Adv. Appl. Prob. **48,** 137–162 (2016)
*doi:*10.1017/apr.2015.10

SIR EPIDEMICS ON A SCALE-FREE SPATIAL NESTED MODULAR NETWORK

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Abstract

We propose a class of random scale-free spatial networks with nested community structures called SHEM and analyze Reed–Frost epidemics with community related independent transmissions. We show that in a specific example of the SHEM the epidemic threshold may be trivial or not as a function of the relation among community sizes, distribution of the number of communities, and transmission rates.

Keywords: Epidemics; SIR; Reed–Frost; percolation; long range; directed; scale free; modular; nested; communities; hierarchical; threshold; spatial; SHEM

2010 Mathematics Subject Classification: Primary 60K35
Secondary 92D30; 93A13; 82B43

1. Introduction

There are a number of both deterministic [12], [26] and random [9], [19], and [37] mathematical models of disease spreading, each formulated to overcome some shortcomings of the previous models. One of the current issues is to combine several of the improved features into a unified model.

Among the random models, in particular, the basic Bernoulli, or Erdős Rényi, random graph [30] has been modified to take care of spatial features [9] analyzed via percolation [32]. Other modifications, such as preferential attachment [5] and [16], are instead scale free, i.e. with a polynomial tail, mostly of exponent between 1 and 3, of the degree distribution [2] and [16]; one interesting finding is that some scale-free random models have a zero critical threshold for large-scale disease spreading [25], [31], and [41]. However, combining spatiality with scale-free properties requires some effort [15], [40]; see also [1], [17], and [24] for works connecting preferential attachment with a metric space; and [23] and [39], studied only on a numerical base due to their intrinsic complications. Other models, such as random intersection [10], exhibit network modularity, i.e. the gathering of individuals in communities with higher transmission rates [3], [4], [6], [7]. Yet another characteristic of some real-world networks is the nested structure of communities (see, for example, [11], [18], [35], and [38]), a feature missing in the networks generated by random intersection and similar mechanisms. Finally, transmission rates depend realistically from the type of community, and two individuals might have a complex intertransmission rate depending on the communities to which both belong.

Received 12 July 2011; revision received 3 February 2015.
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In this paper we introduce a new class of epidemic models, which we call *spatial hierarchical epidemic model* (SHEM), which exhibit many of the features mentioned above as they, at the same time, have spatial features, are scale-free, and possess a community structure with transmission rates between two individuals depending on the communities to which they both belong.

We now describe a quite general version of SHEM, and later specialize to a specific example. The basic construction of a SHEM is as follows. Consider an infinite set V whose elements $v \in V$ represent either individuals or their fixed locations, and which will be the vertex set of several graphs. To start, consider the set of all edges $\mathbb{B} = \{\{u, v\} \mid u, v \in V\}$ and a set of basic edges $\mathbb{B}_1 \subseteq \mathbb{B}$ determining the basic graph $\mathcal{G}_1 = (V, \mathbb{B}_1)$; basic edges represent some primitive relation between individuals, identify a basic metric structure, and, typically, refer to a spatial structure. Next, we take a hierarchical structure of partitions of V into larger and larger blocks representing communities. Then consider random sets S_u, $u \in V$, of nonnegative integers; S_u indicates the set of communities to which u belongs. Two individuals are in contact if they are linked by a basic edge in $\mathbb{B}_1$ or there is at least one community to which they both belong. The first random network we study is the random graph $\mathcal{G}_{\alpha,z} = (V, \mathbb{B}_{\alpha,z})$, where an edge $\{u, v\}$ is in $\mathbb{B}_{\alpha,z}$ if the two individuals u and v are in contact; here α and z are two parameters which are later specified. In a real population $\mathcal{G}_{\alpha,z}$ would be the network of the perceived contacts among individuals and we call it a connectivity graph or connectivity network. Note that the above construction introduces two distances on V: one is the distance d in the basic graph $\mathcal{G}_1$, the other is the distance $d_{\mathcal{H}}$ generated by the hierarchical structure (see Section 2 and [27] for details); we use both of them. The description of the model is then completed by the assumption that disease transmission takes place independently for each link between two individuals, either via the basic edges or via a community, with a rate which is decreasing as the community gets larger. Two additional parameters are natural to describe infections: a transmission rate λ_0 to neighbors, and a decrease rate ρ of the transmission rate for large communities.

By the above scheme, different versions of SHEM can be produced, possibly worthy of further investigation. We start here by studying in detail one specific example, in which, to focus on the role of network randomness, the disease transmission mechanism is kept as simple as possible. We call the specific example an SIR (susceptible $\to$ infected $\to$ removed) epidemic on a hypercubic lattice nested SHEM, as its basic graph is $V = \mathbb{Z}^d$ endowed with nearest-neighbor edges, communities are nested, and epidemic spreading is described by an SIR mechanism. To keep things simple, we consider constant infectious times $T = 1$, with the infection starting from a single infected individual, typically at the origin. Each infected individual u randomly transmits the disease to each susceptible individual with which u is in contact, independently in each community they share, or along the basic nearest-neighbor edges of $\mathbb{Z}^d$. The nearest-neighbor transmission rate is some $\lambda_0 > 0$, so that there is a probability $p = 1 - \exp(-\lambda_0) \in [0, 1]$ of transmission along nearest-neighbor edges. Transmission rates λ_k for a shared community at level k are taken to be exponentially decreasing, so that there exists $\rho \in [0, 1]$ such that $\lambda_k \approx p\rho^k$ in such a way that the transmission probability in a community at level k is $1 - \exp(-\lambda_k) = p\rho^k$.

The main focus of interest in epidemics is on large outbreaks from the initially infected individuals, which we study as a function of the parameters of the SIR epidemics on the hypercubic lattice nested SHEM. It is well known [19], [28], and [32] however that for SIR epidemics with constant transmission times, the set of infected individuals during an outbreak equals the percolation cluster of the initially infected individual in a bond percolation model with bond percolation probabilities equal to transmission probabilities. This allows us to use either percolation or epidemics terminology and techniques. In this paper the former is used mainly to develop the detailed mathematical arguments, and the latter to discuss the meaning of the obtained results. In particular, percolation models have the advantage of having no time variables. In our case the bond percolation probability, i.e. the transmission probability between two individuals, is then the probability that there is a transmission in one of the shared communities or via the basic edges.

The SIR epidemic on the hypercubic lattice nested SHEM will depend on five parameters:

(i) d, indicating space dimension;

(ii) z, determining the growth factor z^d of community sizes;

(iii) $\alpha \geq 1$, determining the distribution of the number of communities to which an individual belongs;

(iv) p, indicating the transmission probabilities to neighbors;

(v) ρ, modulating the decrease in transmission probabilities for large communities.

Our main results concern, on the one hand, the connectivity of the basic graph and, on the other hand, the occurrence of a large outbreak transition. For the connectivity of the basic graph, we show that the degree distribution D_v of any vertex $v \in V$ satisfies $\mathbb{P}(D_v \geq h) \approx h^{-\gamma+1}$, where $\gamma - 1 = \log_z \alpha/(d - \log_z \alpha)$, so that the network is scale-free for all $\alpha \in (1, z^d)$; in particular, for $z^{d/2} \leq \alpha \leq z^{2d/3}$ the network exhibits the typical value of $\gamma - 1 \in (1, 3)$. For the epidemics outbreak, we study the onset of a large outbreak as a function of p, and find that there is a well defined critical point p_{c} (which is independent of the random realization of the connectivity network) such that for $p > p_{\mathrm{c}}$ there is a large outbreak, while there is not for $p < p_{\mathrm{c}}$. We then find that p_{c} is trivial ($p_{\mathrm{c}} = 0$) or not depending on the decrease rate ρ of the transmission probabilities: if $\alpha \in [1, z^d]$ and $\rho < \alpha/z^d$, or $\alpha > z^d$, then $p_{\mathrm{c}} > 0$; however, if $\alpha \in [1, z^d)$ and $\rho > \alpha/z^d$ then $p_{\mathrm{c}} = 0$. This fully determines the (α, ρ) phase diagram in terms of the remaining parameters.

The large ρ phase means that with a slow decrease rate in the transmission probabilities a large epidemic outbreak occurs no matter how small these transmission probabilities are. Thus, our results reveal that the triviality of the critical value can indeed occur in scale-free networks [31] even in the presence of a very structured population with realistic features, outlining the possibility of such a highly undesirable situation. On the other hand, the fact that we identify a transition to the more common nontrivial threshold suggests lines of intervention in terms of reduction of transmission rates in selected highly pivotal communities: we leave this matter for future investigation. The phase diagram is plotted in Figure 1.

In Section 2 we give precise definitions and state the main results, and in Section 3 we discuss the relation with other models. In Section 4 we study the connectivity graph proving the results about the asymptotic degree distribution. In Section 5 we study the community dependent SIR epidemic and identify the parameter range where the critical point is trivial. Finally, in Section 6 we bound our model with a toy model in which each variable X_u is replaced by a collection of independent variables $\{X_{(u,v)}\}_{v \in \mathbb{Z}^d}$ on directed edges with the same distribution but with α replaced by $\sqrt{\alpha}$. In spite of the seemingly inaccurate bound, this enables us to identify the exact region in which $p_{\mathrm{c}} > 0$. A potential relation with one-dependent processes and Shearer's distribution and a hint as to why the bounds end up being so sharp is briefly discussed in Section 3.

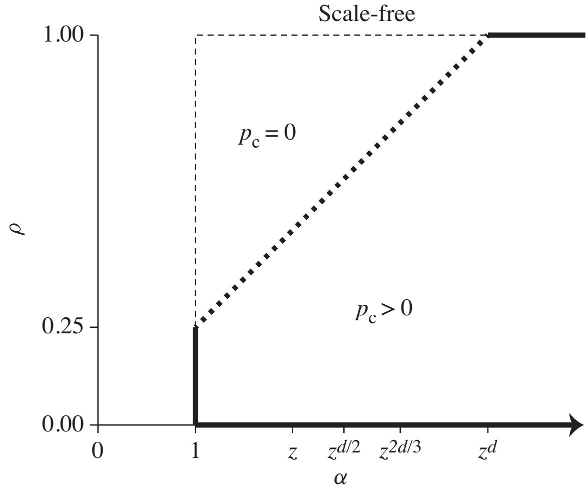

FIGURE 1: The $\alpha - \rho$ phase diagram of the nested hierarchical modular spatial hypercubic lattice nested SHEM in dimension d.

2. Definitions and main results

In this section we introduce the hypercubic lattice nested SHEM in detail and present the main results of this paper. Detailed proofs are carried out in the remaining sections, starting from the connectivity network first, and then moving to the SHEM.

As a preliminary step, though, we must point out that several proofs are obtained by comparing the SHEM and the connectivity network to other, either simpler or well-known, models. In so doing, we end up discussing a total of five different models plus a series of interpolating ones. To unify the presentation we introduce a common notation.

We consider the graph $(\mathbb{Z}^d, \mathbb{B}^d)$ with $\mathbb{B}^d$ the set of all bonds of $\mathbb{Z}^d$ (thus not limited to its nearest neighbor), or its directed version $(\mathbb{Z}^d, \vec{\mathbb{B}}^d)$, where $\vec{\mathbb{B}}^d$ is the set of all directed bonds of $\mathbb{Z}^d$. All models, including the SHEM, consist of distributions $\mathbb{P}$ on (the Borel σ-algebra of) $H = \{0, 1\}^{\mathbb{B}^d}$ or $\vec{H} = \{0, 1\}^{\vec{\mathbb{B}}^d}$. They are defined by taking an initial configuration space $\bar{X}$ (different for different models), a probability μ on the Borel σ-algebra of $\bar{X}$, a map $\phi\colon \bar{X} \to H$, or $\vec{H}$, so that $\mathbb{P} := \phi(\mu) = \mu\phi^{-1}$. As μ and ϕ depend on some parameters, so will $\mathbb{P}$. To avoid confusion, all of these elements are decorated in each model by a reference to the model name and, when it is relevant or there is risk of misunderstandings, the indication of the parameters (with the exception of d). For instance, in the SHEM $\mathbb{P} = \mathbb{P}^{\mathrm{SHEM}} = \mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p} = \phi^{\mathrm{SHEM}}_z(\mu^{\mathrm{SHEM}}_{\alpha,\rho,p})$, where $\mu = \mu^{\mathrm{SHEM}} = \mu^{\mathrm{SHEM}}_{\alpha,\rho,p}$ is a distribution on $\bar{X} = \bar{X}^{\mathrm{SHEM}} = \mathbb{N}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{B}^d \times \mathbb{N}} \times \{0, 1\}^{\mathbb{B}_1}$ with $\mathbb{B}_1 = \{\{u, v\} \mid u, v \in \mathbb{Z}^d, d(u, v) = 1\} \subseteq \mathbb{B}^d$ being the set of nearest-neighbor bonds. Elements of $\bar{X}$ are denoted by x and random variables taking values in $\bar{X}$ by X; the restriction of x to a subset is denoted by using the subset itself as index. For instance, if $x \in \bar{X}^{\mathrm{SHEM}}$ then $x_{\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ is its restriction to $\mathbb{Z}^d$ and $x_{\{u,v\}}$ and $x_{\mathbb{B}^d \times \{1,2,3\}}$ are other possible restrictions. Finally, if $\eta \in H$ or $\vec{H}$ is distributed according to the probability $\mathbb{P}$ of some model, the subset of $\mathbb{B}^d$ or $\vec{\mathbb{B}}^d$ in which $\eta = 1$ is denoted by $\mathbb{B}$ or $\vec{\mathbb{B}}$ decorated by the model name, and occasionally the parameters; this generates the random graph $\mathcal{G} = (\mathbb{Z}^d, \mathbb{B})$, accordingly decorated. In the SHEM model, for instance, it becomes $\mathbb{B}^{\mathrm{SHEM}} = \mathbb{B}^{\mathrm{SHEM}}_{\alpha,z,\rho,p}$ and $\mathcal{G}^{\mathrm{SHEM}} = (\mathbb{Z}^d, \mathbb{B}^{\mathrm{SHEM}})$. As a final remark, note that throughout this paper we are going to denote expectations of suitable random variables X with respect to a probability $\mathbb{P}$ simply by $\mathbb{P}(X)$.

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

We now give a detailed definition of the hypercubic lattice nested SHEM. As mentioned, we introduce, in order, $\bar{X}^{\text{SHEM}}$, μ^{SHEM}, ϕ^{SHEM}, $\mathbb{P}^{\text{SHEM}}$ on H, and, finally, $\mathcal{G}^{\text{SHEM}} = (\mathbb{Z}^d, \mathbb{B}^{\text{SHEM}})$. It all depends on the five parameters: the dimension d (the first parameter) and z, α, ρ, p, the remaining parameters introduced below.

