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On strongly decreasing solutions of cyclic systems of second-order nonlinear differential equations

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The n-dimensional cyclic system of second-order nonlinear differential equations

$$(p_i(t)|x_i'|^{\alpha_i-1}x_i')' = q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1}, \quad i = 1, \dots, n, \; x_{n+1} = x_1,$$

is analysed in the framework of regular variation. Under the assumption that α_i and β_i are positive constants such that $\alpha_1 \cdots \alpha_n > \beta_1 \cdots \beta_n$ and p_i and q_i are regularly varying functions, it is shown that the situation in which the system possesses decreasing regularly varying solutions of negative indices can be completely characterized, and moreover that the asymptotic behaviour of such solutions is governed by a unique formula describing their order of decay precisely. Examples are presented to demonstrate that the main results for the system can be applied effectively to some classes of partial differential equations with radial symmetry to provide new accurate information about the existence and the asymptotic behaviour of their radial positive strongly decreasing solutions.

Keywords: systems of differential equations; positive solutions; asymptotic behaviour; regularly varying functions

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1. Introduction

In this paper we use the framework of regularly varying functions (in the sense of Karamata) in combination with fixed-point techniques to establish the existence and precise asymptotic behaviour of positive solutions, all components of which decay to zero as $t \to \infty$, for the cyclic system of second-order nonlinear differential equations

$$(p_i(t)|x_i'|^{\alpha_i-1}x_i')' = q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1}, \quad i = 1, 2, \dots, n, \; x_{n+1} = x_1, \qquad \text{(A)}$$

where α_i and β_i, $i = 1, 2, \dots, n$, are positive constants such that

$$\alpha_1\alpha_2 \cdots \alpha_n > \beta_1\beta_2 \cdots \beta_n, \qquad (1.1)$$

$p_i, q_i \colon [a, \infty) \to (0, \infty)$, $i = 1, 2, \dots, n$, are continuous functions and all the p_i simultaneously satisfy either

$$\int_a^\infty p_i(t)^{-1/\alpha_i}\,dt = \infty, \quad i = 1, 2, \dots, n, \tag{1.2}$$

or

$$\int_a^\infty p_i(t)^{-1/\alpha_i}\,dt < \infty, \quad i = 1, 2, \dots, n. \tag{1.3}$$

Here, by a positive solution of (A) we mean a vector function $(x_1, x_2, \dots, x_n)$ consisting of positive continuous functions x_i, $i = 1, 2, \dots, n$, that are continuously differentiable together with $p_i|x_i'|^{\alpha_i - 1}x_i'$ on an interval of the form $[T, \infty)$ and satisfy the system of differential equations (A) over that interval.

Systems of the form of (A) with $p_i(t) = t^{N-1}$ and $q_i(t) = t^{N-1}f_i(t)$, $N \geqslant 2$, $i = 1, \dots, n$, arise in the study of positive radial solutions in exterior domains in $\mathbb{R}^N$ for the system of p-Laplacian equations

$$\Delta_p u_i \equiv \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) = f_i(|x|)|u_{i+1}|^{\gamma_i - 1}u_{i+1}, \quad i = 1, \dots, n, \ u_{n+1} = u_1, \tag{B}$$

where $p > 1$ and $\gamma_i > 0$ are constants, $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^N$ and f_i, $i = 1, \dots, n$, are positive continuous functions on $[a, \infty)$. Nonlinear elliptic system (B) is the stationary version of a reaction–diffusion system with power-type reaction terms and cyclic interconnection structure.

Existence and non-existence of entire positive radial solutions of quasi-linear elliptic systems of the form

$$\Delta_p u = f(|x|)v^\alpha, \qquad \Delta_q v = g(|x|)u^\beta$$

in $\mathbb{R}^N$, where $N \geqslant 2$, $p > 1$, $q > 1$, α and β are positive constants, and $f, g \colon [0, \infty) \to [0, \infty)$ are continuous functions (and the special case where $p = q = 2$) has been studied by many authors (see, for example, [1, 3, 4, 7, 15–17, 20–24] and references therein).

In contrast to [22], where the existence of non-negative non-trivial radial entire solutions was characterized in the case when the coefficients f_i in (B) behave like pure powers at infinity (that is, $\lim_{t\to\infty} f_i(t)/t^{\sigma_i} = \text{const.} > 0$ for some σ_i, $i = 1, \dots, n$), our results apply to a larger class of coefficients f_i that are *regularly varying* at infinity in the sense that they satisfy $\lim_{t\to\infty} f_i(\lambda t)/f_i(t) = \lambda^{\sigma_i}$ for every $\lambda > 0$ and some σ_i, $i = 1, \dots, n$. (For more details concerning the concept of regularly varying functions and their basic properties, see § 2.) Also, the results in [22] were proved under the superhomogeneity condition $\alpha_1 \cdots \alpha_n < \beta_1 \cdots \beta_n$, while our study is focused on the subhomogeneous case where the exponents in (A) satisfy (1.1).

It should be noted that a positive solution $(x_1, \dots, x_n)$ of (A) may exhibit a variety of asymptotic behaviours at infinity because for the case in which (1.2) holds, each component x_i is either increasing and satisfies

$$\lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = \infty \quad \text{or} \quad \lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = \text{const.} > 0,$$

where $P_i(t) = \int_a^t p_i(s)^{-1/\alpha_i}\,ds$, or decreasing and satisfies

$$\lim_{t\to\infty} x_i(t) = \text{const.} > 0 \quad \text{or} \quad \lim_{t\to\infty} x_i(t) = 0,$$

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while for the case in which (1.3) holds, each component x_i is either increasing and satisfies

$$\lim_{t\to\infty} x_i(t) = \infty \quad \text{or} \quad \lim_{t\to\infty} x_i(t) = \text{const.} > 0,$$

or is decreasing and satisfies

$$\lim_{t\to\infty} x_i(t) = \text{const.} > 0 \quad \text{or} \quad \lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = \text{const.} > 0 \quad \text{or} \quad \lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = 0,$$

where $\pi_i(t) = \int_t^\infty p_i(s)^{-1/\alpha_i}\,\mathrm{d}s$.

In this paper we are concerned exclusively with positive solutions $(x_1, \dots, x_n)$ of (A), all components of which are decreasing and satisfy

$$\lim_{t\to\infty} x_i(t) = 0, \quad i = 1, 2, \dots, n, \text{ for the case in which (1.2) holds,} \tag{1.4}$$

$$\lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = 0, \quad i = 1, 2, \dots, n, \text{ for the case in which (1.3) holds.} \tag{1.5}$$

Such solutions of (A) are often referred to as *strongly decreasing solutions* (or *ground states*) of system (A).

It is natural to ask: does system (A) really possess strongly decreasing positive solutions? If such such solutions exist, is it possible to determine their precise asymptotic behaviour as $t \to \infty$? Needless to say, these questions are very difficult to answer for the case where p_i and q_i are general continuous functions. However, if we limit ourselves to the system (A) with *regularly varying* p_i and q_i and focus our attention on its *regularly varying solutions*, then with the help of the theory of regular variation we are able to acquire almost complete information about the existence and asymptotic behaviour of strongly decreasing solutions of (A) that are regularly varying of negative indices.

