alpha,\beta)}, \tag{5.25}$$

and (2.17) follows from (1.9).

Since $(I_i, J_i)$ defined by (2.19) has the probability generating function

$$\frac{bx+ay+ixy}{a+b+i},$$

the distributional identity (2.18) follows from (2.17). Thus

$$\mathbb{E}\,N_\alpha = \sum_{i=0}^{n-1}\mathbb{E}\,I_i, \quad \operatorname{Var} N_\alpha = \sum_{i=0}^{n-1}\operatorname{Var} I_i \quad \text{and} \quad \operatorname{Cov}(N_\alpha,N_\beta) = \sum_{i=0}^{n-1}\operatorname{Cov}(I_i,J_i),$$

which yield (2.21)–(2.23).

The case when $\alpha=\infty$ or $\beta=\infty$, or both, follows by taking limits. □

**Proof of Theorem 2.11.** The estimates (2.24)–(2.26) follow from (2.21)–(2.23).

The central limit theorem (2.27)–(2.28) follows from the representation (2.18) in Theorem 2.10 as in the proof of Theorem 2.7; note that (2.26) implies $\operatorname{Cov}(N_\alpha,N_\beta)/\log n \to 0$, which yields the independence of the limits in (2.27)–(2.28). □

**Proof of Theorem 2.15.** The generating function (*i.e.*, the total weight) for $\alpha/\beta$-staircase tableaux with $A=k$ and $B=n-k$ is, using (2.7) and (1.9),

$$\mathbb{P}(A_{n,\alpha,\beta}=k)Z_n(\alpha,\beta) = v_{a,b}(n,k)\alpha^n\beta^n.$$

On the other hand, using the notation of Carlitz and Scoville [9] (see Remark 4.6, and their [9, Theorem 9]),

$$\begin{aligned} v_{a,b}(n,k) &= A(n-k,k \mid a,b) = A(k,n-k \mid b,a)\\ &= \sum_{t,u} P(k,n-k;t,u)b^{t-1}a^{u-1} = \sum_{t,u} P(k,n-k;t,u)\alpha^{1-u}\beta^{1-t}, \end{aligned}$$

where $P(k,n-k;t,u)$ is the number of permutations with $k$ ascents, $n-k$ descents, $t$ left records and $u$ right records. (Note that [9] counts one ascent (rise) and one descent (fall) more than we do.) The result follows by identifying the coefficients of $\alpha^r\beta^s$. □

## 6. Subtableaux

We number the rows and columns of a staircase tableau by $1,\dots,n$ starting at the NW corner (as in a matrix); the boxes are thus labelled by $(i,j)$ with $i,j \geqslant 1$ and $i+j \leqslant n+1$.

The diagonal boxes are $(i, n+1-i)$, $i = 1, \dots, n$, going from NE to SW. We denote the symbol in box $(i, j)$ of a staircase tableau $S$ by $S(i, j)$, with $S(i, j) = 0$ if the box is empty.

If we delete the first rows or columns from a staircase tableau, we obtain a new, smaller, staircase tableau. For $S \in \mathcal{S}_n$ and a box $(i, j)$ in $S$ (so $i + j \leqslant n+1$), let $S[i, j]$ be the subtableau with $(i, j)$ as its top left box, *i.e.*, the subtableau obtained by deleting the first $i-1$ rows and the first $j-1$ columns. Note that $S[i, j] \in \mathcal{S}_{n-i-j+2}$. (Conditions (Si)–(Siv) are clearly satisfied.)