We start from a system of partitions of $\mathbb{Z}^d$ into blocks. Let $z \geq 2$ be a fixed integer (the second parameter). For each $k = 0, 1, 2, \dots$, partition $\mathbb{Z}^d$ into blocks $\Lambda_{z,k}(w) = \Lambda_{z,k}(w_1, \dots, w_d) = \{v = (u_1, \dots, u_d) \in \mathbb{Z}^d \colon z^k w_j \leq u_j \leq (w_j + 1)z^k - 1 \text{ for all } j = 1, \dots, d\}$ for $w \in \mathbb{Z}^d$. Blocks represent a system of nested potential communities; here potential indicates that individual u has a chance to belong to all communities $\Lambda_{z,k}(w)$ for which $u \in \Lambda_{z,k}(w)$; whether it actually does belong to them is randomly decided as described further below. Note that blocks are hypercubes of linear size z^k, and that each partition is such that some of the hypercubes have a vertex at the origin; k is called the level of the community. Note, also, that vertices separated by coordinate hyperplanes always lie in different communities; the community structure is thus confined to orthants, and vertices in different orthants are connected only through nearest-neighbor connections: this constraint is not an unrealistic feature, however, as it might represent very rigid borders or seas.

Let $d_{\mathcal{H}}(u, v) = \min\{k \mid \text{there exists } w \in \mathbb{Z}^d \text{ such that } u, v \in \Lambda_{z,k}(w)\} \in \{0, 1, 2, \dots, \infty\}$ indicate the distance determined by the hierarchical structure of the $\Lambda_{z,k}$s; $d_{\mathcal{H}}(u, v)$ indicates the level of the smallest potential community to which both individuals u and v could simultaneously belong and takes the value ∞ if there is no such community. Note that $k \geq d_{\mathcal{H}}(u, v)$ if and only if there exists $w \in \mathbb{Z}^d$ such that $u, v \in \Lambda_{z,k}(w)$. Note also that there are, in fact, two metrics, the Euclidean distance d and the graph distance $d_{\mathcal{H}}$, which play a role in defining the SHEM and that both are stationary (see [27] for a discussion on $d_{\mathcal{H}}$).

The configuration space $\bar{X}^{\text{SHEM}}$ is built in steps. First, $x_u \in \mathbb{N}$, $u \in \mathbb{Z}^d$, describes actual participation to a community: u belongs to all communities $\Lambda_{z,k}(w)$ such that $u \in \Lambda_{z,k}(w)$ and $x_u \geq k$; such a set of k's is the set S_u mentioned in the introduction. Next, $x_{\{u,v\},k} \in \{0, 1\}$ indicates whether disease transmission takes place between two individuals u, v if they belong to the same community at level k. Finally, $x_{\{u,v\}} \in \{0, 1\}$, $u, v \in \mathbb{Z}^d$, $d(u, v) = 1$, indicates if transmission takes place between two nearest neighbors. Therefore, we take

$$\bar{X}^{\text{SHEM}} = \mathbb{N}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{B}^d \times \mathbb{N}} \times \{0, 1\}^{\mathbb{B}_1}.$$

A random variable $X = X_{\mathbb{Z}^d, \mathbb{B}^d \times \mathbb{N}, \mathbb{B}_1} \in \bar{X}$ with distribution μ^{SHEM} describes the random elements $x \in \bar{X}$. We define μ^{SHEM} as follows.

For $\alpha \geq 1$ (the third parameter), let $\mu_\alpha = \prod_{u \in \mathbb{Z}^d} \mu_{\alpha,u}$ with $\mu_{\alpha,u}(X_u \geq k) = \alpha^{-k}$, $k = 0, 1, \dots$; for $\alpha = 1$ the definition is extended by assuming that $X_u = +\infty$ for all u. Note that with this distribution the overlap, i.e. the average number of communities to which an individual belongs, is $\sum_{k=1}^{\infty} \alpha^{-k} = (\alpha - 1)^{-1}$, and that for human communities this is a realistic number for values of α approximately in the interval $[\frac{5}{4}, \frac{10}{4}]$; see [21]. Note also that for $1 \leq \alpha \leq z^d$ all communities are populated, with an average number $(z^d/\alpha)^k$ of individuals.

Next, given $\rho \in [0, 1]$ (the fourth parameter) and $p \in [0, 1]$ (the fifth parameter), the distribution of $X_{\{u,v\},k}$, $\{u, v\} \in \mathbb{B}^d$, $k \in \mathbb{N}$, is described by a Bernoulli probability $\mu'_{\rho,p} = \prod_{\{u,v\} \in \mathbb{B}^d, k \in \mathbb{N}} \mu'_{\rho,p,\{u,v\},k}$, where $\mu'_{\rho,p,\{u,v\},k}(X_{\{u,v\},k} = 1) = p\rho^{-k}$. Finally, for p as above, the distribution of $X_{\{u,v\}}$, $\{u, v\} \in \mathbb{B}_1$, $d(u, v) = 1$, is described by a Bernoulli probability $\mu''_p = \prod_{\{u,v\} \in \mathbb{B}^d, d(u,v)=1} \mu''_{p,\{u,v\}}$, where $\mu''_{\rho,p,\{u,v\}}(X_{\{u,v\}} = 1) = p$.

Finally, we take $\mu^{\text{SHEM}} = \mu^{\text{SHEM}}_{\alpha,\rho,p} = \mu_\alpha \times \mu'_{\rho,p} \times \mu''_p$. The map ϕ^{SHEM} completes the construction by indicating that u and v are connected in the percolation transcription of epidemic

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

diffusion if they share a community or are nearest-neighbors and transmit disease to each other at that level. Hence, $\phi^{\text{SHEM}} = \phi_z^{\text{SHEM}} : \bar{X}^{\text{SHEM}} \to H$ is such that

$$(\phi_z^{\text{SHEM}}(x))_{\{u,v\}} = \mathbf{1}_{(\text{there exists } k\in\mathbb{N}:\ d_{\mathcal{H}}(u,v)\le k\le \min(x_u,x_v) \text{ and } x_{\{u,v\},k}=1, \text{ or } d(u,v)=1 \text{ and } x_{\{u,v\}}=1)}\,.$$

Note that $(\phi_z^{\text{SHEM}}(x))_{\{u,v\}}$ depends not only on $x_{\{u,v\}}$ but also on x_u, x_v, and $x_{\{u,v\},k}$.

The graph of the actual connections is then $\mathcal{G}^{\text{SHEM}} = (\mathbb{Z}^d, \mathbb{B}^{\text{SHEM}})$, where $\mathbb{B}^{\text{SHEM}} = \mathbb{B}^{\text{SHEM}}(X) = \{\{u,v\} \in \mathbb{B}^d : (\phi^{\text{SHEM}}(X))_{\{u,v\}} = 1\}$. Consequently, $\mathbb{B}^{\text{SHEM}}$ has distribution $\mathbb{P}^{\text{SHEM}} = \mathbb{P}^{\text{SHEM}}_{z,\alpha,\rho,p} = \phi_z^{\text{SHEM}}(\mu^{\text{SHEM}}_{\alpha,\rho,p})$.

With this definition, there is a probability $p\rho^k$ of infectious contact (or connection in percolation terminology) between each pair of individuals when they simultaneously belong to a community at level k, and a probability p of the same occurrence if they are nearest-neighbors. Note that two individuals may have several occasions of disease transmission if they share more communities and/or are also neighbors.

The main result of this paper is about the occurrence of a large outbreak. By the equivalence with percolation mentioned in the introduction, a large outbreak is equivalent to the occurrence of percolation. For $u \in \mathbb{Z}^d$ and $\eta \in H$, let $V_u = V_u(\eta)$ be the set of vertices connected to u, let $A_{u,\infty} = \{\eta \mid |V_u(\eta)| = \infty\}$ and let $A_\infty = \bigcup_{u\in\mathbb{Z}^d} A_{u,\infty}$. We say that a large outbreak or percolation occurs if $\mathbb{P}^{\text{SHEM}}(A_{u,\infty}) > 0$ or, equivalently, $\mathbb{P}^{\text{SHEM}}(A_\infty) = 1$ (see the proof of Theorem 2.1 below), and does not if such probabilities are 0. We are interested in the set of parameters d, α, z, ρ, and p for which a large outbreak or percolation occurs.

In Section 5, by fairly standard arguments, we prove the following theorem.

Theorem 2.1. *For every $d \ge 2$, $z \ge 2$, $\alpha \ge 1$, and $\rho \in [0,1]$, there exists a value $p_c(\alpha,\rho) = p_c(d,z,\alpha,\rho)$ such that, for all $u \in \mathbb{Z}^d$, $\mathbb{P}^{\text{SHEM}}_{z,\alpha,\rho,p}(A_{u,\infty}) > 0$ for all $p > p_c(\alpha,\rho)$ and $\mathbb{P}^{\text{SHEM}}_{z,\alpha,\rho,p}(A_{u,\infty}) = 0$ for all $p < p_c(\alpha,\rho)$.*

In terms of epidemics, when $p_c(\alpha,\rho) > 0$ no large outbreak occurs for small p, i.e. for small values of the transmission parameter λ_0, while a large outbreak occurs for large p, i.e. for large transmission rates λ_0. On the other hand, if $p_c(\alpha,\rho) = 0$ then there is a large outbreak no matter how small the transmission rates are.

In Section 5 we obtain the main result of this paper by determining the entire phase diagram of a SIR epidemic on the hypercubic lattice nested SHEM. We show that both regimes can occur, depending on the relation among the remaining four parameters, as stated in the next theorem.

Theorem 2.2. *For $\alpha \in [1, z^d]$,*

$$p_c(\alpha,\rho) \begin{cases} = 0 & \text{if } \rho > \alpha/z^d, \\ > 0 & \text{if } \rho < \alpha/z^d. \end{cases}$$

For $\alpha > z^d$, we have $p_c(\alpha,\rho) > 0$ for all $\rho \in [0,1]$.

As a final remark, let us mention that a third possible regime, namely $p_c(\alpha,\rho) = 1$ never occurs: since nearest-neighbor edges are included, $p_c(\alpha,\rho) \le p_c(d) < 1$, where $p_c(d)$ is the critical point for independent edge percolation in $\mathbb{Z}^d$.

Before proving these results, we explore in Section 3 the connectivity network. This is just the random graph determined by spatial and community relations, and amounts to taking $\rho = p = 1$ in the SHEM. It is described by the probability distribution $\mathbb{P}^{\text{CN}} = \mathbb{P}^{\text{CN}}_{z,\alpha} = \mathbb{P}^{\text{SHEM}}_{z,\alpha,1,1}$

on H. The random network is then $\mathcal{G}^{\mathrm{CN}} = (\mathbb{Z}^d, \mathbb{B}^{\mathrm{CN}})$, where $\mathbb{B}^{\mathrm{CN}} = \mathbb{B}^{\mathrm{CN}}_{z,\alpha} = \mathbb{B}^{\mathrm{SHEM}}_{z,\alpha,1,1}$ is the random set of 1s in a configuration $\eta \in H$ distributed according to $\mathbb{P}^{\mathrm{CN}}$. To realize such a distribution we can take $\bar{X}^{\mathrm{CN}} = \mathbb{N}^{\mathbb{Z}^d}$,

$$\phi_z^{\mathrm{CN}}(x)_{\{u,v\}} = \mathbf{1}_{\{\text{there exists } k\in\mathbb{N}\colon\, d_{\mathcal{H}}(u,v)\le k\le \min(x_u,x_v) \text{ or } d(u,v)=1\}},$$

and $\mu^{\mathrm{CN}} = \mu^{\mathrm{CN}}_{\alpha} = \mu_\alpha$, the same distribution used in the SHEM as one of the marginals in the definition of μ^{SHEM}.

The main result about the connectivity network concerns the asymptotic behavior of the distribution of the degree $D_u = D_u(\eta) = |\{v \in \mathbb{Z}^d \mid \{u, v\} \in \mathbb{B}^{\mathrm{CN}}\}|$.

Theorem 2.3. *For all* α, z *such that* $1 \le \alpha < z^d$,

$$\begin{aligned}\left(\frac{z^d-1}{z^d-\alpha}\right)^{\gamma-1} &\ge \limsup_{h\to\infty} \mathbb{P}^{\mathrm{CN}}_{z,\alpha}(D_u > h)h^{\gamma-1}\\ &\ge \liminf_{h\to\infty} \mathbb{P}^{\mathrm{CN}}_{z,\alpha}(D_u > h)h^{\gamma-1}\\ &\ge \frac{1}{\alpha}\left(\frac{z^d-1}{z^d-\alpha}\right)^{\gamma-1} \quad \textit{for all } u\in\mathbb{Z}^d,\end{aligned}$$

where $\gamma - 1 = \log_z \alpha/(d - \log_z \alpha)$.

By Theorem 2.3, for large h, $\mathbb{P}^{\mathrm{CN}}(D = h) \approx h^{-\gamma}$ with γ as above. Thus, the connectivity network $\mathcal{G}^{\mathrm{CN}}$ is scale-free for each $\alpha \in (1, z^d)$. The typical degree of most realistic networks is obtained for $2 \le \gamma \le 3$, which is $z^{d/2} \le \alpha \le z^{2d/3}$. The relation with overlap is also quite satisfactory: for $d = 2$ and $z = 2$, the typical scale-free range is $[2, 2^{4/3}]$ which has nonempty intersection with the region $[\frac{5}{4}, \frac{10}{4}]$ where the overlap is realistic.

From the combined analysis of epidemics and of the connectivity network, the transition between absence to nontriviality of a critical threshold emerges, as discussed in the introduction, even in the scale-free parameter region.

3. Relation with other models

Before commencing with formal proofs we comment on the diversity of models appearing in this work and on the relation with other models appearing in the literature. There are three reasons behind this diversity:

(a) while we focus here on a well-determined construction, it is a richness of the model that the general scheme of the SHEM can be detailed in many different ways;

(b) several steps in our proofs end up comparing a SIR epidemic on the hypercubic lattice nested SHEM to modified (sometimes unrealistic) models; note that, signalling a likely scarce influence of specific details, some of the results and proofs shown here hold for some other of these models as well;

(c) it turns out that hypercubic lattice nested SHEM reproduces, for limiting values of its parameters, several other models appearing in the literature, either exactly or approximately. Thus, although the frequent change of model burdens notation and reduces the clarity of exposition, it is actually a strength of this work.

We now briefly review the relation to other models, in particular, with those obtained as limits as mentioned in (c) above. Parameters used by the cited papers are indexed by the initials of the authors of each paper, in order to avoid confusing them with the parameters used here.

(c_1) For $\rho = 0$, we simply obtain the classic nearest-neighbor edge percolation on $\mathbb{Z}^d$ (see, for example, [22]), in which only the parameter p remains.

