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LOG DEL PEZZO SURFACES WITH SIMPLE AUTOMORPHISM GROUPS

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Abstract In the present paper we classify del Pezzo surfaces with log terminal singularities admitting an action of a finite simple group.

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1. Introduction

The Cremona group $\mathrm{Cr}_k(n)$ over a field k is the group of birational automorphisms of the projective space $\mathbb{P}^n_k$. The Cremona group $\mathrm{Cr}_k(1)$ is the group of automorphisms of the projective line, and hence is isomorphic to $\mathrm{PSL}_k(2)$. In this paper we consider the Cremona group over the field of complex numbers, denoted by $\mathrm{Cr}(n)$. Even for $n = 2$, the group $\mathrm{Cr}(n)$ is not well understood. Finite subgroups of $\mathrm{Cr}(2)$ were classified by Dolgachev and Iskovskikh (see [**8**]). Very little is known about the Cremona group in higher-dimensional space. Serre [**12**] posed the problem: does there exist a finite group that is not embeddable in $\mathrm{Cr}(3)$? In [**11**], Prokhorov gave an affirmative answer to this question.

Theorem 1.1 (Prokhorov [11, Theorem 1.3]). *Let $G \subset \mathrm{Cr}(3)$ be a non-abelian simple finite subgroup. Then, G is isomorphic to one of the following groups:*

$$\mathfrak{A}_5,\ \mathfrak{A}_6,\ \mathfrak{A}_7,\ \mathrm{PSL}_2(7),\ \mathrm{SL}_2(8),\ \mathrm{PSp}_4(3).$$

All the possibilities occur.

Moreover, in [**11**], Prokhorov found all conjugacy classes of the subgroups $\mathfrak{A}_7$, $\mathrm{SL}_2(8)$, $\mathrm{PSp}_4(3)$ in $\mathrm{Cr}(3)$. But his method does not work for $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathrm{PSL}_2(7)$.

In [**7**], Cheltsov and Shramov describe some non-conjugate embeddings of $\mathfrak{A}_6$. Moreover, Prokhorov proved the following theorem.

Theorem 1.2 (Cheltsov and Shramov [7, Theorem B.1]). *Let X be a 3-fold with at worst terminal singularities such that the group $\mathrm{Aut}(X)$ has a subgroup $G \cong \mathfrak{A}_6$, and let $\pi\colon X \to \mathbb{P}^1$ be a G-Mori fibration. Then, $X \cong \mathbb{P}^1 \times \mathbb{P}^2$, and π is the projection to the first factor.*

On the other hand, there are some non-isomorphic examples of G-Mori fibration with $G \cong \mathrm{PSL}_2(7)$.*

Example 1.3 (Cheltsov and Shramov). Let $X \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $G = \mathrm{PSL}_2(7)$. Assume that G acts on second factors (see [**8**]). There is an invariant quartic, the so-called *Klein quartic* $C := \{x^3y + y^3z + z^3x = 0\}$. Set $D = R + F_1 + F_2 + \cdots + F_{2n}$, where $R = C \times \mathbb{P}^1$ and $F_i \simeq \mathbb{P}^2$ is a fibre of natural projection $X \to \mathbb{P}^1$. Set $C_i = R \cap F_i$. Let $\pi\colon Y \to X$ be a double cover ramified along D and let $\bar{F}_i = \pi^{-1}(F_i)$, $\bar{C}_i = \pi^{-1}(C_i)$. Note that the singular locus of Y is $\bar{C}_1 \cup \bar{C}_2 \cup \cdots \cup \bar{C}_{2n}$. Let $g\colon Y' \to Y$ be the resolution of the singular locus of Y and let $\tilde{F}_i$ be the strict transforms of $\bar{F}_i$. Let $\phi\colon Y \to Z$ be the contraction of $\tilde{F}_i$. Then, Z admits a G-action.

Conjecture 1.4 (Cheltsov and Shramov). *Let X be a 3-fold with at worst terminal singularities such that the group $\mathrm{Aut}(X)$ has a subgroup $G \cong \mathrm{PSL}_2(7)$, and let $\pi\colon X \to \mathbb{P}^1$ be a G-Mori fibration. Then, $X \cong \mathbb{P}^1 \times \mathbb{P}^2$, and π is the projection to the first factor, or X is isomorphic to Z from Example 1.3.*

Let X be a 3-fold with at worst terminal singularities, let $G \cong \mathrm{PSL}_2(7)$ and let $\pi\colon X \to \mathbb{P}^1$ be a G-Mori fibration. Since G is a simple group, we see that G acts on a fibre of π. Note that every fibre of π is a del Pezzo surface with at worst rational singularities.

In this paper we consider del Pezzo surfaces with only log terminal singularities admitting an action of a finite simple group G. Any del Pezzo surface with log terminal singularities is rational (see, for example, [**3**]). Hence, such a group G is contained in $\mathrm{Cr}(2)$, the plane Cremona group. Finite subgroups of $\mathrm{Cr}(2)$ are classified (see [**8**]). By [**8**], there are only three finite simple subgroups of $\mathrm{Cr}(2)$: $\mathfrak{A}_5$, $\mathfrak{A}_6$ and $\mathrm{PSL}_2(7)$, where $\mathrm{PSL}_2(7)$ is the simple group of order 168 and $\mathfrak{A}_n$ is the alternating group. The groups $\mathrm{PSL}_2(7)$, $\mathfrak{A}_6$ and $\mathfrak{A}_5$ are referred to as Klein's simple group, the Valentiner group and the icosahedral group, respectively. In this paper we classify del Pezzo surfaces with log terminal singularities admitting an action of one of these groups.

Example 1.5 (Dolgachev and Iskovskikh [8]). The Klein group $G = \mathrm{PSL}_2(7)$ has an irreducible three-dimensional representation, so G acts on the projective plane. There exists an invariant quartic, the so-called *Klein quartic* $C := \{x^3y + y^3z + z^3x = 0\}$. Consider the double cover $S_2 \to \mathbb{P}^2$ ramified along C. Then, S_2 is a smooth del Pezzo surface of degree 2. The action of G lifts naturally to S_2. A smooth del Pezzo surface of degree 2 with a $\mathrm{PSL}_2(7)$-action is unique up to isomorphism and $\rho(S_2)^G = 1$.

Example 1.6. Let n be a positive integer and let $k \in \{12, 20, 30, 60\}$. The group $G = \mathfrak{A}_5$ acts naturally on the Hirzebruch surface $\mathbb{F}_{2n}$. Thus, G acts naturally on

* Example 1.3 and Conjecture 1.4 were shown to the author by Cheltsov and Shramov at a workshop in China. They are currently unpublished.

$\mathbb{P}(1,1,2n)$. Note that there exists an embedding $f\colon \mathbb{P}(1,1,2n) \to \mathbb{P}^{2n+1}$. Hence, G acts naturally on $\mathbb{P}^{2n+1}$ with the invariant point. Then, G has an invariant hyperplane. Therefore, there exists an invariant section M with $M^2 = 2n$. Let M and D be disjoint sections with $M^2 = 2n$, $D^2 = -2n$. Let $p_1\colon Z_1 \to \mathbb{F}_{2n}$ be the blow-up of an orbit consisting of k points on M. Let $p_2\colon Z_2 \to Z_1$ be the blow-up of an orbit consisting of k points on the proper transform of M, and so on. After a steps we obtain the surface Z_a with one $(-(ka-2n))$-curve, one $(-2n)$-curve and k chains of $a-1$ (-2)-curves. Let $r\colon Z_a \to F_{2n,ak-2n,a}$ be the contraction of the curves with self-intersection number less than -1. Then, $F_{2n,ak-2n,a}$ is a del Pezzo surface admitting a non-trivial action of G, and $\rho(F_{2n,ak-2n,a})^G = 1$. Note that $F_{2n,ak-2n,a}$ has two singular points of types $(1/2n)(1,1)$, $(1/(ak-2n))(1,1)$ and k Du Val singularities of type A_{a-1}. It is possible that $a=1$, and then the singular locus of $F_{2n,k-2n,1}$ consists of two points.

