EXTENSIONS RELATIVE TO A SERRE CLASS

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(Received 26th April 1974)

Consider a class C of projective R-modules, where R is a commutative ring with identity, which satisfies the conditions of (2), namely that C is closed under the operations of direct sum and isomorphism and C contains the zero module. Following (2) a module M is said to have C-cotype n (respectively C-type n) if it has a projective resolution $\ldots \to P_n \to \ldots \to P_0 \to M \to 0$ with $P_i \in C$ for $i > n$ (respectively $P_i \in C$ for $i \leqq n$). Let S be the class of modules of C-cotype -1, equivalently of C-type infinity. It is assumed throughout that S is a Serre Class. We define an abelian category $\mathscr{S}$ of modules with the property that C-cotype is homological dimension in $\mathscr{S}$, while in the case $C = \mathbf{0}$, S is just the category of R-modules. It follows that all categorical results on homological dimension also hold for cotype.

In Theorem 12 the restriction to a Serre class S is expressed in terms of the coherence of the ring R. Some examples of such classes are given.

Repeated use is made of the following result of (2).

Theorem 1. *Suppose* $0 \to L \to M \to N \to 0$ *is an exact sequence of R-modules. Then for all* $n \geqq -1$,

(i) *if L has cotype* $(n-1)$ *and M has cotype n, then N has cotype n,*

(ii) *if L has cotype n and N has cotype n, then M has cotype n,*

(iii) *if M has cotype n and N has cotype* $(n+1)$, *then L has cotype n.*

Corollary. *If* $0 \to L \to M \to N \to 0$ *is exact and any two of L, M, N belong to* S *then so does the third. Also* $0 \in S$.

It follows that S is a Serre Class if and only if it is closed under submodules; or equivalently, closed under quotient modules.

1. Definition of $\mathscr{S}$

The category $\mathscr{S}$ has as objects all R-modules. The morphisms from A to B are equivalence classes of S-homomorphisms in the sense of Serre (3). The definitions are as follows. Let G be a submodule of $A \oplus B$ and let p: $G \to A$, q: $G \to B$ be the projections. G is an *S-homomorphism from A to B* if Ker p and Coker p both belong to S (that is, if p is an S-isomorphism in the sense of (3)). A relation is defined on S-homomorphisms from A to B by $G \sim H$ if and only if the inclusions $G \cap H \to G$ and $G \cap H \to H$ are S-isomorphisms;

https://doi.org/10.1017/S0013091500010488 Published online by Cambridge University Press

that is, if and only if $G/G\cap H$ and $H/G\cap H$ both belong to S. Hence $G\sim H$ if and only if G and H are S-equal.

Suppose $G\sim H$ and $H\sim K$. Now $G/G\cap K$ is an extension of a submodule of $H/H\cap K$ by a quotient of $G/G\cap H$. Since S is a Serre class this implies that $G/G\cap K\in S$. This proves that $G\sim K$. We therefore define $\mathrm{Hom}_{\mathscr{S}}(A, B)$ as the set of equivalence classes of S-homomorphisms from A to B. For simplicity of notation we will write G for both an S-homomorphism and its equivalence class.

The identity of $\mathrm{Hom}_{\mathscr{S}}(A, A)$ is the diagonal subset D of $A\oplus A$ with the composition of $G\in \mathrm{Hom}_{\mathscr{S}}(A, B)$ and $H\in \mathrm{Hom}_{\mathscr{S}}(B, C)$ defined by

$$HG = \{(a, c)\in A\oplus C\colon \exists b\in B \text{ with } (a, b)\in G \text{ and } (b, c)\in H\}.$$

Lemma 2. *HG is an S-homomorphism and the composition is well defined on equivalence classes. Thus $\mathscr{S}$ is a category.*

Proof. If

$$\begin{array}{ccc} K & \longrightarrow & G \\ \downarrow & & \downarrow q \\ H & \underset{p}{\longrightarrow} & B \end{array}$$

is a pullback square, $K\to G$ is an S-isomorphism and hence so is the composition $K\to A$ in the following commutative square.

$$\begin{array}{ccc} K & \longrightarrow & HG \\ \downarrow & & \downarrow p \\ G & \underset{p}{\longrightarrow} & A \end{array}$$

Here, $K\to HG$ is the canonical epimorphism. Hence $p\colon HG\to A$ is an S-isomorphism.