(c_2) The $\alpha = 1$ case is such that $S_u = \mathbb{N}$ and each individual u belongs to all communities containing the vertex u. Except for the nearest-neighbor edges, this is exactly the long-range percolation (LRP) model on the hierarchical lattice studied in [27], with their $p_{\mathrm{KMT},k_1} = (1 - \exp(-\alpha_{\mathrm{KMT}}/\beta_{\mathrm{KMT}}^{k_1})) = 1 - \prod_{h \geq k_1}(1 - p\rho^h)$. Since the right-hand side is approximately $(k_2 p/(1 - \rho^{k_2}))\rho^{k_1}$ for some k_2 (see (7.2), (7.3), and (7.7) below) and $1 - \exp(-\alpha_{\mathrm{KMT}}/\beta_{\mathrm{KMT}}^{k_1}) \approx \alpha_{\mathrm{KMT}}/\beta_{\mathrm{KMT}}^{k_1}$, our parameters p and ρ play the role of α_{KMT} and $\beta_{\mathrm{KMT}}^{-1}$, respectively, while their N_{KMT} is equal to our z^d. Koval *et al.* [27] determined three phases in terms of the critical $\alpha_{\mathrm{KMT},c}$ $(= p_{\mathrm{c}}(\alpha, \rho))$; the first has $\alpha_{\mathrm{KMT},c} = 0$ for $\beta_{\mathrm{KMT}} \leq N_{\mathrm{KMT}}$, which corresponds to our $\rho \geq z^{-d}$ $(= \alpha z^{-d}$ for $\alpha = 1)$, and the second has $\alpha_{\mathrm{KMT},c} > 0$ for $N_{\mathrm{KMT}} \leq \beta_{\mathrm{KMT}} \leq N_{\mathrm{KMT}}^2$, which corresponds to our $z^{-2d} \leq \rho \leq z^{-d}$. The third phase has $\alpha_{\mathrm{KMT},c} = \infty$ (which would be $p_{\mathrm{c}} = 1$), but our model does not exhibit such a phase as we have included all nearest-neighbor edges so that $p_{\mathrm{c}}(\alpha, \rho) < 1$. There is an analogous relation with the results in [13] which are for $\alpha = 1$ and $N_{\mathrm{DG}} = N_{\mathrm{KMT}} = z^d \to \infty$, and with those in [14] which are for $\alpha = 1$, and $p_{\mathrm{DG},k} = c_{\mathrm{DG},k}/N_{\mathrm{DG}}^{k(1+\delta_{\mathrm{DG}})} \approx (k_2 p/(1 - \rho)^{k_2})\rho^k$ for $k = d_{\mathcal{H}}(u, v)$), so that $N_{\mathrm{DG}}^{(1+\delta_{\mathrm{DG}})}$ plays the role of our ρ with δ_{DG} the relevant parameter. We point out, however, that the formulation of [14, Theorem 3.1] considers large enough $c_{\mathrm{DG},k}$ (which plays the role of our p) and thus the transition at their $\delta_{\mathrm{DG}} = 0$ (which amounts to our $\rho = z^{-d}$) goes unnoticed in that paper.

(c_3) For $\rho = p = 1$, in which case we are merely observing the connectivity network, the SIR epidemic on the hypercubic lattice nested SHEM compares to the Yukich model [40], which describes a model with two parameters s_{Y} and δ_{Y}: $\{U_v\}_{v \in \mathbb{Z}^d}$ are independent and identically distributed (i.i.d.) uniform [0, 1] random variables and u, v are connected if and only if $d(u, v) \leq \delta_{\mathrm{Y}} \min(U_u^{-s_{\mathrm{Y}}}, U_v^{-s_{\mathrm{Y}}})$. It is actually possible to show that for any increasing event A concerning edge connections, such as $\{D_v > h\}$, the probability of A in the SIR epidemic on the hypercubic lattice-nested SHEM with parameters d, z, α (and $\rho = p = 1$) is bounded by the probability of the same event in the Yukich model with parameters $s_{\mathrm{Y}} = (\log_z \alpha)^{-1}$ and $\delta_{\mathrm{Y}} = \sqrt{d}$. Since Yukich [40] computed the exact asymptotic degree distribution, we used those results to state a proof of Theorem 2.3. The proof has now been improved upon, as a result of a referee's suggestion, and the comparison with Yukich's network is no longer needed.

(c_4) Still, the random disk model, which we introduce in order to prove Lemma 7.1 below, is very close to a modified version of Yukich's network in which each connection in Yukich's network is independently removed with a probability depending on the distance. From our results it is not difficult to extract some information about the phase diagram of such a weakened version of Yukich's network, but this seems of no interest here.

(c_5) The scale-free percolation model in [15] is also based on i.i.d. random variables, denoted by W_u, with distribution satisfying $F(w) \approx 1 - w^{-(\tau_{\mathrm{DHH}}-1)}$, and the probabilities of connection are of the form

$$p_{\mathrm{DHH},u,v} = p_{\mathrm{DHH}}(W_u, W_v, d(u,v)) = 1 - \exp\biggl(-\lambda_{\mathrm{DHH}}\biggl(\frac{W_u W_v}{d(u,v)^{\alpha_{\mathrm{DHH}}}}\biggr)\biggr).$$

There is a correspondence with our parameterization as λ_{DHH} plays the same role as our p, $\alpha_{\mathrm{DHH}} - 2$ plays the same role as our $-\log_z \rho$, W_u corresponds to $\sqrt{d} z^{X_u}$ so that $\tau_{\mathrm{DHH}} - 1$ corresponds to $\log_z \alpha$. However, the formulation in [15] and our use of $d_{\mathcal{H}}(u,v)$ instead of $d(u,v)$ determine different behaviors of percolation in the two models. In particular, Deijfen *et al.* [15] showed that the critical $\lambda_{\mathrm{DHH},c}$ is nonzero if and only if the degree distribution of any vertex (which is independent of λ_{DHH}) has a finite second moment. Such phenomenon does not occur in our case: we compute the asymptotic of the degree distribution only for the connectivity graph (which is for $\rho = p = 1$), but already in that case percolation occurs only for $\alpha > z^d$ by Theorem 2.2, when the degree distribution has all moments.

(c_6) It is also interesting to simultaneously compare the hypercubic lattice SHEM to [8], [15], and [27]. In [8] a very general family of inhomogeneous random graphs was proposed, based on i.i.d. random variables like our X_u, such that the connection probability, when just the first n vertices are considered, is of the form $p_{u,v} = p(X_u, X_v, n)$; [15] generalizes it to a graph which can be immediately infinite and $p_{u,v} = p(X_u, X_v, d(u,v))$. On the other hand, Koval *et al.* [27] considered the hierarchical structure and sets $p_{\mathrm{DHH},u,v} = p(d_{\mathcal{H}}(u,v))$. In this perspective, our work is developing the most general case in which $p_{u,v} = p(X_u, X_v, d(u,v), d_{\mathcal{H}}(u,v))$.

(c_7) As $\alpha \to \infty$, the SIR epidemic on the hypercubic lattice-nested SHEM tends again to nearest-neighbor bond percolation in $\mathbb{Z}^d$, and already for $\alpha > z^d$ we expect that the model has the same overall characteristics in terms of the percolation-large outbreak transition.

Finally, we comment on the stochastic properties of $\mathbb{P}^{\mathrm{SHEM}}$.

(d) Under $\mathbb{P}^{\mathrm{SHEM}}$ the η variables, indicating effective disease transmission, are such that if $\{u_i, v_i\} \cap \{u_j, v_j\} = \varnothing$ for $i \neq j$, then the $\eta_{\{u_i,v_i\}}$, $i = 1, \ldots, n$, are collectively independent. This is the edge analogy of the property of a Bernoulli random field with a dependency graph or one-dependent process ([20] and [36]). From this point of view, $\mathbb{P}^{\mathrm{SHEM}}$ is a Bernoulli random field with a dependency hypergraph HG; more precisely, vertices of HG correspond to the edges $\mathbb{B}^d$ and hyperedges of HG correspond to the vertices of $\mathbb{Z}^d$.

For a Bernoulli random field $\mathbb{P}$ one finds, under some conditions, a Bernoulli independent distribution, called Shearer's measure, which stochastically bounds $\mathbb{P}$ and has the same marginals; see [34] and [36]. In our proof we also end up bounding $\mathbb{P}^{\mathrm{SHEM}}$ with a Bernoulli independent distribution, see $\mathbb{P}^{\mathrm{DRD}}$ in Section 7 below, which is optimal in some sense (at least in the sense that it captures the precise phase transition diagram). This somehow suggests that $\mathbb{P}^{\mathrm{DRD}}$ could be some sort of generalized Shearer's measure. There are too many differences, however, to draw direct conclusions from the theory of Bernoulli random fields; for instance, the use of a hypergraph instead of a graph, and the fact that $\mathbb{P}^{\mathrm{DRD}}$ is on variables attached to directed edges (a concept which is absent for the vertex variables of a one-dependent process). But at least this relation hints as to why the seemingly inaccurate bounds of Section 7 end up capturing the phase diagram completely.

4. Degree distribution of the connectivity network

The probability $\mathbb{P}^{\text{CN}}$ is not stationary and the distribution of D_u depends on u, but the dependency is very weak. The hierarchical part is, in fact, stationary as vertices u lie in identical sequences of $\Lambda_{z,k}(w_{k,u})$ for suitable $w_{k,u}$; only the nearest-neighbor relations in the SHEM do not match the hierarchical structure exactly, due to the nonequivalence of $d(u, v)$ and $d_{\mathcal{H}}(u, v)$. Since in the connectivity network each vertex has $2d$ nearest neighbors, the number of nearest neighbors which are added to the set of neighbors observed via the communities can vary from 0 to $2d$ at most. Hence, for all $u, v \in \mathbb{Z}^d$ and $h > 0$,

$$\mathbb{P}^{\text{CN}}(D_u \geq h + 2d) \leq \mathbb{P}^{\text{CN}}(D_v \geq h) \leq \mathbb{P}^{\text{CN}}(D_u \geq h - 2d). \tag{4.1}$$

In order to prove Theorem 2.3, we first compute the expected degree $D = D_0$, with 0 being the origin of $\mathbb{Z}^d$, for each fixed value k of X_0.

Lemma 4.1. *For all $\alpha \in (1, z^d)$, it holds that*

$$\mathbb{E}_k := \mu_\alpha^{\text{CN}}(D \mid X_0 = k) = c_1(d, z, \alpha)\left(\frac{z^d}{\alpha}\right)^k + c_2(d, z, \alpha),$$

where $c_1(d, z, \alpha) = (z^d - 1)/(z^d - \alpha)$ and $c_2(d, z, \alpha)$ is such that

$$-\frac{z^d - 1}{z^d - \alpha} \leq c_2(d, z, \alpha) \leq -\frac{z^d - 1}{z^d - \alpha} + 2d.$$

Proof. Consider the boxes $\Lambda_{z,\ell}(0)$, $\ell \in \mathbb{N}$. Then

$$\begin{aligned}
\mathbb{E}_k &= \sum_{\ell=1}^{k} \sum_{v \in \Lambda_{z,\ell}(0) \setminus \Lambda_{z,\ell-1}(0)} \mu_\alpha^{\text{CN}}(X_v \geq \ell) + \sum_{v\colon d(0,v)=1} (\mathbf{1}_{\{X_v < d_{\mathcal{H}}(0,v) \leq k\}} + \mathbf{1}_{\{k < d_{\mathcal{H}}(0,v)\}}) \\
&= \sum_{\ell=1}^{k} \frac{z^{d\ell} - z^{d(\ell-1)}}{\alpha^\ell} + c_3(k, \alpha) \\
&= \frac{z^d - 1}{z^d - \alpha} \frac{z^{d\cdot k} - \alpha^k}{\alpha^k} + c_3(k, \alpha) \\
&= \frac{z^d - 1}{z^d - \alpha}\left(\frac{z^d}{\alpha}\right)^k - \frac{z^d - 1}{z^d - \alpha} + c_3(k, \alpha) \\
&= c_1(d, z, \alpha)\left(\frac{z^d}{\alpha}\right)^k + c_2(d, z, \alpha),
\end{aligned}$$

where $0 \leq c_3(k, \alpha) \leq 2d$, so that c_1 and c_2 are as claimed. □

Proof of Theorem 2.3. For $\alpha = 1$ the result is trivial, as all terms equal 1. For $\alpha > 1$, in view of (4.1), it is enough to consider $u = 0$. For $h > 0$, let $\bar{k}(h) = \max\{k\colon c_1(z^d/\alpha)^k + c_2 \leq h\}$, and, for $\varepsilon > 0$, consider $k_1(h) = \bar{k}(h - h^{1/2+\varepsilon} - 4d)$ and $k_2(h) = \bar{k}(h + h^{1/2+\varepsilon} + 6d)$. Note that $k_2(h) - k_1(h) \leq 1$ for large h, but can actually be 1.

Denote $\mathbb{E}_\Sigma = \mu_\alpha^{\text{CN}}(\sum_{\ell=1}^{k} \sum_{v \in \Lambda_{z,\ell}(0) \setminus \Lambda_{z,\ell-1}(0)} \mathbf{1}_{\{X_v \geq \ell\}})$; the random variables appearing in the sum are binary and independent, hence, their maximum value is 1 and $\mathbb{E}_\Sigma$ is also the sum

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

of their second moments. Note also that $|\mathbb{E}_\Sigma - \mathbb{E}_k| \le 2d$. By the Bernstein inequality, for all $k \le k_1(h)$, we have

$$\begin{aligned}
\mu_\alpha^{\mathrm{CN}}(D > h \mid X_0 = k) &\le \mu_\alpha^{\mathrm{CN}}\Bigg(\Bigg(\sum_{\ell=1}^{k} \sum_{v \in \Lambda_{z,\ell}(0)\setminus\Lambda_{z,\ell-1}(0)} \mathbf{1}_{\{X_v \ge \ell\}} \\
&\qquad + \sum_{v\colon d(0,v)=1} (\mathbf{1}_{\{X_v < d_{\mathcal{H}}(0,v) \le k\}} + \mathbf{1}_{\{k < d_{\mathcal{H}}(0,v)\}})\Bigg) > h\Bigg) \\
&\le \mu_\alpha^{\mathrm{CN}}\Bigg(\sum_{\ell=1}^{k} \sum_{v \in \Lambda_{z,\ell}(0)\setminus\Lambda_{z,\ell-1}(0)} \mathbf{1}_{\{X_v \ge \ell\}} - \mathbb{E}_\Sigma > h - 4d - \mathbb{E}_k\Bigg) \\
&\le \exp\bigg(-\frac{(h - 4d - \mathbb{E}_k)^2/2}{\mathbb{E}_\Sigma + (h - 4d - \mathbb{E}_k)/3}\bigg) \\
&\le \exp\bigg(-\frac{(h^{1/2+\varepsilon})^2/2}{4h/3}\bigg) \\
&\le \exp\bigg(\frac{-3h^{2\varepsilon}}{8}\bigg)
\end{aligned}$$

since for $k \le k_1(h)$, $h - \mathbb{E}_k - 4d \ge h^{1/2+\varepsilon}$, and $\mathbb{E}_\Sigma \le \mathbb{E}_k + 2d \le h$. Thus, as $k_1(h) \le c_4 \log(h)$,

$$\begin{aligned}
\sum_{k \le k_1(h)} \mu^{\mathrm{CN}}(D > h \mid X_0 = k)\mu^{\mathrm{CN}}(X_0 = k)h^{\gamma-1} &\le k_1(h)\exp\bigg(\frac{-3h^{2\varepsilon}}{8}\bigg)h^{\gamma-1} \\
&\to 0 \quad \text{as } h \to \infty.
\end{aligned}$$

