Proceedings of the Royal Society of Edinburgh, **152**, 28–53, *2022*
DOI:10.1017/prm.2020.91

Fast and slow decay solutions for supercritical fractional elliptic problems in exterior domains

Weiwei Ao and Chao Liu
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
(wwao@whu.edu.cn; chaoliuwh@whu.edu.cn)

Liping Wang
Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, 200241, China
(lpwang@math.ecnu.edu.cn)

(MS Received 12 November 2019; accepted 26 November 2020)

We consider the fractional elliptic problem:

$$\begin{cases}(-\Delta)^s u - u^p = 0,\ u > 0 & \text{in } \mathbb{R}^N \setminus \overline{B_1},\\ u = 0 \ \text{ in } \ \overline{B_1}, & \lim_{|x|\to\infty} u(x) = 0\end{cases}$$

where B_1 is the unit ball in $\mathbb{R}^N$, $N \geqslant 3$, $s \in (0,1)$ and $p > (N+2s)/(N-2s)$. We prove that this problem has infinitely many solutions with slow decay $O(|x|^{-2s/(p-1)})$ at infinity. In addition, for each $s \in (0,1)$ there exists $P_s > (N+2s)/(N-2s)$, for any $(N+2s)/(N-2s) < p < P_s$, the above problem has a solution with fast decay $O(|x|^{2s-N})$. This result is the extension of the work by Dávila, del Pino, Musso and Wei (2008, Calc. Var. Partial Differ. Equ. 32, no. 4, 453–480) to the fractional case.

Keywords: Fractional elliptic problems; supercritical cases; existence

2010 *Mathematics subject classification:* 35J61; 35R11; 53A30

1. Introduction

In this paper we construct classical solutions of the following supercritical fractional exterior problem:

$$\begin{cases}(-\Delta)^s u - u^p = 0, & u > 0 \quad \text{in } \ \mathbb{R}^N \setminus \overline{B_1},\\ u = 0 \quad \text{in } \ \overline{B_1}, & \lim_{|x|\to\infty} u(x) = 0,\end{cases} \tag{1.1}$$

where $s \in (0,1)$, $p > (N+2s)/(N-2s)$ and B_1 is the unit ball whose centre is the original point in $\mathbb{R}^N$ for $N \geqslant 3$. As usual, the operator $(-\Delta)^s$ is the fractional Laplacian, defined at any point $x \in \mathbb{R}^N$ as

$$\begin{aligned}(-\Delta)^s u(x) &:= C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\ &= C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y.\end{aligned}$$

Here $P.V.$ is a commonly used abbreviation for 'in the principal value sense' and $C(N,s)$ is a constant dependent of N and s. We refer to [**13**, **19**].

Non-local equations have attracted a great deal of interest in recent twenty years since they are of central importance in many fields, from the points of view of both pure analysis and applied modelling. In addition, series of non-local PDE's theories have been founded by many authors; such as regularity theory [**21**, **22**], maximum principle in [**5**–**7**], uniqueness in [**14**, **16**, **17**] and so on. In particular, fractional elliptic problems have been extensively studied. See for example [**8**, **9**, **15**] for subcritical case, and [**1**, **2**, **18**] for critical exponent, [**3**] for the supercritical case and the reference therein.

Let us now go back to the problem we are going to consider. For classical case, namely $s = 1$, Dávila, del Pino and Musso [**10**] have proved (1.1) has infinitely many solutions with slow decay $O(|x|^{-2/(p-1)})$ at infinity with either $N \geqslant 4$ and $p > (N+1)/(N-3)$, or $N \geqslant 3$, $p > (N+2)/(N-2)$ and the interior domain Ω is symmetric with respect to N coordinate axes. Later, this result has been extended to $p > (N+2)/(N-2)$ and Ω is a smooth bounded domain by Dávila, del Pino, Musso and Wei [**11**]. For fractional case, we will prove that this result also holds when $s \in (0,1)$, $p > (N+2s)/(N-2s)$ and $\Omega = B_1$ is the unit ball in $\mathbb{R}^N$. For problem (1.1) in general exterior domain, there are some obstacles, see remark 1.3 below.

More precisely, our main results can be stated as follows:

THEOREM 1.1. *For any $s \in (0,1)$ and $p > (N+2s)/(N-2s)$, $N \geqslant 3$, there exists a continuum of solutions u_λ, $\lambda > 0$, to problem (1.1) such that*

$$u_\lambda(x) = \beta^{1/(p-1)} |x|^{-2s/(p-1)} (1 + o(1)) \qquad \text{as} \quad |x| \to \infty$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \overline{B_1}$.

THEOREM 1.2. *For any $s \in (0,1)$, there exists a number $P_s > (N+2s)/(N-2s)$, such that for any $p \in ((N+2s)/(N-2s), P_s)$, problem (1.1) has a fast decay solution u_p, $u_p(x) = O(|x|^{2s-N})$ as $|x| \to +\infty$.*

We basically follow the ideas in [**3**] and [**11**] to prove theorems 1.1 and 1.2. More precisely, to prove theorem 1.1, we will take ω as approximation of (1.1) where ω

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

is a smooth, radially symmetric, entire solution of the following problem:

$$(-\Delta)^s\omega - \omega^p = 0,\ \omega > 0 \ \text{ in } \mathbb{R}^N, \quad \omega(0) = 1, \quad \lim_{|x|\to\infty} \omega(x)|x|^{2s/(p-1)} = \beta^{1/(p-1)}. \tag{1.2}$$

Here β is a positive constant chosen such that $\beta^{1/(p-1)}|x|^{-2s/(p-1)}$ is a singular solution to $(-\Delta)^s\omega - \omega^p = 0$ for which the existence and linear theory have been studied recently in [3] for the fractional case.

While the basic idea in the proof of theorem 1.2 is to consider as an initial approximation the function $\lambda^{(N-2s)/2}\omega_{**}(\lambda x + \xi)$ where

$$\omega_{**}(r) = \left(\frac{1}{1 + A_{N,s}r^2}\right)^{(N-2s)/2} \tag{1.3}$$

is the unique positive radial smooth solution of the problem

$$(-\Delta)^s\omega_{**} = \omega_{**}^{(N+2s)/(N-2s)} \ \text{ in } \mathbb{R}^N, \quad \omega_{**}(0) = 1.$$

These scalings will constitute good approximations for small λ if p is sufficiently close to $(N+2s)/(N-2s)$. We prove then adjusting both ξ and λ, produces a solution as desired after addition of a lower order term.

Remark 1.3. To prove theorems 1.1 and 1.2, we will construct solutions of the equivalent problem (2.1) with the form $\widetilde{u} = \omega - \varphi_\lambda + \phi$ and $\widetilde{u} = \omega_{**} - \varphi_\lambda + \phi$ in § 2 and § 6 respectively where φ_λ is the projection to satisfy Dirichlet data and ϕ is a small perturbation. To obtain the decay of $\widetilde{u}$, we need to know the decay of φ_λ, ϕ. To get the a priori estimate, the expression or asymptotic behaviour of the Poisson kernel $P(x, y)$ in $R^N \setminus B_1$ is important. Using this Poisson kernel $P(x, y)$, we first obtain the decay of φ_λ in (2.6). Secondly, we can derive the decay of ϕ by the Green's function $G(x, y)$ in $R^N \setminus B_1$ in § 3 and § 6. But for general exterior domain, there is a lack of the explicit formulas and the decay of the Poisson kernel and the Green's function of fractional Laplace operator$(-\Delta)^s$. That is the reason we only consider $R^N \setminus B_1$ in this paper.

2. The set up for theorem 1.1

In what follows of this paper we will assume $s \in (0, 1)$ and $p > (N + 2s)/(N - 2s)$ for $N \geqslant 3$.

By the change of variables

$$\widetilde{u}(x) = \lambda^{-2/(p-1)}u\left(\frac{x-\xi}{\lambda}\right)$$

and the maximum principle (see [7]), problem (1.1) is equivalent to

$$\begin{cases} (-\Delta)^s\widetilde{u} - |\widetilde{u}|^p = 0, \quad \widetilde{u} \not\equiv 0 & \text{in} \quad \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}, \\ \widetilde{u} = 0 \quad \text{in} \quad \overline{B_{\lambda,\xi}}, & \lim_{|x|\to\infty} \widetilde{u}(x) = 0 \end{cases} \tag{2.1}$$

where $\lambda > 0$ is a small parameter, $\xi \in \mathbb{R}^N$ is to be determined later, and $B_{\lambda,\xi}$ is the shrinking domain

$$B_{\lambda,\xi} = \{\lambda x + \xi \mid x \in B_1\}.$$

We want to consider the function $\omega(x)$ in (1.2) as an approximation of this problem. We need of course a correction so that the boundary condition is satisfied. Denote φ_λ be the unique solution of the following problem:

$$(-\Delta)^s \varphi_\lambda = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}, \quad \varphi_\lambda(x) = \omega(x) \quad \text{in } \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} \varphi_\lambda(x) = 0, \tag{2.2}$$

then naturally $\omega(x) - \varphi_\lambda(x)$ is regarded as the first approximation of problem (1.1). Now $\psi_\lambda(x) = \varphi_\lambda(\lambda x + \xi)$ satisfies

$$\begin{cases} (-\Delta)^s \psi_\lambda(x) = 0 & \text{in } \mathbb{R}^N \setminus \overline{B_1}, \\ \psi_\lambda(x) = \omega(\lambda x + \xi) & \text{in } \overline{B_1}, \quad \lim_{|x|\to\infty} \psi_\lambda(x) = 0. \end{cases} \tag{2.3}$$

By Poisson's representation, we obtain

$$\psi_\lambda(x) = \int_{B_1} P(x, y)\omega(\lambda y + \xi) \mathrm{d}y,$$

where $P(x, y)$ is the Poisson kernel of the following problem

$$(-\Delta)^s u(x) = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}, \quad u(x) = g(x) \qquad \text{in } \overline{B_1}.$$

Moreover, it was obtained in [**19**] that

$$P(x, y) = \begin{cases} \dfrac{\Gamma(N/2)}{\pi^{N/2+1}} \sin(\pi s) \left(\dfrac{|x|^2 - 1}{1 - |y|^2} \right)^s \dfrac{1}{|x - y|^N}, & |y| < 1, \\ 0, & |y| > 1. \end{cases} \tag{2.4}$$

Let $\psi^0(x) := \int_{B_1} P(x, y) \mathrm{d}y$, then it is easy to get that there is a positive constant α such that

$$\lim_{|x|\to\infty} |x|^{N-2s} \psi^0(x) = \alpha.$$

Since ω is a smooth positive function, we have

$$\psi_\lambda(x) = (\omega(\xi) + O(\lambda))\psi^0(x), \qquad \forall x \in \mathbb{R}^N \setminus B_1,$$

from which we can derive that

$$\varphi_\lambda(x) = (\omega(\xi) + O(\lambda))\psi^0 \left(\frac{x - \xi}{\lambda} \right), \qquad \forall x \in \mathbb{R}^N \setminus B_{\lambda,\xi} \tag{2.5}$$

and

$$|\varphi_\lambda(x)| \leqslant C\lambda^{N-2s} |x - \xi|^{2s-N} \qquad \text{for all } x \in \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}. \tag{2.6}$$

Here $O(\lambda)$ means that there exists a positive constant C such that for all $\lambda \in (0, 1)$ and $\forall x \in \mathbb{R}^N \setminus B_1, |O(\lambda)| \leqslant C\lambda$.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Now we look for a solution to problem (2.1) of the form

$$\widetilde{u} = \omega - \varphi_\lambda + \phi,$$

which yields the following equation for ϕ, according to (2.1),

$$\begin{cases} (-\Delta)^s\phi - p\omega^{p-1}\phi = N(\phi) + E_\lambda \quad \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}, \\ \phi(x) = 0 \quad \text{in } \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{2.7}$$

where

$$E_\lambda = -p\omega^{p-1}\varphi_\lambda,\ N(\phi) = |\omega + \phi - \varphi_\lambda|^p - \omega^p - p\omega^{p-1}\phi + p\omega^{p-1}\varphi_\lambda. \tag{2.8}$$

Thus a solution of problem (2.7) for which ϕ is small compared with $\omega - \varphi_\lambda$ yields one of (1.1) as presented by theorem 1.1.