**Theorem 6.1.** *Let $\alpha, \beta \in (0, \infty]$ and $i + j \leqslant n+1$. The subtableau $S_{n,\alpha,\beta}[i, j]$ of $S_{n,\alpha,\beta}$ has the same distribution as $S_{n-i-j+2,\hat\alpha,\hat\beta}$, where $\hat\alpha^{-1} = \alpha^{-1} + i - 1$ and $\hat\beta^{-1} = \beta^{-1} + j - 1$.*

**Proof.** Consider first the case $i = 1$ and $j = 2$, where we only delete the first (leftmost) column. Let $S \in \bar{\mathcal{S}}_{n-1}$. The probability that $S_{n,\alpha,\beta}[1, 2] = S$ is proportional to the sum of the weights of all extensions of $S$ to a staircase tableau in $\bar{\mathcal{S}}_n$. By the proof of Lemma 5.2, with $x = z = 1$, this sum equals (see (5.7))

$$\begin{aligned}(\beta+\alpha)\alpha^{N_\alpha}\beta^{N_\beta}(1+\beta)^r &= (\beta+\alpha)\alpha^{N_\alpha}\beta^{N_\beta}(1+\beta)^{n-N_\beta} \\ &= (\beta+\alpha)(1+\beta)^n\alpha^{N_\alpha}\left(\frac{\beta}{1+\beta}\right)^{N_\beta},\end{aligned} \tag{6.1}$$

so $\mathbb{P}(S_{n,\alpha,\beta}[1, 2] = S)$ is proportional to $\alpha^{N_\alpha}\hat\beta^{N_\beta}$ with $\hat\beta := \beta/(\beta+1)$, *i.e.*, $\hat\beta^{-1} = \beta^{-1} + 1$. Hence, $S_{n,\alpha,\beta}[1, 2] \overset{\mathrm{d}}{=} S_{n-1,\alpha,\hat\beta}$, so the theorem holds in this case.

Next, the case $i = 2$, $j = 1$, where we delete the top row, follows by symmetry: see Remark 1.3.

Finally, the general case follows by induction, deleting one row or column at a time. □

## 7. The positions of the symbols

We have so far considered the numbers of the symbols $\alpha$ and $\beta$ in a random $\alpha/\beta$-staircase tableau, and the numbers of them on the diagonal. Now we consider the position of the symbols. We begin by considering the symbols on the diagonal, where every box is filled with $\alpha$ or $\beta$.

**Theorem 7.1.** *Let $\alpha, \beta \in (0, \infty]$ and let $a := \alpha^{-1}$, $b := \beta^{-1}$. The probability that the ith diagonal box contains $\alpha$ is*

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i, n+1-i) = \alpha\big) = \frac{n-i+b}{n+a+b-1}, \qquad 1 \leqslant i \leqslant n. \tag{7.1}$$

**Proof.** If $n = 1$, this follows directly from the definition and $\alpha/(\alpha+\beta) = b/(a+b)$.

In general, we use Theorem 6.1 with $j = n+1-i$, which shows that $S_{n,\alpha,\beta}[i, n+1-i] \overset{\mathrm{d}}{=} S_{1,\hat\alpha,\hat\beta}$ with $\hat a := \hat\alpha^{-1} = a + i - 1$, $\hat b := \hat\beta^{-1} = b + n - i$, which yields

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i, n+1-i) = \alpha\big) = \mathbb{P}\big(S_{1,\hat\alpha,\hat\beta}(1, 1) = \alpha\big) = \frac{\hat\alpha}{\hat\alpha+\hat\beta} = \frac{n-i+b}{n+a+b-1}. \qquad \square$$

The probability of an $\alpha$ thus decreases linearly as we go from NE to SW, from approximately 1 to approximately 0 for large $n$. Hence the top part of the diagonal contains mainly $\alpha$s and the bottom part mainly $\beta$s. (This is very reasonable, since these choices give fewer restrictions by (Siii) and (Siv).)

Non-diagonal boxes are often empty. The distribution of a given box is as follows.

