Canad. Math. Bull. Vol. **44** (1), 2001 pp. 3–11

The Generating Degree of $\mathbb{C}_p$

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Abstract. The generating degree $\operatorname{gdeg}(A)$ of a topological commutative ring A with $\operatorname{char} A = 0$ is the cardinality of the smallest subset M of A for which the subring $\mathbb{Z}[M]$ is dense in A. For a prime number p, $\mathbb{C}_p$ denotes the topological completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. We prove that $\operatorname{gdeg}(\mathbb{C}_p) = 1$, *i.e.*, there exists t in $\mathbb{C}_p$ such that $\mathbb{Z}[t]$ is dense in $\mathbb{C}_p$. We also compute $\operatorname{gdeg}\big(A(U)\big)$ where $A(U)$ is the ring of rigid analytic functions defined on a ball U in $\mathbb{C}_p$. If U is a closed ball then $\operatorname{gdeg}\big(A(U)\big) = 2$ while if U is an open ball then $\operatorname{gdeg}\big(A(U)\big)$ is infinite. We show more generally that $\operatorname{gdeg}\big(A(U)\big)$ is finite for any *affinoid* U in $\mathbb{P}^1(\mathbb{C}_p)$ and $\operatorname{gdeg}\big(A(U)\big)$ is infinite for any *wide open* subset U of $\mathbb{P}^1(\mathbb{C}_p)$.

1 Introduction

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\bar{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\bar{\mathbb{Q}}_p$ with respect to the unique extension of the p-adic valuation v on $\mathbb{Q}_p$.

Some insight into the structure of closed subfields of $\mathbb{C}_p$ is provided by the Galois theory in $\mathbb{C}_p$ as developed by Tate [T], Sen [S] and Ax [A]. In particular, there is a canonical one-to-one correspondence between the closed subfields E of $\mathbb{C}_p$ and the subfields $\mathbb{Q}_p \subseteq L \subseteq \bar{\mathbb{Q}}_p$ via the maps (see [I-Z1, Th. 1]):

$$(*) \qquad E \mapsto E \cap \bar{\mathbb{Q}}_p = L \quad \text{and} \quad L \mapsto \tilde{L} = E,$$

where $\tilde{L}$ denotes the topological closure of L in $\mathbb{C}_p$. These maps pave the way for transfering information from subfields of $\bar{\mathbb{Q}}_p$ to closed subfields of $\mathbb{C}_p$.

In practice, when working in such a field L the situation is much improved if $L/\mathbb{Q}_p$ is finite. For one thing, the elements of L can be expressed in terms of a primitive element α of L, which moreover can be chosen in convenient ways, *e.g.* like being a uniformizer. If however $L/\mathbb{Q}_p$ is not finite then no such primitive element exists and in this case one needs to adjoin to $\mathbb{Q}_p$ infinitely many elements $\alpha_1, \alpha_2, \ldots$ from L to control the entire field L and so to produce a dense subfield in E.

With these in mind, Iovita and Zaharescu [I-Z1] investigated the possibility of obtaining something dense in E by adjoining *fewer* elements from E. They showed that it is enough to adjoin one element: there exists t in E such that $\mathbb{Q}_p(t)$ is dense in E.

In [A-P-Z] Alexandru, Popescu and Zaharescu took this matter one step further, by showing how one can actually express the elements of E in terms of this t. It is proven that:

Received by the editors December 11, 1998; revised May 9, 1999.
AMS subject classification: 11S99.

(i) For any element t in $\mathbb{C}_p$ the ring $\mathbb{Q}_p[t]$ and the field $\mathbb{Q}_p(t)$ have the same topological closure. Thus for any closed subfield E of $\mathbb{C}_p$ there exists t such that $\mathbb{Q}_p[t]$ is dense in E.

(ii) The theory of saturated distinguished chains for elements in $\bar{\mathbb{Q}}_p$ developed in [P-Z] naturally extends from $\bar{\mathbb{Q}}_p$ to $\mathbb{C}_p$. This provides us for any $t \in \mathbb{C}_p$ with distinguished sequences of polynomials $\{f_n(X)\}_n$ together with an infinite set of (metric) invariants for t.