To solve problem (2.7), we first consider the following projected problem,

$$\begin{cases} (-\Delta)^s\phi - p\omega^{p-1}\phi = N(\phi) + E_\lambda + \sum_{i=1}^N c_i Z_i \quad \text{in} \quad \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}, \\ \phi(x) = 0 \quad \text{in} \quad \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{2.9}$$

where the c_i's are constants, which are part of the unknown, and

$$Z_i(x) = \frac{\partial \omega}{\partial x_i}(x), \quad i = 1, \ldots, N.$$

Through an application of the Banach fixed point theorem in a suitable L^∞ weight space, we shall prove in § 4 that problem (2.9) is indeed solvable, within a class of ϕ's in the form $\phi = \phi(\lambda, \xi)$, $c_i = c_i(\lambda, \xi)$ where the dependence on the parameter is continuous. We then obtain a solution of problem (2.7) if

$$c_i(\lambda, \xi) = 0 \quad \text{for all} \quad i = 1, \ldots, N.$$

We will show in § 5 that for each sufficiently small λ there is indeed a point ξ such that the above equalities hold true.

In § 3 we will consider the following linear problem corresponding to problem (2.9)

$$\begin{cases} (-\Delta)^s\phi - p\omega^{p-1}\phi = h + \sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}, \\ \phi(x) = 0 \quad \text{in } \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \end{cases} \tag{2.10}$$

The norms on functions ϕ and h are defined on $\mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}$ as the following

$$||\phi||_{*,\xi} := \sup_{|x-\xi|\leqslant 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s/(p-1)} |\phi(x)|,$$

$$||h||_{**,\xi} := \sup_{|x-\xi|\leqslant 1} |x-\xi|^{\sigma+2s} |h(x)| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{(2s/(p-1))+2s} |h(x)|,$$

where $\sigma \in (0, N-2s)$ is a constant to be determined later. In particular, when $\xi = 0$ we denote the above norms by $\|\phi\|_*$ and $\|h\|_{**}$ respectively.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

We have the validity of the following results which will be proved in §3.

PROPOSITION 2.1. *Assume* $(N+2s)/(N-2s) < p < (N+2s-1)/(N-2s-1)$ *and* $\Lambda > 0$. *Then there exist constants* C *and* λ_0 *such that for any* $|\xi| \leqslant \Lambda$ *and any* $0 < \lambda < \lambda_0$ *the following holds: for any* h *with* $||h||_{**,\xi} < \infty$, *there exists a solution of problem* (2.10)

$$(\phi, c_1, c_2, \ldots, c_N) = \Gamma_\lambda(h)$$

which defines a linear operator Γ_λ *of* h, *such that*

$$||\phi||_{*,\xi} + \max_{1\leqslant i\leqslant N} |c_i| \leqslant C||h||_{**,\xi}.$$

PROPOSITION 2.2. *Assume* $N \geqslant 2$ *and* $p > (N+2s-1)/(N-2s-1)$. *Then there exist constants* C *and* λ_0 *such that for any* $0 < \lambda < \lambda_0$ *the following holds: for any* h *with* $||h||_{**} < \infty$, *there exists a solution of problem* (2.10)

$$\phi = \Gamma_\lambda(h) \quad \textit{and} \quad c_i = 0, \quad i = 1, \ldots, N,$$

which defines a linear operator Γ_λ *of* h, *such that*

$$||\phi||_* \leqslant C||h||_{**}.$$

If $p = (N+2s-1)/(N-2s-1)$, the proof of theorem 1.1 is based on a result similar to proposition 2.1 but for slightly different norms, see remark 5.1.

A very similar scheme is followed for the proof of theorem 1.2, having as its basic cell the function ω_{**} in (1.3). In this case, the relevant projected problem also involves the generator of dilations, and both the point ξ and the number λ must be determined as the functions of the small parameter given by the difference $p - (N+2s)/(N-2s)$. It is done in § 6.

3. The proof of propositions 2.1 and 2.2

Keeping the notations of the previous section, we first consider the following linear problem in entire space,

$$(-\Delta)^s\phi - p\omega^{p-1}\phi = h + \sum_{i=1}^{N} c_i Z_i \quad \text{in} \quad \mathbb{R}^N, \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{3.1}$$

First let us recall the result in [3]:

PROPOSITION 3.1 (proposition 5.1 of [3]). *Let* h *satisfy* $||h||_{**,\xi} < +\infty$. *For problem* (3.1), *we have:*

1. *if* $p > (N+2s-1)/(N-2s-1)$, *then there exists a solution* ϕ *with* $c_i = 0, i = 1, \ldots, N$ *satisfying*

$$||\phi||_* \leqslant C||h||_{**} \tag{3.2}$$

for some $C > 0$.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

2. if $(N+2s)/(N-2s) < p < (N+2s-1)/(N-2s-1)$, there exists a solution $(\phi, c_1, \ldots, c_N)$ and it satisfies

$$||\phi||_{*,\xi} + \sum_{i=1}^{N} |c_i| \leqslant C||h||_{**,\xi} \tag{3.3}$$

for some $C > 0$.

Next we will use the above result to prove propositions 2.1 and 2.2. We shall fix $\Lambda > 0$ large and work with $|\xi| \leqslant \Lambda$. All the estimates will depend on only Λ. Let $0 < R_0 < R_1$ be fixed such that $3R_0 < R_1$ and $B_1 \subset B_{R_0}$. Define $\rho \in C^\infty(\mathbb{R}^N)$, $0 \leqslant \rho \leqslant 1$ be such that

$$\rho(x) = 0 \quad \text{for} \quad |x| \leqslant 1, \qquad \rho(x) = 1 \quad \text{for} \quad |x| \geqslant 2$$

and set

$$\eta_\lambda(x) = \rho\left(\frac{x-\xi}{\lambda R_0}\right), \qquad \zeta_\lambda(x) = \rho\left(\frac{x-\xi}{\lambda R_1}\right). \tag{3.4}$$

We look for a solution to problem (2.10) of the form

$$\phi = \eta_\lambda \varphi + \psi,$$

where φ, ψ are two unknown functions. Thus, we need to solve the following system

$$\begin{cases} (-\Delta)^s \varphi - p\omega^{p-1}\varphi = p\omega^{p-1}\zeta_\lambda \psi + \zeta_\lambda h + \sum_{i=1}^{N} c_i \zeta_\lambda Z_i & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} (-\Delta)^s \psi - p(1-\zeta_\lambda)\omega^{p-1}\psi = f(x) + (1-\zeta_\lambda)h + \sum_{i=1}^{N} c_i(1-\zeta_\lambda)Z_i & \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}} \\ \psi(x) = 0 \quad \text{in } \overline{B_{\lambda,\xi}}, \qquad \lim_{|x|\to\infty} \psi(x) = 0, \end{cases} \tag{3.6}$$

where

$$f(x) = -C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\eta_\lambda(x) - \eta_\lambda(y)}{|x-y|^{N+2s}} \varphi(y) \mathrm{d}y.$$

Here we regard x as the only variable of function f because in the first step of the proof, $(\varphi, c_1, \ldots, c_N)$ is given. Then we define Banach space

$$X := \left\{(\varphi, c_1, \ldots, c_N) \middle| \varphi \in L^\infty \text{ with } ||\varphi||_{*,\xi} < \infty \text{ and } c_i \in \mathbb{R},\ 1 \leqslant i \leqslant N\right\}$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

with the norm

$$\begin{aligned} ||(\varphi, c_1, \ldots, c_N)||_X &:= \sup_{|x-\xi|\leqslant \lambda R_1} |\varphi(x)| + \sup_{\lambda R_1 \leqslant |x-\xi| \leqslant 1} |x-\xi|^{\sigma} |\varphi(x)| \\ &\quad + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s/(p-1)} |\varphi(x)| \\ &\quad + \sup_{x,y\in B_{2\lambda R_1}(\xi)\setminus B_{\frac{\lambda R_0}{2}}(\xi)} \lambda^s \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} + \sum_{i=1}^N |c_i|. \end{aligned} \tag{3.7}$$

Given $(\varphi, c_1, \ldots, c_N) \in X$, we first note that (3.6) has a solution for λ small enough because $||p(1-\zeta_\lambda)\omega^{p-1}||_{L^{N/2s}(R^N \setminus B_{\lambda,\xi})} \to 0$ as $\lambda \to 0$ and the operator $(-\Delta)^s\psi - p(1-\zeta_\lambda)\omega^{p-1}$ is coercive. Let $\psi(\varphi, c_1, \ldots, c_N)$ denote this solution. Moreover, $\psi(x) = O(|x|^{2s-N})$ as $|x| \to \infty$. So the right-hand side of (3.5) has finite $||\cdot||_{**,\xi}$ norms, by proposition 3.1, (3.5) has a solution $(\bar\varphi, \bar c_1, \ldots, \bar c_N)$.

Proposition 2.1 and 2.2 will be shown through the fixed point theorem in X. In particular, for the proof of proposition 2.2, we just choose $\xi = 0$ because (3.1) is solvable with $c_i = 0, 1 \leqslant i \leqslant N$, according to proposition 3.1.

