THE GENERALIZED KULIKOV CRITERION

CHARLES MEGIBBEN

In 1941, Kulikov (**5**) showed that a p-primary abelian group G is a direct sum of cyclic groups if and only if G is the union of an ascending sequence of subgroups each of which has a finite bound on the heights of its elements. An easy reformulation of the Kulikov criterion is: A p-primary abelian group G is a direct sum of cyclic groups if and only if $G[p] = \bigoplus_{n<\omega} S_n$ where, for each n, the non-zero elements of S_n have precisely height n. This statement suggests the consideration of reduced p-groups G such that $G[p] = \bigoplus_{\alpha<\lambda} S_\alpha$ where, for each α, $S_\alpha - \{0\} \subseteq p^\alpha G - p^{\alpha+1}G$. We shall call such p-groups *summable* (the term *principal p-group* has been used by Honda (**4**)). Recall that the length of a reduced p-group G is the first ordinal λ such that $p^\lambda G = 0$. Thus, our reformulation of the Kulikov criterion becomes: A p-group of length ω is a direct sum of cyclic groups if and only if the group is summable.

Our aim in this paper is to establish a generalization of the Kulikov criterion for direct sums of countable reduced p-groups. Now if G is such a group, then $G/p^\alpha G$ is a direct sum of countable groups for all ordinals α. Since it is easy to construct summable groups G of length greater than ω such that $G/p^\omega G$ is not a direct sum of cyclic groups (and, therefore, not a direct sum of countable groups), summability is not enough. Apparently, we must require $G/p^\alpha G$ to be a direct sum of countable groups for all α less than the length of G in order to ensure that G is a direct sum of countable groups. This condition is, of course, free for groups of length ω. Now if α is a countable ordinal and if both $G/p^\alpha G$ and $p^\alpha G$ are direct sums of reduced countable p-groups, then G itself is a direct sum of countable groups. (This result is due to Nunke (**7**) and will also be proved below). Thus, a generalized Kulikov criterion will be of interest only for groups of limit length. My first formulation of such a generalization appeared in the joint paper (**3**). In that paper, the barest outline of a proof of the following result was sketched.

THEOREM A. *Let λ be a countable limit ordinal. Then a p-primary abelian group G of length λ is a direct sum of countable groups if and only if G is summable and $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \lambda$.*

Now, obviously, a direct sum of countable reduced p-groups has length at most Ω and, as is proved in (**3**; **4**), the same applies to summable groups. Therefore, Theorem A has the aesthetic fault of not covering the case $\lambda = \Omega$.

Received April 29, 1968. This work was supported in part by the National Science Foundation Research Grant GP–7252.

This is unavoidable; for Hill (**2**) has just recently constructed an example that shows the theorem fails for $\lambda = \Omega$, as does Theorem B below. The insight in Hill's example is the easily seen fact that an uncountable reduced p-group with countable Ulm invariants cannot itself be a direct sum of countable groups. On the other hand, there does exist a summable group G with $f_G(\alpha) = \aleph_0$ for all $\alpha < \Omega$.

The version of the generalized Kulikov criterion stated in Theorem A, though a rather elegant formulation, proved to be inadequate for my generalizations of the classical results about purity, basic subgroups, etc., which are to appear in (**6**). What is needed in (**6**) is the following theorem.

THEOREM B. *Let G be a p-primary abelian group of length λ, where λ is a countable limit ordinal. Then G is a direct sum of countable groups provided G is summable and, for each $\alpha < \lambda$, G contains a p^α-high subgroup which is a direct sum of countable groups.*

In this paper, I shall give detailed proofs of Theorems A and B and shall consider certain applications that do not involve homological techniques. I am, indeed, able to avoid completely concepts that depend on homological algebra for their very statements; for example, p^α-purity and p^α-projectivity. Such notions, of course, are of great importance in the study of abelian groups; but their introduction in the proofs of Theorems A and B (compare the proof of Theorem A sketched in (**3**)) really obscures the simple combinatorial nature of the arguments involved. Having decided to avoid such methods, I have moreover been able to write a paper that is more elementary and more nearly self-contained than would otherwise have been possible. This has, however, necessitated reproving some known results and stating a number of elementary lemmas.

Throughout this paper, all groups are assumed to be p-primary abelian groups for a fixed prime p (though many of our results are valid for all abelian groups). A subgroup H of G will be said to be *isotype* if $H \cap p^\alpha G = p^\alpha H$ for all ordinals α. By a *p^α-high subgroup* of G we mean a subgroup maximal among those subgroups of G that intersect $p^\alpha G$ trivially. p^α-high subgroups are easily shown to be isotype. Another simple observation: if H is a p^α-high subgroup of G and $\alpha \geqq \omega$, then G/H is divisible. By the *height* $h_G(x)$ of a non-zero element x of the reduced p-group G we mean the first ordinal α such that $x \notin p^{\alpha+1}G$.

1. Summability. For a simple proof that summable groups have length at most Ω, see (**3**). Call a subgroup H of G *height-finite* if the heights (as computed in G) of the elements of H assume only finitely many values. If G is a summable group of countable length, then $G[p]$ is the union of an ascending sequence of height-finite subgroups. Indeed, if $G[p] = \bigoplus_{\alpha<\lambda} S_\alpha$, where $S^\alpha - \{0\} \subseteq p^\alpha G - p^{\alpha+1}G$ and if $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ is an enumeration of the ordinals less than λ, then $G[p] = \bigcup_{n=1}^\infty Q_n$, where $Q_n = S_{\alpha_1} \oplus \ldots \oplus S_{\alpha_n}$ is height-finite. Conversely, Honda (**4**), has shown that if $G[p]$ is the union of ascending sequence $\{Q_n\}_{n=1}^{\infty}$ of height-finite subgroups, then G is summable. The proof is rather obvious. By induction, one constructs decompositions $Q_n = \bigoplus_{\alpha<\lambda} S_\alpha^{(n)}$ with $S_\alpha^{(n)} - \{0\} \subseteq p^\alpha G - p^{\alpha+1}G, S_\alpha^{(n)} = 0$ for almost all α and $S_\alpha^{(n)} \subseteq S_\alpha^{(n+1)}$. Then $G[p] = \bigoplus_{\alpha<\lambda} S_\alpha$, where $S_\alpha = \bigcup_{n=1}^{\infty} S_\alpha^{(n)}$. As immediate corollaries we have: (1) *Countable reduced p-groups are summable;* (2) *Direct sums of countable reduced p-groups are summable;* (3) *Isotype subgroups of summable groups of countable length are summable.*