(iii) Given any t such that $\mathbb{Q}_p[t]$ is dense in E and any distinguished sequence of polynomials associated to t, there is a canonical way to obtain from it a sequence $\{M_m(t)\}_{m\geq 0}$ of polynomials in t which as elements in $\mathbb{Q}_p[t]$ form an integral basis of E over $\mathbb{Q}_p$. Thus:

   (1) Any $z \in E$ can be expressed in a unique way in the form: $z = \sum_{m\geq 0} c_m M_m(t)$ where the c_m's are in $\mathbb{Q}_p$ and $c_m \to 0$ as $m \to \infty$, and

   (2) The above z belongs to the ring of integers O_E if and only if all the coefficients c_m are in $\mathbb{Z}_p$.

Some of these results were generalized in [I-Z3] and were applied to the ring B^+_{dR} defined by J.-M. Fontaine in [Fo]. In particular it is proved that there is an element T in B^+_{dR} such that $\mathbb{Q}_p[T]$ is dense in B^+_{dR}. Here one has a canonical projection of B^+_{dR} on $\mathbb{C}_p$ and the image of the above T in $\mathbb{C}_p$ will be an element t for which $\mathbb{Q}_p[t]$ is dense in $\mathbb{C}_p$. It should be stressed that not all the above results for $\mathbb{C}_p$ could be lifted to B^+_{dR}, one of the main obstructions here being the failure of the Galois correspondence in B^+_{dR} (for more details, see [I-Z2]).

The concept of generating degree was introduced in [I-Z3] as a convenient way to formulate various results from [I-Z2] and [I-Z3] (see Section 2 below). These generating degrees are important on their own. Being unchanged under isomorphisms of topological rings, they provide us with some natural invariants of these rings.

For two commutative topological rings $A \subset B$, a subset $M \subset B$ is said to be a *generating set* of B over A if the ring $A[M]$ is dense in B. The *generating degree* of B/A is defined to be

$$\operatorname{gdeg}(B/A) := \min\{|M|, \text{where } M \text{ is a generating set of } B/A\}$$

where $|M|$ denotes the number of elements of M if M is finite and ∞ if M is not finite.

The generating degree of B over $\mathbb{Z}$ if $\operatorname{char} B = 0$, respectively over $\mathbb{F}_p$ if $\operatorname{char} B = p$, will be denoted by $\operatorname{gdeg}(B)$ and will be called the absolute generating degree of B.

Some general properties of generating degrees are presented in Section 2. Our objective is to compute $\operatorname{gdeg}(\mathbb{C}_p)$. This is achieved in Section 3 following an investigation on the structure of closed subrings of $\mathbb{C}_p$. We show that $\operatorname{gdeg}(\mathbb{C}_p) = 1$ and that the same holds true for any of its closed subfields:

Theorem 1 *For any closed subfield E of $\mathbb{C}_p$ there exists t in E such that $\mathbb{Z}[t]$ is dense in E.*

By contrast we note that $\operatorname{gdeg}(O_{\mathbb{C}_p})$ is infinite, where $O_{\mathbb{C}_p}$ denotes the ring of integers in $\mathbb{C}_p$.

https://doi.org/10.4153/CMB-2001-001-1 Published online by Cambridge University Press

In the last section we consider rings $A(U)$ of rigid analytic functions defined on various open sets U of $\mathbb{C}_p$ (for the general theory of rigid analytic functions see [F-P]). We found that if U is an affinoid then $\operatorname{gdeg}\big(A(U)\big)$ is finite. The situation changes dramatically if we replace U by a "wide open set" (in the terminology of Coleman [Co]). In this case $\operatorname{gdeg}\big(A(U)\big)$ is infinite.

For example, if $a \in \mathbb{C}_p$ and $0 < r \in \{|z|; z \in \mathbb{C}_p\}$ then the "closed ball" $B[a,r] := \{z \in \mathbb{C}_p; |z-a| \le r\}$ is an affinoid while the "open ball" $B(a,r) := \{z \in \mathbb{C}_p; |z-a| < r\}$ is a wide open set.