For $(\varphi, c_1, \ldots, c_N) \in X$ we will first establish a point-wise estimate for the solution $\psi(\varphi, c_1, \ldots, c_N)$ of (3.6), namely

$$\begin{aligned} |\psi(x)| \leqslant C\lambda^{N-2s-\sigma}(||h||_{**,\xi} + ||(\varphi, c_1, \ldots, c_N)||_X)|x-\xi|^{2s-N} \\ \text{for all } x \in \mathbb{R}^N \setminus B_{\lambda,\xi}. \end{aligned} \tag{3.8}$$

Indeed, let $\widetilde\psi(z) = \psi(\xi + \lambda z)$, then by (3.6),

$$\begin{cases} (-\Delta)^s\widetilde\psi - p\lambda^{2s}\left(1-\rho\left(\frac{z}{R_1}\right)\right)\omega^{p-1}(\xi+\lambda z)\widetilde\psi = g(z) \ \text{ in } \ \mathbb{R}^N \setminus \overline{B_1}, \\ \widetilde\psi(x) = 0 \quad \text{in} \ \ \overline{B_1}, \qquad\qquad \lim_{|x|\to\infty} \widetilde\psi(x) = 0, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} g(z) &= -C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\rho(z/R_0) - \rho(y/R_0)}{|z-y|^{N+2s}} \varphi(\xi+\lambda y)\mathrm{d}y \\ &\quad + \left(1-\rho\left(\frac{z}{R_1}\right)\right)\lambda^{2s}h(\xi+\lambda z) \\ &\quad + \lambda^{2s}\sum_{i=1}^N c_i\left(1-\rho\left(\frac{z}{R_1}\right)\right)Z_i(\xi+\lambda z). \end{aligned}$$

For any $z \in \mathbb{R}^N \setminus \overline{B_1}$, first we can get that

$$\left|\left(1-\rho\left(\frac{z}{R_1}\right)\right)\lambda^{2s}h(\xi+\lambda z)\right| \leqslant C\lambda^{-\sigma}||h||_{**,\xi}|z|^{-2s-\sigma}\chi_{\{|z|\leqslant 2R_1\}} \tag{3.10}$$

and

$$\begin{aligned} \lambda^{2s}\left|\sum_{i=1}^N c_i\left(1-\rho\left(\frac{z}{R_1}\right)\right)Z_i(\xi+\lambda z)\right| &\leqslant C\lambda^{2s}\sum_{i=1}^N |c_i|\chi_{\{|z|\leqslant 2R_1\}} \\ &\leqslant C\lambda^{2s}||(\varphi, c_1, \ldots, c_N)||_X \ \chi_{\{|z|\leqslant 2R_1\}}. \end{aligned} \tag{3.11}$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Moreover, for $s \in \left(0, \frac{1}{2}\right)$ we obtain

$$
\begin{aligned}
&\left| P.V. \int_{\mathbb{R}^N} \frac{\rho(z/R_0) - \rho(y/R_0)}{|z-y|^{N+2s}} \varphi(\xi + \lambda y) \mathrm{d}y \right| \\
&\leqslant P.V. \int_{\mathbb{R}^N} \frac{|\rho(z/R_0) - \rho(y/R_0)|}{|z-y|^{N+2s}} |\varphi(\xi + \lambda y)| \mathrm{d}y \\
&\leqslant \begin{cases} C P.V. \int_{|z-y| \leqslant 1} \frac{|\nabla \rho|}{|z-y|^{N+2s-1}} |\varphi(\xi + \lambda y)| \mathrm{d}y \\ \quad + C \int_{|z-y|>1} \frac{|\varphi(\xi+\lambda y)|}{|z-y|^{N+2s}} \mathrm{d}y \quad \text{if } |z| \leqslant 3R_1 \\ C \int_{|y| \leqslant 2R_0} \frac{|\varphi(\xi+\lambda y)|}{|z-y|^{N+2s}} \mathrm{d}y \qquad \text{if } \ |z| > 3R_1 \end{cases} \\
&\leqslant \begin{cases} C P.V. \int_{|z-y| \leqslant 1} \frac{||(\varphi, c_1, \ldots, c_N)||_X}{\lambda^\sigma |z-y|^{N+2s-1}} \mathrm{d}y \\ \quad + C \int_{|z-y|>1} \frac{||(\varphi, c_1, \ldots, c_N)||_X}{\lambda^\sigma |z-y|^{N+2s}} \mathrm{d}y \quad \text{if } |z| \leqslant 3R_1 \\ C \int_{|y| \leqslant 2R_0} \frac{||(\varphi, c_1, \ldots, c_N)||_X}{|z-y|^{N+2s} \lambda^\sigma} \mathrm{d}y \qquad \text{if } \ |z| > 3R_1 \end{cases} \\
&\leqslant C \lambda^{-\sigma} ||(\varphi, c_1, \ldots, c_N)||_X \chi_{\{|z| \leqslant 3R_1\}} \\
&\quad + C \lambda^{-\sigma} |z|^{-2s-N} ||(\varphi, c_1, \ldots, c_N)||_X \chi_{\{|z| > 3R_1\}}
\end{aligned} \tag{3.12}
$$

since

$$
|\varphi(\xi + \lambda y)| \leqslant ||(\varphi, c_1, \ldots, c_N)||_X \left\{ \chi_{\{|y| \leqslant R_1\}} + \frac{1}{|\lambda y|^\sigma} \chi_{\{R_1 < |y| \leqslant \frac{1}{\lambda}\}} + \chi_{\left\{\frac{1}{\lambda} < |y|\right\}} \right\}
$$

and we used that, $s \in \left(0, \frac{1}{2}\right)$, $\sigma \in (0, N-2s)$ in the last inequality and χ is a characteristic function in different domains.

For $s \in [\frac{1}{2}, 1)$, using $\nabla \rho(z/R_0) = 0$ for $|z| \leqslant R_0$ and $\int_{\mathbb{R}^N} \frac{\varphi(\xi+\lambda z)[\nabla \rho](z/R_0) \cdot (z-y/R_0)}{|z-y|^{N+2s}} \mathrm{d}y = 0$, we can obtain

$$
\begin{aligned}
&\left| P.V. \int_{\mathbb{R}^N} \frac{\rho(z/R_0) - \rho(y/R_0)}{|z-y|^{N+2s}} \varphi(\xi + \lambda y) \mathrm{d}y \right| \\
&\leqslant \begin{cases} C \int_{|z-y| \leqslant 1} \frac{O(|x-y|^2)|\varphi(\lambda y + \xi)|}{|z-y|^{N+2s}} \mathrm{d}y + \int_{|z-y|>1} \frac{|\varphi(\xi+\lambda y)|}{|z-y|^{N+2s}} \mathrm{d}y \qquad \text{if } |z| \leqslant R_0 \\ C \int_{|z-y| \leqslant \frac{R_0}{2}} \frac{|\nabla \rho(z/R_0)((z-y)/R_0)[\varphi(\xi+\lambda z) - \varphi(\xi+\lambda y)]|}{|z-y|^{N+2s}} + \frac{O(|x-y|^2)|\varphi(\lambda y+\xi)|}{|z-y|^{N+2s}} \mathrm{d}y \\ +2 \int_{|z-y| > \frac{R_0}{2}} \frac{|\varphi(\xi+\lambda y)|}{|z-y|^{N+2s}} \mathrm{d}y \qquad \text{if } R_0 < |z| \leqslant R_1 \\ C \int_{|y| \leqslant 2R_0} \frac{|\varphi(\xi+\lambda y)|}{|z-y|^{N+2s}} \mathrm{d}y \qquad \text{if } \ |z| > R_1. \end{cases}
\end{aligned}
$$

Using the definition of $||\cdot||_X$ in (3.7), we get

$$\leqslant \begin{cases} C\int_{|z-y|\leqslant 1}\dfrac{||(\varphi,c_1,\ldots,c_N)||_X}{\lambda^\sigma|z-y|^{N+2s-2}}\mathrm{d}y \\ \quad +C\int_{|z-y|>1}\dfrac{||(\varphi,c_1,\ldots,c_N)||_X}{\lambda^\sigma|z-y|^{N+2s}}\mathrm{d}y \qquad \text{if } |z|\leqslant R_0 \\ C\int_{|z-y|\leqslant \frac{R_0}{2}}\dfrac{||(\varphi,c_1,\ldots,c_N)||_X}{|z-y|^{N+s-1}}+\dfrac{||(\varphi,c_1,\ldots,c_N)||_X}{\lambda^\sigma|z-y|^{N+2s-2}}\mathrm{d}y \\ \quad +C\int_{|z-y|>\frac{R_0}{2}}\dfrac{||(\varphi,c_1,\ldots,c_N)||_X}{\lambda^\sigma|z-y|^{N+2s}}\mathrm{d}y \qquad \text{if } R_0<|z|<R_1 \\ C\int_{|y|\leqslant 2R_0}\dfrac{||(\varphi,c_1,\ldots,c_N)||_X}{|z-y|^{N+2s}\lambda^\sigma}\mathrm{d}y \qquad \text{if } |z|>R_1 \end{cases}$$

$$\begin{aligned} &\leqslant C\lambda^{-\sigma}||(\varphi,c_1,\ldots,c_N)||_X\chi_{\{|z|\leqslant R_1\}} \\ &\quad + C\lambda^{-\sigma}|z|^{-2s-N}||(\varphi,c_1,\ldots,c_N)||_X\chi_{\{|z|>R_1\}} \\ &\leqslant C\lambda^{-\sigma}||(\varphi,c_1,\ldots,c_N)||_X\chi_{\{|z|\leqslant 3R_1\}} \\ &\quad + C\lambda^{-\sigma}|z|^{-2s-N}||(\varphi,c_1,\ldots,c_N)||_X\chi_{\{|z|>3R_1\}}. \end{aligned} \tag{3.13}$$

According to the above estimates (3.10)–(3.13), we obtain

$$\begin{aligned} |g(z)| &\leqslant C\lambda^{-\sigma}\{||(\varphi,c_1,\ldots,c_N)||_X+||h||_{**,\xi}|z|^{-2s-\sigma}\}\chi_{\{|z|\leqslant 3R_1\}} \\ &\quad + C||(\varphi,c_1,\ldots,c_N)||_X\lambda^{-\sigma}|z|^{-N-2s}\chi_{\{|z|>3R_1\}}. \end{aligned} \tag{3.14}$$

Let $G(x,y)$ be the Green's function satisfying

$$\begin{cases} (-\Delta)^s G(x,y)=\delta(x-y) & \text{if } x \text{ and } y\in\mathbb{R}^N\backslash\overline{B_1}, \\ G(x,y)=0 & \text{if } x \text{ or } y\in\mathbb{R}^N\backslash\overline{B_1}. \end{cases} \tag{3.15}$$

Moreover, using the Poisson kernel in (2.4), we can derive that

$$G(x,y)=\frac{C(s,N)}{|x-y|^{N-2s}}-\int_{|x|<1}\frac{P(x,y)}{|z-y|^{N-2s}}\mathrm{d}z,$$

where $C(s,N)$ is a constant depending on s and N, which satisfies that $(C(s,N))/(|x-y|^{N-2s})$ is the Green's function of $(-\Delta)^s$ in R^N. By similar arguments as theorem 1.3 in [**4**], we can derive that

$$G(x,y)=\frac{C(s,N)}{|x-y|^{N-2s}}-\frac{\sin(\pi s)}{\pi}\frac{2}{|x-y|^{N-2s}}\int_1^\infty\frac{\rho(|x^*|^2-1)^s}{(\rho^2-1)^s(\rho^2-|x^*|^2)}\mathrm{d}\rho,$$

where x^* satisfies $|x-y||x^*-y|=1-|y|^2$. Thus we derive that

$$|G(x,y)|\leqslant A(s,N)\frac{1}{|x-y|^{N-2s}},\quad x,y\in\mathbb{R}^N\backslash\overline{B_1}, \tag{3.16}$$

for some constant $A(s,N)$ depending on s and N.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