Now suppose $G\sim G'$ and K, K' are the corresponding pullbacks. Then $K/K\cap K'\to G/G\cap G'$ is a monomorphism and there is an epimorphism from $K/K\cap K'$ to $HG/HG\cap HG'$. Thus $HG/HG\cap HG'\in S$ and $HG\sim HG'$.

Similar techniques may be used to prove that $\mathscr{S}$ is an abelian category with the following definitions, where $G, H\in \mathrm{Hom}_{\mathscr{S}}(A, B)$.

(i) Let $G+H = \{(a, b_1+b_2)\colon (a, b_1)\in G \text{ and } (a, b_2)\in H\}$. Then $\mathrm{Hom}_{\mathscr{S}}(A, B)$ is an abelian group with zero $N = \{(a, 0)\colon a\in A\}$ and inverse $-G = \{(a, b)\colon (a, -b)\in G\}$.

(ii) Let $X = p(G\cap N)$. X is the submodule of elements a in A such that $(a, 0)\in G$. Let $i\colon X\to A$ be the inclusion and $K = \{(x, i(x))\colon x\in X\}$. Then $K\in \mathrm{Hom}_{\mathscr{S}}(X, A)$ is a kernel of G.

(iii) Let $Y = B/q(G)$ and let $j\colon B\to Y$ be the projection. Define $C = \{(b, j(b))\colon b\in B\}$. Then $C\in \mathrm{Hom}_{\mathscr{S}}(B, Y)$ is a cokernel of G.

(iv) Let $Z = q(G)$ with inclusion $k\colon Z \to B$. Define $I = \{(a, z)\colon (a, k(z)) \in G\}$ and $J = \{(z, k(z))\colon z \in Z\}$. Then $I \in \mathrm{Hom}_{\mathscr{S}}(A, Z)$ is an image of G and $J \in \mathrm{Hom}_{\mathscr{S}}(Z, B)$ is a coimage of G.

We therefore have the following analysis of G.

$$\begin{array}{ccccccc} X & \xrightarrow{K} & A & \xrightarrow{G} & B & \xrightarrow{C} & Y. \\ & & & {\scriptstyle I}\searrow \quad \nearrow{\scriptstyle J} & & & \\ & & & Z & & & \end{array}$$

Theorem 3. *$G \in \mathrm{Hom}_{\mathscr{S}}(A, B)$ is a monomorphism (epimorphism) in $\mathscr{S}$ if and only if $q\colon G \to B$ is an S-monomorphism (epimorphism).*

Proof. G is a monomorphism in $\mathscr{S}$ if and only if $K \sim N$ as submodules of $X \oplus A$. But $K \cap N = 0$ so $K \sim N$ if and only if $K \in S$ and $N \in S$. K and N are both isomorphic to X which is isomorphic to $\mathrm{Ker}\,(q\colon G \to B)$. Hence $K \in S$ and $N \in S$ if and only if $\mathrm{Ker}\, q \in S$. The dual result is proved similarly.

It follows that A and B are S-isomorphic in the sense of Serre if and only if there is an isomorphism $G \in \mathrm{Hom}_{\mathscr{S}}(A, B)$. The null objects of the category $\mathscr{S}$ are the modules in S and we have

Proposition 4. $\mathrm{Hom}_{\mathscr{S}}(A, B) = 0$ *for all B if and only if $A \in S$.*

Proof. If $\mathrm{Hom}_{\mathscr{S}}(A, A) = 0$, then $D \sim N$; but $D \cap N = 0$ and A is isomorphic to N. Therefore $A \in S$. Conversely if $A \in S$ and $G \in \mathrm{Hom}_{\mathscr{S}}(A, B)$ then $p\colon G \to A$ is an S-isomorphism so $G \in S$. It follows that $G \sim N$.

Finally note that if $S = \mathbf{0}$ then $\mathscr{S}$ reduces to the category of R-modules and module homomorphisms.