Example 1.7. Let $k \in \{12, 20, 30, 60\}$. The group $G = \mathfrak{A}_5$ acts naturally on $\mathbb{P}^2$. Let C be a (unique) G-invariant conic on $\mathbb{P}^2$. Let $p_0\colon Z_0 \to \mathbb{P}^2$ be the blow-up of an orbit consisting of k points $P_1, \dots, P_k$ on C, and let C_0 be the proper transform of C. Let $p_1\colon Z_1 \to Z_0$ be the blow-up of an orbit of points on C_0 that correspond to $P_1, \dots, P_k$. Repeating this procedure $s+1$ times we obtain a smooth surface Z_s with one $(-(k(s+1)-4))$-curve and k chains of s (-2)-curves. Let $r\colon Z_s \to \tilde{\mathbb{P}^2}_{k,s}$ be the contraction of all rational curves whose self-intersection number is at most -2. Then, $\tilde{\mathbb{P}^2}_{k,s}$ is a del Pezzo surface admitting a non-trivial action of G, and $\rho(\tilde{\mathbb{P}^2}_{k,s})^G = 1$. The singular locus of $\tilde{\mathbb{P}^2}_{k,s}$ consists of one (fixed) point of type $(1/(k(s+1)-4))(1,1)$ and k Du Val singular points of type A_s. It is possible that $s=0$, and then the singular locus of $\tilde{\mathbb{P}^2}_{k,0}$ consists of one (fixed) point.

The main result of this paper is the following.

Theorem 1.8. *Let X be a del Pezzo surface with log terminal singularities and let $G \subset \mathrm{Aut}(X)$ be a finite simple group.*

(i) *If $G \simeq \mathfrak{A}_5$ and $\rho(X)^G = 1$, then the following cases hold:*

- $X \simeq \mathbb{P}^2$*;*
- $X \simeq S_5$*, where S_5 is a smooth del Pezzo surface of degree 5;*
- $X \simeq \mathbb{P}(1,1,2n)$*, a cone over a rational normal curve of degree $2n$;*
- $X \simeq F_{2n,ak-2n,a}$ *(see Example 1.6);*
- $X \simeq \tilde{\mathbb{P}^2}_{k,s}$ *(see Example 1.7).*

(ii) *If G is the Klein group, then $X \simeq \mathbb{P}^2$ or $X \simeq S_2$.*

(iii) *If G is the Valentiner group, then $X \simeq \mathbb{P}^2$.*

Remark 1.9. Unfortunately, this theorem does not hold if we consider del Pezzo surfaces with rational singularities (see Example 1.3). So, the hope is that by classifying del Pezzo surfaces with rational singularities this theorem can be used to prove Conjecture 1.4.

Note that there is no classification of finite subgroups of $\mathrm{Cr}(3)$. Our main theorem gives some modern approaches to studying $\mathrm{Cr}(3)$.

2. Preliminaries

Notation 2.1. We work over $\mathbb{C}$. Throughout this paper G is one of the groups $\mathfrak{A}_5$, $\mathfrak{A}_6$ or $\mathrm{PSL}_2(7)$. X denotes a del Pezzo surface with at worst log terminal singularities admitting a non-trivial action of G. We also employ the following notation.

- $\mathbb{F}_n$ denotes the Hirzebruch surface, $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.
- $\mathbb{P}(a, b, c)$ denotes a weighted projective plane.
- S_d denotes a del Pezzo surface of degree d.
- $\rho(X)$ denotes the Picard number.
- A G-surface is a surface V with a given embedding $G \subset \mathrm{Aut}(V)$.
- $\rho(X)^G$ denotes the G-invariant Picard number.
- An (n)-curve is a smooth rational curve whose self-intersection number is equal to n.

Definition 2.2. Let S be a normal projective surface and let $f\colon \tilde{S} \to S$ be a resolution. Let $D = \sum_i D_i$ be the exceptional divisor. There then exists a unique $\mathbb{Q}$-divisor $D^\sharp = \sum_i \alpha_i D_i$ such that $f^* K_S \equiv K_{\tilde{S}} + D^\sharp$. The numbers α_i are called the *codiscrepancy* of D_i.

Lemma 2.3. *Let V be a G-surface with at worst log terminal singularities, and let $P \in V$ be a fixed point. Then, P is singular and $G \simeq \mathfrak{A}_5$. Moreover, P has type $(1/r)(1, 1)$, where r is even.*

Proof. Assume that P is a smooth point. Then, G acts on the Zariski tangent space $T_{P,V}$. Since G is a finite simple group, we see that G has no non-trivial two-dimensional representations. Hence, P is singular. Let $\tilde{V} \to V$ be the minimal resolution of P, and let $D = \sum D_i$ be the exceptional divisor. Then, G acts on D. Since G does not admit any embeddings to $\mathfrak{S}_k$, where $k \leqslant 4$, we see that D consists of one irreducible component. Hence, P has type $(1/r)(1, 1)$. On the other hand, the Klein group and the Valentiner group do not admit a non-trivial action on a smooth rational curve. Hence, $G \simeq \mathfrak{A}_5$.

Finally, the action of $\mathfrak{A}_5$ on $\tilde{V}$ induces an action of $\mathfrak{A}_5$ on the total space of the conormal bundle $N^\vee_{D/\tilde{V}} \simeq \mathcal{O}_{\mathbb{P}^1}(r)$. In particular, the group $\mathfrak{A}_5$ naturally acts on

$$H^0(D, N^\vee_{D/\tilde{V}}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) \simeq S^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)).$$

This is possible if and only if r is even. □

Lemma 2.4 (Cheltsov [6, Theorem B.12]). *Let $G \simeq \mathfrak{A}_5$ and let $f\colon S \to \mathbb{P}^1$ be a smooth relatively minimal conic bundle with an action of G. Then $S \simeq \mathbb{F}_{2n}$. Moreover, there are two possibilities:*

- *$S = \mathbb{P}^1 \times \mathbb{P}^1$ with non-trivial action on each factor,*
- *$S = \mathbb{F}_{2n}$, $n \geqslant 0$, there is an invariant section and this case occurs for every $n \geqslant 0$.*

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Proof. Let

$$\alpha\colon G \to O(\mathrm{Pic}(S)), \quad g \to (g^*)^{-1}$$

be the natural representation of G in the orthogonal group of $\mathrm{Pic}(S)$. By [**8**, Theorem 5.7] we have that $\ker(\alpha) \neq \{e\}$. Since G is a simple group, we see that $\ker(\alpha) = G$. Hence, f has no singular fibres. Then, $S = \mathbb{F}_r$.

Since the case $r = 0$ is trivial, we assume that $r > 0$. Consider the contraction $\varphi\colon \mathbb{F}_r \to V_r$ of the negative section. Here, V_r is a cone in $\mathbb{P}^{r+1}$ over a rational normal curve $C_r \subset \mathbb{P}^r$ of degree r or, equivalently, the weighted projective plane $\mathbb{P}(1,1,r)$. Clearly, φ is $\mathfrak{A}_5$ equivariant, so $\mathfrak{A}_5$ acts non-trivially on V_r. By Lemma 2.3, r is even. On the other hand, one can write down an action of $\mathfrak{A}_5$ on $\mathbb{P}(1,1,2n)$ explicitly. □

Lemma 2.5. *Let $P \in X$ be a log terminal singularity and let $f\colon \tilde{X} \to X$ be its minimal resolution. Let $\sum \alpha_i D_i$ be a $\mathbb{Q}$-divisor such that*

$$f^*K_X \equiv K_{\tilde{X}} + \sum \alpha_i D_i.$$

Assume that $\alpha_i < \frac{2}{5}$ for every i. Then, P is either a Du Val singularity or P has type $\frac{1}{3}(1,1)$, i.e. the exceptional divisor of f consists of a single (-3)-curve.

Proof. Assume that there is a component D_j of $\sum D_i$ such that $D_j^2 \leqslant -4$. Then, by [**2**, Lemma 2.17] we have $\alpha_j \geqslant \frac{1}{2}$, a contradiction.

Hence, $D_i^2 \geqslant -3$ for every i. Assume that there exist a component D_j with $D_j^2 = -3$ and a component D_k with $D_j \cdot D_k = 1$. Then, by [**2**, Lemma 2.17] we have $\alpha_j \geqslant \frac{2}{5}$. Again we have a contradiction.

Therefore, P is either a Du Val singularity or the exceptional divisor $\sum D_i$ has only one component D_1 with $D_1^2 = -3$. □

The following lemma is a consequence of the classification of log terminal singularities (see [**5**]).