According to the Green's function $G(x, y)$ in (3.15), the equations (3.9), (3.14) and (3.16), we have

$$|\tilde{\psi}(z)| \leqslant C\lambda^{-\sigma}\{||(\varphi, c_1, \dots, c_N)||_X + ||h||_{**,\xi}\}|z|^{-N+2s} \quad \text{for all } z \in \mathbb{R}^N \setminus \overline{B_1}. \tag{3.17}$$

Thus, we obtain from (3.8)

$$|\psi(x)| \leqslant C\lambda^{N-2s-\sigma}\left(||h||_{**,\xi} + ||(\varphi, c_1, \dots, c_N)||_X\right)|x-\xi|^{2s-N}, \quad \text{for all } x \in \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}.$$

Let $(\varphi, c_1, \dots, c_N) \in X$, $\psi = \psi(\varphi, c_1, \dots, c_N)$ be the solution to (3.6) and $F(\varphi, c_1, \dots, c_N) = (\bar{\varphi}, \bar{c}_1, \dots, \bar{c}_N)$. By proposition 3.1, we have

- if $p > (N+2s-1)/(N-2s-1)$,
$$||\bar{\varphi}||_* \leqslant C(||p\omega^{p-1}\zeta_\lambda\psi||_{**} + ||\zeta_\lambda h||_{**}); \tag{3.18}$$
- if $(N+2s-1)/(N-2s-1) > p > (N+2s)/(N-2s)$,
$$||\bar{\varphi}||_{*,\xi} + \sum_{i=1}^{N}|\bar{c}_i| \leqslant C\left(||p\omega^{p-1}\zeta_\lambda\psi||_{**,\xi} + ||\zeta_\lambda h||_{**,\xi}\right). \tag{3.19}$$

Using (3.8) we can estimate $||p\omega^{p-1}\zeta_\lambda\psi||_{**,\xi}$ as follows:

$$\begin{aligned}
&\sup_{|x-\xi|\leqslant 1}|x-\xi|^{2s+\sigma}\omega^{p-1}\zeta_\lambda|\psi| \\
&\quad\leqslant C\lambda^{N-2s-\sigma}(||h||_{**,\xi} + ||(\varphi, c_1, \dots, c_N)||_X) \times \sup_{\lambda R_1\leqslant|x-\xi|\leqslant 1}|x-\xi|^{4s-N+\sigma} \\
&\quad\leqslant C\lambda^{\gamma}(||h||_{**,\xi} + ||(\varphi, c_1, \dots, c_N)||_X)
\end{aligned}$$

where

$$\gamma = \min\{2s, N-2s-\sigma\} > 0.$$

On the other hand

$$\begin{aligned}
&\sup_{|x-\xi|\geqslant 1}|x-\xi|^{2s+(2s/(p-1))}\omega^{p-1}\zeta_\lambda|\psi| \\
&\leqslant C\lambda^{N-2s-\sigma}\left(||h||_{**,\xi} + ||(\varphi, c_1, \dots, c_N)||_X\right) \times \sup_{|x-\xi|\geqslant 1}|x-\xi|^{2s-N+(2s/(p-1))} \\
&\leqslant C\lambda^{N-2s-\sigma}\left(||h||_{**,\xi} + ||(\varphi, c_1, \dots, c_N)||_X\right).
\end{aligned}$$

Hence we obtain

$$||p\omega^{p-1}\xi_\lambda\psi||_{**,\xi} \leqslant C\lambda^{\gamma}(||h||_{**,\xi} + ||(\varphi, c_1, \dots, c_N)||_X). \tag{3.20}$$

Since $||\zeta_\lambda h||_{**,\xi} \leqslant ||h||_{**,\xi}$, from (3.18)–(3.20) we deduce

- if $p > (N+2s-1)/(N-2s-1)$,
$$||\bar{\varphi}||_* \leqslant C\lambda^{\gamma}||(\varphi, c_1, \dots, c_N)||_X + C||h||_{**},$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

- if $(N+2s-1)/(N-2s-1) > p > (N+2s)/(N-2s)$,

$$||\bar{\varphi}||_{*,\xi} + \sum_{i=1}^{N} |\bar{c}_i| \leqslant C\lambda^{\gamma}||(\varphi, c_1, \dots, c_N)||_X + C||h||_{**,\xi}.$$

Hence

$$||F(\varphi, c_1, \dots, c_N)||_{*,\xi} \leqslant C(\lambda^{\gamma}||(\varphi, c_1, \dots, c_N)||_X + ||h||_{**,\xi}).$$

Since the right-hand side of (3.5) is bounded for $|x| < \lambda R_1$, by regularity estimates of solutions in ball regions (refer to lemma 12.3.2 in [7]), we derive

$$||F(\varphi, c_1, \dots, c_N)||_X \leqslant C(\lambda^{\gamma}||(\varphi, c_1, \dots, c_N)||_X + ||h||_{**,\xi}).$$

This estimate shows that F has a unique fixed point $(\varphi, c_1, \dots, c_N)$ in X for $\lambda > 0$ suitably small, and

$$||(\varphi, c_1, \dots, c_N)||_X \leqslant C||h||_{**,\xi}.$$

Finally we make a remark on how to recognize $c_i = 0$ in equation (2.10) for $(N+2s)/(N-2s) < p < (N+2s-1)/(N-2s-1)$.

LEMMA 3.2. *Assume $(N+2s)/(N-2s) < p < (N+2s-1)/(N-2s-1)$. There is $\lambda_0 > 0$ small such that for $\lambda < \lambda_0, ||h||_{**,\xi} < \infty$, and ϕ is a solution to (2.10) with $||\phi||_{*,\xi} < +\infty$, then $c_i = 0$ for all $i = 1, \dots, N$ if only if*

$$\int_{\mathbb{R}^N \setminus B_{\lambda,\xi}} (-\Delta)^s \phi Z_i - (-\Delta)^s Z_i \phi = \int_{\mathbb{R}^N \setminus B_{\lambda,\xi}} h Z_i.$$

Proof. Since $\partial\omega/\partial x_j$ satisfies the linear homogeneous equation: $(-\Delta)^s \phi - p\omega^{p-1}\phi = 0$ in $\mathbb{R}^N$, multiplying (3.1) by $Z_j = \partial\omega/\partial x_j$, yields

$$\int_{\mathbb{R}^N \setminus B_{\lambda,\xi}} (-\Delta)^s \phi Z_i - (-\Delta)^s Z_i \phi = \int_{\mathbb{R}^N \setminus B_{\lambda,\xi}} h Z_i + \sum_{j=1}^{N} c_j \int_{\mathbb{R}^N \setminus B_{\lambda,\xi}} Z_j Z_i.$$

For $\lambda > 0$ sufficiently small the matrix with entries $\int_{\mathbb{R}^N \setminus B_{\lambda,\xi}} Z_j Z_i$ is closed to $\int_{\mathbb{R}^N} Z_j Z_i$ which is invertible. Thus it implies the desired conclusion. □

4. The nonlinear projection problem

To solve problem (2.9), we need to establish some estimates of E_λ and $N(\phi)$. Recall that

$$E_\lambda = -p\omega^{p-1}\varphi_\lambda, \quad N(\phi) = |\omega + \phi - \varphi_\lambda|^p - \omega^p - p\omega^{p-1}\phi + p\omega^{p-1}\varphi_\lambda.$$

LEMMA 4.1. *Let $p > (N+2s)/(N-2s)$, then we have*

- *for any fixed $0 < \sigma < N-2s$,*

$$||E_\lambda||_{**,\xi} \leqslant C\lambda^{\min\{N-2s, \sigma+2s\}}, \tag{4.1}$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

- *for any fixed* $0 < \sigma \leqslant \min\{2s/(p-1), 2s\}$ *and* $||\phi||_{*,\xi} \leqslant 1$,

$$||N(\phi)||_{**,\xi} \leqslant C\left(||\phi||_{*,\xi}^2 + ||\phi||_{*,\xi}^p + \lambda^{\min\{\sigma+2s,N-2s\}}\right). \tag{4.2}$$

Proof. According to the definition of the norms, we do estimates term by term. Recall that $|\varphi_\lambda(x)| \leqslant C\lambda^{N-2s}|x-\xi|^{2s-N}$ for all $x \in \mathbb{R}^N \setminus B_{\lambda,\xi}$, then

$$\begin{aligned}\sup_{|x-\xi|\leqslant 1,\ x\notin B_{\lambda,\xi}} & |x-\xi|^{2s+\sigma}\omega^{p-1}(x)\varphi_\lambda(x)\\ &\leqslant C\lambda^{N-2s}||\omega^{p-1}||_\infty \sup_{\lambda\leqslant|x-\xi|\leqslant 1}|x-\xi|^{4s-N+\sigma}\\ &\leqslant C\lambda^{\min\{\sigma+2s,\ N-2s\}}.\end{aligned} \tag{4.3}$$

Also

$$\begin{aligned}&\sup_{|x-\xi|\geqslant 1}|x-\xi|^{2s+(2s/(p-1))}\omega^{p-1}(x)\varphi_\lambda(x)\\ &\leqslant C\lambda^{N-2s}\sup_{|x-\xi|\geqslant 1}|x-\xi|^{2s-N+2s/(p-1)} \leqslant C\lambda^{N-2s}.\end{aligned} \tag{4.4}$$

Next we will prove (4.2).
For $p \geqslant 2$. Assuming $0 < \sigma \leqslant 2s/(p-1)$ and using

$$|N(\phi)| \leqslant C\omega^{p-2}(|\phi|^2 + |\varphi_\lambda|^2) + C(|\phi|^p + |\varphi_\lambda|^p), \tag{4.5}$$

we have for all $x \in \mathbb{R}^N \setminus B_{\lambda,\xi}$

$$\sup_{|x-\xi|\leqslant 1}|x-\xi|^{2s+\sigma}|\phi|^2 \leqslant C||\phi||_{*,\xi}^2 \sup_{\lambda\leqslant|x-\xi|\leqslant 1}|x-\xi|^{2s-\sigma} \leqslant C||\phi||_{*,\xi}^2, \tag{4.6}$$

$$\begin{aligned}\sup_{|x-\xi|\leqslant 1}|x-\xi|^{2s+\sigma}|\phi|^p &\leqslant C||\phi||_{*,\xi}^p \sup_{\lambda\leqslant|x-\xi|\leqslant 1}|x-\xi|^{2s-\sigma(p-1)}\\ &\leqslant C||\phi||_{*,\xi}^p \leqslant C||\phi||_{*,\xi}^2\end{aligned} \tag{4.7}$$

with $||\phi||_{*,\xi} \leqslant 1$ in the last inequality. Similar calculation gives out that

$$\sup_{|x-\xi|\leqslant 1}|x-\xi|^{2s+\sigma}|\varphi_\lambda|^2 \leqslant C\lambda^{\min\{\sigma+2s,N-2s\}} \tag{4.8}$$

and

$$\sup_{|x-\xi|\leqslant 1}|x-\xi|^{2s+\sigma}|\varphi_\lambda|^p \leqslant C\lambda^{\min\{\sigma+2s,N-2s\}}. \tag{4.9}$$