Call a direct decomposition $H = \bigoplus_{i\in I} H_i$ of the subgroup H of G a *normal decomposition* if

$$h_G(x_1 + x_2 + \ldots + x_n) = \min[h_G(x_1), h_G(x_2), \ldots, h_G(x_n)]$$

whenever the distinct x_j's belong to distinct H_i's. It is then trivial to show that a reduced p-group G is summable if and only if there exists a normal decomposition $G[p] = \bigoplus_{i\in I} T_i$ with $|T_i| \leqq \aleph_0$ for each i. One, perhaps, requires here the obvious observation that the above statement about unions of height-finite subgroups applies to subgroups of $G[p]$: If S is a subgroup of $G[p]$ such that S is the union of an ascending sequence of height-finite subgroups, then there exists a direct decomposition $S = \bigoplus_{\alpha<\lambda} S_\alpha$ with $S_\alpha - \{0\} \subseteq p^\alpha G - p^{\alpha+1}G$ for each α. This suggests the obvious extension of the notion of summability to what we might call *summable subsocles.*

2. A decomposition theorem. This section is devoted to a proof and some consequences of the following result.

THEOREM 2.1. *If H is a subgroup of the p-group G such that*
(1) $p^\alpha(G/H) = p^\alpha G + H/H$ *for all* $\alpha < \beta$,
(2) $p^\alpha G \cap H = p^\alpha H$ *for all* $\alpha \leqq \beta$, *and*
(3) G/H *is countable of length at most* β,
then H is a direct summand of G.

First we need to show that the properties (1) and (3) of Theorem 2.1 are inherited by finite subextensions of H.

LEMMA 2.2. *If S/H is a finite subgroup of G/H and if H satisfies conditions* (1) *and* (3) *of Theorem* 2.1, *then S also satisfies these conditions.*

Proof. To show that condition (1) is satisfied by S we proceed by induction. Assume that $\gamma < \beta$ and that $p^\alpha(G/S) = p^\alpha G + S/S$ for all $\alpha < \gamma$. Let $x + S \in p^\gamma(G/S)$. If $\gamma = \alpha + 1$, then $x + S = p(y + S)$ for some

$$y + S \in p^\alpha(G/S) = p^\alpha G + S/S.$$

Thus, we may assume that $y \in p^\alpha G$, and therefore $x + S = py + S \in p^\gamma G + S/S$. Suppose, however, that γ is a limit. Then, for each $\alpha < \gamma$, we write $x + S = z_\alpha + S$, where $z_\alpha \in p^\alpha G$. Then $x - z_\alpha = s_\alpha + h_\alpha$, where

$s_\alpha \in S$, $h_\alpha \in H$, and the s_α's assume only finitely many values. Therefore, there is a fixed $s \in S$ and a set $A \subseteq \{\alpha : \alpha < \gamma\}$ such that $\sup A = \gamma$ and $x - z_\alpha = s + h_\alpha$ with $h_\alpha \in H$ for each $\alpha \in A$. Then $x - s + H = z_\alpha + H \in p^\alpha G + H/H$ for all $\alpha \in A$. Since $\sup A = \gamma$, $x - s + H \in p^\gamma(G/H) = p^\gamma G + H/H$ and we can write $x - s + H = z + H$ with $z \in p^\gamma G$. But then $x + S = z + S \in p^\gamma G + S/S$.

The proof that $p^\beta(G/S) = 0$ is similar, requiring separate consideration for the two cases (i) $\beta = \alpha + 1$ and (ii) β a limit ordinal. In either case, beginning with an $x + S \in p^\beta(G/S)$, we find an $s \in S$ such that

$$x - s + H \in p^\beta(G/H) = 0,$$

and therefore $x - s \in H$, or $x + S = 0$.

Proof of Theorem 2.1. We call a mapping $\phi\colon S \to T$ between subsets of G *height-increasing* if $h_G(\phi(x)) \geqq h_G(x)$ for all $x \in S$. Since G/H is countable, it suffices to prove the following: If

(i) S/H is a finite subgroup of G/H,

(ii) $px \in S$, and

(iii) $\phi\colon S \to H$ is a height-increasing homomorphism such that $\phi|H = 1_H$,

then ϕ extends to a height-increasing homomorphism $\bar{\phi}\colon \langle S, x\rangle \to H$. Assume (i), (ii), (iii), and $x \notin S$. Then $h_{G/S}(x + S) = \alpha$ for some $\alpha < \beta$. Therefore $x + S = y + S$ with $y \in p^\alpha G$. Since $py \in S$ and $\langle x, S\rangle = \langle y, S\rangle$, we may assume that $x \in p^\alpha G$. We then have $h_G(x + s) \leqq h_{G/S}(x + S) = h_G(x) = \alpha$ for all $s \in S$. Since $\alpha + 1 \leqq \beta$, we have an $h \in p^\alpha H$ such that

$$ph = \phi(px) \in p^{\alpha+1}G \cap H = p^{\alpha+1}H.$$

Define $\bar{\phi}$ on $\langle S, x\rangle$ by $\bar{\phi}(nx + s) = nh + \phi(s)$ whenever $n \in Z$ and $s \in S$. It is then routine to show that $\bar{\phi}$ is a well-defined, height-increasing homomorphism that extends ϕ.

As a corollary to Theorem 2.1, we have the following theorem, which has been proved in greater generality in (**1**; **3**).

THEOREM 2.3. *Let H be a subgroup of the p-group G such that*

(1) *$(G/H)/p^\beta(G/H)$ is countable,*

(2) *$H + p^\beta G/p^\beta G$ is a direct summand of $G/p^\beta G$,*

(3) *$H \cap p^\beta G = p^\beta H$, and*

(4) *$p^\beta G = p^\beta H \oplus L$.*

Then $G = H \oplus K$ with $K \supseteq L$.