2. Extensions in $\mathscr{S}$

The functors Ext^n can be defined in any abelian category which has sufficient projectives. We show that every projective R-module is a projective object of $\mathscr{S}$. Hence every R-module has a projective resolution in $\mathscr{S}$.

Lemma 5. *If A and B are S-isomorphic then they have the same cotype.*

Proof. It is sufficient to prove that if $f\colon A \to B$ is an S-isomorphism then A and B have the same cotype. But now let

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & & & & \searrow \quad \nearrow & & & & & \\ & & & & & I & & & & & \end{array}$$

be exact with $K, C \in S$. The result follows immediately from Theorem 1.

https://doi.org/10.1017/S0013091500010488 Published online by Cambridge University Press

Theorem 6. *A module A of C-cotype zero is a projective object in $\mathcal{S}$.*

Proof. Suppose $G \in \mathrm{Hom}_{\mathcal{S}}(A, C)$ and let $H \in \mathrm{Hom}_{\mathcal{S}}(B, C)$ be an epimorphism is $\mathcal{S}$. We must produce $K \in \mathrm{Hom}_{\mathcal{S}}(A, B)$ such that $HK \sim G$. The first step reduces the problem to the case in which $q\colon H \to C$ is epi.

Let $X = q(H)$ and $i\colon X \to C$ the inclusion. Then i is an S-isomorphism since H is an epimorphism in $\mathcal{S}$ and $q\colon H \to X$ is epi. Also H can be considered as $H' \in \mathrm{Hom}_{\mathcal{S}}(B, X)$. Let $G' = \{(a, x) \in A \oplus X\colon (a, i(x)) \in G\}$. Then $G' \to G$ is mono and so is the map $G/G' \to C/i(X)$ induced by the projections. Hence $G' \to G$ is an S-isomorphism; therefore so is the composition $G' \to G \to A$. Thus $G' \in \mathrm{Hom}_{\mathcal{S}}(A, X)$ while $H' \in \mathrm{Hom}_{\mathcal{S}}(B, X)$ is such that $q\colon H' \to X$ is epi. Moreover, if $K \in \mathrm{Hom}_{\mathcal{S}}(A, B)$ satisfies $H'K \sim G'$, then $HK = iH'K \sim iG' = G$. It is therefore sufficient to prove the result in the case where $q\colon H \to C$ is epi.

Now by Lemma 5, G has cotype 0 since A does; so by definition there is an exact sequence $0 \to X \to P \xrightarrow{\phi} G \to 0$ with P projective and $X \in S$. Hence there exists a map $\alpha\colon P \to H$ such that the following diagram is commutative.

$$\begin{array}{ccccc} P & \xrightarrow{\phi} & G & \xrightarrow{p_G} & A \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle q_G} & & \\ H & \xrightarrow{q_H} & C & & \\ {\scriptstyle p_H}\downarrow & & & & \\ B & & & & \end{array}$$

For clarity the various projection maps are here distinguished by subscripts. Let $\pi = p_G\phi$, $\theta = p_H\alpha$ and $\beta = q_G\phi = q_H\alpha$. Now define $K \subseteq A \oplus B$ as $K = (\pi, \theta)(P)$. The diagram

$$\begin{array}{ccc} P & \xrightarrow{(\pi,\theta)} & K \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle p} \\ & & A \end{array}$$

is commutative with π an S-isomorphism and (π, θ) epi. Therefore $K \in \mathrm{Hom}_{\mathcal{S}}(A, B)$.

It remains to verify that $HK \sim G$. If $(a, c) \in G$, choose $x \in P$ with $\phi(x) = (a, c)$. Then $\alpha(x) = (\theta(x), c) \in H$ and $\pi(x) = a$, so $(a, \theta(x)) \in K$. Hence $(a, c) \in HK$; that is $G \subseteq HK$. Let $Y = \mathrm{Ker}\, p_H$ which belongs to S. Then $q = q_H \mid Y\colon Y \to C$ is mono so $q(Y) \in S$. Hence the quotient Z of $q(Y)$ modulo the subgroup $q(Y) \cap q_G(\mathrm{Ker}\, p_G)$ also belongs to S. We define a map $\psi\colon HK \to Z$ with kernel G. This will complete the proof by showing that HK/G is isomorphic to a subgroup of Z and hence belongs to S.