The inequalities (4.6)–(4.9) yield, for $p \geqslant 2$, $0 < \sigma \leqslant 2s/(p-1)$ and $||\phi||_{*,\xi} \leqslant 1$,

$$\sup_{|x-\xi|\leqslant 1}|x-\xi|^{2s+\sigma}|N(\phi)| \leqslant C\left(||\phi||_{*,\xi}^2 + \lambda^{\min\{\sigma+2s,N-2s\}}\right). \tag{4.10}$$

Now we consider $|x-\xi| \geqslant 1$. By the definition of $||\cdot||_{*,\xi}$ and the assumption $||\phi||_{*,\xi} \leqslant 1$,

$$|\phi(x)| = |\phi(x)| \times |x-\xi|^{2s/(p-1)}|x-\xi|^{-2s/(p-1)} \leqslant |x-\xi|^{-2s/(p-1)} \leqslant C\omega(x).$$

For $\lambda > 0$ small, $x \in \mathbb{R}^N \setminus B_{\lambda,\xi}$,

$$\begin{aligned} |\varphi_\lambda(x)| &\leqslant C\lambda^{N-2s}|x-\xi|^{2s-N} = C\lambda^{N-2s}|x-\xi|^{2s-N+(2s/(p-1))}|x-\xi|^{-2s/(p-1)} \\ &\leqslant C\omega(x)\lambda^{N-2s}|x-\xi|^{2s-N+(2s/(p-1))} \leqslant C\omega(x)\lambda^{N-2s}. \end{aligned} \tag{4.11}$$

Thus instead of (4.5) we can estimate $N(\phi)$ by

$$|N(\phi)| \leqslant C\omega^{p-2}(\phi^2 + \varphi_\lambda^2).$$

Using this inequality and the estimate $\omega(x) \leqslant C(1+|x|)^{-2s/(p-1)}$ we have

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}\omega^{p-2}|\phi|^2 \leqslant C||\phi||_{*,\xi}^2 \tag{4.12}$$

and

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}\omega^{p-2}|\varphi_\lambda|^2 \leqslant C\lambda^{2(N-2s)}. \tag{4.13}$$

Thus (4.12) and (4.13) yield

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+\sigma}|N(\phi)| \leqslant C\left(||\phi||_{*,\xi}^2 + \lambda^{\min\{\sigma+2s,\ N-2s\}}\right).$$

Thus this estimate together with (4.10) proves (4.2) in the case of $p \geqslant 2$.

For $1 < p < 2$ and $0 < \sigma \leqslant 2s$, a similar calculation using

$$|N(\phi)| \leqslant C(|\phi|^p + |\varphi_\lambda|^p)$$

implies

$$\sup_{|x-\xi|\leqslant 1} |x-\xi|^{2s+\sigma}|N(\phi)| \leqslant C\left(||\phi||_{*,\xi}^p + \lambda^{\min\{N-2s,2s+\sigma\}}\right). \tag{4.14}$$

To estimate $|x-\xi|^{2s+\sigma}$ for $|x-\xi| \geqslant 1$ we write

$$\begin{aligned} N(\phi) &= |\omega + \phi - \varphi_\lambda|^p - \omega^p - p\omega^{p-1}(\phi - \varphi_\lambda) \\ &= N_1 + N_2 + p\omega^{p-1}\varphi_\lambda, \end{aligned} \tag{4.15}$$

where

$$N_1 = |\omega + \phi - \varphi_\lambda|^p - |\omega + \phi|^p, \quad N_2 = |\omega + \phi|^p - \omega^p - p\omega^{p-1}\phi. \tag{4.16}$$

We note that since $||\phi||_{*,\xi} \leqslant 1$ and $|\phi| \leqslant C\omega(x)$ for $|x-\xi| \geqslant 1$, together with (4.11), we can obtain

$$|N_1| = ||\omega + \phi - \varphi_\lambda|^p - |\omega + \phi|^p| \leqslant C\omega^{p-1}\varphi_\lambda.$$

Then

$$\begin{aligned} \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}|N_1| &\leqslant C \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}\omega^{p-1}\varphi_\lambda \\ &\leqslant C\lambda^{\min\{N-2s,\ \sigma+2s\}} \end{aligned} \tag{4.17}$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

as (4.4) shows. Next we can estimate N_2 as follows

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}|N_2| \leqslant C \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}|\phi|^p \leqslant C||\phi||^p_{*,\xi}. \tag{4.18}$$

Thus, by (4.16)–(4.18) and (4.4) for the last term in (4.15) we deduce

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}|N(\phi)| \leqslant C\left(||\phi||^p_{*,\xi} + \lambda^{\min\{\sigma+2s,\ N-2s\}}\right).$$

This inequality and (4.14) prove (4.2) in the case of $1 < p < 2$. □

According to lemma 4.1 and the fixed point argument, we can derive the following lemmas:

Lemma 4.2. *Let $\sigma \in (0, 2s/(p-1))$, $(N+2s)/(N-2s) < p < (N+2s-1)/(N-2s-1)$ and $\Lambda > 0$. Then there are positive numbers λ_0, C such that for $|\xi| < \Lambda$ and $0 < \lambda < \lambda_0$ there exists $(\phi_\lambda(\xi), c_1(\lambda,\xi), \dots, c_N(\lambda,\xi))$ solution to problem (2.9) such that*

$$||\phi_\lambda(\xi)||_{*,\xi} + \max_{1\leqslant i\leqslant N} |c_i(\lambda,\xi)| \leqslant C\lambda^{\min\{2s+\sigma, N-2s\}}. \tag{4.19}$$

Lemma 4.3. *Let $\sigma \in (0, 2s/(p-1))$ and $p > (N+2s-1)/(N-2s-1)$, taking $\xi = 0$, then there are positive numbers λ_0 and C, such that for any $0 < \lambda < \lambda_0$, there exists ϕ_λ solution to problem (2.9) with $c_i = 0$, $1 \leqslant i \leqslant N$ such that*

$$||\phi_\lambda||_* \leqslant C\lambda^{\min\{2s+\sigma, N-2s\}}. \tag{4.20}$$

Proof of lemma 4.2. Fix $0 < \sigma \leqslant \min\{2s, (2s/(p-1))\}$, we define for small $\rho > 0$

$$\mathcal{F} = \{\phi : \mathbb{R}^N \setminus B_{\lambda,\xi} \to \mathbb{R} \;\big|\; ||\phi||_{\lambda,\xi} \leqslant \rho\}$$

and the operator $\bar\phi = \mathcal{A}(\phi)$ where $(\bar\phi, c_1, \dots, c_N)$ is the solution in proposition 2.1 to

$$\begin{cases} (-\Delta)^s\phi - p\omega^{p-1}\phi = N(\phi) + E_\lambda + \sum\limits_{i=1}^N c_i Z_i & \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}} \\ \phi(x) = 0 \quad \text{in } \overline{B_{\lambda,\xi}}, & \lim\limits_{|x|\to\infty} \phi(x) = 0, \end{cases}$$

where $N(\phi)$ and E_λ are given by (2.8).

We prove that $\mathcal{A}$ has a fixed point in $\mathcal{F}$. From proposition 2.1 we have the estimate

$$||\mathcal{A}(\phi)||_{*,\xi} \leqslant C(||N(\phi)||_{**,\xi} + ||E_\lambda||_{**,\xi})$$

and by lemma 4.1,

$$\begin{aligned} ||\mathcal{A}(\phi)||_{*,\xi} &\leqslant C\left(||\phi)||^2_{*,\xi} + ||\phi)||^p_{*,\xi} + \lambda^{\min\{2s+\sigma, N-2s\}}\right) \\ &\leqslant C\left(\rho^2 + \rho^p + \lambda^{\min\{2s+\sigma, N-2s\}}\right) < \rho \end{aligned}$$

if $\rho > 0$ is fixed suitably small and then one consider $\lambda \to 0$. This proves $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$.

Now we show that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$. Take ϕ_1, ϕ_2 in $\mathcal{F}$, then

$$||\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)||_{*,\xi} \leqslant C||N(\phi_1) - N(\phi_2)||_{**,\xi}. \tag{4.21}$$

Write

$$N(\phi_1) - N(\phi_2) = D_{\bar{\phi}} N(\bar{\phi})(\phi_1 - \phi_2)$$

where $\bar{\phi}$ lies in the segment joining ϕ_1 and ϕ_2. Then, for $|x-\xi| \leqslant 1$,

$$|x-\xi|^{2s+\sigma}|N(\phi_1) - N(\phi_2)| = |x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})| \; ||\phi_1 - \phi_2||_{*,\xi},$$

while, for $|x-\xi| \geqslant 1$,

$$|x-\xi|^{2s+(2s/(p-1))}|N(\phi_1) - N(\phi_2)| = |x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})| \; ||\phi_1 - \phi_2||_{*,\xi}.$$

Then we have

$$||N(\phi_1) - N(\phi_2)||_{**,\xi} \leqslant C \sup_{x \in \mathbb{R}^N \setminus B_{\lambda,\xi}} (|x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})|)||\phi_1 - \phi_2||_{*,\xi}. \tag{4.22}$$

Directly from the definition of $N(\phi)$, we compute

$$D_{\bar{\phi}} N(\bar{\phi}) = -p\big[(\omega + \bar{\phi} - \varphi_\lambda)^{p-1} - \omega^{p-1}\big].$$

If $p \geqslant 2$, $0 < \sigma \leqslant 2s/(p-1)$, using $|D_{\bar{\phi}} N(\bar{\phi})| \leqslant C(\omega^{p-2}(|\overline{\phi}| + \varphi_\lambda) + |\overline{\phi}|^{p-1} + \varphi_\lambda^{p-1})$,

$$\begin{aligned}
\sup_{|x-\xi|\leqslant 1} |x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})| &\leqslant C \sup_{|x-\xi|\leqslant 1} |x-\xi|^{2s} \left(\omega^{p-2}(|\overline{\phi}| + \varphi_\lambda) + |\overline{\phi}|^{p-1} + \varphi_\lambda^{p-1}\right) \\
&\leqslant C \left(||\phi_1||_{*,\xi} + ||\phi_2||_{*,\xi} + \lambda^{\min\{2s, N-2s\}}\right) \\
&\leqslant C \left(\rho + \lambda^{\min\{2s, N-2s\}}\right).
\end{aligned} \tag{4.23}$$

In the region $|x-\xi| \geqslant 1$ we can use $|D_{\bar{\phi}} N(\bar{\phi})| \leqslant C\omega^{p-2}(|\overline{\phi}| + \varphi_\lambda)$ and obtain

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})| \leqslant C \left(\rho + \lambda^{\min\{2s, N-2s\}}\right). \tag{4.24}$$

Similarly, if $1 < p < 2$ and $0 < \sigma \leqslant 2s/(p-1)$ then for all $x \in \mathbb{R}^N \setminus B_{\lambda,\xi}$

$$\begin{aligned}
|x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})| &\leqslant C|x-\xi|^{2s} \left(|\overline{\phi}|^{p-1} + \varphi_\lambda^{p-1}\right) \\
&\leqslant C\lambda^{2s} \left(||\phi_1||_{*,\xi}^{p-1} + ||\phi_2||_{*,\xi}^{p-1} + \lambda^2\right) \\
&\leqslant C(\rho^{p-1} + \lambda^2).
\end{aligned} \tag{4.25}$$

Estimates (4.23)–(4.25) show that

$$\sup_{x \in \mathbb{R}^N \setminus B_{\lambda,\xi}} |x-\xi|^{2s}|D_{\bar{\phi}} N(\bar{\phi})| \leqslant C \left(\rho + \rho^{p-1} + \lambda^{\min\{2s, N-2s\}}\right). \tag{4.26}$$

Gathering (4.21), (4.22) and (4.26) we conclude that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$ provided $\rho > 0$ suitably small, and hence it has a unique fixed point in this set.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Claim: Let $\phi_\lambda \in \mathcal{F}$ denote the fixed point of $\mathcal{A}$ found in previous step. For any fixed $0 < \sigma < 2s/(p-1)$ we have

$$||\phi_\lambda||_{*,\xi,\sigma} \leqslant C\lambda^{\min\{2s+\sigma,\ N-2s\}} \tag{4.27}$$

where for convenience, we emphasize the dependence on σ in the notation of the norm $||\ ||_{*,\xi}$.