A prototype of the results we are going to prove says that if f and g are regularly varying functions of indices λ and μ, respectively, and $p \geqslant N$, then the two-dimensional system

$$\Delta_p u = f(|x|)v^\alpha, \qquad \Delta_p v = g(|x|)u^\beta, \tag{B$_2$}$$

where $\alpha\beta < (p-1)^2$, possesses positive radial solutions (u, v), the components of which are regularly varying functions of indices $\rho < 0$ and $\sigma < 0$, respectively, if and only if

$$(p-1)(p+\lambda) + \alpha(p+\mu) < 0 \quad \text{and} \quad \beta(p+\lambda) + (p-1)(p+\mu) < 0.$$

In this case, ρ and σ are uniquely determined by

$$\rho = \frac{p-1}{(p-1)^2 - \alpha\beta}\left[p + \lambda + \frac{\alpha}{p-1}(p+\mu)\right], \tag{1.6$\,a$}$$

$$\sigma = \frac{p-1}{(p-1)^2 - \alpha\beta}\left[\frac{\beta}{p-1}(p+\lambda) + p + \mu\right] \tag{1.6$\,b$}$$

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and the asymptotic behaviour of any such solution (u, v) as $|x| \to \infty$ is governed by the decay law

$$u(|x|) \sim |x|^{\rho}\left[\frac{\varphi(|x|)^{1/(p-1)}}{D(\rho)}\left(\frac{\psi(|x|)^{1/(p-1)}}{D(\sigma)}\right)^{\alpha/(p-1)}\right]^{(p-1)^2/((p-1)^2-\alpha\beta)}, \tag{1.7 a}$$

$$v(|x|) \sim |x|^{\sigma}\left[\left(\frac{\varphi(|x|)^{1/(p-1)}}{D(\rho)}\right)^{\beta/(p-1)}\frac{\psi(|x|)^{1/(p-1)}}{D(\sigma)}\right]^{(p-1)^2/((p-1)^2-\alpha\beta)}, \tag{1.7 b}$$

where φ and ψ are the slowly varying parts of f and g, respectively, and $D(\tau) = (p - N - (p-1)\tau)^{1/(p-1)}(-\tau)$ for $\tau < 0$.

In the case $p < N$, system (B_2) possesses decreasing positive radial solutions (u, v), the components of which are regularly varying of indices $\rho < (p-N)/(p-1)$ and $\sigma < (p-N)/(p-1)$, respectively, if and only if

$$N+\lambda+\frac{\alpha}{p-1}\left[p+\mu+\frac{\beta}{p-1}(p-N)\right] < 0, \quad \frac{\beta}{p-1}\left[p+\lambda+\frac{\alpha}{p-1}(p-N)\right]+N+\mu < 0,$$

in which case ρ and σ are uniquely determined by (1.6 a) and (1.6 b). The asymptotic behaviour of any such solution (u, v) as $|x| \to \infty$ is governed by (1.7 a) and (1.7 b).

The main result of this paper will be presented in § 4. Under the assumption that p_i and q_i are regularly varying, the existence of strongly decreasing regularly varying solutions of (A) is proved by solving the system of integral equations

$$x_i(t) = \int_t^{\infty}\left(\frac{1}{p_i(s)}\int_s^{\infty} q_i(r)x_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i}\mathrm{d}s, \quad i = 1, 2, \dots, n, \tag{1.8}$$

with the help of fixed-point techniques combined with basic properties of regularly varying functions. Furthermore, the asymptotic behaviour of the obtained solutions is determined accurately. For this purpose an essential role is played by the fact that one can obtain thorough knowledge of regularly varying solutions of the following system of asymptotic relations associated with (A):

$$x_i(t) \sim \int_t^{\infty}\left(\frac{1}{p_i(s)}\int_s^{\infty} q_i(r)x_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i}\mathrm{d}s, \quad t \to \infty,\ i = 1, 2, \dots, n, \tag{1.9}$$

which is regarded as an 'approximation' of (1.8). Here the symbol $\sim$ is used to mean the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty}\frac{g(t)}{f(t)} = 1.$$

The exposition of the analysis of system (1.9) by means of regular variation is given in § 3, which is preceded by § 2, in which the definition and some basic properties of regularly varying functions are summarized for the reader's benefit. The proof of our main results on strongly decreasing regularly varying solutions of system (A) with regularly varying coefficients p_i and q_i, $i = 1, \dots, n$, can be found in § 4. First, we construct strongly decreasing solutions of (A) by solving, by means of fixed-point techniques, the system of integral equations (1.8) in a function class that is slightly larger than that of regularly varying functions, and then verify that all the solutions obtained are actually regularly varying functions having accurate asymptotic behaviour at infinity. The results of § 3 play a crucial role throughout the proof. Finally, in § 5 it is shown that our main results on (A) can be effectively applied to some classes of partial differential equations including metaharmonic equations (see [5, 6] for related results) and systems involving p-Laplace operators on exterior domains in $\mathbb{R}^N$.

Since the publication of the monograph [18] of Marić in the year 2000 there has been an increasing interest in the study of differential equations by means of regularly varying functions and, as a consequence, the theory of regular variation has proved to be a powerful tool in the asymptotic analysis of differential equations, giving rise to detailed and accurate information about the existence, the asymptotic behaviour and the structure of positive solutions of various types of ordinary differential equations, which may well be called generalized Emden–Fowler and Thomas–Fermi equations (see, for example, [9–14]).

2. Regularly varying functions

For the reader's convenience we summarize here the definition and some basic properties of regularly varying functions that will be needed in developing our main results in §§ 3, 4 and 5.

Definition 2.1. A measurable function $f\colon [0,\infty) \to (0,\infty)$ is called *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The set of all regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV to denote $\mathrm{RV}(0)$ and call members of SV *slowly varying functions*. Any function $f \in \mathrm{RV}(\rho)$ is expressed as $f(t) = t^\rho g(t)$ with $g \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. One of the most important properties of regularly varying functions is the following *representation theorem*.

Proposition 2.2. *$f \in \mathrm{RV}(\rho)$ if and only if f is represented in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, \mathrm{d}s \right\}, \quad t \geqslant t_0, \tag{2.1}$$

for some $t_0 > 0$ and for some measurable functions c and δ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If, in particular, $c(t) \equiv c_0$ in (2.1), then f is referred to as a *normalized* regularly varying function of index ρ.

Typical examples of slowly varying functions are all functions tending to some positive constant as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{ \prod_{n=1}^{N} (\log_n t)^{\beta_n} \right\}, \quad \beta_n \in (0,1),$$

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where $\log_n t$ denotes the nth iteration of the logarithm. It is known that the function

$$L(t) = \exp\{(\log t)^\theta \cos(\log t)^\theta\}, \quad \theta \in (0, \tfrac{1}{2}),$$

is a slowly varying function that is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0.$$

The following result illustrates operations that preserve slow variation.

Proposition 2.3. *Let L, L_1, L_2 be slowly varying. Then, L^α for any $\alpha \in \mathbb{R}$, $L_1 + L_2$, $L_1 L_2$ and $L_1(L_2)$ (if $L_2(t) \to \infty$ as $t \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited, as is shown in the following proposition.

Proposition 2.4. *Let $f \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^\varepsilon f(t) = \infty, \qquad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for determining the regularity of differentiable positive functions is given by the following proposition.

Proposition 2.5. *A differentiable positive function f is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t\frac{f'(t)}{f(t)} = \rho.$$

The following result, called Karamata's integration theorem, is of the highest importance in handling slowly and regularly varying functions analytically and will be used throughout the paper.

Proposition 2.6. *Let $L \in \mathrm{SV}$. Then*

(i) *for $\alpha > -1$,*

$$\int_a^t s^\alpha L(s)\,\mathrm{d}s \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *for $\alpha < -1$,*

$$\int_t^\infty s^\alpha L(s)\,\mathrm{d}s \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *for $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s}\,\mathrm{d}s \in \mathrm{SV};$$

and if $\int_a^\infty s^{-1} L(s)\,\mathrm{d}s < \infty$, then

$$m(t) = \int_t^\infty \frac{L(s)}{s}\,\mathrm{d}s \in \mathrm{SV}.$$

DEFINITION 2.7. A vector function $(x_1, \ldots, x_n)$ is said to be regularly varying of index $(\rho_1, \ldots, \rho_n)$ if $x_i \in \mathrm{RV}(\rho_i)$ for $i = 1, \ldots, n$. If all ρ_i are positive (or negative), then $(x_1, \ldots, x_n)$ is called regularly varying of positive (or negative) index $(\rho_1, \ldots, \rho_n)$. The set of all regularly varying vector functions of index $(\rho_1, \ldots, \rho_n)$ is denoted by $\mathrm{RV}(\rho_1, \ldots, \rho_n)$.