Lemma 2.6. *Let X be a projective normal surface. Let $P \in X$ be a log terminal non-Du Val singularity and let $f\colon \tilde{X} \to X$ be its minimal resolution. Let $\sum \alpha_i D_i$ be a codiscrepancy $\mathbb{Q}$-divisor over P. Assume that there exist a (-1)-curve E and a morphism $g\colon \tilde{X} \to Z$ such that $g(E + \sum D_i)$ is a smooth point. Then, $E \cdot \sum \alpha_i D_i \geqslant \frac{2}{11}$.*

Proof. Consider minimal resolution of log terminal singularities [**5**] case by case. For example, if $P \in X$ has type $\frac{1}{7}(1,3)$, then $E \cdot \sum \alpha_i D_i = \frac{2}{7}$. □

Proposition 2.7. *In Notation 2.1, assume that X has at worst Du Val singularities. We then have one of the following cases.*

(i) *G is the Valentiner group and $X \simeq \mathbb{P}^2$.*

(ii) *G is the Klein group and $X \simeq \mathbb{P}^2$ or $X \simeq S_2$.*

(iii) *$G \simeq \mathfrak{A}_5$ and either X is smooth or $X \simeq \mathbb{P}(1,1,2)$. If, moreover, $\rho(X)^G = 1$, then X is isomorphic to $\mathbb{P}^2$, S_5 or $\mathbb{P}(1,1,2)$.*

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Proof. We use some elementary facts on del Pezzo surfaces with Du Val singularities (see, for example, [**10**]). Recall that $9 \geqslant K_X^2 \geqslant 1$ and

$$\dim H^0(X, \mathcal{O}(-K_X)) = K_X^2 + 1.$$

So, we have the following cases.

Case 1 ($\boldsymbol{K_X^2 = 1}$). In this case, $\dim|-K_X| = 1$ and there exists a non-singular fixed point $\{p\} = \operatorname{Bs}|-K_X|$. On the other hand, by Lemma 2.3, every fixed point is singular, a contradiction.

Case 2 ($\boldsymbol{K_X^2 = 2}$). In this case, $\dim|-K_X| = 2$. The linear system $|-K_X|$ defines a double cover $\phi = \phi_{|-K_X|}\colon X \to \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be the ramification divisor of ϕ. We have $\deg B = 4$. Since G is simple, we see that B is irreducible. Since the number of singular points of B is at most 3, we see that B is smooth. So is X. By [**8**] we have $G \simeq \mathrm{PSL}_2(7)$ and $X \simeq S_2^{\mathrm{k}}$.

Case 3 ($\boldsymbol{K_X^2 = 3}$). In this case, $\dim|-K_X| = 3$ and $X = X_3 \subset \mathbb{P}^3$ is a cubic surface. Here, G has a faithful representation in $H^0(X, -K_X) = \mathbb{C}^4$. Assume that G is the Klein group or the Valentiner group. Then, G has no irreducible four-dimensional representations. So, the representation on $H^0(X, -K_X)$ is reducible. Hence, there exists a G-invariant hyperplane H. The intersection $H \cap X$ is a (G-invariant) smooth elliptic curve, because otherwise we get a fixed point $P \in H \cap X$, which is impossible. Since a simple group cannot act on an elliptic curve, we get a contradiction. Hence, $G \simeq \mathfrak{A}_5$.

We claim that the natural representation of G on $H^0(X, -K_X) = \mathbb{C}^4$ is irreducible. Indeed, otherwise there exists an invariant hyperplane $H \subset \mathbb{P}^3$ and, as above, we get a fixed point $P \in H \cap X$. Consider the representation of G on the Zariski tangent space $T_{P,X}$. Since $\mathfrak{A}_5$ has no irreducible two-dimensional representations, $\dim T_{P,X} = 3$, i.e. the point $P \in X$ is singular (and Du Val). Take a G-equivariant local embedding $(X, P) \hookrightarrow \mathbb{C}^3 = T_{P,X}$ into the corresponding affine chart. Let $f = f_2 + f_3$ be the local equation of X at P, where f_i is a homogeneous polynomial of degree i. We have $f_i \neq 0$, and $\mathfrak{A}_5$ acts on $\mathbb{C}^3$ so that f_i are invariants. Therefore, $\mathfrak{A}_5$ acts on $\mathbb{P}^2$ so that the locus $\{f_2 = 0\}$ is an invariant conic and $\{f_2 = 0\} \cap \{f_3 = 0\}$ is an invariant subset consisting of six points or less. On the other hand, any orbit of $\mathfrak{A}_5$ on a smooth rational curve contains at least 12 points. The contradiction proves our claim. Hence, there are no fixed points of G on X.

Thus, the representation of G on $H^0(X, -K_X) = \mathbb{C}^4$ is irreducible. This representation can be regarded as an invariant hyperplane $\sum x_i = 0$ in $\mathbb{C}^5$, where $\mathfrak{A}_5$ acts on $\mathbb{C}^5$ by permutations of coordinates. The ring of invariants $\mathbb{C}[x_1, \dots, x_5]^{\mathfrak{A}_5}$ is generated by $\sigma_1, \dots, \sigma_5, \delta$, where σ_i is the symmetric polynomial of degree i and δ is the discriminant. Therefore, the equation of our cubic surface $X \subset \mathbb{P}^3 \subset \mathbb{P}^4$ can be written as $\sigma_3 = \sigma_1^3 = 0$. This surface is smooth.

Case 4 ($\boldsymbol{K_X^2 \geqslant 4}$). In this case, the linear system $|-K_X|$ defines an embedding $X \hookrightarrow \mathbb{P}^{K_X^2}$. Let $\pi\colon X_0 \to X$ be the minimal resolution. Then, $\rho(X_0) = 10 - K_{X_0}^2 = 10 - K_X^2 \leqslant 6$. So, the number of singular points of X is at most 5. Moreover, if X has exactly five singular points, then $K_X^2 = 4$ and $\rho(X) = 1$. On the other hand, such a surface X does not exist (see, for example, [**4**, **9**]). Hence, X has at most four singular points.

Assume that G is the Klein group or the Valentiner group. Run the G-equivariant minimal model program (MMP) on X_0. We finally obtain a del Pezzo surface S with $\rho(S)^G = 1$ and $K_S^2 \geqslant 4$. So, $S \simeq \mathbb{P}^2$ [**8**]. Let $s := \rho(X/X_0)$, the number of exceptional curves of $X \to S$. Since the Klein group and the Valentiner group do not admit any embeddings to $\mathfrak{S}_5$ and do not act non-trivially on a rational curve, we see that $s \geqslant 6$. Therefore,

$$K_X^2 = K_{X_0}^2 = K_S^2 - s \leqslant 9 - s < 4,$$

a contradiction.

Thus, $G \simeq \mathfrak{A}_5$. Assume that X is singular and $X \not\simeq \mathbb{P}(1,1,2)$. Since X has at most four singular points, there exists a singular fixed point P of G on X. Note that there is a line $E_1 \subset X \subset \mathbb{P}^d$ passing through P, an image of a (-1)-curve $\ell \subset X_0$. Therefore, there is an orbit of lines $E_1, \dots, E_p$ passing through P, where $p \geqslant 5$. On the other hand, $X \subset \mathbb{P}^{K_X^2}$ is an intersection of quadrics (see, for example, [**10**]). Hence, there are at most four lines on $X \subset \mathbb{P}^d$ passing through P, a contradiction.

The last assertion follows by [**8**]. □

Definition 2.8. Let S be a normal projective surface and let Δ be an effective $\mathbb{Q}$-divisor on S. We say that (S, Δ) is a *weak log del Pezzo surface* if the pair (S, Δ) is Kawamata log terminal (klt) and the divisor $-(K_S + \Delta)$ is nef and big.

Remark 2.9.

(i) Let (S, Δ) be a weak log del Pezzo surface and let $\varphi\colon S \to S'$ be a birational contraction to a normal surface S'. Then, $(S', \varphi_*\Delta)$ is also a weak log del Pezzo surface.

(ii) For any weak log del Pezzo surface (S, Δ), the Mori cone $\mathrm{NE}(S)$ is polyhedral and generated by contractible extremal rays.

Construction 2.10. Under Notation 2.1, let $\pi\colon X_0 \to X$ be the minimal resolution. Run the G-equivariant MMP on X_0. We obtain a sequence of birational contractions of smooth surfaces $\phi_i\colon X_i \to X_{i+1}$. At the last step $\phi_p\colon X_p \to X_{p+1}$, we have either a conic bundle over $X_{p+1} \simeq \mathbb{P}^1$ or a contraction to a del Pezzo surface X_{p+1} with $\rho(X_{p+1})^G = 1$ (see [**8**]). Write $\pi^* K_X \equiv K_{X_0} + \Delta_0$, where Δ_0 is an effective π-exceptional $\mathbb{Q}$-divisor. Note that (X_0, Δ_0) is a weak log del Pezzo surface. Define, by induction, $\Delta_i = \phi_{i-1*}\Delta_{i-1}$. On each step of the MMP the above property is preserved: (X_i, Δ_i) is also a weak log del Pezzo surface. Since $\rho(X_p)^G = 2$, we see that there exists a G-equivariant extremal contraction $g\colon X_p \to Y$ such that g is different from ϕ_p. Thus, we get the following sequence of G-equivariant contractions:

$$\begin{tikzcd} X_0 \arrow[r, "\phi_0"] & X_1 \arrow[r, "\phi_1"] & \cdots \arrow[r, "\phi_{p-1}"] & X_p \arrow[r, "\phi_p"] \arrow[d, "g"'] & X_{p+1} \\ & & & Y & \end{tikzcd}$$

We distinguish the following cases.