**Theorem 7.2.** *Let $\alpha$, $\beta$ and $a, b$ be as in Theorem 7.1. The probability that the non-diagonal box $(i, j)$ contains $\alpha$ or $\beta$ is*

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i,j)=\alpha\big)=\frac{j-1+b}{(i+j+a+b-1)(i+j+a+b-2)}, \tag{7.2}$$

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i,j)=\beta\big)=\frac{i-1+a}{(i+j+a+b-1)(i+j+a+b-2)}, \tag{7.3}$$

*and thus*

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i,j)\neq 0\big)=\frac{1}{i+j+a+b-1}. \tag{7.4}$$

*For $\alpha=\beta=\infty$ and $i=j=1$, we interpret* (7.2) *and* (7.3) *as* $1/2$.

**Proof.** Consider first the case $i=j=1$. By Theorem 2.10, the expected total number of symbols $\alpha$ in $S=S_{n,\alpha,\beta}$ is

$$\mathbb{E}\,N_\alpha=\sum_{i=0}^{n-1}\left(1-\frac{a}{a+b+i}\right). \tag{7.5}$$

If we delete the first column, the remaining part $S[1,2]$ is by Theorem 6.1 an $S_{n-1,\alpha_1,\beta_1}$ with $a_1:=\alpha_1^{-1}=a$ and $b_1:=\beta_1^{-1}=b+1$; hence Theorem 2.10 shows that the expected number of symbols in $S[1,2]$ is

$$\sum_{i=0}^{n-2}\left(1-\frac{a_1}{a_1+b_1+i}\right)=\sum_{i=0}^{n-2}\left(1-\frac{a}{a+b+1+i}\right)=\sum_{i=1}^{n-1}\left(1-\frac{a}{a+b+i}\right). \tag{7.6}$$

Taking the difference of (7.5) and (7.6), we see that

$$\mathbb{E}\big(\#\alpha\text{ in the first column}\big)=1-\frac{a}{a+b}=\frac{b}{a+b}. \tag{7.7}$$

Now delete the first row of $S$. By Theorem 6.1, the remainder $S[2,1]$ is an $S_{n-1,\alpha_2,\beta_2}$ with $a_2:=\alpha_2^{-1}=a+1$ and $b_2:=\beta_2^{-1}=b$. Hence (7.7) applied to this subtableau shows that

$$\mathbb{E}\big(\#\alpha\text{ in boxes }(2,1),\dots,(n,1)\big)=\frac{b_2}{a_2+b_2}=\frac{b}{a+b+1}, \tag{7.8}$$

and taking the difference of (7.7) and (7.8) we obtain

$$\mathbb{P}\big(S_{n,\alpha,\beta}(1,1)=\alpha\big)=\frac{b}{a+b}-\frac{b}{a+b+1}=\frac{b}{(a+b)(a+b+1)}. \tag{7.9}$$

(This argument is valid also for $n=2$, since (7.7) also holds for $n=1$, by Theorem 7.1 or by noting that (7.6) also holds, trivially, for $n=1$.)

We have shown (7.9), which is (7.2) for $i=j=1$. The general case of (7.2) follows by Theorem 6.1, (7.3) follows by symmetry (Remark 1.3) and (7.4) follows by summing. ☐

**Example 7.3.** For $2 \leqslant k \leqslant n$, the expected total number of symbols in the boxes on the line $i+j=k$ parallel to the diagonal is

$$\sum_{i=1}^{k-1} \frac{1}{k+a+b-1} = \frac{k-1}{k+a+b-1}. \tag{7.10}$$

Thus, for $k$ large there is on average about one symbol on each such line that is not too short. (In the case $\alpha=\beta=\infty$, the expectation equals 1 for every such line.) We do not know the distribution of symbols on the line $i+j=k$, and leave that as an open problem. We conjecture that the distribution is asymptotically Poisson as $n,k \to \infty$.