The most complete exposition of the theory of regular variation and its applications can be found in the book of Bingham *et al*. [2] (see also [19]). For a comprehensive survey of results up to the year 2000 on the asymptotic analysis of second-order ordinary differential equations by means of regular variation, the reader is referred to the monograph of Marić [18].

3. Asymptotic relations associated with (A)

It is assumed here that $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$ and that they are expressed as

$$p_i(t) = t^{\lambda_i} l_i(t), \quad q_i(t) = t^{\mu_i} m_i(t), \quad l_i, m_i \in \mathrm{SV}, \; i = 1, 2, \ldots, n, \tag{3.1}$$

and we seek positive decreasing solutions $x_i \in \mathrm{RV}(\rho_i)$ of system (A) represented in the form

$$x_i(t) = t^{\rho_i} \xi_i(t), \quad \xi_i \in \mathrm{SV}, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

We note that condition (1.2) is satisfied if either

$$\lambda_i < \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-1/\alpha_i} \, \mathrm{d}t = \infty, \tag{3.3}$$

while condition (1.3) is satisfied if either

$$\lambda_i > \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-1/\alpha_i} \, \mathrm{d}t < \infty. \tag{3.4}$$

In analysing strongly decreasing solutions of system (A), it is convenient to distinguish the case in which the p_i satisfy (1.2) from the case in which the p_i satisfy (1.3). For the case of (1.2), which is equivalent to (3.3) holding for $i = 1, \ldots, n$, the solutions $(x_1, \ldots, x_n)$ will be sought in the class $\mathrm{RV}(\rho_1, \ldots, \rho_n)$ with $\rho_i < 0$, $i = 1, \ldots, n$. For the case of (1.3), however, our attention will be focused on the two extreme cases

$$\text{(a) } \lambda_i = \alpha_i, \quad i = 1, \ldots, n, \quad \text{and} \quad \text{(b) } \lambda_i > \alpha_i, \quad i = 1, \ldots, n,$$

which imply, respectively, that

$$\pi_i(t) = \int_t^\infty s^{-1} l_i(s)^{-1/\alpha_i} \, \mathrm{d}s \in \mathrm{SV}$$

and

$$\pi_i(t) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} t^{(\alpha_i - \lambda_i)/\alpha_i} l_i(t)^{-1/\alpha_i} \in \mathrm{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right),$$

and an attempt will be made to detect solutions belonging to $\mathrm{RV}(\rho_1, \ldots, \rho_n)$ with $\rho_i < 0$, $i = 1, \ldots, n$, or to $\mathrm{RV}(\rho_1, \ldots, \rho_n)$ with $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \ldots, n$, according to whether (a) or (b) holds, respectively.

Let $(x_1, \dots, x_n)$ be a strongly decreasing solution of (A) on $[T, \infty)$. It then holds that

$$x_i(t) = \int_t^\infty \left(\frac{1}{p_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} \, \mathrm{d}r \right)^{1/\alpha_i} \mathrm{d}s, \quad t \geqslant T, \ i = 1, \dots, n, \tag{3.5}$$

which clearly applies to both cases (1.2) and (1.3). Our task is to solve this system of integral equations by means of regularly varying functions. This can be accomplished on the basis of the analysis of regularly varying solutions of the system of integral asymptotic relations

$$x_i(t) \sim \int_t^\infty \left(\frac{1}{p_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} \, \mathrm{d}r \right)^{1/\alpha_i} \mathrm{d}s, \quad t \to \infty, \ i = 1, \dots, n, \tag{3.6}$$

in the framework of regular variation. It will turn out that one can acquire thorough knowledge of all possible regularly varying solutions of negative indices of (3.6).

We begin by considering system (3.6) with p_i satisfying condition (1.2). Suppose that (3.6) has a positive solution $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ on $[T, \infty)$ with $\rho_i < 0$, $i = 1, \dots, n$. Using (3.1) and (3.2), we have

$$\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} \, \mathrm{d}s = \int_t^\infty s^{\mu_i + \beta_i \rho_{i+1}} m_i(s) \xi_{i+1}(s)^{\beta_i} \, \mathrm{d}s, \quad t \geqslant T, \ i = 1, \dots, n. \tag{3.7}$$

Here $\mu_i + \beta_i \rho_{i+1} \leqslant -1$ because of the convergence of the integrals, but the equality is not allowed for any i. In fact, if the equality holds for some i, then (3.7) implies that

$$\left(\frac{1}{p_i(t)} \int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i} = t^{-\lambda_i/\alpha_i} l_i(t)^{-1/\alpha_i} \left(\int_t^\infty s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i} \in \mathrm{RV}\left(-\frac{\lambda_i}{\alpha_i} \right). \tag{3.8}$$

Since (3.8) is integrable over $[T, \infty)$, it is only possible that $\lambda_i = \alpha_i$, in which case integration of (3.8) on $[t, \infty)$ shows, in view of (3.5), that $x_i \in \mathrm{SV} = \mathrm{RV}(0)$. This contradicts the hypothesis that $x_i \in \mathrm{RV}(\rho_i)$ with negative ρ_i. Therefore, we must have $\mu_i + \beta_i \rho_{i+1} < -1$ for $i = 1, \dots, n$. By applying Karamata's integration theorem to the right-hand side of (3.7), we obtain

$$\left(\frac{1}{p_i(t)} \int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i} \sim \frac{t^{(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{[-(\mu_i + \beta_i \rho_{i+1} + 1)]^{1/\alpha_i}}, \quad t \to \infty, \tag{3.9}$$

for $i = 1, \dots, n$. Note that each relation in (3.9) is integrable on $[T, \infty)$. We claim that

$$\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} < -1, \quad i = 1, \dots, n. \tag{3.10}$$

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In fact, if the equality would hold for some i in (3.10), then $\mu_i + \beta_i\rho_{i+1} + 1 = \lambda_i - \alpha_i < 0$, and from (3.9) and (3.5) it would follow that

$$x_i(t) \sim (\alpha_i - \lambda_i)^{-1/\alpha_i} \int_t^\infty s^{-1} l_i(s)^{-1/\alpha_i} m_i(s)^{1/\alpha_i} \xi_{i+1}(s)^{\beta_i/\alpha_i}\, ds \in \mathrm{SV}, \quad t \to \infty,$$

an impossibility. Using (3.10) and applying Karamata's integration theorem, we find that

$$x_i(t) \sim \frac{t^{(-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1)/\alpha_i+1} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{[-(\mu_i+\beta_i\rho_{i+1}+1)]^{1/\alpha_i}[-((-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1)/\alpha_i+1)]}, \quad t \to \infty. \tag{3.11}$$

This means that

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} + 1, \quad i = 1, \dots, n, \tag{3.12}$$

or

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = 1, \dots, n. \tag{3.13}$$

To solve the algebraic linear system (3.13) in ρ_i, $i = 1, \dots, n$, it suffices to note that the coefficient matrix

$$A = A\left(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n}\right) = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{\beta_{n-1}}{\alpha_{n-1}} \\ -\frac{\beta_n}{\alpha_n} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.14}$$

is invertible because

$$|A| = \frac{A_n - B_n}{A_n} > 0, \quad \text{where } A_n = \alpha_1\alpha_2\cdots\alpha_n,\ B_n = \beta_1\beta_2\cdots\beta_n, \tag{3.15}$$

and its inverse is given explicitly by

$$A^{-1} = \frac{A_n}{A_n - B_n} \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \cdots & \cdots & \frac{\beta_1\beta_2\cdots\beta_{n-1}}{\alpha_1\alpha_2\cdots\alpha_{n-1}} \\ & 1 & \frac{\beta_2}{\alpha_2} & \frac{\beta_2\beta_3}{\alpha_2\alpha_3} & \cdots & \frac{\beta_2\beta_3\cdots\beta_{n-1}}{\alpha_2\alpha_3\cdots\alpha_{n-1}} \\ & & 1 & \frac{\beta_3}{\alpha_3} & \cdots & \frac{\beta_3\cdots\beta_{n-1}}{\alpha_3\cdots\alpha_{n-1}} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & \frac{\beta_{n-1}}{\alpha_{n-1}} \\ & & & & & 1 \end{pmatrix}, \tag{3.16}$$