From the definition of $k_1(h)$, we have

$$k_1(h) + 1 \ge \log\bigg(\frac{h - h^{1/2+\varepsilon} + 4d - c_2}{c_1}\bigg)\frac{1}{\log(z^d/\alpha)};$$

therefore, as $\mathbb{P} = \mu(\phi^{-1})$,

$$\begin{aligned}
&\mathbb{P}_{z,\alpha}^{\mathrm{CN}}(D > h)h^{\gamma-1} \\
&\quad = h^{\gamma-1}\sum_{k=k_1(h)}^{\infty} \mu_\alpha^{\mathrm{CN}}((\phi_z^{\mathrm{CN}})^{-1}(D > h) \mid X_0 = k)\mu_\alpha^{\mathrm{CN}}(X_0 = k) + o(h) \\
&\quad \le h^{\gamma-1}\alpha^{-(k_1(h)+1)} + o(h) \\
&\quad \le h^{\gamma-1}\exp\bigg(-\log\alpha\log\bigg(\frac{h - h^{1/2+\varepsilon} + 4d - c_2}{c_1}\bigg)\frac{1}{\log(z^d/\alpha)}\bigg) + o(h) \\
&\quad \le h^{\gamma-1}c_1^{(\log_z\alpha/(d-\log_z\alpha))}\exp\bigg(-\frac{\log_z\alpha}{d - \log_z\alpha}\log(h - h^{1/2+\varepsilon} + 4d - c_2)\bigg) + o(h) \\
&\quad \le h^{\gamma-1}c_1^{\gamma-1}\frac{1}{(h - h^{1/2+\varepsilon} + 4d - c_2)^{\gamma-1}} + o(h) \\
&\quad \to c_1^{\gamma-1} \quad \text{as } h \to \infty.
\end{aligned}$$

Proceeding in the same way, if $k \geq k_2(h) + 1$, we have

$$\begin{aligned}\mu_\alpha^{\mathrm{CN}}(D \leq h \mid X_0 = k) &\leq \mu_\alpha^{\mathrm{CN}}\bigg(\mathbb{E}_\Sigma - \sum_{\ell=1}^{k} \sum_{v \in \Lambda_{z,\ell}(0) \setminus \Lambda_{z,\ell-1}(0)} \mathbf{1}_{\{X_v \geq \ell\}} > \mathbb{E}_k - 4d - h\bigg) \\ &\leq \exp\bigg(-\frac{(\mathbb{E}_k - 6d - h)^2/2}{\mathbb{E}_\Sigma + (\mathbb{E}_k - 6d - h)/3}\bigg) \\ &\leq \exp\bigg(-\frac{3}{8}(\mathbb{E}_k)^\varepsilon\bigg)\end{aligned}$$

since $\mathbb{E}_\Sigma \leq \mathbb{E}_k + 2d$ and $\mathbb{E}_k \geq h + h^{1/2+\varepsilon}$ imply that

$$\frac{(\mathbb{E}_k - 6d - h)^2/2}{\mathbb{E}_\Sigma + (\mathbb{E}_k - 6d - h)/3} \geq \frac{(\mathbb{E}_k - 6d - h)^2/2}{\mathbb{E}_k + \mathbb{E}_k/3} \geq \frac{3}{8}(\mathbb{E}_k)^\varepsilon$$

for large h (depending on ε). Hence,

$$\begin{aligned}&\mathbb{P}_{z,\alpha}^{\mathrm{CN}}(D > h)h^{\gamma-1} \\ &\quad\geq h^{\gamma-1} \sum_{k=k_2(h)+1}^{\infty} \mu^{\mathrm{CN}}(D > h \mid X_0 = k)\mu_\alpha(X_0 = k) \\ &\quad\geq h^{\gamma-1}\bigg(\sum_{k=k_2(h)}^{\infty} \mu_\alpha(X_0 = k) - \sum_{k=k_2(h)+1}^{\infty} \exp\bigg(-\frac{3}{8}\mathbb{E}_k^\varepsilon\bigg)\alpha^{-k}(\alpha - 1)\bigg) \\ &\quad\geq h^{\gamma-1}\bigg(\alpha^{-(k_2(h)+1)} - \sum_{k=k_2(h)+1}^{\infty} \exp\bigg(-\frac{3}{8}\bigg(c_1\bigg(\frac{z^d}{\alpha}\bigg)^k + c_2\bigg)^\varepsilon\bigg)\alpha^{-k}(\alpha - 1)\bigg)\end{aligned}$$

since $c_1(z^d/\alpha)^{k_2(h)} + c_2 \geq h + h^{1/2+\varepsilon} + 6d \geq h$. The second term is bounded by

$$\begin{aligned}h^{\gamma-1}\exp\bigg(-\frac{3}{8}\bigg(c_1\bigg(\frac{z^d}{\alpha}\bigg)^{(k_2(h)+1)} + c_2\bigg)^\varepsilon\bigg)\alpha^{-k_2(h)} &\leq h^{\gamma-1}\exp\bigg(-\frac{3}{8}h^\varepsilon\bigg)\alpha^{\log((h-c_2)/c_1)1/\log(z^d/\alpha)} \\ &\leq \exp\bigg(-\frac{3}{8}h^\varepsilon\bigg)\bigg(\frac{h - c_2}{c_1}\bigg)^{\gamma-1} h^{\gamma-1} \\ &\to 0 \quad \text{as } h \to \infty.\end{aligned}$$

From the definition of $k_2(h)$, we have $k_2(h) \leq \log((h + h^{1/2+\varepsilon} - c_2)/c_1)1/\log(z^d/\alpha)$; so, finally, the first term above satisfies

$$\begin{aligned}h^{\gamma-1}\alpha^{-(k_2(h)+1)} &\geq h^{\gamma-1}\frac{1}{\alpha}\exp\bigg(-\log\alpha\log\bigg(\frac{h + h^{1/2+\varepsilon} + 4d - c_2}{c_1}\bigg)\frac{1}{\log(z^d/\alpha)}\bigg) \\ &\geq h^{\gamma-1}\frac{1}{\alpha}c_1^{\gamma-1}\frac{1}{(h + h^{1/2\varepsilon} + 4d - c_2)^{\gamma-1}} \\ &\to \frac{1}{\alpha}c_1^{\gamma-1} \quad \text{as } h \to \infty.\end{aligned}$$

This concludes the proof as $c_1 = (z^d - 1)/(z^d - \alpha)$. □

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

5. Epidemics: existence of a threshold

Lemma 5.1. *Fix d, z, α, and ρ. For $\mu^{\mathrm{SHEM}}_{\alpha,\rho,p}$-almost all $x_{\mathbb{Z}^d} \in \bar{X}_{\mathbb{Z}^d}$, $\mu^{\mathrm{SHEM}}_{\alpha,\rho,p}(\phi_z^{-1}(A_\infty) \mid x_{\mathbb{Z}^d}) \in \{0, 1\}$ and is nondecreasing in p; therefore, there exists $p_{\mathrm{c}} = p_{\mathrm{c}}(d, z, x_{\mathbb{Z}^d}, \rho)$ such that $\mathbb{P}^{\mathrm{SHEM}}(A_\infty) = 0$ for $p < p_{\mathrm{c}}$ and $\mathbb{P}^{\mathrm{SHEM}}(A_\infty) = 1$ for $p > p_{\mathrm{c}}$. Furthermore, the random variable $p_{\mathrm{c}}(d, z, x_{\mathbb{Z}^d}, \rho)$ is constant for μ_α-almost all $x_{\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$.*

Proof. We have

$$\mu^{\mathrm{SHEM}}_{\alpha,\rho,p}(\phi_z^{-1}(A_\infty) \mid x_{\mathbb{Z}^d}) = \mu'_{\rho,p} \times \mu''_p(A_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1}(x_{\mathbb{Z}^d})),$$

where, for $x_{\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$, $A_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1}(x_{\mathbb{Z}^d}) = \{x_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1} \in \{0,1\}^{\mathbb{B}^d\times\mathbb{N}} \times \{0,1\}^{\mathbb{B}_1} : \phi_z(x_{\mathbb{Z}^d} \times x_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1}) \in A_\infty\}$; note that $\mu'_{\rho,p} \times \mu''_p$ is Bernoulli. The event $A_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1}(x_{\mathbb{Z}^d})$ does not depend on any finite set of $(\{u,v\},k)$ in $\mathbb{B}^d \times \mathbb{N}$ and $\{u,v\}$ in $\mathbb{B}_1$, so it is a tail event and its probability is in $\{0,1\}$. Moreover, for each $x_{\mathbb{Z}^d}$ the event $A_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1}(x_{\mathbb{Z}^d})$ is monotone in the semiorder of $\{0,1\}^{\mathbb{B}^d\times\mathbb{N}} \times \{0,1\}^{\mathbb{B}_1}$, and for $p' > p$ $\mu'_{\rho,p'} \times \mu''_{p'}$ dominates in the Fortuin–Kasteleyn–Ginibre (FKG) sense $\mu'_{\rho,p} \times \mu''_p$. This shows the existence of p_{c} as claimed.

Next, the random variable $p_{\mathrm{c}}(d, z, x_{\mathbb{Z}^d}, \rho)$ is in the tail σ-algebra of $\mathbb{N}^{\mathbb{Z}^d}$, as changes of a finite number of x_u do not affect the set $A_{\mathbb{B}^d\times\mathbb{N},\mathbb{B}_1}(x_{\mathbb{Z}^d})$ hence the value of p_{c}, and $\mu^{\mathrm{SHEM}}_\alpha$ is Bernoulli. $\square$

Proof of Theorem 2.1. If $p < p_{\mathrm{c}}(\alpha,\rho)$ then $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p}(A_\infty) = 0$; hence, $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p}(A_{u,\infty}) = 0$. If $p > p_{\mathrm{c}}(\alpha,\rho)$ then $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p}(A_\infty) > 0$. By σ-additivity there is a vertex $v \in \mathbb{Z}^d$ such that $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p}(A_{v,\infty}) > 0$; for another fixed $u \in \mathbb{Z}^d$, the event B that $X_u, X_v > d_{\mathcal{H}}(u,v)$ and $X_{\{u,v\},k} = 1$ for some $k > d_{\mathcal{H}}(u,v)$ is an increasing event in the semiorder in $\bar{X}$ and has $\mathbb{P}^{\mathrm{SHEM}}(B) > 0$, so by FKG, $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p}(A_{u,\infty}) \geq \mathbb{P}^{\mathrm{SHEM}}(A_{v,\infty})\mathbb{P}^{\mathrm{SHEM}}(B) > 0$. $\square$

6. Parameter region with trivial threshold

Lemma 6.1. *For $\alpha \in [1, z^d)$ and $\rho > \alpha/z^d$, we have $p_{\mathrm{c}}(\alpha,\rho) = 0$.*

Proof. We begin with a dynamic construction of $\mathbb{P}^{\mathrm{SHEM}}$. Starting from the origin 0, consider the sequence of boxes $\Lambda_{z,k} = \Lambda_{z,k}(0)$; recall that $\Lambda_{z,0} = \{0\}$ and let $\Lambda_{z,k'} = \varnothing$ for $k' < 0$; for $k = 0, \ldots,$ sequentially generate the following variables:

…

(k_a) X_v, $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$;

(k_b) $X_{\{v,u\},j}$, $u, v \in \Lambda_{z,k-1} \setminus \Lambda_{z,k-2}$, $j = 1, \ldots, k-1$, $u \neq v$;

(k_c) $X_{\{v,u\},k}$, $u \in \Lambda_{z,k-1}$, $v \in \Lambda_{z,k}$;

…

(Last) $X_{\{u,v\},0}$ for all nearest-neighbor pairs $\{u,v\}$.

Note the following.

(i) In every step we generate variables which were not generated in the previous steps: this is clear for X_u and $X_{\{u,v\},0}$, while for $X_{\{v,u\},k}$ we need to verify that three pairs of sets are disjoint.

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- First, for $k \neq \ell$, $\{(\{v, u\}, j) \mid u, v \in \Lambda_{z,k-1} \setminus \Lambda_{z,k-2}, j = 1, \ldots, k-1\} \cap \{(\{v, u\}, j) \mid u, v \in \Lambda_{z,\ell-1} \setminus \Lambda_{z,\ell-2}, j = 1, \ldots, \ell-1\} = \varnothing$ as the sets of bonds are disjoint.
- Second, for $k \neq \ell$, $\{(\{v, u\}, k), u \in \Lambda_{z,k-1}, v \in \Lambda_{z,k}\} \cap \{(\{v, u\}, \ell), u \in \Lambda_{z,\ell-1}, v \in \Lambda_{z,\ell}\} = \varnothing$ as the community levels are different.
- Finally, for all k and ℓ, $\{(\{v, u\}, j) \mid u, v \in \Lambda_{z,k-1} \setminus \Lambda_{z,k-2}, j = 1, \ldots, k-1\} \cap \{(\{v, u\}, \ell), u \in \Lambda_{z,\ell-1}, v \in \Lambda_{z,\ell}\} = \varnothing$ since, for $\ell \leq k-1$, bonds are different (this should be checked with care for $\ell = k-1$, as in that case the second set has $v \in \Lambda_{z,k-1}$, but then $u \in \Lambda_{z,k-2}$); and, for all $\ell \geq k$, community levels are different.

(ii) The last step can be performed at any time, possibly subdivided into several steps.

(iii) All the generated variables are measurable with respect to $\bar{X} = \bar{X}_{\mathbb{Z}^d, \mathbb{B}^d \times \mathbb{N}, \mathbb{B}_1}$.

(iv) As a side remark, although unnecessary for the current proof but potentially useful in simulations, observe that all relevant variables are generated; in fact, if $u' \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$ and $v' \in \Lambda_{z,k+r} \setminus \Lambda_{z,k+r-1}$, $r \geq 1$, then $x_{\{v',u'\},j}$ for $j = 1, \ldots, k+r-1$ is not generated but it is also not relevant in the process.

Following this construction we can show that for $\alpha \in (1, z^d)$, $\rho > \alpha/z^d$, and any $p > 0$ there is an infinite cluster. We generate a sequence i_k, $k \in \mathbb{N}$, of either vertices in $\Lambda_{z,k} \setminus \Lambda_{z,k-1}$ or empty sets with the following procedure, in which the definition of i_k depends on three events which may occur depending on the status of i_{k-1}.

Step 1. If $x_0 \geq 1$ then $i_0 = 0$, else $i_0 = \varnothing$;

- if $i_{k-1} \in \Lambda_{z,k-1} \setminus \Lambda_{z,k-2}$ then
  - if there exists $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$ such that $x_v \geq k+1$ and $x_{\{i_{k-1},v\},k} = 1$, then i_k equals one of such vertices v (the first in some fixed order);
  - if there exists $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1} : x_v \geq k+1$ but for all such v $x_{\{i_{k-1},v\},k} = 0$, then i_k equals one of vertices v with the first property (the first in some fixed order);
  - if for all $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$, we have $x_v < k+1$ then $i_k = \varnothing$;
- if $i_{k-1} = \varnothing$ then
  - if there exists $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1} : x_v \geq k+1$ then i_k equals one of such vertices v (the first in some fixed order);
  - if for all $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$, we have $x_v < k+1$ then $i_k = \varnothing$.