Proof. We write $G/p^\beta G = (H + p^\beta G)/p^\beta G \oplus M/p^\beta G$. By an inductive argument, the details of which we leave to the reader, we establish that $(H + p^\alpha G) \cap M = p^\alpha G \cap M$ for all $\alpha \leqq \beta$. Next we show that $p^\alpha G \cap H = p^\alpha H$ for all $\alpha \leqq \beta$. Indeed, suppose that $p^\alpha G \cap H = p^\alpha H$ for all $\alpha < \gamma \leqq \beta$. If γ is a limit, we immediately have $p^\gamma G \cap H = p^\gamma H$. Suppose then that $\gamma = \alpha + 1$ and let $x \in p^\gamma G \cap H$. We can write $x = p(h + m)$, where $h \in H$, $m \in M$, and $h + m \in p^\alpha G$. But then $m \in (H + p^\alpha G) \cap M = p^\alpha G \cap M$, and therefore $h = (h + m) - m \in p^\alpha G \cap H = p^\alpha H$. Thus, $x - ph = pm \in H \cap M = H \cap p^\beta G = p^\beta H$ and $x = (x - ph) + ph \in p^\gamma H$. To show that $p^\alpha(G/H) = p^\alpha G + H/H$ for all $\alpha \leqq \beta$, it suffices to show that $p^\alpha[(G/H)/p^\alpha G + H/H] \cong p^\alpha(G/p^\alpha G + H) = 0$ for all $\alpha \leqq \beta$. It is enough then to show that $G/p^\alpha G + H$ is isomorphic to a subgroup of $G/p^\alpha G$. However,

$$G/H + p^\alpha G = (H + p^\alpha G) + M/(H + p^\alpha G) \cong M/M \cap (H + p^\alpha G)$$
$$= M/p^\alpha G \cap M \cong p^\alpha G + M/p^\alpha G \subseteq G/p^\alpha G$$

for all $\alpha \leqq \beta$.

We wish to apply Theorem 2.1 not to H but rather to $H \oplus L/L = H + p^\beta G/L$. We observe that:

(1) $$p^\alpha[(G/L)/(H \oplus L/L)] \cong p^\alpha(G/H \oplus L) = p^\alpha(G/H + p^\beta G)$$
$$\cong p^\alpha((G/H/p^\beta(G/H)) = p^\alpha(G/H)/p^\beta(G/H) \cong p^\alpha G + H/p^\beta G + H$$
$$\cong (p^\alpha G + H/L)/(p^\beta G + H/L) = (p^\alpha(G/L) + H \oplus L/L)/(H \oplus L/L).$$

Since the composition of the four isomorphisms in the above sequence of equations is just the identity on $(G/L)/(H \oplus L/L)$, we have equality between the end terms for all $\alpha < \beta$.

(2) $$p^\alpha(G/L) \cap H \oplus L/L = p^\alpha G/L \cap H \oplus L/L = (p^\alpha G \cap H) \oplus L/L$$
$$= p^\alpha H \oplus L/L \subseteq p^\alpha(H \oplus L/L) \quad \text{for all } \alpha \leqq \beta.$$

(3) $$(G/L)(H \oplus L/L) \cong G/H + p^\beta G \cong (G/p^\beta G)/(H + p^\beta G/p^\beta G)$$
$$= (G/H)/p^\beta(G/H)$$

is countable of length at most β. Applying Theorem 2.1, we obtain $G/L = H \oplus L/L \oplus K/L$ and therefore $G = H \oplus K$.

To apply Theorem 2.1 to the proof of Theorem B, we require a few more lemmas.

LEMMA 2.4. *If $p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H$ for all $\alpha < \lambda$ and if λ is a limit ordinal, then $p^\alpha(G/H) = p^\alpha G + H/H$ for all $\alpha < \lambda$.*

Proof. Suppose that $\beta < \lambda$ and that $p^\alpha(G/H) = p^\alpha G + H/H$ for all $\alpha < \beta$.

Case 1. $\beta = \alpha + 1$. Let $x + H \in p^\beta(G/H)$ and write $x + H = p(y + H)$ with $y \in p^\alpha(G/H) = p^\alpha G + H/H$. We may then assume that $y \in p^\alpha G$, and therefore $x + H = py + H \in p^\beta G + H/H$, as desired.

Case 2. β is a limit. Assume that we have established

$$p^\beta(G/H)[p^n] \subseteq p^\beta G + H/H$$

and let $x + H \in p^\beta(G/H)[p^{n+1}]$. Then $p^n(x + H) \in p^{\beta+n}(G/H)[p] = (p^{\beta+n}G)[p] + H/H$ since $\beta + n < \gamma$. Thus $p^n x + H = p^n z + H$, where $z \in (p^\beta G)[p^{n+1}]$ and $(x - z) + H \in p^\beta(G/H)[p^n] \subseteq p^\beta G + H/H$. Therefore $x - z + H = w + H$ with $w \in p^\beta G$ and $x + H = (z + w) + H \in p^\beta G + H/H$.

LEMMA 2.5. *If K/H is a divisible subgroup of G/H and if* $p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H$, *then* $p^\alpha(G/K)[p] = (p^\alpha G)[p] + K/K$.

Proof. We can write $G/H = K/H \oplus M/H$. Let $x + K \in p^\alpha(G/K)[p]$ and write $x = k + m$ with $k \in H$ and $m \in M$. There is an obvious isomorphism $\phi: G/K \to M/H$ such that $\phi(x + K) = m + H$. Therefore

$$m + H \in p^\alpha(M/H)[p] \subseteq p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H.$$

Thus, there is a $z \in (p^\alpha G)[p]$ such that $m + H = z + H$, and consequently $x + K = m + K = z + K \in (p^\alpha G)[p] + K/K$.

LEMMA 2.6. *If K is a $p^{\alpha+n}$-high subgroup of G where $n < \omega$, then* $G[p^i] \subseteq K[p^i] + p^\alpha G$ *for all* $i \leqq n + 1$.

Proof. $G[p] = K[p] \oplus (p^{\alpha+n}G)[p] \subseteq K[p] + p^\alpha G$. Suppose that

$$G[p^i] \subseteq K[p^i] + p^\alpha G,$$

where $i \leqq n$ and let $x \in G[p^{i+1}]$. Then $p^i x = k + p^n z$, where $k \in K[p]$ and $z \in p^\alpha G$. However, $k = p^i x - p^n z \in p^i G \cap K = p^i K$, and therefore $k = p^i k_1$ for some $k_1 \in K$. We then have $x - k_1 - p^{n-i} z \in G[p^i] \subseteq K[p^i] + p^\alpha G$, from which it follows that $x \in K[p^{i+1}] + p^\alpha G$.