Note that from the previous step we see that $||\phi_\lambda||_{*,\xi,\sigma} \leqslant C\lambda^{\min\{2s+\sigma,\ N-2s\}}$ for $\sigma > 0$ suitable small. Now we fix $0 < \sigma < 2s/(p-1)$. In order to improve the estimate of the fixed point ϕ_λ we need to estimate better $N(\phi_\lambda)$. First we observe that ϕ_λ is uniformly bounded. Indeed, the function $u_\lambda = \omega - \varphi_\lambda + \phi_\lambda$ solves

$$\begin{cases} (-\Delta)^s u_\lambda - u_\lambda^p = \sum_{i=1}^{N} c_i(\lambda,\xi) Z_i \quad \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}, \\ u_\lambda = 0 \ \text{ in } \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} u_\lambda(x) = 0. \end{cases} \tag{4.28}$$

For x with $|x-\xi| = 1$, $u_\lambda(x)$ remains bounded since $|\phi_\lambda(x)| \leqslant C$ for $|x-\xi| = 1$. Then a uniform upper bound for u_λ follows from (4.28) and $||u_\lambda^p||_{L^q(B_1(\xi)\setminus B_{\lambda,\xi})}$ remaining bounded as $\lambda \to 0$ for some $q > N/2s$. In fact,

$$\int_{B_1(\xi)\setminus B_{\lambda,\xi}} u_\lambda^{pq} \leqslant C \int_{B_1} \omega^{pq} + |\phi_\lambda|^{pq} \leqslant C + C \int_{B_1(\xi)\setminus B_{\lambda,\xi}} |x|^{-\sigma pq} \mathrm{d}x \leqslant C$$

if we choose $\sigma < 2s/p$, as we have done. Hence

$$|u_\lambda(x)| \leqslant C \quad \text{for all } |x-\xi| \leqslant 1. \tag{4.29}$$

It follows from (4.29) that

$$|\phi_\lambda(x)| \leqslant C \quad \text{for all } x \in \mathbb{R}^N \setminus B_{\lambda,\xi}.$$

We shall estimate $||\phi_\lambda||_{*,\xi,\theta}$ for some $\theta > \sigma$. Since ϕ_λ is a fixed point of $\mathcal{A}$, for $0 < \theta < N-2s$ we have, by (4.1)

$$\begin{aligned} ||\phi_\lambda||_{*,\xi,\theta} = ||\mathcal{A}(\phi_\lambda)||_{*,\xi,\theta} &\leqslant C\big(||N(\phi_\lambda)||_{**,\xi,\theta} + ||E_\lambda||_{**,\xi,\theta}\big) \\ &\leqslant C\left(||N(\phi_\lambda)||_{**,\xi,\theta} + \lambda^{\min\{2s+\theta,\ N-2s\}}\right). \end{aligned} \tag{4.30}$$

Since ϕ_λ is uniformly bounded, when $p \geqslant 2$

$$|N(\phi_\lambda)| \leqslant C\left(|\phi_\lambda|^2 + |\varphi_\lambda|^2\right). \tag{4.31}$$

Taking $0 < \theta < N-2s$ such that $2s+\theta \geqslant 2\sigma$, by (4.27) we have

$$\begin{aligned} \sup_{\lambda\leqslant|x-\xi|\leqslant 1} |x-\xi|^{2s+\theta}|\phi_\lambda(x)|^2 &\leqslant C||\phi_\lambda||^2_{*,\xi,\sigma} \sup_{\lambda\leqslant|x-\xi|\leqslant 1} |x-\xi|^{2s+\theta-2\sigma} \\ &\leqslant C\lambda^{2\min\{2s+\sigma,\ N-2s\}}. \end{aligned} \tag{4.32}$$

On the other hand

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s+(2s/(p-1))}|\phi_\lambda(x)| \leqslant C||\phi_\lambda||^2_{*,\xi,\sigma} \leqslant C\lambda^{2\min\{2s+\sigma,\ N-2s\}}. \tag{4.33}$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Thus, from (4.31)–(4.33) and (4.13) we see that

$$||N(\phi_\lambda)||_{**,\xi,\theta} \leqslant C\lambda^{2\min\{2s+\sigma,\ N-2s\}}.$$

This and (4.30) imply

$$||\phi_\lambda||_{*,\xi,\theta} \leqslant C\lambda^{\min\{4s+2\sigma,\ 2s+\theta,\ N-2s\}}.$$

Repeating this argument with finite times we deduce the validity of (4.19) in the case of $p \geqslant 2$.

If $1 < p < 2$ instead of (4.31), using

$$|N(\phi_\lambda)| \leqslant C|\phi_\lambda|^p,$$

the same argument as before yields

$$||N(\phi_\lambda)||_{**,\xi,\theta} \leqslant C\lambda^{\min\{2s+\sigma,\ N-2s,\ p(2s+\sigma)\}}.$$

Then we finish the proof of the claim. □

REMARK 4.4. Let $\xi = 0$ and $c_i = 0, 1 \leqslant i \leqslant N$ in the above argument, we can derive that lemma 4.3 holds similarly.

5. The proof of theorem1.1

According to lemma 4.3, if $p > (N+2s-1)/(N-2s-1)$, there exists ϕ_λ solution to problem (2.9) with $c_i = 0,\ 1 \leqslant i \leqslant N$ such that

$$||\phi_\lambda||_* \leqslant C\lambda^{\min\{2s+\sigma,N-2s\}} \quad \text{for all} \ \ 0<\lambda<\lambda_0,\ \text{for some}\ C, \lambda_0 > 0. \tag{5.1}$$

Thus, problem (2.1) has a solution

$$\widetilde{u} = \omega - \varphi_\lambda + \phi_\lambda.$$

Furthermore, using (2.6) and the change of variables

$$u(y) = \lambda^{2s/(p-1)}\widetilde{u}(\lambda y)$$

we derive that theorem 1.1 holds for $p > (N+2s-1)/(N-2s-1)$.

Next we will present the detailed proof of theorem 1.1 for $(N+2s)/(N-2s) < p < (N+2s-1)/(N-2s-1)$. The case of $p = (N+2s-1)/(N-2s-1)$ is referred to remark 5.1.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

We have found a solution $(\phi_\lambda(\xi), c_1(\lambda,\xi), \ldots, c_N(\lambda,\xi))$ to problem (2.9) satisfying

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} (-\Delta)^s\phi_\lambda Z_i - (-\Delta)^s Z_i\phi_\lambda = \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \left(E_\lambda + N(\phi_\lambda) + \sum_{j=1}^N c_j Z_j\right) Z_i,$$

for $1\leqslant i\leqslant N$. Thus, for all λ small, we need to find $\xi=\xi_\lambda$ such that $c_i=0$, $1\leqslant i\leqslant N$, which is

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} (-\Delta)^s Z_i\phi_\lambda - (-\Delta)^s\phi_\lambda Z_i + \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} (E_\lambda + N(\phi_\lambda))\, Z_i = 0. \tag{5.2}$$

Let us define

$$G_i(\xi) := \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} (-\Delta)^s Z_i\phi_\lambda - (-\Delta)^s\phi_\lambda Z_i \mathrm{d}x + \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \big(E_\lambda + N(\phi_\lambda)\big) Z_i \mathrm{d}x. \tag{5.3}$$

The function G_i is continuous for any $1\leqslant i\leqslant n$, as it follows from local uniqueness, the fixed point characterization of ϕ_λ and elliptic estimates. We claim that

$$G_i(\xi) = \omega(\xi)\alpha\lambda^{N-2s}\int_{\mathbb{R}^N} |x-\xi|^{-(N-2s)}\omega^{p-1}(x)\frac{\partial\omega}{\partial x_j}(x)\mathrm{d}x + o(\lambda^{N-2s}) \tag{5.4}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$. First we have

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \left|N(\phi_\lambda)\frac{\partial\omega}{\partial x_j}\right| = o\left(\lambda^{N-2s}\right) \quad \text{as } \lambda\to 0 \tag{5.5}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$. Indeed

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \left|N(\phi_\lambda)\frac{\partial\omega}{\partial x_j}\right| = \int_{B_1(\xi)\setminus B_{\lambda,\xi}}\cdots + \int_{\mathbb{R}^N\setminus B_1(\xi)}\cdots$$

In the case of $p\geqslant 2$, by (4.19) and $|N(\phi_\lambda)|\leqslant C\omega^{p-2}|\phi_\lambda|^2$, we have for $\sigma<N/2$

$$\int_{B_1(\xi)\setminus B_{\lambda,\xi}} \left|N(\phi_\lambda)\frac{\partial\omega}{\partial x_j}\right| \leqslant ||\phi_\lambda||_{*,\xi}^2\int_{B_1(\xi)\setminus B_{\lambda,\xi}} |x-\xi|^{-2\sigma} \leqslant C\lambda^{2\min\{2s+\sigma,\ N-2s\}}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_1(\xi)} \left|N(\phi_\lambda)\frac{\partial\omega}{\partial x_j}\right| &\leqslant C||\phi_\lambda||_{*,\xi}^2\int_{\mathbb{R}^N\setminus B_1(\xi)} |x-\xi|^{-(4s/(p-1))-2s-1}\\ &\leqslant C\lambda^{2\min\{2s+\sigma,\ N-2s\}}.\end{aligned}$$

Choosing $((N-2s)/2)-2s<\sigma<\min\{N/2,\ N-2s\}$, we obtain (5.5). Similarly, if $p<2$ we have for $0<\sigma<N/p$

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \left|N(\phi_\lambda)\frac{\partial\omega}{\partial x_j}\right| = O\left(\lambda^{p\min\{2s+\sigma,\ N-2s\}}\right) \qquad \text{as } \lambda\to 0,$$

and taking $((N-2s)/p)-2s<\sigma<\min\{N/p,\ N-2s\}$ we can still obtain (5.5).