(i) Y is a curve. Then, $Y \simeq \mathbb{P}^1$ and g is a conic bundle with $\rho(X_p/Y)^G = 1$. Moreover, in this case X_p is a smooth del Pezzo surface with $\rho(X_p)^G = 2$. Since the groups $\mathfrak{A}_6$ and $\mathrm{PSL}_2(7)$ cannot act non-trivially on a rational curve, we have $G \simeq \mathfrak{A}_5$.

(ii) Y is a smooth surface. The contraction g is then K-negative. In this case both Y and X_p are smooth del Pezzo surfaces with $\rho(X_p)^G = 2$ and $\rho(Y)^G = 1$. By Proposition 2.7 we have $G \simeq \mathfrak{A}_5$.

(iii) Y is a singular surface. Then, $(Y, g_*\Delta_p)$ is a weak log del Pezzo surface. In particular, Y is a del Pezzo surface with log terminal singularities. The group G transitively acts on $\mathrm{Sing}(Y)$.

Assume that both contractions g and ϕ_p are birational. Let $D = \sum_{i=1}^m D_i$ be the g-exceptional divisor, let $B_i = \phi_p(D_i)$ and let $B = \sum_{i=1}^m B_i$. Since $\rho(X_p)^G = 2$, we see that the group G acts transitively on $\{D_i\}$ and on $\{B_i\}$, so the curves B_i have the same anti-canonical degrees and self-intersection numbers. Since $\rho(X_{p+1})^G = 1$, the divisor B is ample and proportional to $-K_{X_{p+1}}$. Hence, B is connected. Assume that $\mathrm{Sing}(B) = \emptyset$. Then, B is an irreducible curve. Since B is rational, by the genus formula $K_{X_{p+1}} + B$ is negative. This is possible only if $X_{p+1} \simeq \mathbb{P}^2$. Thus, we have the following.

Claim 2.11. *In the above notation, either*

(i) $\mathrm{Sing}(B) \neq \emptyset$ *or*

(ii) $X_{p+1} \simeq \mathbb{P}^2$ *and B is a smooth irreducible curve of degree less than or equal to* 2.

Construction 2.12. Under Notation 2.1, let $\rho(X)^G = 1$. Assume that X is singular. Consider the minimal resolution $\mu \colon Y \to X$ and let $R = \sum_{i=1}^n R_i$ be the exceptional divisor. The action of G lifts naturally to Y. Write

$$\mu^* K_X = K_Y + \sum \alpha_i R_i,$$

where $0 \leqslant \alpha_i < 1$. Fix a component, say $R_1 \subset R$, and let $R' = R_1 + \cdots + R_k$ be its G-orbit. We can contract all the curves in $R - R'$ over X:

$$\mu \colon Y \xrightarrow{\eta} \bar{X} \xrightarrow{\phi} X.$$

Then, $\rho(\bar{X})^G = 2$ and

$$\phi^* K_X = \eta_* K_Y = K_{\bar{X}} + \Delta, \quad \text{where } \Delta := \eta_* \sum \alpha_i R_i.$$

Therefore, $(\bar{X}, \Delta)$ is a weak log del Pezzo surface. Let $\psi\colon \bar{X} \to X'$ be a (unique) $K_{\bar{X}} + \Delta$-negative contraction. Clearly, $\psi \neq \phi$, ψ does not contract any component of Δ, and ψ is also $K_{\bar{X}}$-negative. We get the following G-equivariant diagram:

$$\begin{tikzcd} & \bar{X} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ X & & X' \end{tikzcd}$$

where X' is either a smooth rational curve or a del Pezzo surface with at worst log terminal singularities and $\rho(X')^G = 1$.

For a normal surface V, denote by $\mathrm{d}(V)$ the Picard number of its minimal resolution. In our situation, $\psi \circ \eta$ is a non-minimal resolution of singularities (because $-K_{\bar{X}}$ is ψ-ample). Hence, $\mathrm{d}(X') < \mathrm{d}(X)$.

The following procedure is well known. It is called the '2-ray game'.

Construction 2.13. Apply Construction 2.12 several times. We get the following sequence of G-equivariant birational morphisms:

$$\begin{tikzcd} & \bar{X}_0 \arrow[dl, "\phi_0"'] \arrow[dr, "\psi_0"] & & & & \bar{X}_d \arrow[dl, "\phi_d"'] \arrow[dr, "\psi_d"] & \\ X = X_0 & & X_1 & \cdots & X_d & & X_{d+1} \end{tikzcd}$$

Since $\mathrm{d}(X_i) > \mathrm{d}(X_{i+1})$, the process terminates. Thus, we end up with X_{d+1}, which is either a smooth curve or a smooth del Pezzo surface with $\rho(X_{d+1})^G = 1$. Recall that each X_i for $i = 1, \ldots, d$ is a del Pezzo surface with log terminal singularities and $\rho(X_i)^G = 1$.

Note that at each step the extraction ϕ_i is not unique; this obviously depends on the choice of R' (as in Notation 2.12). For our purposes it is convenient to choose R' in one of the following ways.

(1) R' is the orbit of exceptional curves over non-fixed points with maximal codiscrepancy.

(2) $\phi_i(R')$ is a fixed point $P \in X_i$. By Lemma 2.3, R' is a unique exceptional curve over P.

3. The Valentiner and Klein groups

In this section we prove our main theorem in the case where $G = \mathfrak{A}_6$ or $\mathrm{PSL}_2(7)$ (i.e. G is the Valentiner or Klein group).

Proposition 3.1. *Assume that the surface X is singular and that G is either the Klein group or the Valentiner group. Then, X_p has only cyclic quotient singularities of type $\frac{1}{3}(1,1)$.*

Proof. Apply Construction 2.10. By our assumption we get the case (iii), i.e. the contraction g is birational and Y is a singular del Pezzo surface (with log terminal singularities and $\rho(Y)^G = 1$). Moreover, the contraction ϕ_{p+1} is also birational and the exceptional loci of g and ϕ_p are reducible (because G cannot act non-trivially on a rational curve). Write

$$g^*K_Y = K_{X_p} + \sum_{i=1}^{m} \alpha_i D_i \quad \text{and} \quad D = \sum_{i=1}^{m} D_i,$$

where, as above, the D_i are g-exceptional curves. Since the group G acts transitively on $\{D_i\}$, we have

$$\alpha_1 = \cdots = \alpha_m := \alpha \quad \text{and} \quad -D_1^2 = \cdots = -D_m^2 := n.$$

Furthermore, by the classification of log terminal singularities [**5**], the exceptional divisor over every singular point is either a pair of $(-n)$-curves or a single $(-n)$-curve (otherwise, G cannot interchange the D_i).

We claim that $n \leqslant 3$. Indeed, assume, on the contrary, that $n > 3$. Note that ϕ_p is the blow-up of points in $\operatorname{Sing}(B)$. Let E be a ϕ_p-exceptional curve on X_p. Then,

$$0 > K_Y \cdot g_*E = g^*K_Y \cdot E = \left(K_{X_p} + \sum_{i=1}^{m} \alpha D_i\right) \cdot E \geqslant -1 + 2\alpha.$$

Therefore, $\alpha < \frac{1}{2}$. On the other hand,

$$0 = g^*K_Y \cdot D_j = \left(K_{X_p} + \sum_{i=1}^{m} \alpha D_i\right) \cdot D_j \geqslant n - 2 - \alpha n, \quad n\alpha \geqslant n - 2.$$

Hence, $n \leqslant 3$. Moreover, if $n = 3$, then $\alpha D_j \cdot \sum D_i = -1$, and so $D_j \cdot \sum_{i \neq j} D_i < 1$. This means that $D_i \cap D_j = 0$ for $i \neq j$. Hence, every singular point on Y is either Du Val of type A_l, $l \leqslant 2$, or a cyclic quotient singularity of type $\frac{1}{3}(1,1)$. By Proposition 2.7 we are done. □

By Proposition 2.7 we may assume that the singularities of X are worse than Du Val. Apply Construction 2.10. We get the case (iii). In particular, $\operatorname{Sing}(Y) \neq \emptyset$ and is a del Pezzo surface with log terminal singularities and $\rho(Y)^G = 1$. Moreover, $X_{p+1} \simeq \mathbb{P}^2$ or S_2^{k}, and the latter is possible only for $G = \mathrm{PSL}_2(7)$ (see [**8**]). As in the proof of Proposition 3.1, let $D = \sum D_i$ be the g-exceptional divisor and let $B_i := \phi_p(D_i)$. By Proposition 3.1, every singular point on Y is of type $\frac{1}{3}(1,1)$, i.e. D consists of disjoint (-3)-curves.