In the following by a *closed ball* in $\mathbb{P}^1(\mathbb{C}_p)$ we mean either a set of the form $B[a,r]$ as above or a set of the form $\mathbb{P}^1(\mathbb{C}_p) \setminus B(a,r)$. Similarly subsets of the form $B(a,r)$ or $\mathbb{P}^1(\mathbb{C}_p) \setminus B[a,r]$ will be called *open balls*. An affinoid in $\mathbb{P}^1(\mathbb{C}_p)$ is a subset U of the form $U = \mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{j=1}^{g} B_j$ where each B_j is an open ball in $\mathbb{P}^1(\mathbb{C}_p)$. A subset U as above, $U = \mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{j=1}^{g} B_j$ where the B_j's are balls and at least one of them is a closed ball is called a wide open set in $\mathbb{P}^1(\mathbb{C}_p)$. With these notations and terminology we have the following:

Theorem 2

(i) *If U is a wide open set in $\mathbb{P}^1(\mathbb{C}_p)$ then* $\operatorname{gdeg}\big(A(U)\big)$ *is infinite.*

(ii) *Let $U = \mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{j=1}^{g} B_j$, where the B_j's are distinct, be an affinoid in $\mathbb{P}^1(\mathbb{C}_p)$. Then* $\operatorname{gdeg}\big(A(U)\big) \le g+1$.

(iii) *If U is a closed ball in $\mathbb{P}^1(\mathbb{C}_p)$ then* $\operatorname{gdeg}\big(A(U)\big) = 2$.

2 Generating Degrees

Recall the definitions from the Introduction:

For two commutative topological rings $A \subset B$, a subset $M \subset B$ is said to be a *generating set* of B over A if the ring $A[M]$ is dense in B. The *generating degree* of B/A, $\operatorname{gdeg}(B/A) \in \mathbf{N} \cup \infty$ is defined to be

$$\operatorname{gdeg}(B/A) := \min\{|M|, \text{ where } M \text{ is a generating set of } B/A\}$$

where $|M|$ denotes the number of elements of M if M is finite and ∞ if M is not finite.

Thus A is dense in B if and only if $\operatorname{gdeg}(B/A) = 0$.

Define the absolute generating degree $\operatorname{gdeg}(B)$ of B by $\operatorname{gdeg}(B) = \operatorname{gdeg}(B/\mathbb{Z})$ if $\operatorname{char} B = 0$, respectively $\operatorname{gdeg}(B) = \operatorname{gdeg}(B/\mathbb{F}_p)$ if $\operatorname{char} B = p$.

Some very simple properties of generating degrees are summarized in the following

Proposition 3

a) $\operatorname{gdeg}(B/A)$ *is invariant with respect to isomorphisms of topological rings.*

b) *If $A \subset B \subset C$ then* $\operatorname{gdeg}(C/A) \ge \operatorname{gdeg}(C/B)$.

c) *If $A \subset B \subset C$ then* $\operatorname{gdeg}(C/A) \le \operatorname{gdeg}(B/A) + \operatorname{gdeg}(C/B)$.

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d) If $A \subset B$ and $\psi\colon B \to C$ is a continuous morphism of rings then for any generating set M of B over A, $\psi(M)$ will be a generating set of $\psi(B)$ over $\psi(A)$. In particular: $\operatorname{gdeg}\big(\psi(B)/\psi(A)\big) \leq \operatorname{gdeg}(B/A)$ *and* $\operatorname{gdeg}\big(\psi(B)\big) \leq \operatorname{gdeg}(B)$.

e) If $A \subset B$ is a finite separable extension of fields then we have $\operatorname{gdeg}(B/A) \leq 1$.

Remark It is not true that for any $A \subset B \subset C$ one has $\operatorname{gdeg}(C/A) \geq \operatorname{gdeg}(B/A)$. For example $\operatorname{gdeg}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = \infty$ while $\operatorname{gdeg}(\mathbb{C}_p/\mathbb{Q}_p) = 1$.