The next lemma should be worth the reader's effort to prove.

LEMMA 2.7. *If* $p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H$ *for all* $\alpha < \beta$, *then* $p^\alpha G \cap H = p^\alpha H$ *for all* $\alpha \leqq \beta$. *For* $\beta \leqq \omega$, *the converse holds.*

LEMMA 2.8. *If H is a direct summand of a p^α-high subgroup of G, then* $p^\gamma(G/H)[p] = (p^\gamma G)[p] + H/H$ *for all* $\gamma \leqq \alpha$.

Proof. Now H is necessarily a pure subgroup of G, and therefore the desired equality follows for all $\gamma < \omega$ by Lemma 2.7. We may therefore assume that $\alpha \geqq \omega$. Let $\alpha = \beta + n$, where $n < \omega$ and β is a limit ordinal. Suppose that $H \oplus M$ is a p^α-high subgroup of G. We assume that the lemma has been established for all ordinals $\delta < \alpha$. Now if $\omega \leqq \delta < \alpha$ and if A is a p^δ-high subgroup of H, then A is the direct summand of a p^δ-high subgroup of G. Indeed, $A \oplus B$ is p^δ-high in G if B is p^δ-high in M. Thus, by our inductive assumption, $p^\delta(G/A)[p] = (p^\delta G)[p] + A/A$. However, H/A is divisible and therefore, by Lemma 2.5, $p^\delta(G/H)[p] = (p^\delta G)[p] + H/H$. We conclude then that we have the desired equality for all $\gamma < \alpha$.

Showing that $p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H$ is obviously equivalent to showing that $p^\beta(G/H)[p^{n+1}] = (p^\beta G)[p^{n+1}] + H/H$. Let

$$x + H \in p^\beta(G/H)[p^{n+1}].$$

By the purity of H, we may assume that $x \in G[p^{n+1}]$ and, by Lemma 2.6, we can write $x = h + m + z$, where $h \in H[p^{n+1}]$, $m \in M[p^{n+1}]$, and $z \in p^\beta G$.

Since β is a limit, we may (by Lemma 2.4) write, for each $\gamma < \beta$, $x + H = z_\gamma + H$, where $z_\gamma \in p^\gamma G$. Thus, for each $\gamma < \beta$, we have an equation $x = h + m + z = h_\gamma + z_\gamma$ with $h_\gamma \in H$. Therefore

$$h - h_\gamma + m \in p^\gamma G \cap (H \oplus M) = p^\gamma H \oplus p^\gamma M \quad \textit{and} \quad m \in \bigcap_{\gamma<\beta} p^\gamma M = p^\beta M.$$

Consequently, we have the desired result

$$x + H = (m + z) + H \in (p^\beta G)[p^{n+1}] + H/H.$$

3. Proof of Theorem B. We approach the proof of Theorem B somewhat cautiously, choosing to prove some very technical preliminary lemmas.

LEMMA 3.1. *Let $K = \bigoplus_{i\in I} K_i$ be a subgroup of G such that each K_i is countable. If H is a subgroup of G such that $H \cap K = \bigoplus_{i\in I'} K_i$ and if A/H is a countable subgroup of G/H, then there is a subgroup M of G such that $A \subseteq M$, M/H* is *countable, and $M \cap K = \bigoplus_{i\in J} K_i$ for some subset J of I. If, moreover, G/K and $H/H \cap K$ are divisible, then M can be chosen so that $M/M \cap K$ is also divisible.*

Proof. $A \cap K/H \cap K$ is countable, and therefore there is a countable subset I'' of I such that $\bigoplus_{i\in I''} K_i$ contains a complete set of representatives of $A \cap K/H \cap K$. We need only set $M = A + \bigoplus_{i\in I''} K_i = A + \bigoplus_{i\in J} K_i$, where $J = I' \cup I''$.

Suppose, however, that we also have G/K and $H/H \cap K$ divisible. We first prove that if B/H is countable, then there exists an $N \supseteq B$ such that N/H is countable and $B \subseteq pN + K$. Indeed, if $b_1, b_2, \ldots, b_n, \ldots$ is a complete set of representatives of B/H, we choose elements $g_1, g_2, \ldots, g_n, \ldots$ such that $b_n - pg \in K$. If N is generated by B and the g_n's, then N/H is clearly countable and $B \subseteq pN + K$. It is then evident that we can choose two ascending sequences of subgroups

$$A \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots \quad \text{and} \quad N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots,$$

where M_n/H and N_n/H are countable for each n, $M_n \cap K = \bigoplus_{i\in I_n} K_i$, $M_n \subseteq N_n \subseteq M_{n+1}$, and $M_n \subseteq pN_n + K$ for each n. We then need only set

$$M = \bigcup_{n=1}^{\infty} M_n = \bigcup_{n=1}^{\infty} N_n \quad \text{and} \quad J = \bigcup_{n=1}^{\infty} I_n.$$

The conditions $M = \bigcup_{n=1}^{\infty} N_n$ and $N_n \subseteq pN_{n+1} + K$ ensure that $M/M \cap K$ is divisible.

We now extend Lemma 3.1 to take care of countably many K's.

LEMMA 3.2. *Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of subgroups of G and suppose, for each n, that $K_n = \bigoplus_{i\in I_n} K_n^{(i)}$, where each $K_n^{(i)}$ is countable. Let P be a subset of the positive integers such that G/K_n is divisible for each $n \in P$. If H is a subgroup of G such that $H \cap K_n = \bigoplus_{i\in I_n'} K_n^{(i)}$ for each n and $H/H \cap K_n$ is* *divisible whenever* $n \in P$ *and if* A/H *is a countable subgroup of* G/H, *then there is a subgroup* M *of* G *such that* $A \subseteq M$, M/H *is countable*, $M \cap K_n = \bigoplus_{i \in J_n} K_n^{(i)}$ *for each* n, *and* $M/M \cap K_n$ *is divisible whenever* $n \in P$.