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where the lower triangular elements are omitted for economy of notation. Let (M_{ij}) denote the matrix on the right-hand side of (3.16). We then see that the ith row of (M_{ij}) is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \dots, \frac{\beta_i\beta_{i+1}\cdots\beta_{i+(n-2)}}{\alpha_i\alpha_{i+1}\cdots\alpha_{i+(n-2)}}\right), \quad \alpha_{n+1} = \alpha_1,\ \beta_{n+1} = \beta_1,$$

$i-1$ times to the right cyclically, so that the lower triangular elements M_{ij}, $i > j$, satisfy the relations

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2\cdots\beta_n}{\alpha_1\alpha_2\cdots\alpha_n}, \quad i > j,\ i = 1, 2, \dots, n. \tag{3.17}$$

Then the unique solution ρ_i, $i = 1, \dots, n$, of (3.13) is given explicitly by

$$\rho_i = \frac{A_n}{A_n - B_n} \sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = 1, \dots, n, \tag{3.18}$$

from which it follows that all ρ_i are negative if

$$\sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0, \quad i = 1, \dots, n. \tag{3.19}$$

We observe that (3.11) can be rewritten in the form

$$x_i(t) \sim \frac{t^{(\alpha_i+1)/\alpha_i} p_i(t)^{-1/\alpha_i} q_i(t)^{1/\alpha_i} x_{i+1}(t)^{\beta_i/\alpha_i}}{D_i}, \quad t \to \infty, \tag{3.20}$$

where

$$D_i = (\alpha_i - \lambda_i - \alpha_i\rho_i)^{1/\alpha_i}(-\rho_i) \tag{3.21}$$

for $i = 1, \dots, n$. It is a matter of elementary computation to derive from (3.20) the following independent explicit asymptotic formulae for each x_i:

$$x_i(t) \sim \left[\prod_{j=1}^{n} \left(\frac{t^{(\alpha_j+1)/\alpha_j} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{D_j}\right)^{M_{ij}}\right]^{A_n/(A_n-B_n)}, \qquad t \to \infty,\ i = 1, \dots, n. \tag{3.22}$$

Notice that (3.22) is transformed into

$$x_i(t) \sim t^{\rho_i} \left[\prod_{j=1}^{n} \left(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j}\right)^{M_{ij}}\right]^{A_n/(A_n-B_n)}, \quad t \to \infty,\ i = 1, \dots, n. \tag{3.23}$$

Let us now assume that (3.19) holds, define the constants ρ_i by (3.18) and define the functions $X_i \in \mathrm{RV}(\rho_i)$ on $[a, \infty)$ by

$$X_i(t) = \left[\prod_{j=1}^{n} \left(\frac{t^{(\alpha_j+1)/\alpha_j} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{D_j}\right)^{M_{ij}}\right]^{A_n/(A_n-B_n)}, \quad i = 1, \dots, n. \tag{3.24}$$

It can then be shown that the X_i satisfy the system of asymptotic relations (3.6), i.e.

$$\int_t^\infty \left(\frac{1}{p_i(s)}\int_s^\infty q_i(r)X_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i}\mathrm{d}s \sim X_i(t), \quad t\to\infty,\ i=1,\dots,n, \tag{3.25}$$

where $X_{n+1}(t)=X_1(t)$. In fact, using the following expression for $X_i(t)$,

$$X_i(t)=t^{\rho_i}\varXi_i(t), \quad \varXi_i(t)=\left[\prod_{j=1}^n\left(\frac{l_j(t)^{-1/\alpha_j}m_j(t)^{1/\alpha_j}}{D_j}\right)^{M_{ij}}\right]^{A_n/(A_n-B_n)},$$

we have, via Karamata's integration theorem,

$$\left(\frac{1}{p_i(t)}\int_t^\infty q_i(s)X_{i+1}(s)^{\beta_i}\,\mathrm{d}s\right)^{1/\alpha_i} \sim \frac{t^{\rho_i-1}l_i(t)^{-1/\alpha_i}m_i(t)^{1/\alpha_i}\varXi_{i+1}^{\beta_i/\alpha_i}}{(\alpha_i-\lambda_i-\alpha_i\rho_i)^{1/\alpha_i}}$$

and

$$\int_t^\infty \left(\frac{1}{p_i(s)}\int_t^\infty q_i(r)X_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i}\mathrm{d}s \sim \frac{t^{\rho_i}l_i(t)^{-1/\alpha_i}m_i(t)^{1/\alpha_i}\varXi_{i+1}(t)^{\beta_i/\alpha_i}}{D_i} \tag{3.26}$$

as $t\to\infty$. Since a simple calculation, with the help of the relations

$$M_{i+1,i}\frac{\beta_i}{\alpha_i}=\frac{B_n}{A_n}, \quad M_{i+1,j}\frac{\beta_i}{\alpha_i}=M_{ij} \quad \text{for } j\neq i$$

between the ith and $(i+1)$th rows of the matrix A, shows that

$$\begin{aligned}
&\frac{l_i(t)^{-1/\alpha_i}m_i(t)^{1/\alpha_i}}{D_i}\varXi_{i+1}(t)^{\beta_i/\alpha_i}\\
&\quad=\frac{l_i(t)^{-1/\alpha_i}m_i(t)^{1/\alpha_i}}{D_i}\left[\prod_{j=1}^n\left(\frac{l_j(t)^{-1/\alpha_j}m_j(t)^{1/\alpha_j}}{D_j}\right)^{M_{i+1,j}(\beta_i/\alpha_i)}\right]^{A_n/(A_n-B_n)}\\
&\quad=\left[\prod_{j=1}^n\left(\frac{l_j(t)^{-1/\alpha_j}m_j(t)^{1/\alpha_j}}{D_j}\right)^{M_{ij}(\beta_i/\alpha_i)}\right]^{A_n/(A_n-B_n)}\\
&\quad=\varXi_i(t),
\end{aligned}$$

we conclude from (3.26) that the X_i, $i=1,\dots,n$, satisfy the asymptotic relations (3.25), as desired.

Summarizing the above discussions, we obtain the following noteworthy result providing complete information about the existence and asymptotic behaviour of regularly varying solutions with negative indices of (3.6).

THEOREM 3.1. *Suppose that $p_i\in\mathrm{RV}(\lambda_i)$ and $q_i\in\mathrm{RV}(\mu_i)$, $i=1,\dots,n$, and that the p_i satisfy condition (1.2). The system of asymptotic relations (3.6) has regularly varying solutions $(x_1,\dots,x_n)\in\mathrm{RV}(\rho_1,\dots,\rho_n)$ with $\rho_i<0$, $i=1,\dots,n$, if and only if (3.19) holds, in which case the ρ_i are uniquely determined by (3.18) and the asymptotic behaviour of any such solution is governed by the unique formula (3.22).*

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We next turn to the case in which the p_i satisfy condition (1.3), and show that for the two particular cases (i) $\lambda_i = \alpha_i$ and (ii) $\lambda_i > \alpha_i$, for all $i = 1, 2, \dots, n$, complete analysis can be made of the system of asymptotic relations (3.6) from the viewpoint of regular variation.

THEOREM 3.2. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$ for $i = 1, \dots, n$. Suppose that the p_i satisfy condition (1.3).*

(i) *Suppose that $\lambda_i = \alpha_i$, $i = 1, \dots, n$. System (3.6) has regularly varying solutions $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i < 0$, $i = 1, \dots, n$, if and only if*

$$\sum_{j=1}^{n} M_{ij} \frac{\mu_j + 1}{\alpha_j} < 0, \quad i = 1, \dots, n, \tag{3.27}$$

in which case the ρ_i are uniquely determined by

$$\rho_i = \frac{A_n}{A_n - B_n} \sum_{j=1}^{n} M_{ij} \frac{\mu_j + 1}{\alpha_j}, \quad i = 1, \dots, n, \tag{3.28}$$

and the asymptotic behaviour of any such solution is governed by the unique set of formulae (3.22) with $D_j = (\alpha_j)^{1/\alpha_j}(-\rho_j)^{(\alpha_j+1)/\alpha_j}$, $j = 1, \dots, n$.