**Example 7.4.** The expected number of $\alpha$s on the line $i+j=k$, with $2 \leqslant k \leqslant n$, is

$$\sum_{i=1}^{k-1} \frac{j-1+b}{(k+a+b-1)(k+a+b-2)} = \frac{(k-1)(k+2b-2)}{2(k+a+b-1)(k+a+b-2)}, \tag{7.11}$$

which is about $1/2$ for large $k$ (with equality when $\alpha=\beta=\infty$). Again, we do not know the distribution, but we conjecture that it is asymptotically Poisson as $n,k \to \infty$.

We can also consider the joint distribution for several boxes. We consider only boxes on the diagonal, leaving non-diagonal boxes as an open problem. Our key tool is the following simple lemma. Compare to Theorem 6.1 with no conditioning and (in this case) a shift of $\beta$.

**Lemma 7.5.** *If we condition $S_{n,\alpha,\beta}$ on the bottom box $S_{n,\alpha,\beta}(n,1)=\alpha$, then the subtableau $S_{n,\alpha,\beta}[1,2]$ obtained by deleting the first column has the distribution of $S_{n-1,\alpha,\beta}$.*

**Proof.** If $S$ is an $\alpha/\beta$-staircase tableau such that the bottom box $S(n,1)=\alpha$, then the first column is otherwise empty by (Siv), and the remainder, *i.e.* $S[1,2]$, is an arbitrary $\alpha/\beta$-staircase tableau of size $n-1$. Introducing weights (1.3), if we condition $S_{n,\alpha,\beta}$ on $S_{n,\alpha,\beta}(n,1)=\alpha$ and then delete the first column, then we obtain a copy of $S_{n-1,\alpha,\beta}$ as asserted. ☐

The following theorem gives a complete description of the distribution of the boxes on the diagonal. For convenience, we use a simplified notation, letting $S_n(j)$ be the symbol of the random $S_{n,\alpha,\beta}$ in the diagonal box in *column* $j$, *i.e.*,

$$S_n(j) := S_{n,\alpha,\beta}(n+1-j,j). \tag{7.12}$$

**Theorem 7.6.** *Let $\alpha$, $\beta$ and $a,b$ be as in Theorem 7.1, and let $1 \leqslant j_1 < \cdots < j_\ell \leqslant n$. Then*

$$\mathbb{P}\big(S_n(j_1)=\cdots=S_n(j_\ell)=\alpha\big) = \prod_{k=1}^{\ell} \frac{j_k-k+b}{n-k+a+b}. \tag{7.13}$$

For $\ell=1$, this is Theorem 7.1.

**Proof.** We use induction on $n$. (Induction on $\ell$ is also possible.)

If $j_1 > 1$, we may delete the first column, which decreases $n$ and each $j_k$ by 1 and, by Theorem 6.1, increases $b$ by the same amount. Thus (7.13) follows by the inductive hypothesis.

If $j_1 = 1$, we use Lemma 7.5 and obtain by Theorem 7.1 and induction

$$\begin{aligned}\mathbb{P}\big(S_n(j_1) = \cdots = S_n(j_\ell) = \alpha\big) \\ &= \mathbb{P}(S_n(1) = \alpha)\,\mathbb{P}\big(S_n(j_2) = \cdots = S_n(j_\ell) = \alpha \mid S_n(1) = \alpha\big) \\ &= \mathbb{P}(S_n(1) = \alpha)\,\mathbb{P}\big(S_{n-1}(j_2 - 1) = \cdots = S_{n-1}(j_\ell - 1) = \alpha\big) \\ &= \frac{b}{n+a+b-1}\prod_{k=1}^{\ell-1}\frac{j_{k+1}-1-k+b}{n-1-k+a+b},\end{aligned}$$

which shows (7.13) in this case too. □

The case $\ell = 2$ can also be expressed as a covariance formula.