There is a connection between generating degrees and continuous derivations of B over A. Let $A \subset B$ be two topological commutative rings. A derivation of B over A is a map $D\colon B \to B$ which satisfies the usual rules:

$$D(u+v) = D(u) + D(v), \quad D(uv) = uD(v) + vD(u)$$

and whose restriction to A is trivial. Assume at this point that B is an integral domain and denote by F and E the field of fractions of A and B respectively. Then any such D has a unique extension to a derivation of E over F, given by:

$$D\left(\frac{u}{v}\right) = \frac{vD(u) - uD(v)}{v^2}$$

and the set $D(B/A)$ of all such derivations becomes a vector space over E. Let us denote by $D_{\text{cont}}(B/A)$ the subspace of $D(B/A)$ spanned by derivations $D\colon B \to B$ which are continuous with respect to the topology of B. With these notations, we have the following:

Proposition 4 $\dim_E D_{\text{cont}}(B/A) \leq \operatorname{gdeg}(B/A)$.

There is also a connection between the generating degrees and chains of open prime ideals of B. Recall that the height $h(\mathcal{P})$ of a prime ideal $\mathcal{P}$ of a commutative ring B is defined to be the largest integer n for which there is a chain of prime ideals in B:

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n = \mathcal{P}.$$

Then one defines the Krull dimension of B to be

$$\dim B := \sup\{h(\mathcal{P})\}$$

where $\mathcal{P}$ runs over the set of prime ideals in B.

If now B is a topological commutative ring we can define its topological Krull dimension $\dim_\top B$ by counting only open prime ideals, as follows. Define the topological height $h_\top(\mathcal{P})$ of an open prime ideal $\mathcal{P}$ of B to be the largest integer n for which there is a chain

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n = \mathcal{P}$$

of open prime ideals of B. Then set:

$$\dim_\top B = \sup\{h_\top(\mathcal{P})\}$$

where $\mathcal{P}$ runs over the set of open prime ideals in B.

Note that $\dim_\top B \leq \dim B$ and if B is endowed with the discrete topology then $\dim_\top B = \dim B$. With the above notations we also have the following:

Proposition 5 *For any topological commutative ring B one has:*

$$\dim_{\top}(B) \leq \operatorname{gdeg} B.$$

We skip the details of the proofs of the above results and mention only that:

1) In the proof of Proposition 4 the point is that if M is a generating set of B/A then any continuous derivation D of B is uniquely determined by its restriction to M, and

2) For the proof of Proposition 5 intersect an arbitrary chain of open prime ideals

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n$$

with $\mathbb{Z}[M]$ where M is an arbitrary generating set of B to get a chain $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n$ of open prime ideals in $\mathbb{Z}[M]$. Now the point is that the sets $\mathcal{P}_j \setminus \mathcal{P}_{j-1}$ being open and $\mathbb{Z}[M]$ being dense in B there will be points from $\mathbb{Z}[M]$ in $\mathcal{P}_j - \mathcal{P}_{j-1}$ thus $\mathcal{J}_0 \subset \mathcal{J}_1 \subset \cdots \subset \mathcal{J}_n$, so n is bounded by $\dim_{\top} \mathbb{Z}[M]$ which is bounded by $|M|$.

Now let us see some examples of generating degrees in $\mathbb{C}_p$ and in B_{dR}^+. Galois theory in $\mathbb{C}_p$ shows that for any algebraic extension L of Q_p we have $(\mathbb{C}_p)^{G_L} = \tilde{L}$, where $G_L = \mathrm{Gal}(\bar{\mathbb{Q}}_p/L) = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/L)$. In other words:

$$\operatorname{gdeg}\big((\mathbb{C}_p)^{G_L}/L\big) = 0.$$

As was mentioned in the introduction the Galois correspondence fails in B_{dR}^+. Thus in general an algebraic extension L is not dense in $(B_{dR}^+)^{G_L}$, although $\bar{\mathbb{Q}}_p$ itselt is dense in B_{dR}^+ as was proved in [F-C]. We do have however the following result:

If $K := \mathbb{Q}_p^{ur} \subseteq L \subseteq \bar{\mathbb{Q}}_p$ and L is not a *deeply* ramified extension of K (in the sense of Coates-Greenberg [C-G]) then

$$\operatorname{gdeg}\big((B_{dR}^+)^{G_L}/L\big) = 0.$$

It is proved in [I-Z3] that for any algebraic extension L of K one has:

$$\operatorname{gdeg}\big((B_{dR}^+)^{G_L}/L\big) \leq 1.$$

A characterization of deeply ramified extensions L of K satisfying the equation $\operatorname{gdeg}\big((B_{dR}^+)^{G_L}/L\big) = 0$ is obtained in [I-Z2]. Concerning generating degrees over $\mathbb{Q}_p$ we have the following result:

Let $\mathbb{Q}_p \subset L \subseteq \bar{\mathbb{Q}}_p$ and let E be the topological closure of L in $\mathbb{C}_p$ (respectively in B_{dR}^+). Then (in both cases) we have:

$$\operatorname{gdeg}(E/\mathbb{Q}_p) = 1.$$

Note that, by contrast, one has:

$$\operatorname{gdeg}(O_{\mathbb{C}_p}/\mathbb{Z}_p) = \infty.$$

Indeed, for any finite subset M of $\mathbb{C}_p$ the image of $\mathbb{Z}_p[M]$ in the residue field $\bar{\mathbb{F}}_p$ of $O_{\mathbb{C}_p}$ will be a finite field. Then any element of $O_{\mathbb{C}_p}$ whose image in $\bar{\mathbb{F}}_p$ lies outside this finite field will be at distance 1 from $\mathbb{Z}_p[M]$, so $\mathbb{Z}_p[M]$ is not dense in $O_{\mathbb{C}_p}$ and M is not a generating set of $O_{\mathbb{C}_p}/\mathbb{Z}_p$.

Let now L be a finite extension of $\mathbb{Q}_p$, $L \neq \mathbb{Q}_p$. It is well known that L has a maximal unramified subextension, say F, that $O_F = \mathbb{Z}_p[u]$ and $O_L = O_F[\pi]$ where u is a unit in O_F whose image in $\bar{\mathbb{F}}_p$ generates the residue field of L and π is a uniformiser of O_L. Hence $\{u, \pi\}$ is a generating set of O_L and $\operatorname{gdeg}(O_L) \leq 2$. It is proved in [Se, Ch. III, Proposition 12] that there is an α in O_L such that $O_L = \mathbb{Z}_p[\alpha]$. Thus in fact one has:

$$\operatorname{gdeg}(O_L/\mathbb{Z}_p) = 1.$$

3 Closed Subrings of $\mathbb{C}_p$

By a closed subring of $\mathbb{C}_p$ we mean a subring of $\mathbb{C}_p$ which is closed with respect to the topology induced from $\mathbb{C}_p$.

Lemma 6 *Let E be a closed subring of C_p. Then either $E \subseteq O_{\mathbb{C}_p}$ or $\mathbb{Q}_p \subseteq E$.*

Proof Assume E is not contained in O_{C_p}. Choose $t \in E$ with $v(t) < 0$. Raise t to an integer power $r \geq 1$ such that $v(t^r)$ is an integer $-m$. Then $t^r = p^{-m}u$, where $m > 0$ and u is a unit in $O_{\mathbb{C}_p}$. Now raise u to a power $k \geq 1$ such that u^k is a principal unit. Hence $u^k = 1 - x$ with $v(x) > 0$. Let $y = \frac{1}{1-x} = 1 + x + \cdots + x^n + \cdots$. Since $u = p^m t^r \in E$ it follows that $x = 1 - u^k \in E$ and so $y \in E$. Therefore $\frac{1}{p} = t^{kr} p^{(mk-1)} y \in E$ and then clearly $\mathbb{Q}_p \subseteq E$. ■

Theorem 7 *Let E be a closed subring of $\mathbb{C}_p$, not contained in $O_{\mathbb{C}_p}$. Then E is a field.*

We note the following consequence of Theorem 7:

Corollary 8 *For any $z_1, z_2, \ldots, z_n \in \mathbb{C}_p$ the ring $\mathbb{Q}_p[z_1, z_2, \ldots, z_n]$ and the field $\mathbb{Q}_p(z_1, z_2, \ldots, z_n)$ have the same topological closure.*

Indeed, the closure of $\mathbb{Q}_p[z_1, z_2, \ldots, z_n]$ is a ring E which is not contained in $O_{\mathbb{C}_p}$ thus by Theorem 7 it follows that E is a field so it contains $\mathbb{Q}_p(z_1, z_2, \ldots, z_n)$.