(ii) *Suppose that $\lambda_i > \alpha_i$, $i = 1, \dots, n$. System (3.6) has regularly varying solutions $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, if and only if*

$$\sum_{j=1}^{n} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right) < 0, \quad i = 1, \dots, n, \tag{3.29}$$

where $\alpha_{n+1} = \alpha_1$, $\lambda_{n+1} = \lambda_1$, in which case the ρ_i are uniquely determined by (3.18) and the asymptotic behaviour of any such solution is governed by the unique set of formulae (3.22).

Proof. (i) Let a solution $(x_1, \dots, x_n)$ of (3.6) be a member of $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with negative indices. It is easy to confirm that starting from (3.7) one can proceed exactly as in the proof of theorem 3.1 to reach the conclusion that (3.19) holds, the ρ_i are given by (3.18) and the x_i obey the unique decay law (3.22). Since $\lambda_i = \alpha_i$, (3.19) and (3.18) are simplified to (3.27) and (3.28), respectively. This proves the 'only if' part. To prove the 'if' part we need only repeat the argument showing that in the present case the functions (3.24) satisfy the asymptotic relations (3.25).

(ii) Suppose that (3.6) has a solution $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$. If $\mu_i + \beta_i \rho_{i+1} = -1$ in (3.7), then, integrating (3.8) on $[t, \infty)$ and using Karamata's integration theorem, we have

$$x_i(t) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} t^{(\alpha_i - \lambda_i)/\alpha_i} l_i(t)^{-1/\alpha_i} \left(\int_t^{\infty} s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i} \in \mathrm{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i} \right),$$

contrary to the hypothesis that $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$. Therefore, $\mu_i + \beta_i\rho_{i+1} < -1$, and we obtain (3.9), which is integrable on $[t, \infty)$. Hence, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \leqslant -1$. But here the equality is excluded because if the equality holds, then $0 < \lambda_i - \alpha_i = \mu_i + \beta_i\rho_{i+1} + 1 < -1$, which is impossible. Thus, (3.10) must hold. Then, integrating (3.9) from t to ∞, we obtain (3.11), which implies that the ρ_i must satisfy (3.13) so that they are determined explicitly by the formulae (3.18). Putting $\sigma_i = \rho_i - (\alpha_i - \lambda_i)/\alpha_i < 0$, we transform (3.13) into

$$\sigma_i - \frac{\beta_i}{\alpha_i}\sigma_{i+1} = \frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i\alpha_{i+1}}, \quad i = 1, \dots, n. \tag{3.30}$$

It is clear that the solution σ_i of (3.30) is given by

$$\sigma_i = \frac{A_n}{A_n - B_n}\sum_{j=1}^{n} M_{ij}\left(\frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i\alpha_{i+1}}\right), \quad i = 1, \dots, n,$$

from which (3.29) follows immediately. Finally, after expressing (3.11) in the form (3.20), we are able to establish the asymptotic formulae (3.22) for the x_i. On the other hand, it can be verified that if (3.29) holds and the ρ_i are defined by (3.18), then the functions X_i given by (3.24) satisfy the asymptotic relations (3.25). This completes the proof. □

REMARK 3.3. As is easily seen, theorem 3.1 and theorem 3.2(i) can be unified into the following theorem.

THEOREM 3.4. *Suppose that $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. Suppose in addition that $\lambda_i \leqslant \alpha_i$, $i = 1, \dots, n$. System (3.6) has regularly varying solutions $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i < 0$, $i = 1, \dots, n$, if and only if (3.19) holds, in which case the ρ_i are uniquely determined by (3.18) and the asymptotic behaviour of any such solution is governed by the unique formula (3.22).*

Note that this result applies to those systems of the form of (A) in which some or all of the p_i such that $\lambda_i = \alpha_i$ satisfy $\int_a^\infty p_i(t)^{-1/\alpha_i}\,\mathrm{d}t < \infty$.

4. Strongly decreasing solutions of (A)

We now turn to the problem of existence of strongly decreasing solutions for system (A) in the framework of regularly varying functions. Our main results are the following two theorems. Use is made of the notation and properties of the matrix (3.14) and its inverse (3.16).

THEOREM 4.1. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and let $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. Suppose that $\lambda_i \leqslant \alpha_i$, $i = 1, \dots, n$. Then system (A) possesses strongly decreasing solutions in $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i < 0$, $i = 1, \dots, n$, if and only if (3.19) holds, in which case the ρ_i are given by (3.18) and the asymptotic behaviour of any such solution $(x_1, \dots, x_n)$ is governed by the unique formula (3.22).*

THEOREM 4.2. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and let $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. Suppose that $\lambda_i > \alpha_i$, $i = 1, \dots, n$. Then system (A) possesses strongly decreasing solutions in $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, if and only if (3.29) holds,*

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in which case the ρ_i are given by (3.18) and the asymptotic behaviour of any such solution $(x_1, \dots, x_n)$ is governed by the unique formula (3.22).

We remark that the 'only if' parts of these theorems follow immediately from the corresponding parts of theorem 3.4 and theorem 3.2(ii). The 'if' parts will be proved via the following results (which are of interest in their own right) for systems of the form of (A) with nearly regularly varying coefficients p_i and q_i in the sense defined below.

DEFINITION 4.3. Let f be regularly varying of index σ and suppose that g satisfies $kf(t) \leqslant g(t) \leqslant Kf(t)$ for some positive constants k, K and for all large t. Then g is said to be a *nearly regularly varying function of index* σ. Such a relation between f and g is denoted by $g(t) \asymp f(t)$ as $t \to \infty$.

THEOREM 4.4. *Let p_i and q_i be nearly regularly varying of indices λ_i and μ_i, respectively, that is, there exist $\tilde{p}_i \in \mathrm{RV}(\lambda_i)$ and $\tilde{q}_i \in \mathrm{RV}(\mu_i)$ such that*

$$p_i(t) \asymp \tilde{p}_i(t), \quad q_i(t) \asymp \tilde{q}_i(t), \quad t \to \infty, \ i = 1, \dots, n. \tag{4.1}$$

Suppose in addition that $\lambda_i \leqslant \alpha_i$, $i = 1, \dots, n$, and that (3.19) holds. Then system (A) possesses strongly decreasing solutions $(x_1, \dots, x_n)$ that are nearly regularly varying of negative index $(\rho_1, \dots, \rho_n)$ in the sense that

$$x_i(t) \asymp \left[\prod_{j=1}^{n} \left(\frac{t^{(\alpha_j+1)/\alpha_j} \tilde{p}_j(t)^{-1/\alpha_j} \tilde{q}_j(t)^{1/\alpha_j}}{D_j} \right)^{M_{ij}} \right]^{A_n/(A_n - B_n)}, \quad t \to \infty, \ i = 1, \dots, n, \tag{4.2}$$

where ρ_i and D_j are defined by (3.18) and (3.21), respectively.