Given the vertices i_ks, we can define the following events:

- $A_k = \{$there exists $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1} : x_v \geq k+1, x_{\{i_{k-1},v\}}, k = 1\}$
- $C_k = \{$there exists $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1} : x_v \geq k+1$ but either $i_{k-1} = \varnothing$ or for all such vs $x_{\{i_{k-1},v\},k} = 0\}$
- $F_k = \{$for all $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$, it holds that $x_v < k+1\}$,

where clearly A_k is not defined if $i_{k-1} = \varnothing$. Note that all the events A_k, C_k, and F_k are defined in terms of the variables at steps (k_a) and (k_c) of the construction outlined above.

This implies that such events are defined in terms of variables which, once i_{k-1} is given, are independent from those involved in defining A_i, C_i, and F_i for $i = 1, \dots, k-1$. Moreover, for each k the three events form a partition of the probability space. Therefore, the sequence $Z_k = a_k(c_k, f_k$, respectively) if $A_k(C_k, F_k$, respectively) occurs, is a (nonhomogeneous) Markov chain, whose transition matrix can be estimated in terms of the x variables. In fact,

$$\begin{aligned}\mathbb{P}^{\mathrm{SHEM}}(Z_k = a_k \mid Z_{k-1} = a_{k-1}) &= \mathbb{P}^{\mathrm{SHEM}}(Z_k = a_k \mid Z_{k-1} = c_{k-1}) \\ &= 1 - \bigg(1 - \frac{p\rho^k}{\alpha^{k+1}}\bigg)^{z^{dk} - z^{d(k-1)}} \\ &\geq 1 - \exp\bigg(-\frac{p\rho^k(z^{dk} - z^{d(k-1)})}{\alpha^{k+1}}\bigg), \\ \mathbb{P}^{\mathrm{SHEM}}(Z_k = c_k \mid Z_{k-1} = f_{k-1}) &= 1 - \bigg(1 - \frac{1}{\alpha^{k+1}}\bigg)^{z^{dk} - z^{d(k-1)}} \\ &\geq 1 - \exp\bigg(-\frac{(z^{dk} - z^{d(k-1)})}{\alpha^{k+1}}\bigg),\end{aligned}$$

and all other conditional probabilities are smaller than $\exp(-p\rho^k(z^{dk} - z^{d(k-1)})/\alpha^{k+1})$ if $Z_{k-1} = a_{k-1}$ or $Z_{k-1} = c_{k-1}$, and smaller than $\exp(-(z^{dk} - z^{d(k-1)})/\alpha^{k+1})$ if $Z_{k-1} = f_{k-1}$.

Since $p, \rho \leq 1$, we have

$$\begin{aligned}\mathbb{P}^{\mathrm{SHEM}}(Z_k = f_k) &= \sum_{z \in \{a_{k-1}, c_{k-1}, f_{k-1}\}} \mathbb{P}^{\mathrm{SHEM}}(Z_k = f_k \mid Z_{k-1} = z)\mathbb{P}^{\mathrm{SHEM}}(Z_{k-1} = z) \\ &\leq \exp\bigg(-\frac{p\rho^k(z^{dk} - z^{d(k-1)})}{\alpha^{k+1}}\bigg)\end{aligned}$$

and, similarly,

$$\mathbb{P}^{\mathrm{SHEM}}(Z_k = c_k) \leq \exp\bigg(-\frac{z^{dk} - z^{d(k-1)}}{\alpha^{k+1}}\bigg) + \exp\bigg(-\frac{p\rho^{k-1}(z^{d(k-1)} - z^{d(k-2)})}{\alpha^{k+1}}\bigg),$$

so that if $\rho > \alpha/z^d$, both $\sum_{k=1}^{\infty}\mathbb{P}^{\mathrm{SHEM}}(Z_k = f_k) < \infty$ and $\sum_{k=1}^{\infty}\mathbb{P}^{\mathrm{SHEM}}(Z_k = c_k) < \infty$. By the first Borel–Cantelli lemma, F_k and C_k occur only a finite number of times, so that with probability 1 the sequence terminates with one C_k and then A_h for $h > k$. In such a case, the vertex i_k is connected to an infinite cluster containing all vertices i_h for $h > k$. Since there are countably many vertices there must be one k and one vertex $v \in \Lambda_{z,k} \setminus \Lambda_{z,k-1}$ which is the starting vertex of an infinite cluster with probability $c_1 > 0$. Note that the infinite cluster is using edges in communities at a level of at least k. Such a vertex can be connected to the origin using nearest-neighbor edges, which are independent from the previous construction as they were involved only in the last step of the dynamic joint generation of graph and epidemic, with some probability $c_2 > 0$. In the end, the probability of percolation from the origin is at least $c_1c_2 > 0$. □

7. Parameter region with nontrivial threshold

To identify the region with nontrivial threshold we are going to give an upper bound to the probability $\mathbb{P}^{\mathrm{SHEM}}(A_{u,\infty})$ in terms of the corresponding probability in a sequence of models. The three main ones are the random disks (RD) probability $\mathbb{P}^{\mathrm{RD}}$, the directed random disks

(DRD) probability $\mathbb{P}^{\mathrm{DRD}}$, and a long range percolation (LRP) probability $\mathbb{P}^{\mathrm{LRP}}$. The first two are probably of little relevance elsewhere; the last one is well known in the literature; see [33]. In addition, we introduce an infinite sequence of interpolating models between the RD and the DRD; they are called the h-directed random disks (h-DRD) models, with their distributions denoted by $\mathbb{P}^{h\text{-DRD}}$, $h \in \mathbb{N}$.

In the proofs of this section nearest-neighbor edge variables are just a nuisance. Therefore, we are going to fix one realization $\bar{x}_{\mathbb{B}_1}$ of the variables in $\mathbb{B}_1$ such that percolation does not occur in $(\mathbb{Z}^d, \mathbb{B}_1)$, as can be done with probability 1 for small enough p; $\bar{x} = \bar{x}_{\mathbb{B}_1}$ is kept *fixed for the whole section* except in the proof of Theorem 2.2. Let us denote by $\mathbb{B}_1^*$ the set of edges $\{u, v\}$ such that $d(u, v) = 1$ and $\bar{x}_{\{u,v\}} = 1$; we will always have $\eta_{\{u,v\}} = 1$. We then consider $\mathbb{P}^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}} = \mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}} = \phi^{\mathrm{SHEM}}_{z,\bar{x}_{\mathbb{B}_1}}(\mu^{\mathrm{SHEM}}_{\alpha,\rho,p})$, where $\phi^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}} \colon \bar{X}^{\mathrm{SHEM}} \to H$ is such that

$$(\phi^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}}(x))_{\{u,v\}} = \mathbf{1}_{\{\text{there exists } k \in \mathbb{N} \colon d_{\mathcal{H}}(u,v) \le k \le \min(x_u, x_v) \text{ and } x_{\{u,v\},k} = 1, \text{ or } \{u,v\} \in \mathbb{B}_1^*\}}\,.$$

Similarly to the general SHEM construction, now $\mathcal{G}^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}} = (\mathbb{Z}^d, \mathbb{B}^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}})$, where $\mathbb{B}^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}}$ is the set of edges where $\phi^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}} = 1$; $\mathbb{B}^{\mathrm{SHEM}}$ has distribution $\mathbb{P}^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}} = \mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}} = \phi^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}}(\mu^{\mathrm{SHEM}}_{\alpha,\rho,p})$.

Note that, equivalently, $\mathbb{P}^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}} = \phi^{\mathrm{SHEM}}(\mu^{\mathrm{SHEM}}_{\alpha,\rho,p}(\cdot \mid \bar{x}_{\mathbb{B}_1}))$, where $\mu^{\mathrm{SHEM}}(\cdot \mid \bar{x}_{\mathbb{B}_1})$ is just $\mu = \mu^{\mathrm{SHEM}}$ conditioned to the fixed configuration on $\mathbb{B}_1$; $\mu^{\mathrm{SHEM}}(\cdot \mid \bar{x}_{\mathbb{B}_1})$ is then a distribution on $\mathbb{N}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{B}^d \times \mathbb{N}}$; and

$$\mu^{\mathrm{SHEM}} = \int_{\{0,1\}^{\mathbb{B}_1}} \mu^{\mathrm{SHEM}}(\cdot \mid \bar{x}_{\mathbb{B}_1}) \mu^{\mathrm{SHEM}}(\mathrm{d}\bar{x}_{\mathbb{B}_1});$$

hence,

$$\mathbb{P}^{\mathrm{SHEM}} = \int_{\{0,1\}^{\mathbb{B}_1}} \mathbb{P}^{\mathrm{SHEM}}_{\bar{x}_{\mathbb{B}_1}} \mu^{\mathrm{SHEM}}(\mathrm{d}\bar{x}_{\mathbb{B}_1}). \tag{7.1}$$

The models used to bound the percolation probability are also defined in terms of the given $\bar{x}_{\mathbb{B}_1}$ following the general scheme discussed in Section 2; each will depend on various parameters. We prove that for suitable choices of the parameters and $h \ge 1$,

$$\begin{aligned}
\mathbb{P}^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(A_{u,\infty}) &\le \mathbb{P}^{\mathrm{RD}}(A_{u,\infty}) \\
&\le \mathbb{P}^{(h-1)\text{-DRD}}(A_{u,\infty}) \\
&\le \mathbb{P}^{h\text{-DRD}}(A_{u,\infty}) \\
&\le \mathbb{P}^{\mathrm{DRD}}(A_{u,\infty}) \\
&\le \mathbb{P}^{\mathrm{LRP}}(A_{u,\infty}),
\end{aligned}$$

where we have not yet explicitly indicated the parameters of the other distributions. In the end, we easily identify the parameter region with a nontrivial threshold for the long-range model, and translate it back into a parameter region for the SHEM. It is somewhat surprising that we obtain the complete description of the phase space with this procedure, in particular, as the inequality between RD and DRD requires a choice of parameters which does not seem optimal at first sight.

We begin by introducing the RD model. It is based on random variables X_u with the same distribution as the corresponding ones in SHEM, but now the connectivity graph is based just on distances. Let $\bar{X}^{\mathrm{RD}} = \mathbb{N}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{B}^d}$. For $\alpha \geq 1$ and $\xi \colon \mathbb{B}^d \to [0,1]$, with $\xi_{\{u,v\}} = 1$ for $\{u,v\} \in B_1^*$, let $\mu^{\mathrm{RD}} = \mu^{\mathrm{RD}}_{\alpha,\xi} = \mu_\alpha \times \mu_\xi$, where μ_α is like in the SHEM and μ_ξ is inhomogeneous Bernoulli, i.e. $\mu_\xi = \prod_{\{u,v\}\in\mathbb{B}^d} \mu_{\xi,\{u,v\}}$ with $\mu_{\xi,\{u,v\}}(x_{\{u,v\}} = 1) = \xi_{\{u,v\}} = 1 - \mu_{\xi,\{u,v\}}(x_{\{u,v\}} = 0)$. Then, let $\phi^{\mathrm{RD}} = \phi^{\mathrm{RD}}_{z,\delta,\bar{x}_{\mathbb{B}_1}} \colon \bar{X}^{\mathrm{RD}} \to H$ be such that

$$\phi^{\mathrm{RD}}_{z,\delta}(x)_{\{u,v\}} = \mathbf{1}_{\{\delta z^{x_u}, \delta z^{x_v} \geq d(u,v) \text{ and } x_{\{u,v\}}=1, \text{ or } \{u,v\}\in\mathbb{B}_1^*)\}},$$

and $\mathbb{P}^{\mathrm{RD}} = \mathbb{P}^{\mathrm{RD}}_{z,\alpha,\delta,\xi} = \phi^{\mathrm{RD}}_{z,\delta}(\mu^{\mathrm{RD}}_{\alpha,\xi})$ which is the distribution of $\mathcal{G}^{\mathrm{RD}} = \mathcal{G}^{\mathrm{RD}}_{z,\alpha,\delta,\xi} = (\mathbb{Z}^d, \mathbb{B}^{\mathrm{RD}})$.

For fixed p and ρ, let

$$k_2(\rho,p) = \min\left\{k_2' \geq 1 \colon k_2' \text{ is odd and } \frac{p\rho^{k_2'}}{(1-\rho)^{k_2'+1}} \leq \frac{1}{2}\right\} \tag{7.2}$$

and for $u, v \in \mathbb{Z}^d$ and $\delta > 1$, define

$$k_{1,\delta}(u,v) = \min\{k_1' \geq 1 \colon \delta z^{k_1'} \geq d(u,v)\}. \tag{7.3}$$

Lemma 7.1. *If, for $\{u,v\} \notin B_1^*$,*

$$\xi_{\{u,v\}}(\delta,p,\rho) = \frac{k_2(\rho,p)p}{(1-\rho)^{k_2(\rho,p)}}\,\rho^{k_{1,\delta}(u,v)} \wedge 1, \tag{7.4}$$

then for all increasing events $A \subseteq H$, $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(A) \leq \mathbb{P}^{\mathrm{RD}}_{z,\alpha,\sqrt{d},\xi(\sqrt{d},p,\rho)}(A)$.

Proof. Now let $x_{\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ be fixed and consider the conditional distributions

$$\mu^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(\cdot \mid x_{\mathbb{Z}^d}), \qquad \mu^{\mathrm{RD}}_{\alpha,\xi(\sqrt{d},p,\rho)}(\cdot \mid x_{\mathbb{Z}^d}).$$

Under these distributions, the events that $\eta_{\{u,v\}} = 1$ are independent by construction, and for $\{u,v\} \notin \mathbb{B}_1^*$, $\mu^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}((\phi^{\mathrm{SHEM}})^{-1}(\eta_{\{u,v\}} = 1 \mid x_{\mathbb{Z}^d})) = 1 - \prod_{k\in I_{x_{\mathbb{Z}^d}}(u,v)}(1-p\rho^k)$, where

$$I_{x_{\mathbb{Z}^d}}(u,v) = \{k \geq 1 \mid \text{there exists } w \in \mathbb{Z}^d \colon u,v \in \Lambda_{k,z}(w) \text{ and } x_u, x_v \geq k\};$$

and $\mu^{\mathrm{RD}}_{\alpha,\xi}((\phi^{\mathrm{RD}}_{\alpha,\sqrt{d}})^{-1}(\eta_{\{u,v\}} = 1 \mid x_{\mathbb{Z}^d})) = \xi_{\{u,v\}}(\sqrt{d},\rho,p)\,\mathbf{1}_{\{\sqrt{d}z^{x_u},\sqrt{d}z^{x_v}\geq d(u,v)\}}$. For all $k \in I_{x_{\mathbb{Z}^d}}(u,v)$ $u, v \in \Lambda_{k,z}(w)$, and $x_u, x_v \geq k$ which yields

$$d(u,v) \leq \sqrt{d}z^k \leq \sqrt{d}z^{x_u}, \sqrt{d}z^{x_v}; \tag{7.5}$$

the first inequality implies that $k_{1,\sqrt{d}}(u,v) \leq \min I_{x_{\mathbb{Z}^d}}(u,v)$ and, hence, $I_{x_{\mathbb{Z}^d}}(u,v) \subset \{k_{1,\sqrt{d}}(u,v), k_{1,\sqrt{d}}(u,v)+1, \ldots, \min(x_u,x_v)\}$. We then have

$$\begin{aligned} 1 - \prod_{k\in I_{x_{\mathbb{Z}^d}}(u,v)}(1-p\rho^k) &\leq 1 - \prod_{k\geq k_{1,\sqrt{d}}(u,v)}(1-p\rho^k) \\ &= p\sum_{k\geq k_{1,\sqrt{d}}(u,v)}\rho^k - p^2\sum_{h>k\geq k_{1,\sqrt{d}}(u,v)}\rho^{k+h} + \cdots \\ &= \sum_{j=1}^{\infty}(-1)^{j+1}\frac{p^j\rho^{jk_{1,\sqrt{d}}(u,v)+j(j-1)/2}}{\prod_{i=1}^{j}(1-\rho^i)} \\ &=: \sum_{j=1}^{\infty}\ell_j. \end{aligned} \tag{7.6}$$