We first consider the case $X \simeq \mathbb{P}^2$. Then, $B_i \cap B_j \neq \emptyset$, and so ϕ_p is a blow-up of points in $B_i \cap B_j$, $i \neq j$. We claim that every curve B_i is smooth and there are at most two components of B passing through every point $P \in \mathbb{P}^2$. Indeed, assume the converse. Then,

$$0 > g^*K_Y \cdot E = \left(K_{X_p} + \frac{1}{3}\sum D_i\right) \cdot E \geqslant -1 + 3 \cdot \tfrac{1}{3} = 0.$$

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Therefore, every B_i is smooth. Furthermore, since the curves B_i are rational, $k \leqslant 2$. If $k = 1$, then the B_i are lines, and on every line we blow up four points. Hence, there are five of these lines, a contradiction.

Finally, consider the case $k = 2$. The B_i are then smooth conics, and on every conic we blow up seven points. It is easy to see that the number of points of intersection of conics is divisible by four, a contradiction.

Now consider the case $X_{p+1} \simeq S_2^{\mathrm{k}}$. Then, G is the Klein group. Let $r := \rho(X_p/X_{p+1})$. Recall that m is the number of the D_i. Then, by Noether's formula,

$$0 < K_Y^2 = g^* K_Y^2 = K_{X_p}^2 - D^3 = 10 - \rho(X_p) + \frac{m}{3} = 2 - r + \frac{m}{3}.$$

Since $m \leqslant r + 7 = \rho(X_p) - 1$, we see that

$$0 < 2 - r + \frac{m}{3} \leqslant -\frac{2m+1}{3} < 0,$$

a contradiction.

4. The icosahedral group

It remains to consider the case $G \simeq \mathfrak{A}_5$. In addition to 2.1 we assume that $\rho(X)^G = 1$. By Proposition 2.7 we may assume also that the singularities of X are worse than Du Val.

By [**8**], there are three cases: $X_{d+1} \simeq \mathbb{P}^1$, $X_{d+1} \simeq \mathbb{P}^2$ or X_{d+1} is a del Pezzo surface S_5 of degree 5.

Lemma 4.1. *Let V be a normal surface and let $C \subset V$ be a smooth curve such that $(K_V + C) \cdot C < 0$. Then, V has at most three singular points on C.*

Proof. By the adjunction formula [**13**] we have that

$$(K_V + C)|_C = K_C + \mathrm{Diff}_C,$$

where Diff_C is the difference, an effective $\mathbb{Q}$-divisor supported in singular points of V lying on C. Moreover, the coefficients of Diff_C are $\geqslant 1/2$. Since, by our conditions, $\deg \mathrm{Diff}_C < -\deg K_C \leqslant 2$, we get that Diff_C is supported in at most three points. □

Lemma 4.2. *For any i, the exceptional divisor of ψ_i has at least five connected components.*

Proof. Let E be the ψ_i-exceptional divisor. Since $\rho(\bar{X}_i)^G = 2$ and $G = \mathfrak{A}_5$ is a simple group, E is either connected or the number of connected components of ψ_i is greater than or equal to 5. Assume that E is connected. Since E is a tree of rational curves, it is irreducible. So, $E \simeq \mathbb{P}^1$. By Lemma 2.3 the action of G on E is non-trivial. If $\bar{X}_i$ is smooth along E, then E is a (-1)-curve and $\psi_i(E)$ is a G-fixed smooth point. This contradicts Lemma 2.3. Therefore, $\bar{X}_i$ has at least five singular points on E. This contradicts Lemma 4.1. □

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Corollary 4.3. *If there exists a G-fixed point on X_i for some i, then there exists a fixed point of G on X_j for any $j \leqslant i$.*

Proof. Assume that X_i has a fixed point of G, say P. By Lemma 4.2, ψ_{i-1} is an isomorphism over P. So, $\phi_{i-1}(\psi_{i-1}(P))$ is a fixed point of G on X_{i-1}. □

Lemma 4.4. *Suppose that $\rho(X)^G = 1$ and G has no fixed points on X. Then, X_{d+1} is not a curve.*

Proof. Assume that $X_{d+1} \simeq \mathbb{P}^1$. By Corollary 4.3, the group G has no fixed points on X_d. We now choose ϕ_i according to Construction 2.13 (1). By Lemma 2.4, the surface $\bar{X}_d$ is singular. Since $\mathfrak{A}_5$ is a simple group and G has no fixed points on X_d, we see that the exceptional divisor of ϕ_d consists of at least five curves $D_1, \dots, D_k$, where $k \geqslant 5$. Let f be a general fibre of ψ_d. Then,

$$0 > f \cdot \left(\pi'^{*} K_{\bar{X}_d} + \alpha \sum_{i=1}^{k} D_i\right) \geqslant -2 + k\alpha,$$

where α is the codiscrepancy of D_i. By Lemma 2.5 and Proposition 2.7 we see that $\alpha = \frac{1}{3}$ and $k = 5$. By [**1**, Theorems 1–5] there exists an irreducible non-singular curve $C \in |-3K_{X_d}| \neq \emptyset$. Set $\bar{C} = \phi_d^{*} C - \sum r_i D_i$, where $r_i \geqslant 0$. Since C is not a rational curve, we see that C is not a section of ψ_d. Then,

$$K_{\bar{X}_d} + \frac{1}{3}\bar{C} + \sum_{i=1}^{k} \left(\alpha + \frac{r_i}{3}\right) D_i \equiv 0.$$

Hence

$$0 = f \cdot \left(K_{\bar{X}_d} + \frac{1}{3}\bar{C} + \sum_{i=1}^{k} \left(\alpha + \frac{r_i}{3}\right) D_i\right) \geqslant -2 + \frac{2}{3} + \frac{5}{3} = \frac{1}{3},$$

a contradiction. □

Claim 4.5. *A smooth del Pezzo surface of degree 5 contains exactly five pencils of conics.*

Proof. Each pencil of conics $|C|$ has exactly three degenerate members that are pairs of meeting lines. Since a del Pezzo surface of degree 5 contains 15 such pair of lines, we are done. □

Lemma 4.6. *Suppose that $\rho(X)^G = 1$ and G has no fixed points on X. Then, $X \simeq \mathbb{P}^2$ or X is a del Pezzo surface S_5 of degree 5.*

Proof. By Lemma 4.4 and [**8**] we have $X_{d+1} \simeq \mathbb{P}^2$ or $X_{d+1} \simeq S_5$. We now choose ϕ_i according to Construction 2.13 (1). By Proposition 2.7 we see that X_d has at least one non-Du Val singularity. Hence, the exceptional divisor of ϕ_d is one orbit $D_1, \dots, D_k$, where $k \geqslant 5$. Let $D = \sum_{i=1}^{k} D_i$ and $m := -D_i^2$. Since the divisor

$$-\phi_d^* K_{X_d} \equiv -K_{\bar{X}_d} - \sum_{i=1}^{k} \alpha D_i$$

is nef and big, so is

$$-\psi_{d*}\phi_d^* K_{X_d} \equiv -K_{X_{d+1}} - \sum_{i=1}^{k} \alpha\psi_d(D_i). \tag{$*$}$$

By Lemma 2.5, $\alpha \geqslant \frac{1}{3}$. Consider the following two cases.

Case 1 ($X_{d+1} \simeq \mathbb{P}^2$). By the above the divisor, $3H - \sum \alpha\psi_d(D_i)$ is nef and big, where H is a line. Assume that $\deg \psi_d(D_i) \geqslant 2$. Then, $3 > 2k\alpha$. Since $k \geqslant 5$ and $\alpha \geqslant \frac{1}{3}$, we see that $2k\alpha \geqslant \frac{10}{3}$, a contradiction.

Hence, $\psi_d(D_i)$ are lines. Then, $k = 5$ or 6. Assume that $k = 5$. There then exists an orbit of five lines on $\mathbb{P}^2$. Note that there exists an invariant conic $C \subset \mathbb{P}^2$. The divisor $\sum \psi_d(D_i)$ meets C in at most 10 points. Hence, there exists an orbit on C consisting of at most 10 points. However, the order of any orbit on $C \simeq \mathbb{P}^1$ is at least 12, a contradiction.