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Next we need to estimate the last term $\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}}(-\Delta)^s Z_i\phi_\lambda - (-\Delta)^s\phi_\lambda Z_i \mathrm{d}x$. Taking cut-off function η_λ given by (3.4), we define $\tilde{Z}_i = \eta_\lambda Z_i$. By the formula (1.5) in [20], we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} & (-\Delta)^s Z_i\phi_\lambda - (-\Delta)^s\phi_\lambda Z_i \mathrm{d}x \\ &= \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \big[(-\Delta)^s Z_i - (-\Delta)^s \tilde{Z}_i\big]\phi_\lambda \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} \big[Z_i - \tilde{Z}_i\big](-\Delta)^s\phi_\lambda \mathrm{d}x := I_1 + I_2,\end{aligned}$$

where I_1, I_2 are defined by the last equality.

To estimate I_1, we first calculate

$$\begin{aligned}\left|(-\Delta)^s Z_i - (-\Delta)^s\tilde{Z}_i\right| &= \left|(1-\eta_\lambda)(-\Delta)^s Z_i + P.V.\int_{\mathbb{R}^N}\frac{\eta_\lambda(x)-\eta_\lambda(y)}{|x-y|^{N+2s}}Z_i(y)\mathrm{d}y\right| \\ &\leqslant (1-\eta_\lambda)\,|(-\Delta)^s Z_i| + C\left|\int_{|x-y|\leqslant\lambda}\frac{\eta_\lambda(x)-\eta_\lambda(y)}{|x-y|^{N+2s}}Z_i(y)\mathrm{d}y\right| \\ &\quad + C\left|\int_{|x-y|\geqslant\lambda}\frac{\eta_\lambda(x)-\eta_\lambda(y)}{|x-y|^{N+2s}}Z_i(y)\mathrm{d}y\right| \\ &\quad \text{(by similar calculations as (3.12) and (3.13))} \\ &\leqslant (1-\eta_\lambda)|(-\Delta)^s Z_i| + C\lambda^{-2s}\chi_{|x|<3R_0\lambda} \\ &\quad + C\frac{\lambda^N}{|x|^{N+2s}}\chi_{|x|\geqslant 3R_0\lambda}\end{aligned}$$

Hence, by (4.19),

$$|I_1| \leqslant C\lambda^{\min\{2s+\sigma,\ N-2s\}+N-2s}. \tag{5.6}$$

We already have that ϕ_λ is uniformly bounded. Moreover, using (2.9), it implies that $|(-\Delta)^s\phi_\lambda| \leqslant C$ in $\mathbb{R}^N\setminus B_{\lambda,\xi}$. Thus, we can derive that

$$|I_2| \leqslant \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} |Z_i|(1-\eta_\lambda)|(-\Delta)^s\phi_\lambda|\mathrm{d}x \leqslant C\lambda^N. \tag{5.7}$$

Combining (5.6), (5.7) and (5.3), (5.4) holds.

Let us consider the vector field

$$G(\xi) = (G_1(\xi),\dots,G_N(\xi)).$$

G is then continuous and, thanks to (5.4)

$$G(\xi)\cdot\xi < 0 \quad \text{for all } |\xi| = R$$

for any fixed small $R > 0$ which means that $G(\xi)$ has never points in the same direction on $\{\xi : |\xi| = R\}$. Then the deformation $G_t(\xi) = tG(\xi) + (t-1)\xi$ and degree

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

theory lead to the existence of ξ such that $c_i = 0$, $1 \leqslant i \leqslant N$. This concludes the proof.

Remark 5.1. The proof of theorem 1.1 for $p = (N+2s-1)/(N-2s-1)$ follows exactly the same lines with the following modified norms:

$$||\phi||_{*,\xi} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^{\sigma}|\phi(x)| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s/(p-1)+\mu}|\phi(x)|$$

$$||h||_{**,\xi} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^{\sigma+2s}|\phi(x)| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2s/(p-1)+\mu+2s}|\phi(x)|$$

where $\mu > 0$ is a small fixed number. With this slightly stronger norms, proposition 2.1 remains valid. Indeed, the stronger decay of h assures that the orthogonality condition $\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} hZ_i = 0$ for all $i = 1, \dots, N$ makes sense. According to remark 7.2 of [**3**], lemma 4.1 (2) also holds when $p = (N+2s-1)/(N-2s-1)$. Then the proof in § 5 carries on. With some minor modifications, we can finally obtain theorem 1.1.

6. The proof of theorem 1.2

In this section we construct fast decay solutions to problem (1.1) when the exponent p is close to the Sobolev critical exponent $(N+2s)/(N-2s)$. For convenience, we denote $p_0 = (N+2s)/(N-2s)$, $p = p_0 + \varepsilon$ where $\varepsilon > 0$ is small.

The basic cell to construct a fast decay solution is the function $\omega_{**} = ((1)/(1+A_{N,s}r^2))^{(N-2s)/2}$. For simplicity, but with slight abuse of notation, we will denote this function simply by ω.

The main difference with the case treated in the previous section is that for the linearized problem, it has $N+1$ dimensional kernels which consists of

$$Z_0 = r\omega'(r) + \frac{N-2s}{2}\omega, \text{ and } Z_i = \frac{\partial \omega}{\partial x_i}.$$

For given $\xi \in \mathbb{R}$ and λ small, we first study existence and estimates for solutions $(\phi, c_0, c_1, \dots, c_N)$ to the problem

$$\begin{cases} (-\Delta)^s\phi - p\omega^{p-1}\phi = N(\phi) + E_\lambda + c_0Z_0 + \sum_{i=1}^{N} c_iZ_i & \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}} \\ \phi(x) = 0 \quad \text{in } \overline{B_{\lambda,\xi}}, & \lim_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{6.1}$$

with

$$\begin{aligned} N(\phi) &= |\omega + \phi - \varphi_\lambda|^{p_0+\varepsilon} - \omega^{p_0+\varepsilon} - (p_0+\varepsilon)\omega^{p_0+\varepsilon-1}(\phi - \varphi_\lambda) \\ &\quad + \left[(p_0+\varepsilon)\omega^{p_0+\varepsilon-1} - p_0\omega^{p_0-1}\right]\phi - \left[(p_0+\varepsilon)\omega^{p_0+\varepsilon-1} - p_0\omega^{p_0-1}\right]\varphi_\lambda \end{aligned} \tag{6.2}$$

and

$$E_\lambda = \omega^{p_0+\varepsilon} - \omega^{p_0} - p_0\omega^{p_0-1}\varphi_\lambda, \tag{6.3}$$

where φ_λ was given by (2.2).

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Approximate norms in the domain $\mathbb{R}^N \setminus \overline{B_{\lambda,\xi}}$ of this case are

$$\begin{aligned} ||\phi||_{*,\xi} &:= \sup_{|x-\xi|\leqslant 1} |x-\xi|^{\sigma}|\phi(x)| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{N-2s}|\phi(x)|, \\ ||h||_{**,\xi} &:= \sup_{|x-\xi|\leqslant 1} |x-\xi|^{2s+\sigma}|h(x)| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{N+s}|h(x)|. \end{aligned} \tag{6.4}$$

In particular, when $\xi = 0$ we denote the above norms by $||\cdot||_*$ and $||\cdot||_{**}$ respectively. We will need to estimate the $||\cdot||_{**,\xi}$-norm of $N(\phi)$ and E.

By a similar argument as in [11], we have the following result:

Proposition 6.1. *If $0 < \sigma < \min\{2s, 2s/(p-1)\}$ there exists a positive constant C such that, with $\nu = \min\{N-2s,\ \sigma+2s\}$,*

$$||N(\phi)||_{**,\xi} \leqslant C\big(||\phi||_{*,\xi}^2 + ||\phi||_{*,\xi}^{p_0} + \lambda^{\nu} + \varepsilon\lambda^{\nu}\big) \tag{6.5}$$

and

$$||E||_{**,\xi} \leqslant C\big(\lambda^{\nu} + \varepsilon\big). \tag{6.6}$$

To solve problem (6.1), we first recall the following nondegeneracy result for the linear problem with critical exponent on whole space in [12]:

Lemma 6.2. *If ϕ is a solution of*

$$(-\Delta)^s\phi - p_0\omega^{p_0-1}\phi = 0 \quad \text{in } \mathbb{R}^N$$

satisfying $\|\phi\|_{\infty} < \infty$, then

$$\phi = \sum_{i=0}^{n} c_i \frac{\partial \omega}{\partial x_i}$$

for some $c_i \in \mathbb{R}$.

Using this non-degeneracy result and following the same argument as in [3], one can get the following solvability:

Proposition 6.3. *Let $|\xi| \leqslant \Lambda$, $p_0 = (N+2s)/(N-2s)$ and $0 < \sigma < N-2s$. There is a linear map $(\phi, c_0, c_1, \ldots, c_N) = \mathcal{T}(h)$ defined whenever $||h||_{**,\xi} < \infty$ such that*

$$(-\Delta)^s\phi - p_0\omega^{p_0-1}\phi = h + c_0 Z_0 + \sum_{i=1}^{N} c_i Z_i \quad \text{in } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} \phi(x) = 0 \tag{6.7}$$

and

$$||\phi||_{*,\xi} + \sum_{i=0}^{N} |c_i| \leqslant C||h||_{**,\xi}. \tag{6.8}$$

Moreover, $c_i = 0$ for all $0 \leqslant i \leqslant N$ if and only if h satisfies

$$\int_{\mathbb{R}^N} h Z_0 = 0, \qquad \int_{\mathbb{R}^N} h \frac{\partial \omega}{\partial x_i} = 0, \qquad \forall 1 \leqslant i \leqslant N.$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

Next we are going to solve problem (6.1):

LEMMA 6.4. *Let $\Lambda > 0$ and $s \in (0,1)$, then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $|\xi| < \Lambda$ and $\lambda < \varepsilon_0$ there exists $(\phi, c_0, c_1, \ldots, c_N)$ solution to*

$$\begin{cases} (-\Delta)^s \phi - p\omega^{p-1}\phi = N(\phi) + E_\lambda + c_0 Z_0 + \sum_{i=1}^{N} c_i Z_i & \text{in } \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}} \\ \phi(x) = 0 \quad \text{in} \quad \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \end{cases} \tag{6.9}$$

In addition,

$$||\phi||_{*,\xi} + \max_{0 \leqslant i \leqslant N} |c_i| \to 0 \quad \text{as } \lambda + \varepsilon \to 0,$$

and

$$||\phi||_{*,\xi} \leqslant C(\lambda^\nu + \varepsilon) \quad \text{for } 0 < \lambda < \lambda_0 \tag{6.10}$$

where

$$\nu = \min\{2s + \sigma,\ N - 2s\}, \qquad 0 < \sigma < N - 2s. \tag{6.11}$$

Proof. The proof follows from the following two facts. **Fact 1.** Assume $s \in (0,1)$, $p_0 = (N+2s)/(N-2s)$, $0 < \sigma < N - 2s$ and let $|\xi| \leqslant \Lambda$. Suppose $||h||_{**,\xi} < \infty$. Then for $\lambda > 0$ sufficiently small the problem

$$\begin{cases} (-\Delta)^s \phi - p\omega^{p-1}\phi = h + c_0 Z_0 + \sum_{i=1}^{N} c_i Z_i & \text{in} \quad \mathbb{R}^N \setminus \overline{B_{\lambda,\xi}} \\ \phi(x) = 0 \quad \text{in} \quad \overline{B_{\lambda,\xi}}, \quad \lim_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{6.12}$$

has a solution $(\phi, c_0, c_1, \ldots, c_N) = \mathcal{T}(h)$ that depends linearly on h and satisfying

$$||\phi||_{*,\xi} + \max_{0 \leqslant i \leqslant N} |c_i| \leqslant C||h||_{**,\xi}.$$

The constant C is independent of λ and ε.