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

If $j > k_2(\rho, p)$,

$$\left|\frac{\ell_{j+1}}{\ell_j}\right| = \frac{p\rho^{k_{1,\sqrt{d}}(u,v)+j}}{(1-\rho^{j+1})} \leq \frac{p\rho^j}{(1-\rho^{j+1})} \leq \frac{1}{2};$$

thus, the ℓ_js are decreasing geometrically with a ratio smaller than or equal to $\frac{1}{2}$ for $j > k_2(\rho, p)$, with alternating signs, starting from a negative term as $k_2(\rho, p)+1$ is even; it follows that the remainder is nonpositive. Hence, since $\mu^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(\cdot \mid x_{\mathbb{Z}^d})$ is a probability,

$$\begin{aligned}
\sum_{j=1}^{\infty} \ell_j \leq \sum_{j=1}^{k_2(\rho,p)} |\ell_j| \wedge 1 &\leq k_2(\rho, p) \max_{j=1,\dots,k_2(p,\rho)} \frac{p^j \rho^{jk_{1,\sqrt{d}}(u,v)+j(j-1)/2}}{\prod_{i=1}^{j}(1-\rho^i)} \wedge 1 \\
&\leq k_2(\rho, p)\frac{\max_{j=1,\dots,k_2(p,\rho)} p^j \rho^{jk_{1,\sqrt{d}}(u,v)+j(j-1)/2}}{\min_{j=1,\dots,k_2(p,\rho)} \prod_{i=1}^{j}(1-\rho^i)} \wedge 1 \\
&\leq \frac{k_2(\rho, p)p}{(1-\rho)^{k_2(\rho,p)}}\rho^{k_{1,\sqrt{d}}(u,v)} \wedge 1 \\
&\leq \xi_{(u,v)}(\sqrt{d}, \rho, p),
\end{aligned} \tag{7.7}$$

where the inequality before the last one holds since $\rho, p \in [0, 1]$ and, thus, $\prod_{i=1}^{k_2}(1-\rho^i) \leq (1-\rho)^{k_2}$. The second inequality in (7.5) implies that if $I_{x_{\mathbb{Z}^d}}(u, v) \neq \varnothing$ then $\mathbf{1}_{\{\sqrt{d}z^{x_u}, \sqrt{d}z^{x_u} \geq d(u,v)\}} = 1$. Together with (7.6), (7.7), and the fact that the distributions coincide on $\{u, v\} \in \mathbb{B}_1^*$, this implies that

$$\mu^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}((\phi^{\mathrm{SHEM}}_z)^{-1}(\eta_{\{u,v\}} = 1) \mid x_{\mathbb{Z}^d}) \leq \mu^{\mathrm{RD}}_{\alpha,\xi(\sqrt{d},p,\rho)}((\phi^{\mathrm{RD}}_{z,\sqrt{d}})^{-1}(\eta_{\{u,v\}} = 1 \mid x_{\mathbb{Z}^d}))$$

and $\mu^{\mathrm{RD}}_{\alpha,\xi(\sqrt{d},p,\rho)}((\phi^{\mathrm{RD}}_{z,\sqrt{d}})^{-1}(\cdot) \mid x_{\mathbb{Z}^d})$ dominates in the FKG sense

$$\mu^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}((\phi^{\mathrm{SHEM}}_z)^{-1}(\cdot) \mid x_{\mathbb{Z}^d}).$$

It follows that, if $A \subseteq H$ is increasing then

$$\begin{aligned}
\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(A) &= \mu^{\mathrm{SHEM}}_{\alpha,\rho,p}((\phi^{\mathrm{SHEM}}_z)^{-1}(A)) \\
&= \int_{X_{\mathbb{Z}^d}} \mu^{\mathrm{SHEM}}_{\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}((\phi^{\mathrm{SHEM}}_z)^{-1}(A) \mid x_{\mathbb{Z}^d})\mu_\alpha(\mathrm{d}x_{\mathbb{Z}^d}) \\
&\leq \int_{X_{\mathbb{Z}^d}} \mu^{\mathrm{RD}}_{\alpha,\xi(\sqrt{d},p,\rho)}((\phi^{\mathrm{RD}}_{z,\sqrt{d}})^{-1}(A) \mid x_{\mathbb{Z}^d})\mu_\alpha(\mathrm{d}x_{\mathbb{Z}^d}) \\
&= \mathbb{P}^{\mathrm{RD}}_{z,\alpha,\sqrt{d},\xi(\sqrt{d},p,\rho)}(A).
\end{aligned}$$

□

Next, we introduce the DRD model. This is the trickier part, as we now want to remove the dependency between $\eta_{u,v}$ and $\eta_{u,v'}$; such dependency is due to the common variable X_u, so we substitute it by independent random variables $X_{(u,v)}$, one for each directed edge (u, v). It is still possible to have an inequality concerning the interesting events (like percolation) if the probabilities for $X_{(u,v)}$ are 'square roots' of those for X_u, in the sense that the scale parameter α is replaced by $\beta = \sqrt{\alpha}$, as described below.

Let $\vec{\mathbb{B}}^d = \{(u, v) \mid u, v \in \mathbb{Z}^d\}$, $\beta \geq 1$ and ξ be as before. Then, let $\bar{X}^{\mathrm{DRD}} = \mathbb{N}^{\vec{\mathbb{B}}^d} \times \{0, 1\}^{\mathbb{B}^d}$, $\mu^{\mathrm{DRD}} = \mu^{\mathrm{DRD}}_{\beta,\xi} = \mu_\beta \times \mu_\xi$, where $\mu_\beta = \prod_{(u,v)\in\vec{\mathbb{B}}^d} \mu_{\beta,(u,v)}$ with $\mu_{\beta,(u,v)}(x_{(u,v)} \geq k) = 1/\beta^k$.

Moreover, for $\delta > 1$, let $\phi^{\mathrm{DRD}} = \phi^{\mathrm{DRD}}_{z,\delta}\colon \bar{X}^{\mathrm{RD}} \to H$ be such that

$$\phi^{\mathrm{DRD}}_{z,\delta}(x)_{\{u,v\}} = \mathbf{1}_{\{\delta z^{x(v,u)}, \delta z^{x(u,v)} \geq d(u,v) \text{ and } x_{\{u,v\}}=1, \text{ or } \{u,v\} \in \mathbb{B}^*_1\}},$$

and let $\mathbb{P}^{\mathrm{DRD}} = \mathbb{P}^{\mathrm{DRD}}_{z,\beta,\delta,\xi} = \phi^{\mathrm{DRD}}_{z,\delta}(\mu^{\mathrm{DRD}}_{\beta,\xi})$ be the distribution of $\mathcal{G}^{\mathrm{DRD}} = \mathcal{G}^{\mathrm{DRD}}_{z,\beta,\delta,\xi} = (\mathbb{Z}^d, \mathbb{B}^{\mathrm{DRD}})$.

We now introduce the sequence of models interpolating between RD and DRD. First, we introduce boxes of more general form than was done to determine the partitions. In general, let

$$\Lambda_n = \left\{ v = (u_1, \ldots, u_d) \in \mathbb{Z}^d\colon -\left\lfloor \frac{n}{2} \right\rfloor \leq u_j \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ for all } j = 1, \ldots, d \right\}.$$

Then we select an ordering of $\mathbb{Z}^d = \{u_1, u_2, \ldots\}$ and, for $h = 0, 1, \ldots,$ we consider the sequence of sets $U(0) = \varnothing, \ldots, U(h) = \{u_1, \ldots, u_h\}$. For later purposes we take the order such that $U(h) \subseteq \Lambda_n$ for all $h \leq n^d$ and $U(n^d) = \Lambda_n$, so the boxes are filled up sequentially.

For a fixed $h \in \mathbb{N}$, let $\bar{X}^{h\text{-DRD}} = \mathbb{N}^{\mathbb{Z}^d \setminus U(h)} \times \mathbb{N}^{(U(h) \times \mathbb{Z}^d) \setminus \{(u,u), u \in U(h)\}} \times \{0,1\}^{\mathbb{B}^d}$; then for $\alpha, \beta > 1$, ξ as before, let $\mu^{h\text{-DRD}} = \mu^{h\text{-DRD}}_{\alpha,\beta,\xi} = \mu^{(h)}_{\alpha,\beta} \times \mu_\xi$, where $\mu^{(h)}_{\alpha,\beta} = \prod_{u \in \mathbb{Z}^d \setminus U(h)} \mu^{(h)}_{\alpha,\beta,u} \times \prod_{(u,v) \in U(h) \times \mathbb{Z}^d \setminus \{(u,u), u \in \mathbb{Z}^d\}} \mu^{(h)}_{\alpha,\beta,(u,v)}$, $\mu^{(h)}_{\alpha,\beta,u}$ is as $\mu_{\alpha,u}$ in RD (or SHEM), and $\mu^{(h)}_{\alpha,\beta,(u,v)}$ is as $\mu_{\beta,(u,v)}$ in DRD. Then let $\phi^{h\text{-DRD}} = \phi^{h\text{-DRD}}_{z,\delta}\colon \bar{X}^{\mathrm{RD}} \to H$ be such that

$$\phi^{h\text{-DRD}}_{z,\delta}(x)_{\{u,v\}} = \mathbf{1}_{\{\delta z^{x_{t(v,u)}}, \delta z^{x_{t(v,u)}} \geq d(u,v) \text{ and } x_{\{u,v\}}=1, \text{ or } \{u,v\} \in \mathbb{B}^*_1\}},$$

where $t(u,v) = u$ if $u \in \mathbb{Z}^d \setminus U(h)$ and $t(u,v) = (u,v)$ if $u \in U(h)$.

Finally, $\mathbb{P}^{h\text{-DRD}} = \mathbb{P}^{h\text{-DRD}}_{z,\alpha,\beta,\delta,\xi} = \phi^{h\text{-DRD}}_{z,\delta}(\mu^{h\text{-DRD}}_{\alpha,\beta,\xi})$ is the distribution of $\mathcal{G}^{h\text{-DRD}} = \mathcal{G}^{h\text{-DRD}}_{z,\beta,\delta,\xi} = (\mathbb{Z}^d, \mathbb{B}^{h\text{-DRD}})$.

Given a box $\Lambda_n \subseteq \mathbb{Z}^d$, let $\mathbb{B}^{(\Lambda_n)} = \{\{v,u\}\colon v, u \in \Lambda_n \cap \mathbb{Z}^d\}$ be the set of unordered edges having both endpoints in Λ_n. To show that $\mathbb{P}^{h\text{-DRD}}$ interpolates between $\mathbb{P}^{\mathrm{RD}}$ and $\mathbb{P}^{\mathrm{DRD}}$ we consider, for $h \leq n^d$ so that $U(h) \subseteq \Lambda_n$, the set

$$M_{n,h} = \{\Lambda_n \setminus U(h)\} \cup (U(h) \times \Lambda_n \setminus \{(v,v), v \in \Lambda_n\}) \cup \mathbb{B}^{(\Lambda_n)}. \tag{7.8}$$

Note that $M_{n,0} = \Lambda_n \cup \mathbb{B}^{(\Lambda_n)}$ and $M_{n,n^d} = (\Lambda_n \times \Lambda_n \setminus \{(v,v), v \in \Lambda_n\}) \cup \mathbb{B}^{(\Lambda_n)}$. For $x \in \bar{X}^{h\text{-DRD}}$, we have that $x_{M_{n,h}}$ is the restriction to $M_{n,h}$ of a configuration of $\bar{X}^{\mathrm{RD}}$ if $h = 0$, while it is the restriction to $M_{n,h}$ of a configuration of $\bar{X}^{\mathrm{DRD}}$ if $h \geq n^d$. By definition, $\mu^{h\text{-DRD}}_{\alpha,\beta,\xi}(x_{M_{n,0}}) = \mu^{\mathrm{RD}}_{\alpha,\xi}(x_{M_{n,h}})$ and $\phi^{0\text{-DRD}}_{z,\delta} = \phi^{\mathrm{RD}}_{z,\delta}$, so that $\mathbb{P}^{0\text{-DRD}}_{z,\alpha,\beta,\delta,\xi} = \mathbb{P}^{\mathrm{RD}}_{z,\alpha,\delta,\xi}$. On the other hand, if $h \geq n^d$ then $\mu^{h\text{-DRD}}_{\alpha,\beta,\xi}(x_{M_{n,h}}) = \mu^{\mathrm{DRD}}_{\beta,\xi}(x_{M_{n,h}})$ and $(\phi^{h\text{-DRD}}_{z,\delta}(x))_{\{u,v\}} = (\phi^{\mathrm{DRD}}_{z,\delta}(x))_{\{u,v\}}$ for all $\{u,v\} \in \mathbb{B}^{(\Lambda_n)}$. So $\mathbb{P}^{h\text{-DRD}}(A) = \mathbb{P}^{\mathrm{DRD}}(A)$ for all A depending only on $\mathbb{B}^{(\Lambda_n)}$. Since $\mathbb{B}^{(\Lambda_n)} \uparrow \mathbb{B}^d$, this implies that $\mathbb{P}^{h\text{-DRD}}_{z,\alpha,\beta,\delta,\xi}$ converges weakly to $\mathbb{P}^{\mathrm{DRD}}_{z,\beta,\delta,\xi}$ as h diverges. In this sense, $\mathbb{P}^{h\text{-DRD}}$ interpolates between $\mathbb{P}^{\mathrm{RD}}$ and $\mathbb{P}^{\mathrm{DRD}}$.

Given a box $\Lambda_n \subseteq \mathbb{Z}^d$ and a fixed $u \in \Lambda_n$, let $\mathbb{B}^{(\Lambda_n,u)} = \{\{v,u\}\colon v \in \Lambda_n \cap \mathbb{Z}^d\}$ be the set of unordered edges in Λ_n having u as an end point. Consider a subset $A \subseteq \mathbb{B}^{(\Lambda_n,u)}$; note that A can be identified either by its edges or by the endpoint different from u of each edge: we occasionally use both ways. For such an A we indicate by $Z_A = \{\eta\colon \eta_{\{v,u\}} = 0 \text{ for all } \{v,u\} \in A\} \subseteq H$ the event that none of the edges of A are open.