**Corollary 7.7.** *If* $1 \leqslant j < k \leqslant n$, *then*

$$\operatorname{Cov}\big(\mathbf{1}\{S_n(j) = \alpha\}, \mathbf{1}\{S_n(k) = \alpha\}\big) = -\frac{(j-1+b)(n-k+a)}{(n+a+b-1)^2(n+a+b-2)}.$$

**Proof.** By Theorem 7.6, the covariance is

$$\begin{aligned}\frac{j-1+b}{n-1+a+b}\cdot\frac{k-2+b}{n-2+a+b} - \frac{j-1+b}{n-1+a+b}\cdot\frac{k-1+b}{n-1+a+b} \\ = \frac{j-1+b}{n-1+a+b}\left(\frac{k-2+b}{n-2+a+b} - \frac{k-1+b}{n-1+a+b}\right),\end{aligned}$$

and the result follows. □

**Remark 7.8.** Barbour and Janson [3] studied the profile of a random permutation tableau, which by the bijection discussed in the Introduction and Example 3.2 is equivalent to studying the sequence of partial sums $\sum_{j=1}^{k}\mathbf{1}\{S_n(j) = \alpha\}$, $k = 1, \ldots, n$, in the case $\alpha = \beta = 1$; it is shown in [3] that after rescaling, this sequence converges to a Gaussian process. This is extended to the case $\alpha = 1$ with arbitrary $\beta > 0$ by Féray [24]. It would be interesting to extend this further to general $\alpha$ and $\beta$.

## 8. The case $\alpha = \beta = \infty$

The limiting case $\alpha = \beta = \infty$ was studied in Example 3.7, where we saw that $S_{n,\infty,\infty}$ is a uniformly random element of $\mathcal{S}_n^{**}$, the set of $\alpha/\beta$-staircase tableau with the maximal number, $2n-1$, of symbols $\alpha$ and $\beta$. We study these $\alpha/\beta$-staircase tableaux further.

**Lemma 8.1.** *A staircase tableau* $S \in \mathcal{S}_n^{**}$ *always has box* $(1,1)$ *filled with a symbol.*

**Proof.** This follows from (7.4) in Theorem 7.2, taking $\alpha = \beta = \infty$ and thus $a = b = 0$, which shows that the random staircase tableau $S_{n,\infty,\infty}$ has a symbol in box $(1,1)$ with probability 1; recall from Example 3.7 that $S_{n,\infty,\infty}$ is uniformly distributed in $\mathcal{S}_n^{**}$.

Alternatively, we can give a combinatorial proof as follows. Suppose that $S \in \mathcal{S}_n^{**}$ has box $(1,1)$ empty. We may replace any $\alpha$ in the first column by $\beta$, and any $\beta$ in the first row by $\alpha$, without violating (Si)–(Siv), and we may then add $\alpha$ (or $\beta$) in box $(1,1)$, yielding a staircase tableau with one more symbol, which is a contradiction since $\mathcal{S}_n^{**}$ consists of the $\alpha/\beta$-staircase tableaux with a maximum number of symbols. □

Given a staircase tableau $S \in \mathcal{S}_n^{**}$, we let as above $S(1,1)$ be the symbol in $(1,1)$, and we let $S'$ be the staircase tableau obtained by removing this symbol from $S$.

**Lemma 8.2.** *If $S \in \mathcal{S}_n^{**}$, then $S'$ has $n-1$ $\alpha$s and $n-1$ $\beta$s.*

*More precisely, $S'$ has an $\alpha$ in each column except the first, and a $\beta$ in each row except the first.*

**Proof.** By (Siv), $S$ has at most one $\alpha$ in each column; moreover, since $(1,1)$ is filled, the first column cannot contain an $\alpha$ in any other box. Hence, $S'$ contains no $\alpha$ in the first column, and at most one $\alpha$ in every other column. Similarly, $S'$ contains no $\beta$ in the first row and at most one in every other row.