To define ψ let $(a, c) \in HK$. Choose $x \in P$ such that $\pi(x) = a$ and $(\theta(x), c) \in H$. Let $z = c - \beta(x)$. Since $(0, z) = (\theta(x), c) - \alpha(x)$, which belongs to H, we have $z \in q(Y)$. Define $\psi(a, c)$ as the equivalence class represented by z. If x' is another choice, then

$$z-z' = \beta(x')-\beta(x) \text{ and } (0, \beta(x')-\beta(x)) = \phi(x')-\phi(x) \in \operatorname{Ker} p_G.$$

Hence z and z' represent the same element of Z. Finally,

$$(a, c) = \phi(p)+(0, c-\beta(p))$$

so $(a, c) \in \operatorname{Ker} \psi$ if and only if $(0, c-\beta(p)) \in G$. It follows that $\operatorname{Ker} \psi = G$. This completes the proof.

Corollary 7. *$\mathscr{S}$ has sufficient projectives.*

Proof. Every projective module has cotype zero and so is a projective object in $\mathscr{S}$. Hence a projective resolution in the module category gives rise to a projective resolution in $\mathscr{S}$.

The functors $\operatorname{Ext}^n_{\mathscr{S}}$ are therefore defined and $\operatorname{Ext}^1_{\mathscr{S}}(A, B) = 0$ for all B if and only if A is a projective object in $\mathscr{S}$.

Lemma 8. *If A has finite cotype and* $\operatorname{Ext}^1_{\mathscr{S}}(A, B) = 0$ *for all B then A has cotype* 0.

Proof. Suppose A has cotype r. There is an exact sequence

$$0 \to K \to P \to A \to 0 \tag{*}$$

with P projective and K of cotype $(r-1)$. A is a projective object of $\mathscr{S}$; so this sequence splits in $\mathscr{S}$ giving an isomorphism $G \in \operatorname{Hom}_{\mathscr{S}}(A \oplus K, P)$. Hence $A \oplus K$ has cotype 0. Theorem 1 (iii) applied to the sequence

$$0 \to A \to A \oplus K \to K \to 0$$

shows that (if $r \geqq 2$) A has cotype $(r-2)$. Hence A has cotype 0.

Theorem 9. If *A has finite cotype, then* $\operatorname{Ext}^{n+1}_{\mathscr{S}}(A, B) = 0$ *for all B if and only if A has cotype n ($n \geqq -1$).*

Proof. Proposition 4 gives the result for $n = -1$. Theorem 6 and Lemma 8 prove the result for $n = 0$. We use induction on $n > 0$. Suppose A has cotype r. The exact sequence (*) gives rise to a long exact sequence

$$\ldots \to \operatorname{Ext}^n_{\mathscr{S}}(P, B) \to \operatorname{Ext}^n_{\mathscr{S}}(K, B) \to \operatorname{Ext}^{n+1}_{\mathscr{S}}(A, B) \to \operatorname{Ext}^{n+1}_{\mathscr{S}}(P, B) \to \ldots$$

for all B. $\operatorname{Ext}^n_{\mathscr{S}}(P, B) = 0$ by Theorem 6 and $\operatorname{Ext}^{n+1}_{\mathscr{S}}(A, B) = 0$ by hypothesis. Hence $\operatorname{Ext}^n_{\mathscr{S}}(K, B) = 0$; so by induction, K has cotype $(n-1)$. It follows that A has cotype n. Conversely, if A has cotype $n (n \geqq 0)$, then K has cotype $(n-1)$ so $\operatorname{Ext}^n_{\mathscr{S}}(K, B) = 0 = \operatorname{Ext}^{n+1}_{\mathscr{S}}(P, B)$ which implies the result.