Proof. Let $n_1, n_2, \ldots, n_k, \ldots$ be a sequence running infinitely through the positive integers, that is, for every pair of positive integers k and n there is a $j > k$ such that $n_j = n$. We construct by Lemma 3.1 an ascending sequence $M_1 \subseteq M_2 \subseteq \ldots \subseteq M_k \subseteq M_{k+1} \subseteq \ldots$ of subgroups of G each containing A and such that, for each k, M_k/H is countable, $M_k \cap K_{n_k} = \bigoplus_{i \in I_{n_k}{}^k} K_{n_k}{}^i$, and $M_k/M_k \cap K_{n_k}$ is divisible whenever $n_k \in P$. Set $M = \bigcup_{k=1}^{\infty} M_k$ and, for each n, let J_n be the union of all $I_{n_k}{}^k$ for which $n_k = n$. Since, for each n, M is the union of those M_k such that $n_k = n$, it is easily verified that M has the desired properties.

We require one further lemma before proceeding to the proof of Theorem B.

LEMMA 3.3. *Let* H *be a subgroup of* G *such that* $p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H$ *for all* $\alpha < \lambda$. *Suppose, further, that there is a normal decomposition* $G[p] = H[p] \oplus S$, *where* S *is a summable subsocle such that* $S \cap p^\lambda G = 0$. *Then* G/H *is a summable group of length at most* λ.

Proof. Let $S = \bigoplus_{\alpha<\lambda} S_\alpha$ with $S_\alpha - \{0\} \subseteq p^\alpha G - p^{\alpha+1} G$ for each α. Then $(G/H)[p] = G[p] + H/H = S \oplus H/H = \bigoplus_{\alpha<\lambda} (S_\alpha \oplus H/H)$. We need only show that, for each $\alpha < \lambda$, the non-zero elements of $S_\alpha \oplus H/H$ have precisely height α in G/H. Since the decomposition $G[p] = H[p] \oplus S$ is normal, $(p^\alpha G)[p] = (H[p] \cap p^\alpha G) \oplus (S \cap p^\alpha G) = (H[p] \cap p^\alpha G) \oplus \bigoplus_{\beta \geqq \alpha} S_\beta$. Therefore

$$p^\alpha(G/H)[p] = (p^\alpha G)[p] + H/H = (S \cap p^\alpha G) + H/H = \bigoplus_{\beta \geqq \alpha} (S_\beta \oplus H/H)$$

$$= (S_\alpha \oplus H/H) \oplus \bigoplus_{\beta \geqq \alpha+1} (S_\beta \oplus H/H) = (S_\alpha \oplus H/H) \oplus p^{\alpha+1}(G/H)[p].$$

Proof of Theorem B. For each α such that $\omega \leqq \alpha < \lambda$, let K_α be a p^α-high subgroup of G which is a direct sum of countable groups and fix direct decompositions $K_\alpha = \bigoplus_{i \in I_\alpha} K_\alpha{}^i$ with each $K_\alpha{}^i$ countable. Let $G[p] = \bigoplus_{i \in I} T_i$ be a normal decomposition with each T_i countable. Choose a well-ordering $\{g_\mu\}$, $\mu < M$, of the elements of G. Since λ is countable, the K_α can be enumerated and therefore, using Lemma 3.2, we construct a well-ordered family $\{H_\mu\}$, $\mu < M$, of subgroups satisfying the following conditions:

(1) $H_0 = 0$, $H_\mu \subseteq H_\sigma$ for $\mu < \sigma$ and $H_\mu = \bigcup_{\sigma<\mu} H_\sigma$ if μ is a limit;
(2) $g_\mu \in H_{\mu+1}$;
(3) $H_\mu \cap K_\alpha = \bigoplus_{i \in I_\alpha{}^\mu} K_\alpha{}^i$ for each α;
(4) $H_{\mu+1}/H_\mu$ is countable;
(5) $H_\mu/H_\mu \cap K_\alpha$ is divisible for each α;
(6) $H_\mu[p] = \bigoplus_{i \in I^\mu} T_i$.

By Lemma 2.8, $p^\alpha(G/H_\mu)[p] = (p^\alpha G)[p] + H_\mu/H_\mu$ for all $\alpha < \lambda$. Therefore,

https://doi.org/10.4153/CJM-1969-132-9 Published online by Cambridge University Press

by Lemma 2.7, each H_μ is isotype and, by Lemma 2.4, $p^\alpha(G/H_\mu) = p^\alpha G + H_\mu/H_\mu$ for all $\alpha < \lambda$. These two conditions combine to yield

$$p^\alpha(H_{\mu+1}/H_\mu) = p^\alpha H_{\mu+1} + H_\mu/H_\mu$$

for all $\alpha < \lambda$. By Lemma 3.3, $H_{\mu+1}/H_\mu$ has length at most λ. Finally, $p^\alpha H_{\mu+1} \cap H_\mu = (p^\alpha G \cap H_{\mu+1}) \cap H_\mu = p^\alpha G \cap H_\mu = p^\alpha H_\mu$ for all $\alpha \leqq \lambda$. Therefore, Theorem 2.1 yields a direct decomposition $H_{\mu+1} = H_\mu \oplus L_\mu$ for each $\mu < M$. Consequently, $G = \bigoplus_{\mu<M} L_\mu$ is a direct sum of countable groups.

4. A theorem of Nunke. Using homological techniques, Nunke (**7**) first proved the following theorem.

THEOREM 4.1. *If β is a countable ordinal and if G is a reduced p-primary abelian group such that both $G/p^\beta G$ and $p^\beta G$ are direct sums of countable groups, then G is a direct sum of countable groups.*

A non-homological proof has been given in (**3**). We shall give another here. First we need the following purification lemmas.

LEMMA 4.2. *If β is a countable ordinal and $x \in p^\beta G$, then there is a countable subgroup N of G such that $x \in p^\beta N$.*

Proof. By induction on β. The result is trivial for $\beta = 0$. Assume that it has been established for all ordinals $\alpha < \beta$.

Case 1. $\beta = \alpha + 1$. Then there exists a $y \in p^\alpha G$ such that $x = py$. By induction, there is a countable subgroup N such that $y \in p^\alpha N$. But then $x = py \in p^{\alpha+1}N = p^\beta N$.

Case 2. β is a limit ordinal. Then $x \in p^\alpha G$ for all $\alpha < \beta$. By induction, there exists, for each $\alpha < \beta$, a countable subgroup N_α such that $x \in p^\alpha N_\alpha$. Let N be the subgroup generated by all the N_α's. Clearly, N is countable and $x \in \bigcap_{\alpha<\beta} p^\alpha N_\alpha \subseteq \bigcap_{\alpha<\beta} p^\alpha N = p^\beta N$.