Proof of Theorem 7 Let E be a closed subring of $\mathbb{C}_p$ not contained in $O_{\mathbb{C}_p}$. From Lemma 6 we know that $\mathbb{Q}_p \subseteq E$. Now let $L = E \cap \bar{\mathbb{Q}}_p$. Then L is a subring of $\bar{\mathbb{Q}}_p$ which contains $\mathbb{Q}_p$. It follows immediately that L is a subfield of $\bar{\mathbb{Q}}_p$. Then $\tilde{L}$ is a complete subfield of $\mathbb{C}_p$. It remains to show that $\tilde{L} = E$. The inclusion $\tilde{L} \subseteq E$ is clear. Assume now that there is an element $z \in E$ such that $z \notin \tilde{L}$. Since $\mathbb{Q}_p[z] \subseteq E$ and E is closed it follows that the topological closure of $\mathbb{Q}_p[z]$, call it H, is also contained in E. From [A-P-Z] we know that H is a field. Moreover from the one-to-one correspondence $(*)$ we know that we can intersect H with $\bar{\mathbb{Q}}_p$ and then we can recover it by completion: $H \cap \bar{\mathbb{Q}}_p = F$ say, $\tilde{F} = H$.

https://doi.org/10.4153/CMB-2001-001-1 Published online by Cambridge University Press

But F is contained in $E \cap \bar{\mathbb{Q}}_p = L$, thus $\tilde{F} \subseteq \tilde{L}$. We obtained a contradiction since z belongs to H but not to $\tilde{L}$, and this completes the proof of Theorem 7.

Proof of Theorem 1 Let E be a closed subfield of $\mathbb{C}_p$. Choose t as in [A-P-Z] such that $\mathbb{Q}_p[t]$ is dense in E. Now divide t by a large power of p to force it out of $O_{\mathbb{C}_p}$: $\frac{t}{p^r} = z \notin O_{\mathbb{C}_p}$. Consider the subring $\mathbb{Z}[z]$ of E. The closure H of $\mathbb{Z}[z]$ will be a closed subring of $\mathbb{C}_p$ which is not contained in $O_{\mathbb{C}_p}$. From Theorem 7 we know that H is a closed subfield of $\mathbb{C}_p$. It now follows easily that $H = E$. ■

4 Proof of Theorem 2

Note first that for any rigid analytic function $F\colon U_1 \to U_2$ we get a map $F^*\colon A(U_2) \to A(U_1)$, given by: $g \mapsto g \circ F$. If U_1 and U_2 are conformal in the sense that there is a one-to-one map $F\colon U_1 \to U_2$ with F and F^{-1} rigid analytic, then $F^*\colon A(U_2) \to A(U_1)$ will be an isomorphism of topological rings. In particular if U_1 and U_2 are conformal then $\operatorname{gdeg}\bigl(A(U_1)\bigr) = \operatorname{gdeg}\bigl(A(U_2)\bigr)$. If now $U = \mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{j=1}^{q} B_j$ is an affinoid or a wide open set one can use a linear fractional transformation $F\colon U \to \mathbb{P}^1(\mathbb{C}_p)$, $F(z) = \frac{az+b}{cz+d}$ with a, b, c, d in $\mathbb{C}_p$, $ad - bc \neq 0$ to send one B_j to or away from the "point at infinity".

Let's now prove (i). By making such a linear fractional transformation, we may assume that

$$U = B(0,1) \setminus \bigcup_{j=1}^{g-1} B_j$$

where $B_j = B(a_j, r_j)$ for $1 \le j \le s$, $B_j = B[a_j, r_j]$ for $s < j \le g-1$ for some integer $1 \le s \le g-1$ and some $a_1, \dots, a_{g-1} \in B(0,1)$ and $0 < r_1, \dots, r_{g-1} < 1$.

Note that any power series $g(z) = \sum_{n=0}^{\infty} a_n z^n$ with coefficients a_n in $O_{\mathbb{C}_p}$ is convergent on $B(0,1)$ and so it belongs to $A(U)$. Moreover, it is easy to see that for such a function f the norm $\|f\| := \{\sup |f(z)|; z \in U\}$ is given by

$$\|f\| = \sup_{n \ge 0} |a_n|$$

As a consequence, two such functions $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $g(z) = \sum_{n=0}^{\infty} a_n z^n$ with $c_n, a_n \in O_{\mathbb{C}_p}$ will be at distance $\|f - g\| = 1$ unless for any n the coefficients c_n and a_n have the same image in the residue field $\bar{\mathbb{F}}_p$ of $O_{\mathbb{C}_p}$.