This theorem shows that, as long as modules have finite cotype, then cotype is simply homological dimension in the category $\mathscr{S}$. We also see that the $\boldsymbol{C}$-cotype of a module depends only on the derived class $\boldsymbol{S}$. Note that $\boldsymbol{C} \subseteq \boldsymbol{S}$; Since $C \in \boldsymbol{C}$ it has a projective resolution $0 \to C \to C \to 0$. $\boldsymbol{C}$ is determined by $\boldsymbol{S}$

provided $\boldsymbol{C}$ is closed under direct summands, since then $\boldsymbol{C} = \boldsymbol{S} \cap \boldsymbol{P}$ where $\boldsymbol{P}$ is the class of projective modules (**2**, Theorem 2). In general the following relations hold between $\boldsymbol{C}$ and $\boldsymbol{S}$.

Lemma 10. *If $\boldsymbol{C}$ and $\boldsymbol{C}'$ both have derived class $\boldsymbol{S}$ then for every $C \in \boldsymbol{C}$ there exists $X \in \boldsymbol{S} \cap \boldsymbol{P}$ and $C' \in \boldsymbol{C}'$ such that $C' = C \oplus X$.*

To prove this, note that if $C \in \boldsymbol{C} \subseteq \boldsymbol{S}$ there is an exact sequence $0 \to X \to C' \to C \to 0$ with $X \in \boldsymbol{S}$ and $C' \in \boldsymbol{C}'$. The result follows since C is projective.

In particular, for a given class $\boldsymbol{C}$ with derived class $\boldsymbol{S}$, let $\boldsymbol{C}' = \boldsymbol{S} \cap \boldsymbol{P}$. This is a class. Let $\boldsymbol{S}'$ be its derived class. $\boldsymbol{C} \subseteq \boldsymbol{C}'$ so $\boldsymbol{S} \subseteq \boldsymbol{S}'$. Conversely if $A \in \boldsymbol{S}'$ there is an exact sequence

$$\ldots \to C'_n \to \ldots \to C'_0 \to A \to 0$$

with $C'_i \in \boldsymbol{S} \cap \boldsymbol{P}$ for all i. Thus if either $\boldsymbol{S}$ is Serre or the sequence is finite we find that $A \in \boldsymbol{S}$ (by the Corollary to Theorem 1). Hence, for a Serre class $\boldsymbol{S}$, $\boldsymbol{S} \cap \boldsymbol{P}$ also has derived class $\boldsymbol{S}$.

Corollary 11. *$\boldsymbol{C} \subseteq \boldsymbol{S} \cap \boldsymbol{P}$ with equality if and only if $\boldsymbol{C}$ is closed under direct summands. If $\boldsymbol{S}$ is Serre, then for every $C' \in \boldsymbol{S} \cap \boldsymbol{P}$ there exists $X \in \boldsymbol{S} \cap \boldsymbol{P}$ such that $C' \oplus X \in \boldsymbol{C}$.*

3. Conditions for S to be a Serre class

The ring R is said to be $(0, \boldsymbol{C})$*-coherent* if every module of $\boldsymbol{C}$-type 0 belongs to $\boldsymbol{S}$.

Theorem 12. *$\boldsymbol{S}$ is a Serre class if and only if R is $(0, \boldsymbol{C})$-coherent.*

Proof. Suppose $\boldsymbol{S}$ is Serre and A has $\boldsymbol{C}$-type 0. Then A has a projective resolution $\ldots \to P \to C \to A \to 0$ with $C \in \boldsymbol{C} \subseteq \boldsymbol{S}$. Hence A, being a quotient of C, belongs to $\boldsymbol{S}$. Conversely, let $A \in \boldsymbol{S}$ and $A \to B$ epi. A has type 0, therefore by (**1**, Lemma 7) B has type 0 and so belongs to $\boldsymbol{S}$. Hence $\boldsymbol{S}$ is closed under quotients.

The following are examples of classes for which $\boldsymbol{S}$ is Serre. For proofs see (**1**).

(i) The class $\boldsymbol{F}$ of finitely generated free modules over a Noetherian ring R. Then $\boldsymbol{S}$ is the class of finitely generated R-modules.

(iii) The class $\boldsymbol{D}$ of free graded R-modules with generators in only a finite number of dimensions, where R is a finite dimensional ring. Then $\boldsymbol{S}$ is the class of R-modules with generators in only a finite number of dimensions.

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