To state the next result we introduce some sets, each a union of vertices, directed edges, and undirected edges, as already was the case in (7.8). Let

$$M^{\mathbb{E}}_h = (\mathbb{Z}^d \setminus U(h)) \cup ((U(h-1) \times \mathbb{Z}^d) \setminus \{(v,v), v \in \mathbb{Z}^d\}) \cup (\mathbb{B}^d \setminus \mathbb{B}^{(\Lambda_n,u_h)}), \tag{7.9}$$

$$M^{\mathbb{E}}_{n,h} = (\Lambda_n \setminus U(h)) \cup ((U(h-1) \times \Lambda_n) \setminus \{(v,v), v \in \Lambda_n\}) \cup (\mathbb{B}^{(\Lambda_n)} \setminus \mathbb{B}^{(\Lambda_n,u_h)}), \tag{7.10}$$

$$M^{\mathrm{IS}}_{n,h} = \{u_h\} \cup \mathbb{B}^{(\Lambda_n,u_h)}, \qquad M^{\mathrm{IM}}_{n,h} = ((u_h \times \Lambda_n) \setminus \{(u_h,u_h)\}) \cup \mathbb{B}^{(\Lambda_n,u_h)}.$$

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Note that $M^{\mathbb{E}}_{n,h} \cup M^{\mathrm{IS}}_{n,h} = M_{n,h-1}$ and $M^{\mathbb{E}}_{n,h} \cup M^{\mathrm{IM}}_{n,h} = M_{n,h}$. Note also that each configuration $x_{M^{\mathbb{E}}_{n,h}}$ is at the same time the restriction of a configuration of $\bar{X}^{(h-1)\text{-DRD}}$ or $\bar{X}^{h\text{-DRD}}$ to $M^{\mathbb{E}}_{n,h}$.

The extension of [29, Theorem 3.1] that we are going to prove uses the following inequality in which we use $\beta^2 = \alpha$.

Lemma 7.2. *For all n, $h \le n^d$ and for all $x_{M^{\mathbb{E}}_{n,h}} \in \bar{X}_{M^{\mathbb{E}}_{n,h}}$,*

$$\mu^{(h-1)\text{-DRD}}_{\alpha,\sqrt{\alpha},\xi}((\phi^{(h-1)\text{-DRD}}_{z,\delta})^{-1}(Z_A \cup Z_B) \mid x_{M^{\mathbb{E}}_{n,h}})) \ge \mu^{h\text{-DRD}}_{\alpha,\sqrt{\alpha},\xi}((\phi^{h\text{-DRD}}_{z,\delta})^{-1}(Z_A \cup Z_B) \mid x_{M^{\mathbb{E}}_{n,h}}))$$

for all pairs of (possibly empty) sets $A, B \subseteq \mathbb{B}^{(\Lambda_n,u)}$.

Proof. For fixed $\Lambda_n \subset \mathbb{Z}^d$ and $u = u_h \in \Lambda_n$, and given $x_{M^{\mathbb{E}}_{n,h}} \in \bar{X}_{M^{\mathbb{E}}_{n,h}}$, if $A \subseteq \mathbb{B}^{(\Lambda_n,u)}$ then $(\phi^{h\text{-DRD}}_{z,\delta})^{-1}(Z_A)$ depends only on the variables in $M^{\mathrm{IS}}_{n,h}$, and $(\phi^{(h-1)\text{-DRD}}_{z,\delta})^{-1}(Z_A)$ depends only on the variables in $M^{\mathrm{IM}}_{n,h}$.

Note that, given $x = x_{M^{\mathbb{E}}_{n,h}}$, to prove the statement it is actually enough to consider the edges in

$$\mathbb{B}^{(\Lambda_n,u,x)} = \left\{\{u,v\} \in \mathbb{B}^{(\Lambda_n,u)} : x_v \ge \log_z\left(\frac{d(u,v)}{\sqrt{d}}\right) \text{ if } v > u \text{ in the fixed order,}\right.$$
$$\left.\text{or } x_{(v,u)} \ge \log_z\left(\frac{d(u,v)}{\sqrt{d}}\right) \text{ if } v < u\right\}$$

as all other edges $\{u, v'\}$ are such that $\eta_{\{u,v'\}} = 0$ for both the $(h-1)$- and h-DRD models. We adopt this assumption from now on. Next, let $A, B \subseteq \mathbb{B}^{(\Lambda_n,u,x)}$, disjoint, with $|A| = r$ and $|B| = m$; and recall that we identify each edge in A or B by its endpoint different from u. We then let $A \cup B = (u_1, u_2, \ldots, u_{m+r})$ indicate such vertices which are endpoints (different from u) of edges in $A \cup B$, ordered according to the distance of the endpoint from u, so that $d(v, u_i) \le d(u, u_{i+1})$. We also indicate $A = \{v_1, v_2, \ldots, v_r\}$ and $B = \{w_1, w_2, \ldots, w_m\}$ ordered in the same way. For simplicity of notation, denote by $d_{u_i} = d(u, u_i)$ the distance from u to u_i; moreover, let $\alpha_{u_i} = \mu_\alpha(X_u \ge \log_z(d_{u_i}/\sqrt{d})) = \alpha^{-\lceil \log_z(d_{u_1}/\sqrt{d})\rceil}$, and the same with α replaced by β. Note that if $\beta = \sqrt{\alpha}$ then $\alpha_{u_i} = (\beta_{u_i})^2$. Next, let $q_{u_i} = \xi_{\{u,u_i\}}$ with $\xi : \mathbb{B}^d \to [0, 1]$ (regardless of h), and let $\mathbb{P}_1 = \mu^{(h-1)\text{-DRD}}_{\alpha,\sqrt{\alpha},\xi}((\phi^{(h-1)\text{-DRD}}_{z,\delta})^{-1}(\cdot) \mid x_{M^{\mathbb{E}}_{n,h}})$; let $\mathbb{P}_2$ indicate the same probability with $h-1$ replaced by h. We want to prove that $\mathbb{P}_1(Z_A \cup Z_B) \ge \mathbb{P}_2(Z_A \cup Z_B)$.

We proceed by induction on the cardinality of A and B. Note that if $|A| = 0$ or $|B| = 0$, then $\mathbb{P}_1(Z_A \cup Z_B) = \mathbb{P}_2(Z_A \cup Z_B) = 1$.

(i) Suppose that $A = \{v\}$, $B = \{w\}$. By symmetry we can assume that $d_w < d_v$; then $\beta_w \ge \beta_v$. Taking complements and using independence under $\mathbb{P}_2$ we easily see that $\mathbb{P}_1(Z_A \cup Z_B) = 1 - \alpha_v q_v q_w$ and $\mathbb{P}_2(Z_A \cup Z_B) = 1 - \beta_v q_v\, \beta_w q_w$. Since $\beta_w \ge \beta_v$, then $\beta_v\beta_w \ge \beta_v^2 = \alpha_v$. The inequality holds, with equality if $d_v = d_w$.

(ii) Now consider $\{u_1, u_2, \ldots, u_{m+r}\} = \{v_1, v_2, \ldots, v_r\} \cup \{w_1, w_2, \ldots, w_m\} = A \cup B$ such that $d_{u_1} \le d_{u_2} \le \cdots \le d_{u_{m+r}}$. As before, consider the probability of $Z_A \cup Z_B$. With respect to $\mathbb{P}_1$, if $X_u < \log_z(d_{v_1}/\sqrt{d})$ then Z_A occurs. Instead, if $\log_z(d_{v_j}/\sqrt{d}) \le X_u < \log_z(d_{v_{j+1}}/\sqrt{d})$ then there exist j connections in the connectivity network and Z_A occurs if all j of them are closed. Analogously for Z_B and $Z_A \cap Z_B$. Thus, $\mathbb{P}_1(Z_A) = (1-\alpha_{v_1}) + \sum_{j=1}^{r-1}(\alpha_{v_j} - \alpha_{v_{j+1}}) \prod_{i=1}^{j}(1 - q_{v_i}) + \alpha_{v_r} \prod_{i=1}^{r}(1 - q_{v_i})$. Analogous expressions hold for $\mathbb{P}_1(Z_B)$ with v_i replaced by w_i and the index running through m, and for $\mathbb{P}_1(Z_A \cap Z_B)$, with v_i replaced by w_i and the index running through $r+m$. With respect to $\mathbb{P}_2$, since $X_{(u,u_i)}$ and $X_{(u,u_j)}$ are independent for $i \neq j$, we have $\mathbb{P}_2(Z_A) = \prod_{i=1}^{r}(1-\beta_{v_i}q_{v_i})$. An analogous expression holds for $\mathbb{P}_2(Z_B)$ with v_i replaced by w_i and the index running through m, while $\mathbb{P}_2(Z_A \cap Z_B) = \mathbb{P}_2(Z_A)\mathbb{P}_2(Z_B)$.

We proceed by induction on $m+r$. We show that if $\mathbb{P}_1(Z_A \cup Z_B) \geq \mathbb{P}_2(Z_A \cup Z_B)$ holds for $m+r-1$ then it also holds for $m+r$. The vertex u_{m+r} can be either in A or in B and we assume with no loss of generality that $u_{m+r} = v_r \in A$. Then we show that if $\mathbb{P}_1(Z_{A'} \cup Z_B) \geq \mathbb{P}_2(Z_{A'} \cup Z_B)$ with $|A'| = r-1$, $|B| = m$ then $\mathbb{P}_1(Z_A \cup Z_B) \geq \mathbb{P}_2(Z_A \cup Z_B)$ with $A = A' \cup \{v_r\}$ and, thus, $|A| = r$, $|B| = m$. It is sufficient to show that

$$\mathbb{P}_1(Z_A \cup Z_B) - \mathbb{P}_1(Z_{A'} \cup Z_B) \geq \mathbb{P}_2(Z_A \cup Z_B) - \mathbb{P}_2(Z_{A'} \cup Z_B). \tag{7.11}$$

By elementary calculation, using the fact that for any probability $\mathbb{P}(Z_A \cup Z_B) = \mathbb{P}(Z_A) + \mathbb{P}(Z_B) - \mathbb{P}(Z_A \cap Z_B)$, and that $\mathbb{P}_1(Z_B) = \mathbb{P}_1(Z_{B'})$, we have

$$\mathbb{P}_1(Z_A \cup Z_B) - \mathbb{P}_1(Z_{A'} \cup Z_B) = -\alpha_{v_r} q_{v_r} \prod_{i=1}^{r-1}(1-q_{v_i})\Bigg[1 - \prod_{j=1}^{m}(1-q_{w_j})\Bigg].$$

Similarly,

$$\mathbb{P}_2(Z_A \cup Z_B) - \mathbb{P}_2(Z_{A'} \cup Z_B) = -\beta_{v_r} q_{v_r} \prod_{i=1}^{r-1}(1-\beta_{v_i}q_{v_i})\Bigg[1 - \prod_{j=1}^{m}(1-\beta_{w_j}q_{w_j})\Bigg].$$

Since $\alpha_{v_r} = \beta_{v_r}^2$, (7.11) is equivalent to

$$\beta_{v_r} \prod_{i=1}^{r-1}(1-q_{v_i})\Bigg[1 - \prod_{j=1}^{m}(1-q_{w_j})\Bigg] \leq \prod_{i=1}^{r-1}(1-\beta_{v_i}q_{v_i})\Bigg[1 - \prod_{j=1}^{m}(1-\beta_{w_j}q_{w_j})\Bigg].$$

As $\beta_{w_i} \leq 1$, it is enough to show that

$$\beta_{v_r}\Bigg[1 - \prod_{j=1}^{m}(1-q_{w_j})\Bigg] \leq \Bigg[1 - \prod_{j=1}^{m}(1-\beta_{w_j}q_{w_j})\Bigg], \tag{7.12}$$

which can be easily performed by induction on m. If $m=1$ then $\beta_{v_r}q_w \leq \beta_w q_w$ because v_r is the vertex at maximal distance from u, so that $\beta_{v_r} \leq \beta_w$. Then, taking again the differences between the increments between the values in $m-1$ and m of both sides of (7.12), it is enough to show that

$$\beta_{w_m} q_{w_m} \prod_{j=1}^{m-1}(1-\beta_{w_j}q_{w_j}) \geq \beta_{v_r} q_{w_m} \prod_{j=1}^{m-1}(1-q_{w_j}),$$

which is easily seen to hold as $\beta_{w_i} \leq 1$ and $\beta_{v_r} \leq \beta_{w_m}$.

There remains to check the case when one of A or B is empty. By symmetry, we need to check one case only, and we thus assume that $B = \varnothing$ and $A = \{v_1, v_2, \ldots, v_r\}$. Note that $\mathbb{P}_1(Z_A)$ and $\mathbb{P}_2(Z_A)$ are as above.

If $r=1$ then $\mathbb{P}_1(Z_A)=(1-\alpha_{v_1})+\alpha_{v_1}(1-q_{v_1})=1-\alpha_{v_1}q_{v_1}\geq 1-\beta_{v_1}q_{v_1}=\mathbb{P}_2(Z_A)$. For general r, using induction as before, and recalling that $\alpha_v=\beta_v^2\leq 1$, we have

$$\begin{aligned}\mathbb{P}_1(Z_A)-\mathbb{P}_1(Z_{A'}) &= -\alpha_{v_r}q_{v_r}\prod_{i=1}^{r-1}(1-q_{v_i})\\ &\geq -\beta_{v_r}q_{v_r}\prod_{i=1}^{r-1}(1-\beta_{v_i}q_{v_i})\\ &= \mathbb{P}_2(Z_A)-\mathbb{P}_2(Z_{A'}).\end{aligned}$$ □

We now want to follow Meester and Trapman's work [29] and we need to recall some definitions in order to make this presentation self-contained. An ordered set of edges of the form $\xi=((v_0,v_1),(v_1,v_2),\dots,(v_{n-1},v_n))$ in some $G\subseteq\vec{\mathbb{B}}^d$ is a directed path from v_0 to v_n; sometimes, to simplify notation, a path is indicated by $\xi=(v_0v_1,v_1v_2,\dots,v_{n-1}v_n)$. A path $\xi=(v_0v_1,v_1v_2,\dots,v_{n-1}v_n,\dots)$ with infinitely many different edges is an infinite path. A path goes through v if for some i, $v_i=v$. Given a finite or infinite path $\xi=((v_0,v_1),(v_1,v_2),\dots,(v_{n-1},v_n))$, we indicate the truncation after k edges as $\xi^s(k)=(v_0v_1,v_1v_2,\dots,v_{k-1}v_k)$ and the tail starting after k edges as $\xi^t(k)=(v_kv_{k+1},\dots)$; for two paths $\xi_1=(v_0v_1,v_1v_2,\dots,v_{n-1}v_n)$ and $\xi_2=(v_nv_{n+1},\dots)$ we denote the conjunction by $(\xi_1,\xi_2)=(v_0v_1,v_1v_2,\dots,v_{n-1}v_n,v_nv_{n+1},\dots)$. Next, let Ξ be a collection of paths. If $G^{(n)}$ is the collection of the first n directed edges of G according to some given enumeration of G, then we indicate by Ξ_n the set of finite paths of Ξ of which all the edges are in $G^{(n)}$ together with all the infinite paths of Ξ truncated at the first instance they leave $G^{(n)}$. In this definition, taken from [29], finite paths are considered but not truncated to be able to handle collections Ξ such as the set of all paths connecting to vertices; for the purposes of this paper we actually need only infinite paths and their truncations. The complete statement of the result would be needed to investigate issues like the individual to individual transmission.