THEOREM 4.5. *Let p_i and q_i be nearly regularly varying of indices λ_i and μ_i, respectively, $i = 1, \dots, n$. Suppose that $\lambda_i > \alpha_i$, $i = 1, \dots, n$, and that (3.29) holds. Then system (A) possesses strongly decreasing solutions $(x_1, \dots, x_n)$ that are nearly regularly varying of negative index $(\rho_1, \dots, \rho_n)$ with $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, and satisfy (4.2), where ρ_i and D_j are defined by (3.18) and (3.21), respectively.*

Proof of theorem 4.4. Assume that the regularly varying functions $\tilde{p}_i$ and $\tilde{q}_i$ in (4.1) are expressed as

$$\tilde{p}_i(t) = t^{\lambda_i} l_i(t) \quad \text{and} \quad \tilde{q}_i(t) = t^{\mu_i} m_i(t), \quad l_i, m_i \in \mathrm{SV}. \tag{4.3}$$

By hypothesis, there exist positive constants h_i, H_i, k_i and K_i such that

$$h_i \tilde{p}_i(t) \leqslant p_i(t) \leqslant H_i \tilde{p}_i(t), \quad k_i \tilde{q}_i(t) \leqslant q_i(t) \leqslant K_i \tilde{q}_i(t), \quad i = 1, \dots, n, \tag{4.4}$$

for all large t. Define the functions X_i by

$$X_i(t) = t^{\rho_i} \left[\prod_{j=1}^{n} \left(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j} \right)^{M_{ij}} \right]^{A_n/(A_n - B_n)}, \quad i = 1, \dots, n. \tag{4.5}$$

It is known that

$$\int_t^\infty \left(\frac{1}{\tilde{p}_i(s)} \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} \, \mathrm{d}r \right)^{1/\alpha_i} \mathrm{d}s \sim X_i(t), \quad t \to \infty, \ i = 1, \dots, n, \tag{4.6}$$

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so there exists $T > a$ such that

$$\tfrac{1}{2}X_i(t) \leqslant \int_t^\infty \left(\frac{1}{\tilde{p}_i(s)} \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i} \mathrm{d}s \leqslant 2X_i(t), \quad t \geqslant T, \tag{4.7}$$

for $i = 1, \dots, n$. Consider the set $\mathcal{X}$ consisting of continuous vector functions $(x_1, \dots, x_n)$ on $[T, \infty)$ satisfying

$$l_i X_i(t) \leqslant x_i(t) \leqslant L_i X_i(t), \quad t \geqslant T,\ i = 1, \dots, n, \tag{4.8}$$

where l_i, L_i are positive constants chosen so that

$$l_i \leqslant \frac{1}{2}\left(\frac{k_i}{H_i}\right)^{1/\alpha_i} l_{i+1}^{\beta_i/\alpha_i}, \quad 2\left(\frac{K_i}{h_i}\right)^{1/\alpha_i} L_{i+1}^{\beta_i/\alpha_i} \leqslant L_i, \quad i = 1, \dots, n, \tag{4.9}$$

where $l_{n+1} = l_1$, $L_{n+1} = L_1$. An example of such choices is

$$l_i = \left[\prod_{j=1}^n \frac{1}{2}\left(\frac{k_i}{H_i}\right)^{M_{ij}/\alpha_i}\right]^{A_n/(A_n-B_n)}, \qquad L_i = \left[\prod_{j=1}^n 2\left(\frac{k_i}{H_i}\right)^{M_{ij}/\alpha_i}\right]^{A_n/(A_n-B_n)}.$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T, \infty)^n$.

Define the mapping $\varPhi$ by

$$\varPhi(x_1, \dots, x_n)(t) = (\mathcal{F}_1 x_2(t), \mathcal{F}_2 x_3(t), \dots, \mathcal{F}_n x_{n+1}(t)), \quad t \geqslant T,\ x_{n+1} = x_1, \tag{4.10}$$

where $\mathcal{F}_i$ stands for the ith integral operator

$$\mathcal{F}_i x(t) = \int_t^\infty \left(\frac{1}{p_i(s)} \int_s^\infty q_i(r) x(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i} \mathrm{d}s, \quad t \geqslant T,\ i = 1, \dots, n, \tag{4.11}$$

and let it act on $\mathcal{X}$. Using (4.3), (4.4), (4.7)–(4.10), one can show that $\varPhi$ is a continuous self-map on $\mathcal{X}$, and sends $\mathcal{X}$ into a relatively compact subset of $C[T, \infty)^n$.

(i) *$\varPhi(\mathcal{X}) \subset \mathcal{X}$:* let $(x_1, \dots, x_n) \in \mathcal{X}$. Then, for $i = 1, \dots, n$,

$$\begin{aligned}
\mathcal{F}_i x_{i+1}(t) &\leqslant \left(\frac{K_i L_{i+1}^{\beta_i}}{h_i}\right)^{1/\alpha_i} \int_t^\infty \left(\frac{1}{\tilde{p}_i(s)} \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i} \mathrm{d}s \\
&\leqslant 2\left(\frac{K_i L_{i+1}^{\beta_i}}{h_i}\right)^{1/\alpha_i} X_i(t) \\
&\leqslant L_i X_i(t), \quad t \geqslant T,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_i x_{i+1}(t) &\geqslant \left(\frac{k_i l_{i+1}^{\beta_i}}{H_i}\right)^{1/\alpha_i} \int_t^\infty \left(\frac{1}{\tilde{p}_i(s)} \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i}\,\mathrm{d}r\right)^{1/\alpha_i} \mathrm{d}s \\
&\geqslant \frac{1}{2}\left(\frac{k_i l_{i+1}^{\beta_i}}{H_i}\right)^{1/\alpha_i} X_i(t) \\
&\geqslant l_i X_i(t), \quad t \geqslant T.
\end{aligned}$$

This shows that $\varPhi(x_1, \dots, x_n) \in \mathcal{X}$, that is, $\varPhi$ maps $\mathcal{X}$ into itself.

(ii) $\Phi(\mathcal{X})$ *is relatively compact:* the inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$ ensures that $\Phi(\mathcal{X})$ is uniformly bounded on $[T, \infty)$. From the inequalities

$$0 \geqslant (\mathcal{F}_i x_{i+1})'(t) \geqslant -L_{i+1}^{\beta_i/\alpha_i} \left(\frac{1}{p_i(t)} \int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i}, \quad t \geqslant T,$$

holding for all $(x_1, \dots, x_n) \in \mathcal{X}$, we see that $\Phi(\mathcal{X})$ is equicontinuous on $[T, \infty)$. The relative compactness of $\Phi(\mathcal{X})$ then follows from the Arzelà–Ascoli theorem.

(iii) Φ *is a continuous map:* let $\{(x_1^\nu, \dots, x_n^\nu)\}$ be a sequence in $\mathcal{X}$ converging as $\nu \to \infty$ to $(x_1, \dots, x_n) \in \mathcal{X}$ uniformly on any compact subinterval of $[T, \infty)$. Using (4.11) we have

$$|\mathcal{F}_i x_{i+1}^\nu(t) - \mathcal{F}_i x_{i+1}(t)| \leqslant \int_t^\infty p_i(s)^{-1/\alpha_i} F_i^\nu(s) \, \mathrm{d}s, \quad t \geqslant T, \tag{4.12}$$

where

$$F_i^\nu(t) = \left| \left(\int_t^\infty q_i(s) x_{i+1}^\nu(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i} - \left(\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{1/\alpha_i} \right|.$$

It is clear that if $\alpha_i \geqslant 1$, then

$$F_i^\nu(t) \leqslant \left(\int_t^\infty q_i(s) |x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i}| \, \mathrm{d}s \right)^{1/\alpha_i}, \tag{4.13}$$

and if $\alpha_i < 1$, then

$$F_i^\nu(t) \leqslant \frac{1}{\alpha_i} \left(\int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} \, \mathrm{d}s \right)^{(1/\alpha_i)-1} \int_t^\infty q_i(s) |x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i}| \, \mathrm{d}s. \tag{4.14}$$

Combining (4.12) with (4.13) or (4.14), we conclude via the Lebesgue dominated convergence theorem that

$$\lim_{\nu \to \infty} \mathcal{F}_i x_{i+1}^\nu(t) = \mathcal{F}_i x_{i+1}(t) \quad \text{uniformly on } [T, \infty), \ i = 1, \dots, n.$$

This proves the continuity of Φ.

Thus, all the hypotheses of the Schauder–Tychonoff fixed-point theorem are fulfilled, and Φ has a fixed point $(x_1, \dots, x_n) \in \mathcal{X}$, which satisfies

$$x_i(t) = \mathcal{F}_i x_{i+1}(t) = \int_t^\infty \left(\frac{1}{p_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} \, \mathrm{d}r \right)^{1/\alpha_i} \mathrm{d}s, \quad t \geqslant T, \ i = 1, \dots, n. \tag{4.15}$$

This clearly implies that $(x_1, \dots, x_n)$ is a solution of system (A) on $[T, \infty)$. Since the solution obtained is a member of $\mathcal{X}$, it is strongly decreasing as well as nearly regularly varying, and enjoys the asymptotic behaviour (4.2). This completes the proof. □

The proof of theorem 4.5 is essentially the same as above, and so it may be omitted.