Proposition 6.3 and the similar proof of proposition 2.1 give out this fact.

Fact 2. Solving (6.1) reduces now to a fixed point problem. Namely, we need to find a fixed point for the map $A(\phi) = \mathcal{T}(N(\phi) + E)$. Define

$$F = \left\{\phi : \mathbb{R}^N \setminus B_{\lambda,\xi} \to \mathbb{R} :\ ||\phi||_{*,\xi} \leqslant M(\lambda^\nu + \varepsilon)\right\}$$

for some $M > 0$ large and $\nu = \min\{N - 2s, \sigma + 2s\}$. Since

$$||A(\phi)||_{*,\xi} \leqslant C(||N(\phi)||_{**,\xi} + ||E||_{**,\xi})$$

and taking into account (6.5) and (6.6), we easily get that $A(F) \subseteq F$ if $0 < \sigma \leqslant \min\{2s + \sigma,\ N - 2s\}$. To show that A is a contraction, we argue as in the proof of

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

lemma 4.2, with

$$D_{\bar{\phi}}N(\bar{\phi}) = (p_0+\varepsilon)\big[(\omega-\varphi_\lambda+\bar{\phi})^{p_0+\varepsilon-1}-\omega^{p_0+\varepsilon-1}\big] + \big[(p_0+\varepsilon)\omega^{p_0+\varepsilon-1}-p_0\omega^{p_0-1}\big]\bar{\phi}$$

to reach that

$$\sup_{x\in\mathbb{R}^N\setminus B_{\lambda,\xi}} |x|^{2s}|D_{\bar{\phi}}N(\bar{\phi})|$$

is infinitesimal as $\lambda+\varepsilon\to 0$. In order to estimate (6.10) in the range of $0<\sigma<N-2s$ we process as in the proof of lemma 4.2. □

Proof of theorem 1.2. Let $(\phi, c_0, \dots, c_N)$ be a solution to problem (6.1). Then it is sufficient to show that the parameter λ and the point ξ can be adjusted so that $c_0, \dots, c_N$ are all equal to zero, which is

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} [(-\Delta)^s\phi Z_i - (-\Delta)^s Z_i\phi] = \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} (E_\lambda + N(\phi))Z_i, \quad \forall\ 0\leqslant i\leqslant N.$$

Define, for $0\leqslant i\leqslant N$,

$$G_i(\xi,\lambda) := \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} (E_\lambda + N(\phi))Z_i - \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} [(-\Delta)^s\phi Z_i - (-\Delta)^s Z_i\phi]\,. \tag{6.13}$$

Arguing as in (5.4) and taking into account that, by symmetry,

$$\int_{\mathbb{R}^N} \omega^{(N+2s)/(N-2s)}(\log\omega)\frac{\partial\omega}{\partial x_i} = 0, \quad \forall i=1,\dots,N,$$

we obtain

$$G_i(\xi,\lambda) = \alpha\omega(\xi)\frac{N+2s}{N-2s}\lambda^{N-2s}\int_{\mathbb{R}^N}|x-\xi|^{-(N-2s)}\omega(x)^{4s/(N-2s)}\frac{\partial\omega}{\partial x_i}(x) + o\left(\lambda^{N-2s}+\varepsilon\right). \tag{6.14}$$

Obviously, for $\xi=0$ the above integral is zero. Since the above integral depends smoothly on ξ, given $\delta>0$ small, for all λ and ε small we can find $\xi\in B_\delta(0)$, depending on λ and ε, so that all $c_i=0$, for $i=1,\dots,N$.

We are now left to show that also $c_0=0$. In order to get this fact, we need to adjust the parameter λ. Using the estimates obtained on ϕ and similar argument as (5.4), we first observe that

$$G_0 = \int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} E_\lambda Z_0 + o\left(\lambda^{N-2s}+\varepsilon\right).$$

Direct computation now yields that

$$\int_{\mathbb{R}^N\setminus B_{\lambda,\xi}} E_\lambda Z_0 = -a\varepsilon + A(\xi)\lambda^{N-2s} + o(\lambda^{N-2s}+\varepsilon) \tag{6.15}$$

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press

where

$$a = \int_{\mathbb{R}^N} \omega^{(N+2s)/(N-2s)}(\log \omega) Z_0,$$
$$A(\xi) = \alpha\omega(\xi)\frac{N+2s}{N-2s}\int_{\mathbb{R}^N} |x-\xi|^{-(N-2s)}\omega^{4s/(N-2s)} Z_0.$$

First we observe that the constant a is positive. Indeed, if we define

$$g(t) = \frac{1}{(p_0+1)^2}\int_{\mathbb{R}^N} \omega_t^{p_0+1} - \frac{1}{p_0+1}\int_{\mathbb{R}^N} \omega_t^{p_0+1}\log(\omega_t)$$

where $\omega_t = ((t)/(1+A_{N,s}t^2|x|^2))^{(N-2s)/2}$, then changing of variables gives that

$$g(t) = a_{N,s} - b_{N,s}\log t$$

for some constants $a_{N,s}$ and $b_{N,s} > 0$, depending on N and s. Observing that $a = -g'(1)$, the conclusion thus follows.

We need now to prove that $A(\xi) > 0$ for ξ close to 0. To do so, it is enough showing that

$$I = \int_{\mathbb{R}^N} |x|^{-(N-2s)}\frac{1-|x|^2}{(1+|x|^2)^{(N/2)+s+1}}\mathrm{d}x > 0.$$

Writing ω_N for the volume of the $N-1$ dimensional unit sphere, we have

$$\begin{aligned} I &= \omega_N \int_0^\infty \frac{1-r^2}{(1+r^2)^{(N/2)+s+1}}\frac{1}{r^{1-2s}}\mathrm{d}r \\ &= \omega_N \int_0^1 \frac{1-r^2}{(1+r^2)^{(N/2)+s+1}}\frac{1}{r^{1-2s}}\mathrm{d}r + \int_1^\infty \frac{1-r^2}{(1+r^2)^{(N/2)+s+1}}\frac{1}{r^{1-2s}}\mathrm{d}r \\ &= \omega_N \int_0^1 \frac{1-r^2}{(1+r^2)^{(N/2)+s+1}}\left(\frac{1}{r^{1-2s}} - r^{N-1}\right)\mathrm{d}r > 0. \end{aligned}$$

From (6.15) we can find λ of order $\varepsilon^{1/(N-2s)}$ so that $c_0 = 0$. This concludes the proof of theorem 1.2. □

7. Financial support

The research of the first author is supported by NSFC 11801421 and the research of the third author is supported by NSFC 11971169.

References

1 V. Ambrosio and G. M. Figueiredo. Ground state solutions for a fractional Schrödinger equation with critical growth. *Asymptot. Anal.* **105** (2017), 159–191.

2 W. Ao, H. Chan, A. DelaTorre, M. A. Fontelos, M. del. M. Gonzalez and J. Wei. On higher dimensional singularities for the fractional Yamabe problem: a non-local Mazzeo-Pacard program. *Duke Math J.* **168** (2019), 3297–3411.

3 W. Ao, H. Chan, M. del. M. Gonzalez and J. Wei. Bound state solutions for the supercritical fractional Schrödinger equation. *Nonlinear Anal.* **193** (2020), 111448.

4 C. Bucur. Some observations on the Green function for the ball in the fractional Laplace framework. *Commun. Pure Appl. Anal.* **15** (2016), 657–699.
5 W. Chen and C. Li. Maximum principles for the fractional p-Laplacian and symmetry of solutions. *Adv. Math.* **335** (2018), 735–758.
6 W. Chen, C. Li and G. Li. Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions. *Calc. Var. Partial Differ. Equ.* **56** (2017), 18. no. 2, Paper No. 29, 18pp.
7 W. Chen, Y. Li and P. Ma. The fractional Laplacian. 2017.
8 M. Cheng. Bound state for the fractional Schrödinger equation with unbounded potential. *J. Math. Phys.* **53** (2012), 043507 7 pp.
9 J. Dávila, M. del Pino, S. Dipierro and E. Valdinoci. Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum. *Anal. PDE* **8** (2015), 1165–1235.
10 J. Dávila, M. del Pino, M. Musso. The supercritical Lane-Emden-Fowler equation in exterior domains. *Comm. Partial Differ. Equ.* **32** (2007), 1225–1243.
11 J. Dávila, M. del Pino, M. Musso and J. Wei. Fast and slow decay solutions for supercritical elliptic problems in exterior domains. *Calc. Var. Partial Differ. Equ.* **32** (2008), 453–480.
12 J. Dávila, M. del Pino and Y. Sire. Nondegeneracy of the bubble in the critical case for non local equations. *Proc. Amer. Math. Soc.* **141** (2013), 3865–3870.
13 E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.
14 M. Fall Mouhamed and E. Valdinoci. Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}^N$ when s is close to 1. *Comm. Math. Phys.* **329** (2014), 383–404.
15 P. Felmer, A. Quaas and J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A* **142** (2012), 1237–1262.
16 R. L. Frank and E. Lenzmann. Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$. *Acta Math.* **210** (2013), 261–318.
17 R. L. Frank, E. Lenzmann and L. Silvestre. Uniqueness of radial solutions for the fractional Laplacian. *Comm. Pure Appl. Math.* **69** (2016), 1671–1726.
18 Q. Guo and X. He. Semiclassical states for fractional Schrödinger equations with critical growth. *Nonlinear Anal.* **151** (2017), 164–186.
19 N. S. Landkof. *Foundations of modern potential theory. Die Grundlehren der mathematischen Wissenschaften180* (Heidelberg: Springer, 1972).
20 X. Ros-Oton and J. Serra. The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.* **213** (2014), 587–628.
21 E. Silvestre Luis. Regularity of the obstacle problem for a fractional power of the Laplace operator. Ph.D.thesis, The University of Texas at Austin.2005.
22 L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* **60** (2007), 67–112.

https://doi.org/10.1017/prm.2020.91 Published online by Cambridge University Press