Now let M be a generating set of $A(U)$. We choose for any power series $h(X) = \sum_{n \ge o} b_n X^n \in \bar{\mathbb{F}}_p[[X]]$ a representative $g(z) = \sum_{n \ge 0} a_n z^n$ with $a_n \in O_{\mathbb{C}_p}$, where b_n is the image of a_n in $\bar{\mathbb{F}}_p$ and then we choose an element $f \in \mathbb{Z}[M]$ such that $\|f - g\| < 1$. Note that for distinct h we have distinct f's, therefore the mapping $h \mapsto f$ gives an injection $\bar{\mathbb{F}}_p[[X]] \hookrightarrow \mathbb{Z}[M]$.

But $\bar{\mathbb{F}}_p[[X]]$ is an uncountable set, therefore M can not be countable, much less finite.

ii) Send the B_j's away from the point at infinity. Thus U will have the form:

$$\mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{j=1}^{g} B(a_j, r_j).$$

Then $A(U)$ consists of functions f of the form (see [F-P]):

$$f(z) = c_0 + \sum_{j=1}^{g} \sum_{n=1}^{\infty} c_{jn}(z - a_j)^{-n}$$

with $c_0, c_{jm} \in \mathbb{C}_p$, and $|c_{jn}|r_j^{-n} \to 0$ as $n \to \infty$ for $1 \leq j \leq g$. Here $\|f\| = \max\{|c_0|, \sup_{jn} |c_{jn}|r_j^{-n}\}$. We have:

$$\lim_{N\to\infty} \Big\| f - c_0 - \sum_{j=1}^{g} \sum_{n=1}^{N} c_{jn}(z - a_j)^{-n} \Big\| = 0.$$

Thus the ring $\mathbb{C}_p[\frac{1}{z-a_1}, \dots, \frac{1}{z-a_g}]$ is dense in $A(U)$, and thus $\operatorname{gdeg}\big(A(U)/\mathbb{C}_p\big) \leq g$. From Theorem 1 and Proposition 3 c) it now follows that $\operatorname{gdeg}\big(A(U)\big) \leq g + 1$.

iii) By making a suitable fractional linear transformation we may assume that $U = B[0, 1]$. From (ii) we know that $\operatorname{gdeg}\big(A(U)\big) \leq 2$. Let's assume that $\operatorname{gdeg}\big(A(U)\big) = 1$ and let f be a generating element of $A(U)$. Now for any $z_0 \in U$ we have a surjective continuous morphism of topological rings $\psi\colon A(U) \to \mathbb{C}_p$ given by $\psi(g) = g(z_0)$. From Proposition 3 d) it follows that $\psi(f) = f(z_0)$ is a generating element of C_p. Thus we arrived at the following question: Is there an $f \in A(U)$ such that $f(z)$ is a generating element of C_p for any z in $B[0, 1]$?

The answer is "no". Indeed, write $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$, with $a_n \in \mathbb{C}_p$ and $a_n \to 0$ as $n \to \infty$. Let us choose an $\alpha \in \bar{\mathbb{Q}}_p$ close enough to a_0 such that $|\alpha - a_0| < \max_{n\geq 1} |a_n|$ and put $g(z) = f(z) - \alpha = (a_0 - \alpha) + a_1 z + \cdots + a_m z^m + \cdots$. Now from the Weierstrass Preparation Theorem (see Lang [L, Ch. 5, Section 2]) we have a decomposition $g(z) = P(z)h(z)$ with $h(z) \in O_{\mathbb{C}_p}[[z]]$ and P polynomial of degree ≥ 1 distinguished in the sense that its leading coefficient is larger than the other coefficients. Here the roots of P are in $B(0, 1)$. If z_1 is such a root then $g(z_1) = 0$ and $f(z_1) = \alpha$ which is not a generating element of $\mathbb{C}_p$. This completes the proof of Theorem 2.

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https://doi.org/10.4153/CMB-2001-001-1 Published online by Cambridge University Press