The 'if' parts of theorems 4.1 and 4.2 can be proved on the basis of theorems 4.4 and 4.5 and with the help of the following generalized L'Hôpital's rule. For the proof see, for example, [8].

LEMMA 4.6. *Let $f, g \in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \textit{ for all large } t,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \textit{and} \quad g'(t) < 0 \textit{ for all large } t.$$

Then,

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leqslant \liminf_{t\to\infty} \frac{f(t)}{g(t)}, \qquad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leqslant \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

Proof of the 'if' part of theorem 4.1. Assume that $p_i \in \mathrm{RV}(\lambda_i)$, $\lambda_i \leqslant \alpha_i$ and $q_i \in \mathrm{RV}(\mu_i)$. Define ρ_i to be the negative constants given by (3.18) and let $X_i \in \mathrm{RV}(\rho_i)$ denote the functions on the right-hand side of (4.2) with $\tilde{p}_i$ and $\tilde{q}_i$ replaced by p_i and q_i, respectively. Then, by theorem 4.4, system (A) has a decreasing nearly regularly varying solution $(x_1, \dots, x_n)$ such that $x_i(t) \asymp X_i(t)$ as $t \to \infty$, $i = 1, \dots, n$. Note that the x_i satisfy the integral equations (4.15).

It remains to verify that the x_i are regularly varying functions, i.e. $x_i \in \mathrm{RV}(\rho_i)$, $i = 1, \dots, n$. Define

$$u_i(t) = \int_t^\infty \left(\frac{1}{p_i(s)} \int_s^\infty q_i(r) X_{i+1}(r)^{\beta_i} \,\mathrm{d}r\right)^{1/\alpha_i} \mathrm{d}s, \quad i = 1, \dots, n, \tag{4.16}$$

and put

$$l_i = \liminf_{t\to\infty} \frac{x_i(t)}{u_i(t)}, \quad L_i = \limsup_{t\to\infty} \frac{x_i(t)}{u_i(t)}, \quad i = 1, \dots, n. \tag{4.17}$$

Since $x_i(t) \asymp X_i(t)$ and

$$u_i(t) \sim X_i(t), \quad t \to \infty,\ i = 1, \dots, n, \tag{4.18}$$

we see that $0 < l_i \leqslant L_i < \infty$, $i = 1, \dots, n$. Using lemma 4.6, we compute

$$\begin{aligned} l_i \geqslant \liminf_{t\to\infty} \frac{x_i'(t)}{u_i'(t)} &= \liminf_{t\to\infty} \frac{(\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} \,\mathrm{d}s)^{1/\alpha_i}}{(\int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} \,\mathrm{d}s)^{1/\alpha_i}} \\ &= \liminf_{t\to\infty} \left(\frac{\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} \,\mathrm{d}s}{\int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} \,\mathrm{d}s}\right)^{1/\alpha_i} \\ &\geqslant \liminf_{t\to\infty} \left(\frac{x_{i+1}(t)}{X_{i+1}(t)}\right)^{\beta_i/\alpha_i} \end{aligned}$$

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$$= \liminf_{t\to\infty} \left(\frac{x_{i+1}(t)}{u_{i+1}(t)}\right)^{\beta_i/\alpha_i}$$
$$= \left(\liminf_{t\to\infty} \frac{x_{i+1}(t)}{u_{i+1}(t)}\right)^{\beta_i/\alpha_i}$$
$$= l_{i+1}^{\beta_i/\alpha_i},$$

where (4.18) has been used in the last step. Thus, we obtain the cyclic inequalities for $\{l_i\}$:

$$l_i \geqslant l_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \dots, n, \ l_{n+1} = l_1. \tag{4.19}$$

Similarly, by taking the upper limits instead of the lower limits, we are led to the cyclic inequalities for $\{L_i\}$:

$$L_i \leqslant L_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \dots, n, \ L_{n+1} = L_1. \tag{4.20}$$

From (4.19) and (4.20) it follows that

$$l_i \geqslant l_i^{\beta_1\cdots\beta_n/\alpha_1\cdots\alpha_n}, \quad L_i \leqslant L_i^{\beta_1\cdots\beta_n/\alpha_1\cdots\alpha_n}, \quad i = 1, \dots, n,$$

whence, using $\beta_1 \cdots \beta_n/\alpha_1 \cdots \alpha_n < 1$, we see that $l_i \geqslant 1$ and $L_i \leqslant 1$, and hence that $l_i = L_i = 1$ or $\lim_{t\to\infty} x_i(t)/u_i(t) = 1$ for $i = 1, \dots, n$. This combined with (4.18) shows that $x_i(t) \sim u_i(t) \sim X_i(t)$ as $t \to \infty$, $i = 1, \dots, n$, implying that every x_i is regularly varying of index ρ_i. Thus, the 'if' part of theorem 4.1 has been proved. We omit the proof of the 'if' part of theorem 4.2. □

5. Application to partial differential equations

The purpose of the final section is to demonstrate that our main results for the system of ordinary differential equations (A) can be applied to some classes of partial differential equations to give birth to new results on the existence and the asymptotic behaviour of their radial positive solutions. Throughout this section, $x = (x_1, \dots, x_N)$ represents the space variable in $\mathbb{R}^N$, $N \geqslant 2$, and $|x|$ denotes the Euclidean length of x. All partial differential equations will be considered in an exterior domain $\Omega_R = \{x \in \mathbb{R}^N : |x| \geqslant R\}$, $R > 0$.

5.1. Systems of *p*-Laplacian equations

Consider the system of nonlinear p-Laplacian equations

$$\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) = f_i(|x|)|u_{i+1}|^{\gamma_i-1}u_{i+1}, \quad i = 1, \dots, n, \ u_{n+1} = u_1, \tag{5.1}$$

for $x \in \Omega_R$, where $p > 1$ and $\gamma_i > 0$ are constants and the f_i are positive continuous functions on $[R, \infty)$ that are regularly varying of indices ν_i, $i = 1, \dots, n$. Our attention will be focused on radial solutions $(u_1, \dots, u_n)$ of (5.1) defined in Ω_R. It is known that $(u_1, \dots, u_n)$ is a (radial) solution of (5.1) in Ω_R if and only if the vector function $(u_1, \dots, u_n)$ is a solution of the system of ordinary differential equations

$$(t^{N-1}|u_i'|^{p-2}u_i')' = t^{N-1}f_i(t)|u_{i+1}|^{\gamma_i-1}u_{i+1}, \quad t \geqslant R, \ i = 1, \dots, n, \ u_{n+1} = u_1, \tag{5.2}$$

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which is a special case of system (A) with

$$\begin{aligned} \alpha_1 = \cdots = \alpha_n = p-1, &\qquad \beta_i = \gamma_i, &\qquad i = 1,\dots,n, \\ \lambda_1 = \cdots = \lambda_n = N-1, &\qquad \mu_i = N-1+\nu_i, &\qquad i = 1,\dots,n. \end{aligned}$$

We assume that

$$\gamma_1 \cdots \gamma_n < (p-1)^n. \tag{5.3}$$

Using the inverse of the matrix $A(\gamma_1/(p-1),\dots,\gamma_n/(p-1))$ associated with (5.2) (see (3.14)), we define

$$(M_{ij}) = \frac{(p-1)^n}{(p-1)^n - \gamma_1\cdots\gamma_n} A\left(\frac{\gamma_1}{p-1},\dots,\frac{\gamma_n}{p-1}\right)^{-1}. \tag{5.4}$$

To analyse (5.2) we need to distinguish between the two cases $p \geqslant N$ and $p < N$, in which conditions (1.2) and (1.3) are satisfied, respectively, for system (5.2).