Thus, $k = 6$. Hence, the lines $\psi_d(D_i)$ are in general position, i.e. every line contains five points of intersection. Therefore, $m \geqslant 4$, and so $3 > k\alpha \geqslant 3$. Again, we have a contradiction.

Case 2 (X_{d+1} is a del Pezzo surface S_5 of degree 5). Assume that $\psi_d(D_i)$ are (-1)-curves. Then, $k = 10$ and, by $(*)$,

$$1 = -K_{S_5} \cdot E_i > 3\alpha - \alpha = 2\alpha.$$

Thus, $\alpha < \frac{1}{2}$. It is well known that on a del Pezzo surface of degree 5 every (-1)-curve meets three other (-1)-curves. Hence, $m \geqslant 4$, a contradiction.

Assume that $(\psi_d(D_i))^2 \geqslant 1$. Then $5 = K_{S_5}^2 > 3k\alpha$. Since $k \geqslant 5$ and $\alpha \geqslant \frac{1}{3}$, we see that $3k\alpha \geqslant 5$, a contradiction.

Therefore, $(\psi_d(D_i))^2 = 0$, i.e. $\psi_d(D_i)$ is a conic. Then $2k\alpha < 5$. Therefore, $m = 3$ and $k = 5$ or 6. By Claim 4.5, there exist only five linear systems of conics. If $\psi_d(D)$ contains two conics of one pencil, then $\psi_d(D)$ has at least 10 components, a contradiction. Therefore, $\psi_d(D)$ consists of five conics contained in different linear systems. Every component of $\psi_d(D)$ meets four other components. Then, ψ_d extracts four points on every component of $\psi_d(D)$. Hence, $m = 4$, a contradiction. □

Lemma 4.7. *Suppose that $\rho(X)^G = 1$. Assume that G has a fixed point P on X. Choose ϕ_0 as in Construction 2.13 (2). Assume that $X_1 = \mathbb{P}^1$. Then, $\bar{X}_0$ is smooth. Moreover, $\bar{X}_0 \simeq \mathbb{F}_n$ and $X \simeq \mathbb{P}(1, 1, n)$.*

Proof. In our case, $\psi_0\colon \bar{X}_0 \to X_1 = \mathbb{P}^1$ is a rational curve fibration. Let D_0 be a unique exceptional curve of ϕ_0. Note that D_0 is contained in the smooth locus of $\bar{X}_0$. Assume that D_0 is a section. There then exist no singular fibres. Hence, by Lemma 2.4, we see that $\bar{X}_0 \simeq \mathbb{F}_n$. So, we may assume that D_0 is not a section of ψ_0.

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Assume that P is not a Du Val singularity. Let $m \geqslant 2$ be the degree of the restriction $\psi_0|_{D_0}\colon D_0 \to \mathbb{P}^1$. Then, by the Hurwitz formula, $-2 = -2m + \deg B$, where B is the ramification divisor. Thus, $\deg B = 2m - 2$. The divisor B is G-invariant. Hence, $\deg B \geqslant 12$ and $m \geqslant 7$. Let f be a general fibre of ψ_0. Then,

$$0 < -\phi_0^* K_X \cdot f = -(K_{\bar{X}} + \alpha D_0) \cdot f = 2 - m\alpha,$$

where α is the codiscrepancy of D_0. Hence, $m < 2/\alpha$. By Lemma 2.5, $\alpha \geqslant \frac{1}{3}$, a contradiction.

Therefore, P is a Du Val singularity. By Proposition 2.7, X also has a non-Du Val singular point. Apply Construction 2.13 (1) to $\bar{X}_0$ over the base $\mathbb{P}^1$:

$$\begin{tikzcd}
& \tilde{X}_0 \arrow[dl, "\xi_0"'] \arrow[dr, "\eta_0"] & & & & \tilde{X}_e \arrow[dl, "\xi_e"'] \arrow[dr, "\eta_e"] & \\
\bar{X}_0 & & \bar{X}_1 & \cdots & \bar{X}_e & & \bar{X}_{e+1}
\end{tikzcd}$$

Here, $\bar{X}_{e+1}$ is smooth and $\bar{X}_e$ is singular. We claim that $\bar{X}_{e+1}$ has no section M with $M^2 = -p$, where $p \geqslant 2$. Indeed, let $\pi\colon Y \to X_0$ be the minimal resolution and let D_0 be a unique exceptional curve over P. Note that η_0 contracts curves meeting D_0. Therefore, $\bar{X}_{e+1}$ has no invariant $(-p)$-curves M such that M is a section, where $p \geqslant 2$.

By Lemma 2.4 we have that $\bar{X}_{e+1} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. By Proposition 2.7, the singularities of $\bar{X}_e$ are worse than Du Val. Let $D_1, \dots, D_k$ be the exceptional curves of ξ_e, where $k \geqslant 5$. Then,

$$0 < -\Big(K_{\tilde{X}_e} + \alpha \sum D_k\Big) \cdot f = 2 - k\alpha,$$

where f is a fibre of the projection $\bar{X}_e \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and α is the codiscrepancy of D_i. By Lemma 2.5 we have $\alpha = \frac{1}{3}$, every non-Du Val singularity on $\bar{X}_e$ has type $\frac{1}{3}(1,1)$, and $k = 5$. According to [**1**, Theorems 1–5] there exists an irreducible non-singular curve $C \in |-3K_{\bar{X}_e}| \neq \emptyset$. Set $\tilde{C} = \xi_e^* C - \sum r_i D_i$, where $r_i \geqslant 0$. Since C is not rational, C is not a section. Then,

$$K_{\tilde{X}_e} + \frac{1}{3}\tilde{C} + \sum \left(\alpha + \frac{r_i}{3}\right) D_i \equiv 0.$$

Hence,

$$0 = f \cdot \left(K_{\tilde{X}_e} + \frac{1}{3}\tilde{C} + \sum \left(\alpha + \frac{r_i}{3}\right) D_i\right) \geqslant -2 + \frac{2}{3} + \frac{5}{3} = \frac{1}{3},$$

a contradiction. □

Lemma 4.8. *Suppose that $\rho(X)^G = 1$ and G has exactly one fixed point on X. Then either $X \simeq \mathbb{P}(1,1,2n)$ or $X \simeq \widetilde{\mathbb{P}^2}_{k,s}$.*

Proof. Let $P \in X$ be the fixed point of G. Then, by Lemma 2.3, $P \in X$ is of type $(1/r)(1,1)$ for some $r \geqslant 2$. Consider the following two cases.

Case 1 ($r \geqslant 11$). We choose ϕ_0 as in Construction 2.13 (2). Then, X_1 has no fixed points. By Lemmas 4.6, 4.7 and 2.4, we may assume that $X_1 \simeq \mathbb{P}^2$ or $X_1 \simeq S_5$. Assume that curves of the exceptional divisor of ψ_0 contain two singular points or one non-Du Val singular point. Let $\pi\colon Y \to X$ be the minimal resolution, let $D = \sum D_i$ be the exceptional divisor and let D_0 be a unique invariant curve of the exceptional divisor. We have that $D_0^2 = -r \leqslant -11$ and there exists a morphism $\pi'\colon Y \to \bar{X}_0$ such that $\pi = \phi_0 \circ \pi'$. Hence, by Lemma 2.6,

$$0 < -K_X \cdot \pi_* E_i = -E_i \cdot (K_Y + D^\sharp) \leqslant 0,$$

where E_i is a (-1)-curve contracted by $\psi_0 \circ \pi'$ and $D^\sharp$ is the codiscrepancy divisor (see 2.2), a contradiction.

Therefore, every contracted curve contains at most one Du Val singular point of type A_p. Assume that X_1 is a del Pezzo surface of degree 5. Let $k := \rho(\bar{X}_0/X_1)$ and let $l = (\psi_* D_0)^2$. Then,

$$K_X^2 = 10 - \rho(Y) + k(p+1) - l - 4 + \frac{4}{k(p+1) - l} = 1 - l + \frac{4}{k(p+1) - l}.$$

So, $l = -1, 0$ or 1. On the other hand, S_5 has no invariant curves with self-intersection number -1, 0 or 1, a contradiction. Therefore, $X_1 \simeq \mathbb{P}^2$. In this case $\psi_* D_0$ is a conic. Hence, $X \simeq \tilde{\mathbb{P}^2}_{k,s}$.