Consequently, $N_\alpha(S') + N_\beta(S') \leqslant (n-1) + (n-1) = 2n-2$. On the other hand, $S$ contains $2n-1$ symbols so $S'$ contains $2n-2$ symbols and we must have equality. □

Conversely, if $S_0 \in \bar{\mathcal{S}}_n$ has $n-1$ $\alpha$s and $n-1$ $\beta$s distributed as described in Lemma 8.2, then box $(1,1)$ is empty and we may add any of $\alpha$ or $\beta$ to $(1,1)$ and obtain a staircase tableau in $\mathcal{S}_n^{**}$. Let $\mathcal{S}_n^{**\prime} := \{S' : S \in \mathcal{S}_n^{**}\}$ be the set of $\alpha/\beta$-staircase tableaux described in Lemma 8.2. The mapping $S \mapsto S'$ is thus a 2–1 map of $\mathcal{S}_n^{**}$ onto $\mathcal{S}_n^{**\prime}$.

Given $\rho \in [0,1]$, we define a random $\alpha/\beta$-staircase tableau $S_{n,\infty,\infty,\rho}$ by picking a random, uniformly distributed, $S' \in \mathcal{S}_n^{**\prime}$ and adding a random symbol, independent of $S'$, in box $(1,1)$, with probability $\rho$ of adding $\alpha$. In particular, $S_{n,\infty,\infty,1/2}$ has the uniform distribution on $\mathcal{S}_n^{**}$, i.e., $S_{n,\infty,\infty,1/2} = S_{n,\infty,\infty}$; see Example 3.7.

**Lemma 8.3.** *Let $\alpha, \beta \in (0,\infty)$. Then the random tableau $S_{n,\alpha,\beta}$ conditioned to have the maximum number $2n-1$ of symbols has the distribution of $S_{n,\infty,\infty,\rho}$ with $\rho = \alpha/(\alpha+\beta)$.*

**Proof.** A staircase tableau $S \in \mathcal{S}_n^{**}$ has weight $\alpha\,\mathrm{wt}(S')$ when $S(1,1) = \alpha$ and $\beta\,\mathrm{wt}(S')$ when $S(1,1) = \beta$. By Lemma 8.2, all staircase tableaux $S' \in \mathcal{S}_n^{**\prime}$ have the same weight $\alpha^{n-1}\beta^{n-1}$, and hence the result follows. □

We have defined $S_{n,\infty,\infty}$ by letting $\alpha = \beta \to \infty$. What happens if we let $\alpha \to \infty$ and $\beta \to \infty$, but with different rates?

**Theorem 8.4.** *Let $\alpha \to \infty$ and $\beta \to \infty$ such that $\alpha/(\alpha+\beta) \to \rho \in [0,1]$, and let $n \geqslant 1$ be fixed. Then $S_{n,\alpha,\beta} \xrightarrow{\mathrm{d}} S_{n,\infty,\infty,\rho}$.*

**Proof.** The weight of every $\alpha/\beta$-staircase tableau in $\bar{\mathcal{S}}_n \setminus \mathcal{S}_n^{**}$ is at most, assuming as we may $\alpha, \beta \geqslant 1$,

$$\alpha^n\beta^{n-2} + \alpha^{n-2}\beta^n = o\big(\alpha^n\beta^{n-1} + \alpha^{n-1}\beta^n\big) = o\big(Z_n(\alpha,\beta)\big). \tag{8.1}$$

Hence $\mathbb{P}(S_{n,\alpha,\beta} \notin \mathcal{S}_n^{**}) \to 0$, so it suffices to consider $S_{n,\alpha,\beta}$ conditioned on being in $\mathcal{S}_n^{**}$, and the result follows by Lemma 8.3. ☐

Thus, although the limiting distribution depends on the size of $\alpha/\beta$, it is only the distribution of the top left symbol $S(1,1)$ that is affected; $S'_{n,\alpha,\beta}$ has a unique limit distribution for all $\alpha, \beta \to \infty$. In particular, the distribution of the symbols on the diagonal has a unique limiting distribution.

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