LEMMA 4.3. *Let β be a countable ordinal and let H be a subgroup of G such that $p^\beta G \cap H = p^\beta H$. If A/H is a countable subgroup of G/H, then there exists a subgroup K of G such that $A \subseteq K$, K/H is countable, and $p^\beta G \cap K = p^\beta K$.*

Proof. Since $A \cap p^\beta G/p^\beta H \cong (A \cap p^\beta G) + H/H \subseteq A/H$, we can choose a sequence $a_1, a_2, \ldots, a_n, \ldots$ that forms a complete set of representatives for $A \cap p^\beta G/p^\beta H$. By Lemma 4.2, there is for each n a countable subgroup N_n such that $a_n \in p^\beta N_n$. Let K_1 be generated by A and all the N_n's. Then K_1/H is countable and, as is easily seen, $A \cap p^\beta G \subseteq p^\beta K_1$. Continuing in this manner we construct a sequence $K_1 \subseteq K_2 \subseteq \ldots$ of subgroups with K_n/H countable and $K_n \cap p^\beta G \subseteq p^\beta K_{n+1}$. Set $K = \bigcup_{n=1}^\infty K_n$.

The proof of Theorem 4.1 requires the same sort of combinatorial game played in the proof of Theorem B. Actually, however, things are somewhat simpler. The requisite lemma is the following.

https://doi.org/10.4153/CJM-1969-132-9 Published online by Cambridge University Press

LEMMA 4.4. *Let β be a countable ordinal and let S be a subgroup of G such that $S = \bigoplus_{i \in I} B_i$ and $G/S = \bigoplus_{j \in J} A_j$, where each B_i and each A_j is countable. Suppose that H is a subgroup of G such that $H \cap S = \bigoplus_{i \in I'} B_i$, $H + S/S = \bigoplus_{j \in J'} A_j$, and $p^\beta G \cap H = p^\beta H$. If A/H is a countable subgroup of G/H, then there exists a subgroup M of G such that $A \subseteq M$, M/H is countable, $M \cap S = \bigoplus_{i \in I''} B_i$, $M + S/S = \bigoplus_{j \in J''} A_j$, and $p^\beta G \cap M = p^\beta M$.*

Proof. Now $A + S/H + S$ is clearly countable. Therefore there is a countable subset J^* of J such that $\bigoplus_{j \in J^*} A_j$ that contains a complete set of representatives of $(A + S/S)/(H + S/S)$. Now if V is generated by a complete set of representatives of $\bigoplus_{j \in J^*} A_j$, then V is a countable subgroup of G such that $V + S/S = \bigoplus_{j \in J^*} A_j$. Set $M_1 = A + N$ and $J_1 = J' \cup J^*$. Then M_1/H is countable and $M_1 + S/S = \bigoplus_{j \in J_1} A_j$. From this construction of M_1 and from Lemmas 4.3 and 3.1, it is then evident that there exist three ascending sequences of subgroups $M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots$, $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$, and $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots$ such that $H \subseteq M_n \subseteq N_n \subseteq K_n \subseteq M_{n+1}$, M_n/H is countable for each n, $M_n + S/S = \bigoplus_{j \in J_n} A_j$, $N_n \cap S = \bigoplus_{i \in I_n} B_i$, and $p^\beta G \cap K_n = p^\beta K_n$. Set $M = \bigcup_{n=1}^\infty M_n$, $I'' = \bigcup_{n=1}^\infty I_n$, and $J'' = \bigcup_{n=1}^\infty J_n$.

Proof of Theorem 4.1. Write $G/p^\beta G = \bigoplus_{j \in J} A_j$ and $p^\beta G = \bigoplus_{i \in I} B_i$, where each A_j and each B_i is countable. Choose a well-ordering $\{g_\mu\}$, $\mu < M$, of the elements of G. Using Lemma 4.4, we construct a well-ordered family $\{H_\mu\}$, $\mu < M$, of subgroups of G satisfying the following conditions:

(1) $H_0 = 0$, $H_\mu \subseteq H_\sigma$ for $\mu < \sigma$, and $H_\mu = \bigcup_{\sigma < \mu} H_\sigma$ if μ is a limit;
(2) $g_\mu \in H_{\mu+1}$;
(3) $H_\mu \cap p^\beta G = \bigoplus_{i \in I_\mu} B_i$;
(4) $H_\mu + p^\beta G/p^\beta G = \bigoplus_{j \in J_\mu} A_j$;
(5) $p^\beta G \cap H_\mu = p^\beta H_\mu$;
(6) $H_{\mu+1}/H_\mu$ is countable.

Clearly $p^\beta H_{\mu+1} \cap H_\mu = p^\beta H_\mu$ and $p^\beta H_\mu = \bigoplus_{i \in I_\mu} B_i$ is a direct summand of $p^\beta H_{\mu+1} = \bigoplus_{i \in I_{\mu+1}} B_i$. Now under the canonical isomorphism

$$H_{\mu+1} + p^\beta G/p^\beta G \to H_{\mu+1}/p^\beta H_{\mu+1},$$

$H_\mu + p^\beta H_{\mu+1}/p^\beta H_{\mu+1}$ is the image of $H_\mu + p^\beta G/p^\beta G$. Since the latter is a direct summand of $H_{\mu+1} + p^\beta G/p^\beta G$, the former is a direct summand of $H_{\mu+1}/p^\beta H_{\mu+1}$. All the conditions of Theorem 2.3 are satisfied and hence, for each $\mu < M$, we have a direct decomposition $H_{\mu+1} = H_\mu \oplus L_\mu$. Therefore $G = \bigoplus_{\mu < M} L_\mu$ is a direct sum of countable groups.

5. A theorem of Hill and the proof of Theorem A.

In addition to proving Theorem A, we give in this section a proof of Hill's theorem on isotype subgroups of direct sums of countable groups that is far simpler than that given in (**1**). As a special case of Hill's theorem, we first establish the following result.

PROPOSITION 5.1. *If α is a countable ordinal and if $G/p^\alpha G$ is a direct sum of*

https://doi.org/10.4153/CJM-1969-132-9 Published online by Cambridge University Press

countable groups, then every p^α-high subgroup of G is a direct sum of countable groups.