Case 2 ($r \leqslant 10$). Assume that X has a singular fixed point and other singular points that are Du Val singular points. We then choose ϕ_0 as in Construction 2.13 (2). Then, $X_1 \simeq \mathbb{P}^2$ or $X_1 \simeq S_5$ and all non-fixed singular points have type A_p. Hence,

$$0 < K_X^2 = 10 - \rho(X_1) - (p+1)k + m - 4 + \frac{4}{m},$$

where $k \in \{12, 20, 30, 60\}$. Therefore, $k = 12$, $p = 0$ and $X_1 \simeq \mathbb{P}^2$. As above, we have $X \simeq \tilde{\mathbb{P}^2}_{k,0}$.

We may now assume that X has a fixed singular point and at least one non-Du Val non-fixed singular point.

Apply Construction 2.13. We may construct ϕ_i as in (1) therein. Then, X_d is a surface with one fixed point of G. Since $r \leqslant 10$, we see that every ψ_i does not contract curves containing the fixed point of G.

Consider the case where $X_d \simeq \mathbb{P}(1,1,2n)$. Assume that there exist non-Du Val singularities on X_{d-1} other than P. Let $D_1, \dots, D_k$ be the exceptional curves of ϕ_{d-1}. Then,

$$-K_{\mathbb{P}(1,1,2n)} = (2n+2)l > \alpha \psi_{d-1*} \sum D_i,$$

where l is a generator of the Weil divisor class group and α is the codiscrepancy of D_i. Since ψ_i does not contract curves containing the fixed point of G, we see that $\psi_d(D_i)$ also does not contain the fixed point of G. Then $l \cdot \psi_{d-1}(D_i) \geqslant 1$. We obtain $n = 1$, $m = 3$ and $\alpha = \frac{1}{3}$. Hence, $\psi_{d-1}(D_i) \geqslant 2l$, a contradiction. Therefore, X_{d-1} has exactly one non-Du Val singularity. Denote it by P.

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We now choose ϕ'_{d-1} according to Construction 2.13 (2). By Proposition 2.7, X'_d is a smooth del Pezzo surface. We obtain $n = 1, 2$ or 3. Assume that $n = 2$ or 3. Then,

$$0 < K^2_{X_{d-1}} \leqslant 10 - \rho(Y) + \tfrac{8}{3} \leqslant -15 + \tfrac{8}{3} < 0,$$

a contradiction. Hence, $n = 1$. Then,

$$0 < K^2_{X_{d-1}} = 10 - \rho(Y) \leqslant 0,$$

a contradiction.

Consider the case where $X_d \neq \mathbb{P}(1, 1, 2n)$. Let $\phi_d\colon \bar{X}_d \to X_d$ be the minimal resolution of X_d and let D be the exceptional curve. Let $\psi_d\colon \bar{X}_d \to X_{d+1}$ be the contraction of another G-equivariant extremal ray. Suppose that $X_{d+1} \simeq S_5$. Then,

$$0 < K^2_{X_d} = 10 - \rho(\bar{X}_d) + l - 4 + \frac{4}{l}.$$

Assume that $l \geqslant 4$. Then, $(\psi_d(D))^2$ is equal to -1, 0 or 1. On the other hand, S_5 has no invariant curves whose self-intersection number is equal to -1, 0 or 1, a contradiction.

Suppose that $l = 3$. Then,

$$0 < K^2_{X_d} = 10 - \rho(\bar{X}_d) + \tfrac{1}{3}.$$

We see that every exceptional curve of ψ_d meets D in at most two points. Note that there exists some orbit $P_1, \dots, P_j$ consisting of points of intersections of exceptional curves with D. Since the order of any orbit on $\mathbb{P}^1$ is at least 12, we see that the number of exceptional curves is at least six. Then, $\rho(\bar{X}_d) \geqslant 11$, a contradiction.

Assume that $l = 2$. Then, $0 < K^2_{X_d} = 10 - \rho(\bar{X}_d)$. We have that $\rho(\bar{X}_d) \geqslant 5 + r$, where $r = \rho(\bar{X}_d/X_{d+1}) \geqslant 5$, a contradiction.

Therefore, $X_{d+1} \simeq \mathbb{P}^2$. Suppose that $l \geqslant 4$. Then, $\psi_d(D)$ is a conic. On the other hand, ψ_d is a blow-up of at least twelve points on $\psi_d(D)$, a contradiction.

Finally, if $\psi_d(D)$ is singular, then ψ_d is the blow-up of singular points $P_1, \dots, P_k$ of the curve $\psi_d(D)$. Let $q := \operatorname{mult}_{P_i}(\psi_d(D)) \geqslant 2$ and $\psi_d(D) \equiv tH$, where H is a line on $\mathbb{P}^2$. Then, $D^2 = t^2 - kq$. By the genus formula, we have that

$$k\frac{q(q-1)}{2} = \frac{(t-1)(t-2)}{2}.$$

Hence, $t^2 = kq^2 + 3t - kq - 2$, and so $D^2 = kq(q-2) + 3t - 2 > 0$, a contradiction. □

Lemma 4.9. *Assume that $\rho(X)^G = 1$ and X has exactly two singular points. Then, $X \simeq F_{2n+k,k-2n,1}$.*

Proof. Since G is a simple group, we see that both singular points are fixed points. By Lemma 2.3 there exist exactly two fixed points P_1 and P_2 on X, and these points have types $(1/r_1)(1, 1)$ and $(1/r_2)(1, 1)$ for some r_1, r_2. We may assume that $r_1 \geqslant r_2$. Apply Construction 2.13, choosing ϕ_0 as in (2) therein with $P = P_1$. By Lemma 4.8 we may assume that $X_1 \simeq \mathbb{P}(1, 1, 2n)$ or $X_1 \simeq \tilde{\mathbb{P}^2}_{k,0}$. Let N be a unique exceptional curve of ϕ_0. If the exceptional curves of ψ_0 contain P_2, then $r_2 \geqslant 5$. Since $r_1 \geqslant r_2$, we see that $\psi_0(N)$ does not contain the singular point. Assume that $X_1 \simeq \tilde{\mathbb{P}^2}_{k,0}$. Let $m = -N^2$. Since $-\phi_0^*(K_X) = -K_{\bar{X}_0} + (m-2)N/m$ is nef and big, we see that $-K_{X_1} > (m-2)\psi_0(N)/m$. Since $\psi_0(N)$ does not contain a singular point, we see that $\psi_0(N) \cdot E_i' \geqslant 1$, where E_i' is the image of the (-1)-curve. Then,

$$-K_{X_1} \cdot E_i' = 1 - \frac{k-6}{k-4} > \frac{m-2}{m}.$$

On the other hand, $m \geqslant k-4 \geqslant 8$, a contradiction.

Therefore, $X_1 \simeq \mathbb{P}(1,1,2n)$. Since ψ_0 is the blow-up of one orbit, we see that $N^2 = 2n-k$, where $k = 12, 20, 30$ or 60. Hence, $X \simeq F_{2n,k-2n,1}$. □

Lemma 4.10. *Assume that $\rho(X)^G = 1$. Then, G has at most two fixed points on X.*

Proof. Assume that G has three fixed points $P_1, P_2, P_3 \in X$. Apply Construction 2.13, choosing ϕ_i as in (1) therein. By Lemma 2.3 we obtain a surface X_{d-1} with exactly three singular points. Let $\pi\colon Y \to X_{d-1}$ be the minimal resolution and let D_1, D_2, D_3 be the exceptional curves. Set $n_i := D_i^2$.

By Lemma 2.3 all n_i are even. On the other hand, there exists a rational curve fibration $\Phi\colon Y \to \mathbb{P}^1$ such that D_1, D_2, D_3 are horizontal curves. Assume that X_{d-1} has no Du Val singularities. Then,

$$0 < -f \cdot \left(K_Y + \frac{n_1-2}{n_1}D_1 + \frac{n_2-2}{n_2}D_2 + \frac{n_3-2}{n_3}D_3\right) = -f \cdot \pi^*(K_{X_{d-1}}),$$

where f is a generic fibre of Φ. Hence,

$$\frac{n_1-2}{n_1} + \frac{n_2-2}{n_2} + \frac{n_3-2}{n_3} < 2.$$

We obtain $n_1 = 4$ and $n_2 \leqslant 6$.

We now apply Construction 2.13, choosing ϕ_{d-1} as in (2) therein with $P = P_3$. We see that the exceptional divisor of ψ_{d-1} does not contain any singular point. Then, X_d is a del Pezzo surface with two fixed points of G. Since $n_1 = 4$ and $n_2 \leqslant 6$, we see that X_d is not isomorphic to $F_{2n,k-2n,1}$, a contradiction.