Furthermore, given a configuration $\eta\in\vec{H}=\{0,1\}^G$, we say that ξ is open in η if for all edges (v_k,v_{k+1}) we have $\eta_{(v_k,v_{k+1})}=1$. Given a collection Ξ of paths, we indicate by A^Ξ the event that at least one path in Ξ is open. We say that Ξ is *hoppable* if

- for any $v\in\mathbb{Z}^d$ and any two paths ξ and ϕ of Ξ going through v, where v is the end vertex of the ith edge of ξ and the starting vertex of the jth edge of ϕ, then $(\xi^s(i),\phi^t(j))\in\Xi$;
- the following limit exists: $\lim_n A^{\Xi_n}=A^\Xi$.

If Ξ is the collection of all infinite paths starting from the origin, which is the only case considered here, then A^{Ξ_n} is decreasing in n and, hence, $\lim_n A^{\Xi_n}=\bigcap_n A^{\Xi_n}$ exists. By a standard percolation argument, saying that out of infinitely many longer and longer finite paths from a vertex one can extract an infinite one, we have $\bigcap_n A^{\Xi_n}=A^\Xi$.

Lemma 7.3. *For every z,α,δ,ξ, every hoppable collection of paths Ξ in $\mathbb{B}^d$, and $h\geq 1$,*

$$\mathbb{P}^{\mathrm{RD}}_{z,\alpha,\delta,\xi}(A^\Xi)\leq\mathbb{P}^{(h-1)\text{-}DRD}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^\Xi)\leq\mathbb{P}^{h\text{-}DRD}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^\Xi)\leq\mathbb{P}^{\mathrm{DRD}}_{z,\sqrt{\alpha},\delta,\xi}(A^\Xi).$$

Proof. We mimic the proof of [29, Theorem 3.1], dividing the argument into three steps.

(i) The first step is to prove the corresponding statement for a fixed Ξ_n and a fixed h. To this purpose recall the definitions of $M_h^{\mathbb{E}}$ and $M_{n,h}^{\mathbb{E}}$ from (7.9) and (7.10), respectively,

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

and note that $\bar{X}^{(h-1)\text{-DRD}}_{M^{\mathbb{E}}_{n,h}} = \bar{X}^{h\text{-DRD}}_{M^{\mathbb{E}}_{n,h}}$; the restrictions of $\mu^{(h-1)\text{-DRD}}$ and $\mu^{h\text{-DRD}}$ to (the Borel σ-algebra generated by the variables in) $M^{\mathbb{E}}_{n,h}$ also coincide. Recall that we assume that the enumeration of the vertices is such that boxes Λ_n are progressively filling up. Therefore, the difference $\mathbb{P}^{h\text{-DRD}}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n}) - \mathbb{P}^{(h-1)\text{-DRD}}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n})$ is the integral (on $(\bar{X}^{h\text{-DRD}}_{M^{\mathbb{E}}_{n,h}}, \mathcal{B}, \mu^{(h-1)\text{-DRD}})$, with $\mathcal{B}$ the Borel σ-algebra) of the difference between $\mu^{h\text{-DRD}}((\phi^{h\text{-DRD}})^{-1}(A^{\Xi_n}) \mid x_{M^{\mathbb{E}}_{n,h}})$ and $\mu^{(h-1)\text{-DRD}}((\phi^{(h-1)\text{-DRD}})^{-1}(A^{\Xi_n}) \mid x_{M^{\mathbb{E}}_{n,h}})$.
Note that there are three possibilities: either $(\phi^{(h-1)\text{-DRD}})^{-1}(A^{\Xi_n})$ and $(\phi^{h\text{-DRD}})^{-1}(A^{\Xi_n})$ both occur in $x_{M^{\mathbb{E}}_{n,h}}$ regardless of $x_{M^{\text{IS}}_{n,h}}$ or $x_{M^{\text{IM}}_{n,h}}$; or they both do not occur regardless of $x_{M^{\text{IS}}_{n,h}}$ or $x_{M^{\text{IM}}_{n,h}}$; or, finally, these occurrences depend on $x_{M^{\text{IS}}_{n,h}}$ or $x_{M^{\text{IM}}_{n,h}}$. In the first two cases the second difference above is 0.
In the third case there are two sets of edges A and B, possibly one of whose is empty, such that $\phi^{-1}(A^{\Xi_n}))$ does not occur in $\phi^{-1}(Z_A \cup Z_B)$, while it occurs in the complement. Denoting, for any event A, $\rho_h(A) = \mu^{h\text{-DRD}}((\phi^{h\text{-DRD}})^{-1}(A) \mid x_{M^{\mathbb{E}}_{n,h}})$, we obtain

$$\rho_h(A^{\Xi_n}) - \rho_{h-1}(A^{\Xi_n}) = \rho_h((Z_A \cup Z_B)^c) - \rho_{h-1}((Z_A \cup Z_B)^c) \geq 0$$

from Lemma 7.2.

(ii) By iteration, as $\mathbb{P}^{\text{RD}}$ and $\mathbb{P}^{\text{DRD}}$ are obtained for $h = 0$ and $h = n^d$, respectively,

$$\mathbb{P}^{\text{RD}}_{z,\alpha,\delta,\xi}(A^{\Xi_n}) \leq \mathbb{P}^{(h-1)\text{-DRD}}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n}) \leq \mathbb{P}^{h\text{-DRD}}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n}) \leq \mathbb{P}^{\text{DRD}}_{z,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n}).$$

(iii) In the last step recall that by definition of a hoppable collection of paths $A^{\Xi_n} \to A^{\Xi}$. Thus, $\lim_{n\to\infty} \mathbb{P}^{\text{RD}}_{z,\alpha,\delta,\xi}(A^{\Xi_n}) = \mathbb{P}^{\text{RD}}_{z,\alpha,\delta,\xi}(A^{\Xi})$. On the other hand, since $\mathbb{P}^{h\text{-DRD}}$ converges weakly to $\mathbb{P}^{\text{DRD}}$ and A^{Ξ_n} is measurable with respect to a finite number of variables, we have $\lim_{n\to\infty} \lim_{h\to\infty} \mathbb{P}^{h\text{-DRD}}_{z,\alpha,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n}) = \lim_{n\to\infty} \mathbb{P}^{\text{DRD}}_{z,\sqrt{\alpha},\delta,\xi}(A^{\Xi_n}) = \mathbb{P}^{\text{DRD}}_{z,\sqrt{\alpha},\delta,\xi}(A^{\Xi})$.

Using steps (i) and (ii) in (iii) the proof is completed. □

Before using Lemma 7.3, we introduce the last model we need, namely long range percolation (LRP). Recall that $\bar{x}_{\mathbb{B}_1}$ is still fixed. An LRP model does not require any space $\bar{X}$ or distribution μ, and we can directly define $\mathbb{P}^{\text{LRP}} = \mathbb{P}^{\text{LRP}}_{\beta,s}$ on (the Borel σ-algebra of) H as a Bernoulli distribution with $\mathbb{P}^{\text{LRP}}_{\beta,s}(\eta_{\{u,v\}} = 1) = \beta/d(u,v)^s \wedge 1$ for $\{u,v\} \notin \mathbb{B}^*_1$, and as $\mathbb{P}^{\text{DRD}}$ for $\{u,v\} \in \mathbb{B}^*_1$.

Lemma 7.4. *Let $\rho < \alpha$. When*

$$s = \log_z\left(\frac{\alpha}{\rho}\right), \qquad \beta = \frac{k_2(\rho,p)p}{(1-\rho)^{k_2(\rho,p)}}\left(\frac{\alpha}{\rho}\right)^{(\log_z d - 1)/2},$$

and $\xi_{\{u,v\}}$ is as in (7.4), it holds that for any event A increasing with respect to the semiorder of H, $\mathbb{P}^{\text{DRD}}_{z,\sqrt{\alpha},\sqrt{d},\xi}(A) \leq \mathbb{P}^{\text{LRP}}_{\beta,s}(A)$.

https://doi.org/10.1017/apr.2015.10 Published online by Cambridge University Press

Proof. If $s = \log_z(\alpha/\rho)$ and $\beta = (k_2(\rho, p)p/(1-\rho)^{k_2(\rho,p)})(\alpha/\rho)^{((\log_z d)/2)+1}$, since $\rho < \alpha$, then for $\{u, v\} \notin \mathbb{B}_1^*$,

$$
\begin{aligned}
\mathbb{P}^{\mathrm{DRD}}_{z,\sqrt{\alpha},\sqrt{d},\xi}(\eta_{\{u,v\}} = 1) &= \mu^{\mathrm{DRD}}_{\sqrt{\alpha},\xi}(x_{(v,u)} \geq k_{1,\sqrt{d}}(u, v), x_{(u,v)} \geq k_{1,\sqrt{d}}(u, v), x_{\{u,v\}} = 1) \\
&= (\sqrt{\alpha})^{-2k_{1,\sqrt{d}}(u,v)}\xi_{\{u,v\}} \\
&= (\sqrt{\alpha})^{-2k_{1,\sqrt{d}}(u,v)}\biggl(\frac{k_2(\rho, p)p\rho^{k_{1,\sqrt{d}}(u,v)}}{(1-\rho)^{k_2(\rho,p)}} \wedge 1\biggr) \\
&\leq \frac{k_2(\rho, p)p}{(1-\rho)^{k_2(\rho,p)}}\biggl(\frac{\rho}{\alpha}\biggr)^{\log_z(d(u,v)/\sqrt{d})-1} \\
&\leq \frac{\beta}{(d(u, v))^s} \\
&= \mathbb{P}^{\mathrm{LRP}}_{\beta,s}(\eta_{\{u,v\}} = 1).
\end{aligned}
$$

Then $\mathbb{P}^{\mathrm{DRD}}_{z,\sqrt{\alpha},\delta,\xi}$, which, by construction, $\mathbb{P}^{\mathrm{DRD}}_{z,\sqrt{\alpha},\delta,\xi}$ is also a Bernoulli distribution, is dominated in the FKG sense by $\mathbb{P}^{\mathrm{LRP}}_{\beta,s}$, and the result follows. □

Lemma 7.5. *When $\rho < \alpha$, $s = \log_z(\alpha/\rho)$ and $\beta = (k_2(\rho, p)p/(1-\rho)^{k_2(\rho,p)})(\alpha/\rho)^{(\log_z d)/2}$, it holds that $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{LRP}}_{\beta,s,\bar{x}_{\mathbb{B}_1}}(A_{0,\infty})$.*

Proof. Let Ξ be the collection of all infinite paths starting at the origin 0. Then $A_{0,\infty} = A^{\Xi}$ is both increasing with respect to the semiorder in H and hoppable; see [29] after Definition 1.2 for a detailed argument. With ξ as in (7.4), recalling that the inequalities in Lemmas 7.1, 7.3, and 7.4 are all for the given $\bar{x}_{\mathbb{B}_1}$, we have $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{RD}}_{z,\alpha,\sqrt{d},\xi}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{DRD}}_{z,\sqrt{\alpha},\sqrt{d},\xi}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{LRP}}_{\beta,s,\bar{x}_{\mathbb{B}_1}}(A_{0,\infty})$. □

Proof of Theorem 2.2. By Lemma 6.1, we need to consider only $\rho < \alpha/z^d$. By Lemma 7.5, we have $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p,\bar{x}_{\mathbb{B}_1}}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{LRP}}_{\beta,s,\bar{x}_{\mathbb{B}_1}}(A_{0,\infty})$ for each $\bar{x}_{\mathbb{B}_1}$, when $s = \log_z(\alpha/\rho)$ and $\beta = (k_2(\rho, p)p/(1-\rho)^{k_2(\rho,p)})(\alpha/\rho)^{((\log_z d)/2)-1}$. Consider now a LRP model $\mathbb{P}^{\mathrm{LRPB}}$ in which, in addition to long-range variables, there are also independent nearest-neighbor variables; these are then distributed like μ^{SHEM}_p restricted to $\mathbb{B}_1$. From (7.1), integrating the above inequality over $\mu^{\mathrm{SHEM}}_p(\mathrm{d}\bar{x}_{\mathbb{B}_1})$, we obtain $\mathbb{P}^{\mathrm{SHEM}}_{z,\alpha,\rho,p}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{LRPB}}_{\beta,s}(A_{0,\infty})$.

Hence, it is enough to show that for $\rho < \alpha/z^d$ and small enough p, $\mathbb{P}^{\mathrm{LRPB}}_{\beta,s}(A_{0,\infty}) = 0$. This can be obtained by a simple comparison with a Galton–Watson tree, whose distribution is indicated by $\mathbb{P}^{\mathrm{GW}}_{\beta,s}$, via a stepwise realization of the percolation cluster. In each step, new edges are added with probability $\beta/d(u, v)^s$ with some constraints; the cluster is then bounded by that realized in the unconstrained growth of a Galton–Watson tree with the same probabilities, so that $\mathbb{P}^{\mathrm{LRP}}_{\beta,s}(A_{0,\infty}) \leq \mathbb{P}^{\mathrm{GW}}_{\beta,s}(A_{0,\infty})$.

A Galton–Watson random tree is subcritical, i.e. $\mathbb{P}^{\mathrm{GW}}_{\beta,s}(A_{0,\infty}) = 0$, if the expected number of descendants, i.e. the expected degree, satisfies $\mathbb{P}^{\mathrm{GW}}_{\beta,s}(D_0) < 1$. Since $k_2(\rho, p) \leq k_2(\rho, 1)$, we have

$$
\begin{aligned}
\mathbb{P}^{\mathrm{GW}}_{\beta,s}(D_0) &= 2dp + \sum_{v\in\mathbb{Z}^d} \frac{k_2(\rho, p)p}{(1-\rho)^{k_2(\rho,p)}}\biggl(\frac{\alpha}{\rho}\biggr)^{((\log_z d)/2)-1}\frac{1}{d(0, v)^{\log_z(\alpha/\rho)}} \\
&\leq 2dp + \sum_{k\in\mathbb{N}} 2dk^{d-1}\frac{k_2(\rho, p)p}{(1-\rho)^{k_2(\rho,p)}}\biggl(\frac{\alpha}{\rho}\biggr)^{((\log_z d)/2)-1}\frac{1}{k^{\log_z(\alpha/\rho)}}
\end{aligned}
$$

$$\leq 2dp + p \sum_{k\in\mathbb{N}} 2d \frac{k_2(\rho, p)p}{(1-\rho)^{k_2(\rho,p)}} \left(\frac{\alpha}{\rho}\right)^{((\log_z d)/2)-1} k^{d-1-\log_z(\alpha/\rho)}$$
$$< \infty$$

if $d - 1 - \log_z(\alpha/\rho) < -1$, which is $d < \log_z(\alpha/\rho)$, i.e. $\rho < \alpha/z^d$. In such a case, $\mathbb{P}^{\mathrm{GW}}_{\beta,s}(D_0) < cp$, for some constant c, and for small enough p, we have $\mathbb{P}^{\mathrm{GW}}_{\beta,s}(D_0) < 1$. $\square$

Acknowledgements

We thank Pieter Trapman for reading and commenting on an early draft and for interesting discussions, Christoph Temmel for pointing out a possible relation with Shearer's measure, and an anonymous referee for suggesting the current proof of Theorem 2.3, which is shorter and more accurate than the one we had in the first draft of the paper.

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