Proof. The proof is by induction on α, where we may surely assume that $\alpha \geqq \omega$. Assume that the proposition is established for all ordinals less than α and write $\alpha = \beta + n$, where β is a limit and $n < \omega$. Let H be a p^α-high subgroup of G. Then by Lemmas 2.8 and 2.4, $G/H = p^\gamma(G/H) = p^\gamma G + H/H$ for all $\gamma < \beta$. Therefore

$$H/p^\gamma H \cong p^\gamma G + H/p^\gamma G = G/p^\gamma G \cong (G/p^\alpha G)/p^\gamma(G/p^\alpha G)$$

is a direct sum of countable groups for all $\gamma < \beta$. Thus

$$(H/p^\beta H)/p^\gamma(H/p^\beta H) \cong H/p^\gamma H$$

is a direct sum of countable groups for all $\gamma < \beta$ and, by our induction hypothesis, each p^γ-high subgroup of $H/p^\beta H$ is a direct sum of countable groups. Since $H/p^\beta H \cong H + p^\beta G/p^\beta G$ and the latter group is isotype in the summable group $G/p^\beta G$, $H/p^\beta H$ is summable. Theorem B then applies to show that $H/p^\beta H$ is a direct sum of countable groups, and therefore, by Theorem 4.1, H itself is a direct sum of countable groups.

Remark 5.2. Proposition 5.1 fails for $\alpha = \Omega$. Indeed, if $p^\Omega G \neq 0$, a p^Ω-high subgroup of G cannot even be summable, much less a direct sum of countable groups. The proof of this assertion is easy. For suppose that H is summable and a p^Ω-high subgroup of G, where $p^\Omega G \neq 0$. For some $n < \omega$, the $(\Omega + n)$th Ulm invariant of G is non-zero, that is, $G[p]$ has an element of height $\Omega + n$. If $K \supseteq H$ is taken to be a $p^{\Omega+n+1}$-high subgroup of G, then K is obviously a summable group of length exceeding Ω, which is known to be impossible.

We now actually have Theorem A proved. The conditions that G be summable and $G/p^\alpha G$ be a direct sum of countable groups are obviously necessary. Proposition 5.1 together with Theorem B show these conditions to be sufficient.

The following theorem is due to Hill (**1**).

THEOREM 5.3. *If H is an isotype subgroup of G having countable length and if G is a direct sum of countable reduced p-groups, then H is also a direct sum of countable groups.*

Proof. Let λ be the length of H. Since $H \oplus p^\lambda G/p^\lambda G$ is isotype in $G/p^\lambda G$, there is no loss of generality in assuming that G also has countable length λ. The proof is by induction on λ. We assume that the theorem has been established for all lengths $\alpha < \lambda$. Then $H + p^\alpha G/p^\alpha G$ is isotype in $G/p^\alpha G$ for all $\alpha < \lambda$ and, by our induction hypothesis, $H/p^\alpha H \cong H + p^\alpha G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \lambda$.

Case 1. $\lambda = \alpha + 1$. Then both $H/p^\alpha H$ and $p^\alpha H$ are direct sums of countable groups, and the conclusion follows from Theorem 4.1.

Case 2. λ is a limit. We have $H/p^\alpha H$ as a direct sum of countable groups for all $\alpha < \lambda$ and that H, being isotype in the summable group G of countable length, is summable. Theorem A then implies that H is a direct sum of countable groups.

6. Further applications. Nunke (**8**), using properties of the functor Tor showed that if one p^{α}-high subgroup of a p-group is a direct sum of countable groups, then they all are direct sums. The use of Tor in establishing such a result strikes me as unnatural and it is therefore pleasing that I can give a group-theoretic proof of this result. First, the following lemma is needed.

LEMMA 6.1. *Let H and K be p^{α}-high subgroups of G, where $\alpha = \beta + n$ with $n < \omega$. Then the correspondence $h + p^{\beta}H \leftrightarrow k + p^{\beta}K$ if and only if $h - k \in p^{\beta}G$ yields an isomorphism between $(H/p^{\beta}H)[p]$ and $(K/p^{\beta}K)[p]$ that preserves heights (as computed in $H/p^{\beta}H$ and $K/p^{\beta}K$).*

Proof. It suffices to show that for each $h \in H$ such that $ph \in p^{\beta}H$ there exists a $k \in K$ such that $h - k \in p^{\beta}G$. Indeed, once this is established, it will be evident that the correspondence is well-defined, one-to-one, additive, and height-preserving. Suppose then that $h \in H$ and $ph \in p^{\beta}H$. We may assume that $h \notin K$. Then $\langle K, h\rangle \cap p^{\alpha}G \neq 0$, and therefore there is a $k_1 \in K$ and a $z_1 \in p^{\beta}G$ such that $k_1 + p^i h = p^n z_1 \neq 0$. Since $K \cap p^{\alpha}G = 0$, $i \leqq n$, and there is a $k_2 \in K$ such that $k_1 = p^i k_2$. By Lemma 2.6,

$$k_2 + h - p^{n-i}z_1 \in G[p^i] \subseteq K[p^i] + p^{\beta}G,$$

and therefore $h = (k_3 - k_2) + p^{n-i}z_1 + z$ for some $k_3 \in K[p^i]$ and $z \in p^{\beta}G$. Let $k = k_3 - k_2$.

THEOREM 6.2. *If one p^{α}-high subgroup of the p-group G is a direct sum of countable groups, then each p^{α}-high subgroup of G is a direct sum of countable groups.*

Proof. By Remark 5.2, we may assume that α is countable. Obviously, we may also take $\alpha \geqq \omega$. Write $\alpha = \beta + n$, where β is a limit and $n < \omega$. Let H and K be p^{α}-high subgroups of G and suppose that H is a direct sum of countable groups. Then $H/p^{\beta}H$ is also a direct sum of countable groups and, by Lemma 6.1, $K/p^{\beta}K$ is at least summable. Now β is a limit, and, as observed in the proof of Proposition 5.1, we must have $H/p^{\gamma}H \cong G/p^{\gamma}G \cong K/p^{\gamma}K$ for all $\gamma < \beta$. Therefore, by Theorem A, $K/p^{\beta}K$ is a direct sum of countable groups. However, $p^{\beta}K$ is bounded and thus, by Theorem 4.1, K itself is a direct sum of countable groups.