Therefore, $n_1 = 2$. Assume that $n_2 = 2$. We now choose ϕ_{d-1} as in Construction 2.13 (2) with $P = P_3$. We see that the exceptional divisor of ψ_{d-1} is contained in the smooth locus. Hence, X_d is a del Pezzo surface with two Du Val singular points, a contradiction with Proposition 2.7. Therefore, $n_3 \geqslant n_2 \geqslant 4$.

We now choose ϕ_{d-1} as in Construction 2.13 (2) with $P = P_2$. We claim that the components of the exceptional divisor of ψ_{d-1} do not contain singular points. Indeed, assume that the exceptional divisor of ψ_{d-1} contains P_1. Then, X_d is a del Pezzo surface with a fixed smooth point, a contradiction.

Assume that the exceptional divisor of ψ_{d-1} contains P_3. Hence, there exists a (-1)-curve E on Y meeting both D_2 and D_3. Therefore,

$$\begin{aligned} E\cdot\pi^*K_{X_{d-1}} &= E\cdot\left(K_Y+\frac{n_2-2}{n_2}D_2+\frac{n_3-2}{n_3}D_3\right)\\ &\geqslant -1+\frac{n_2-2}{n_2}+\frac{n_3-2}{n_3}\\ &\geqslant 0, \end{aligned}$$

a contradiction.

Thus, if we choose ϕ_{d-1} as in Construction 2.13 (2) with $P=P_2$, then the exceptional divisor of ψ_{d-1} does not contain the singular points. The same holds for P_3. Note that, after the contraction of another G-equivariant extremal ray, we obtain $F_{2n,k-2n,1}$. Hence, $n_2, n_3 \in \{10, 18, 28, 58\}$. We now choose ϕ_{d-1} as in Construction 2.13 (2) with $P=P_1$. Then, $X_d \simeq F_{2n,k-2n,1}$. Assume that the exceptional divisor of ψ_{d-1} contains a singular point. Then, $2n=n_2-h$ and $k-2n=n_3$ or $2n=n_2$ and $k-2n=n_3-h$, where $h\in\{12,20,30,60\}$. Since $k\in\{12,20,30,60\}$ and $29\geqslant n\geqslant 1$, this case is impossible. Hence, the exceptional divisor of ψ_{d-1} does not contain any singular point. We obtain $F_{2n,k-2n,1}$, where $2n=10, 18, 28$ or 58. Hence, $n_2=n_3=10$. Then,

$$K^2_{X_{d-1}} = K_Y^2+\frac{(n_2-2)^2}{n_2}+\frac{(n_3-2)^2}{n_3} = 22-\rho(Y)+\frac{4}{5}.$$

We now compute $\rho(Y)$. We have that $\rho(F_{10,10,1})=20$. Then, $\rho(\bar{X}_{d-1})=20+s$, where $s=\rho(\bar{X}_{d-1}/F_{10,10,1})\geqslant 5$ is the number of components of the exceptional divisor of ψ_{d-1}. Then, $\rho(Y)=22+s$. Hence, $K^2_{X_{d-1}}<0$, a contradiction. □

Lemma 4.11. *Assume that $\rho(X)^G=1$ and G has at least two fixed points on X. Then, $X\simeq F_{2n,ak-2n,a}$.*

Proof. By Lemma 4.10 the group G has exactly two fixed points $P_1, P_2\in X$, and by Lemma 4.9 we may assume that X also has a non-fixed singular point, say Q.

First, we consider the case where Q is not Du Val. Apply Construction 2.13, choosing ϕ_i as in (1) therein. Then, $X_d\simeq F_{2n,k-2n,1}$. Let $D_{0,0},\dots,D_{0,l}$ be the exceptional curves of ϕ_0. Then, $-(K_{\bar{X}_0}+\alpha\sum_{j=1}^{l}D_j)$ is nef and big, where $\alpha\geqslant\frac{1}{3}$ and $l\geqslant 5$. Let $\tilde{D}_{i,0},\dots,\tilde{D}_{i,l}$ be the proper transform of $D_{0,0},\dots,D_{0,l}$ on X_i. Hence, the divisor $-(K_{X_i}+\alpha\sum_{j=1}^{l}\tilde{D}_{i,j})$ is nef and big. Assume that a curve of the exceptional divisor of ψ_i contains a fixed point P and meets $D_{i-1,j}$. Since the exceptional curves of ψ_i are contained in one orbit, we see that every curve of the exceptional divisor of ψ_i contains a fixed point P. Hence, P has type $(1/m)(1,1)$, where $m\geqslant 7$. Let $\pi_i\colon Y_i\to X_i$ be the minimal resolution. The divisor $-(K_Y+((m-2)/m)D+\alpha\sum\bar{D}_{i,j})$ is then nef and big, where D is a unique exceptional curve over P and $\bar{D}_{i,j}$ is the proper transform of $\tilde{D}_{i,j}$. On the other hand, there exists a (-1)-curve E on Y_i such that E meets D and $\bar{D}_{i,j}$. Hence, $0<1-(m-2)/m-\alpha<0$, a contradiction. Therefore, $\tilde{D}_{d,0},\dots,\tilde{D}_{d,l}$ do not contain a fixed point. Let $\pi_d\colon Y_d\to F_{2n,k-2n,1}$ be the minimal resolution. Let D_1 and D_2 be exceptional divisors over fixed points of G and let $\bar{D}_{d,0},\dots,\bar{D}_{d,l}$ be the proper transform of $\tilde{D}_{d,0},\dots,\tilde{D}_{d,l}$. There exists a G-equivariant rational curve fibration $\Phi\colon Y_d\to\mathbb{P}^1$ such that D_1 and D_2 are sections. Since the curves $\tilde{D}_{d,0},\dots,\tilde{D}_{d,l}$ do not contain fixed points, we see that $\bar{D}_{d,0},\dots,\bar{D}_{d,l}$ are horizontal curves. Hence,

$$\begin{aligned}0&<-f\cdot\left(K_{Y_d}+\frac{2n-2}{2n}D_1+\frac{k-2n-2}{k-2n}D_2+\alpha\sum_{j=1}^{l}\bar{D}_{d,j}\right)\\&\leqslant 2-\frac{2n-2}{2n}-\frac{k-2n-2}{k-2n}-l\alpha\\&<0,\end{aligned}$$

a contradiction.

Therefore, the singularities of $X\setminus\{P_1,P_2\}$ are Du Val. By Lemma 2.3, points P_1 and P_2 are types $(1/m_1)(1,1)$ and $(1/m_2)(1,1)$, respectively. We may assume that $m_1\geqslant m_2$. If we apply Construction 2.13, choosing ϕ_i as in (1) therein, we obtain $X_d\simeq F_{2n,k-2n,1}$. Hence, $m_1+m_2\geqslant 12$. We now choose ϕ_0 as in Construction 2.13 (2) with $P=P_1$. By Lemma 4.8 we have two possibilities.

(1) ($X_1\simeq\tilde{\mathbb{P}^2}_{k,s}$.) Since $m_1\geqslant m_2\geqslant 8$, we see that the exceptional curves of ψ_0 do not contain the fixed point P_2. We now choose ϕ_1 as in Construction 2.13 (2) with $P=P_2$. We obtain $X_2\simeq\mathbb{P}^2$. Let D_2 be a unique exceptional curve over P_2. Let $\pi\colon Y\to X$ be the minimal resolution and let D_1 be the proper transform of a unique exceptional curve over P_1. If an exceptional curve of ψ_1 meets D_1, then there exists a (-1)-curve E on Y such that E meets $\bar{D}_1$ and $\bar{D}_2$. Hence,

$$0<-E\cdot\left(K_Y+\frac{m_1-2}{m_1}\bar{D}_1+\frac{m_2-2}{m_2}\bar{D}_2\right)=1-\frac{m_1-2}{m_1}-\frac{m_2-2}{m_2}<0,$$

a contradiction. Therefore, the exceptional curves of ψ_1 do not meet D_1. Then, $\psi_1(D_1)$ does not meet $\psi_1(D_2)$, a contradiction.

(2) ($X_1\simeq\mathbb{P}(1,1,2n)$.) As above, we see that the exceptional curves of ψ_0 do not contain the fixed point P_2. Note that every exceptional curve is a rational curve with one Du Val singular point of type A_s. Let $\pi\colon\mathbb{F}_{2n}\to\mathbb{P}(1,1,2n)$ be the minimal resolution and let D_2 be a unique exceptional curve. Then, D_2 does not meet D_1, where D_1 is the proper transform of a unique exceptional curve of ϕ_0. Hence, $D_2^2=-2n$ and $D_1^2=2n$. Therefore, $X\simeq F_{2n,ak-2n,a}$.

□

Theorem 1.8 now follows from Lemmas 4.6, 4.8 and 4.11.

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