Case 1. Suppose that $p \geqslant N$. In this case, applying theorem 4.1 to (5.2), we conclude that system (5.1) possesses decreasing radial solutions $(u_1,\dots,u_n)$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\rho_i < 0$, $i = 1,\dots,n$, if and only if

$$\sum_{j=1}^{n} M_{ij}(p+\nu_j) < 0, \quad i = 1,\dots,n. \tag{5.5}$$

In this case the ρ_i are uniquely determined by

$$\rho_i = \frac{(p-1)^{n-1}}{(p-1)^n - \gamma_1\cdots\gamma_n} \sum_{j=1}^{n} M_{ij}(p+\nu_j) < 0, \quad i = 1,\dots,n, \tag{5.6}$$

and furthermore the asymptotic behaviour of any such solution as $|x| \to \infty$ is governed by the unique decay law

$$u_i(|x|) \sim |x|^{\rho_i}\left[\prod_{j=1}^{n}\left(\frac{\varphi_j(|x|)^{1/(p-1)}}{(p-N-(p-1)\rho_j)^{1/(p-1)}(-\rho_j)}\right)^{M_{ij}}\right]^{(p-1)^n/((p-1)^n-\gamma_1\cdots\gamma_n)}, \quad |x| \to \infty, \tag{5.7}$$

where $\varphi_i \in \mathrm{SV}$ are the slowly varying parts of f_i, i.e. $f_i(t) = t^{\nu_i}\varphi_i(t)$, $i = 1,\dots,n$.

Case 2. Suppose that $p < N$. In this case, from theorem 4.2 applied to (5.2), it follows that system (5.1) possesses decreasing radial solutions $(u_1,\dots,u_n)$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\rho_i < (p-N)/(p-1)$, $i = 1,\dots,n$, if and only if

$$\sum_{j=1}^{n} M_{ij}\left(N+\nu_j+\frac{p-N}{p-1}\gamma_j\right) < 0, \quad i = 1,\dots,n. \tag{5.8}$$

In this case, the ρ_i are uniquely determined by (5.6) and the asymptotic behaviour of any such solution as $|x| \to \infty$ is governed by the unique formulae (5.7).

A few words about the particular case of (5.1), in which $f_i(t) \equiv c_i > 0$, i.e.

$$\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) = c_i|u_{i+1}|^{\gamma_i-1}u_{i+1}, \quad i = 1,\dots,n,\ u_{n+1} = u_1. \tag{5.9}$$

Note that all the ν_i are zero. If $N \leqslant p$, then, since (5.5) is not satisfied, theorem 4.1 implies that (5.9) does not admit positive decreasing radial solutions $(u_1, \dots, u_n)$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\rho_i < 0$, $i = 1, \dots, n$. On the other hand, if $N > p$, then (5.8) is violated under the assumption $\gamma_i < p - 1$, $i = 1, \dots, n$, which is more stringent than (5.3), and so, by theorem 4.2, system (5.9) cannot possess decreasing solutions $(u_1, \dots, u_n)$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\rho_i < (p - N)/(p - 1)$, $i = 1, \dots, n$.

5.2. Nonlinear metaharmonic equations

We next consider the nonlinear metaharmonic equation

$$\Delta^m u = g(|x|)|u|^{\gamma-1}u, \quad x \in \Omega_R, \tag{5.10}$$

where $m \geqslant 2$ and $\gamma > 0$ are constants and g is a positive continuous function on $[R, \infty)$ that is regularly varying of index ν. We are interested in radial positive solutions u of (5.10) such that u and $\Delta^k u$, $k = 1, \dots, m - 1$, are regularly varying of negative indices. It is clear that seeking such solutions of (5.10) is equivalent to seeking radial regularly varying solutions of negative indices of the system

$$\Delta u_i = u_{i+1}, \quad i = 1, \dots, m - 1, \qquad \Delta u_m = g(|x|)|u_{m+1}|^{\gamma-1}u_{m+1}, \quad x \in \Omega_R, \tag{5.11}$$

where $u_{m+1} = u_1$. This system is equivalent to the following system of ordinary differential equations

$$\left.\begin{aligned} (t^{N-1}u_i')' &= t^{N-1}u_{i+1}, \quad i = 1, \dots, m - 1, \\ (t^{N-1}u_m')' &= t^{N-1}g(t)|u_{m+1}|^{\gamma-1}u_{m+1} \end{aligned}\right\} \tag{5.12}$$

for $t \geqslant R$, which is a special case of (A) with

$$\begin{aligned} &\alpha_1 = \cdots = \alpha_m = 1, && \beta_1 = \cdots = \beta_{m-1} = 1, && \beta_m = \gamma, \\ &\lambda_1 = \cdots = \lambda_m = N - 1, && \mu_1 = \cdots = \mu_{m-1} = N - 1, && \mu_m = N - 1 + \nu. \end{aligned}$$

We assume that $\gamma < 1$. The $m \times m$ matrix (3.14) associated with (5.12) reads $A(1, \dots, 1, \gamma)$. Define the matrix (M_{ij}) by

$$(M_{ij}) = (1 - \gamma)A(1, \dots, 1, \gamma)^{-1}. \tag{5.13}$$

It is easy to check that $M_{ij} = 1$ for $1 \leqslant i \leqslant j \leqslant m$, and $M_{ij} = \gamma$ for $1 \leqslant j < i \leqslant m$. Observe that conditions (1.2) and (1.3) for (5.12) reduce, respectively, to $N = 2$ and $N \geqslant 3$.

(i) Let $N = 2$. Let us apply theorem 4.1 to (5.12). Then (3.18) and (3.19) reduce, respectively, to

$$\sum_{j=1}^{m} M_{ij}\frac{\lambda_j - \alpha_j + \mu_j + 1}{\alpha_j} = 2\sum_{j=1}^{m} M_{ij} + \nu M_{im} < 0, \quad i = 1, \dots, m, \tag{5.14}$$

and

$$\rho_j = \frac{2\sum_{j=1}^{m} M_{ij} + \nu M_{im}}{1 - \gamma}, \quad i = 1, \dots, m, \tag{5.15}$$

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whence it follows that

$$\rho_1 = \frac{2m+\nu}{1-\gamma} \quad \text{and} \quad \rho_i = \rho_1 - (i-1) \quad \text{for } i = 2, \dots, m. \tag{5.16}$$

We then conclude that (5.10) has decreasing radial regularly varying solutions of negative index ρ if and only if $\nu < -2m$, in which case ρ is given by $\rho = (2m + \nu)/(1-\gamma)$ and any such solution u enjoys the exact asymptotic behaviour

$$u(|x|) \sim |x|^{\rho} \left[\frac{\psi(|x|)}{((-\rho)(1-\rho)\cdots(m-1-\rho))^2} \right]^{1/(1-\gamma)}, \quad |x| \to \infty, \tag{5.17}$$

where $\psi \in \mathrm{SV}$ denotes the slowly varying part of g, i.e. $g(t) = t^{\nu}\psi(t)$.

(ii) Let $N \geqslant 3$. We compute the constants (3.29) for (5.12):

$$\begin{aligned}\sigma_i &= \sum_{j=1}^{m} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_i(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right) \\ &= \nu + 2m - (2-N)(1-\gamma) + 2(i-1)(1-\gamma), \quad i = 1, \dots, m,\end{aligned}$$

from which we see that

$$\sigma_1 = \nu + 2m - (2-N)(1-\gamma), \quad \sigma_i = \sigma_1 - 2(i-1)(1-\gamma), \quad i = 2, \dots, m.$$

Now applying theorem 4.2 to (5.2), we can assert that (5.10) possesses decreasing radial solutions u that are regularly varying of negative index $\rho < 2-N$ if and only if $\nu < -2m + (2-N)(1-\gamma)$, in which case ρ is given by $\rho = (2m+\nu)/(1-\gamma)$ and the asymptotic behaviour of any such solution is governed by the unique formula

$$u(|x|) \sim |x|^{\rho} \left[\frac{\psi(|x|)}{\prod_{i=1}^{m}(i-1-\rho)(i+1-N-\rho)} \right]^{1/(1-\gamma)}, \quad |x| \to \infty, \tag{5.18}$$

where ψ is as in (5.17).

From the above observation, it follows in particular that if the regularity index ν of g is non-negative, then (5.10) cannot admit strongly decreasing radial solutions u that are regularly varying of negative indices.

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