PROPOSITION 6.3. *If $G/p^{\alpha}G$ is summable for all $\alpha \leqq \lambda$, where λ is countable, then $G/p^{\lambda}G$ is a direct sum of countable groups.*

Proof. By induction on λ. Assume that the proposition has been established for all ordinals less than λ.

Case 1. $\lambda = \mu + 1$. By the inductive hypothesis, $G/p^{\mu}G$ is a direct sum of countable groups. But then both $(G/p^{\lambda}G)/p^{\mu}(G/p^{\lambda}G) \cong G/p^{\mu}G$ and $p^{\mu}(G/p^{\lambda}G)$

are direct sums of countable groups, and therefore the desired conclusion follows from Theorem 4.1.

Case 2. λ is a limit. By our inductive assumption, $(G/p^\lambda G)/p^\alpha(G/p^\lambda G) \cong G/p^\alpha G$ is a direct sum of countable groups for each $\alpha < \lambda$ and, since $G/p^\lambda G$ is summable, Theorem A yields the conclusion that $G/p^\lambda G$ is a direct sum of countable groups.

COROLLARY 6.4. *If G is a p-group of length Ω, then the following conditions are equivalent:*

(1) *$G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \Omega$;*
(2) *$G/p^\alpha G$ is summable for all $\alpha < \Omega$.*

COROLLARY 6.5. *Let λ be a countable limit ordinal and let G be a p-group of length λ. Then the following conditions are equivalent:*

(1) *G is a direct sum of countable groups;*
(2) *G is summable and $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \lambda$;*
(3) *$G/p^\alpha G$ is summable for all $\alpha \leqq \lambda$;*
(4) *G is summable and, for each $\alpha < \lambda$, the p^α-high subgroups of G are direct sums of countable groups.*

We can easily translate these results to statements about Ulm subgroups and Ulm factors. Recall that *the αth Ulm subgroup G^α of G* is defined by $G^\alpha = p^{\omega\alpha}G$ and that *the αth Ulm factor* is the quotient group $G^\alpha/G^{\alpha+1}$. The *type* of a p-group G is the smallest ordinal τ such that $G^\tau = 0$.

LEMMA 6.6. *Let τ be a countable ordinal. Then $G/p^\beta G$ is a direct sum of countable groups for all $\beta < \omega\tau$ if and only if G/G^α is a direct sum of countable groups for all $\alpha < \tau$.*

Proof. If $\beta < \omega\tau$, we can write $\beta = \omega\alpha + n$ with $n < \omega$ and $\alpha < \tau$. Then we have

$$G/G^\alpha = G/p^{\omega\alpha}G \cong (G/p^\beta G)/(p^{\omega\alpha}G/p^\beta G) = (G/p^\beta G)/p^{\omega\alpha}(G/p^\beta G).$$

Thus, $p^{\omega\alpha}(G/p^\beta G)$ is bounded and, if G/G^α is a direct sum of countable groups, $G/p^\beta G$ is a direct sum of countable groups by Theorem 4.1.

THEOREM 6.7. *Let G be a p-primary abelian group of countable type. Then G is a direct sum of countable groups if and only if*

(1) *all Ulm factors of G are direct sums of cyclic groups and*
(2) *G/G^α is summable for all limit ordinals α.*

Proof. Condition (2), in particular, tells us that G is summable. Therefore, whether or not G has limit length, it is clear that G will be a direct sum of countable groups if $G/p^\beta G$ is a direct sum of countable groups for all β less than the length of G. Let τ be the type of G. Then, by Lemma 6.6, it suffices to show that G/G^α is a direct sum of countable groups for all $\alpha < \tau$. Assume then that $\alpha < \tau$ and that we have established that G/G^β is a direct sum of countable groups for all $\beta < \alpha$.

Case 1. $\alpha = \gamma + 1$. Then $(G/G^\alpha)/p^{\omega\gamma}(G/G^\alpha) = (G/G^\alpha)/(G^\gamma/G^\alpha) \cong G/G^\gamma$ is a direct sum of countable groups by our induction hypothesis. Furthermore, $p^{\omega\gamma}(G/G^\alpha) = G^\gamma/G^\alpha = G^\gamma/G^{\gamma+1}$ is a direct sum of cyclic groups by (1). Thus, by Theorem 4.1, we conclude that G/G^α is a direct sum of countable groups.

Case 2. α is a limit ordinal. Then $G/G^\alpha = G/p^{\omega\alpha}G$ is summable by (2). However, by induction, G/G^γ is a direct sum of countable groups for all $\gamma < \alpha$. But then, by Lemma 6.6, $G/p^\beta G$ is a direct sum of countable groups for all $\beta < \omega\alpha$. An application of Theorem A yields the desired conclusion that $G/p^{\omega\alpha}G = G/G^\alpha$ is a direct sum of countable groups.

COROLLARY 6.8. *Let G be a summable p-group of type ω. Then G is a direct sum of countable groups if and only if all of its Ulm factors are direct sums of cyclic groups.*

Finally, we mention that some of the most striking applications of the generalized Kulikov criterion occur in (**6**), though a familiarity with Nunke's paper (**7**) is essential to full understanding of that paper. In particular, it is in (**6**) that one perceives that our Theorem B is the more potent formulation of the generalized Kulikov criterion.

REFERENCES

1. P. Hill, *Isotype subgroups of direct sums of countable groups* (to appear).
2. ——— *A summable C_Ω-group*, Proc. Amer. Math. Soc. (to appear).
3. P. Hill and C. Megibben, *On direct sums of countable groups and generalizations*, pp. 183–206, Études sur les Groupes Abéliens, Paris, 1968.
4. K. Honda, *On the structure of abelian p-groups*, pp. 81–86, Proc. Colloq. on Abelian Groups, Budapest, 1964.
5. L. Kulikov, *Zur Theorie der abelschen Gruppen von beliebizer Machtizheit*, Mat. Sb. *9* (1941), 165–181.
6. C. Megibben, *A generalization of the classical theory of primary groups* (to appear).
7. R. Nunke, *Homology and direct sums of countable abelian groups*, Math. Z. *101* (1967), 182–212.
8. ——— *On the structure of* Tor II, Pacific J. Math. *22* (1967), 453–464.

Vanderbilt University,
Nashville, Tennessee

https://doi.org/10.4153/CJM-1969-132-9 Published online by Cambridge University Press