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ON THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVE

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Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Given a zero of $P'(z)$, we shall determine regions which contains at least one zero of $P(z)$. In particular, it will be shown that if all the zeros of $P(z)$ lie in $|z| \leq 1$ and $w_1, w_2, \ldots, w_{n-1}$ are the zeros of $P'(z)$, then each of the disks $|(z/2)-w_j| \leq \frac{1}{2}$ and $|z-w_j| \leq 1$, $j = 1, 2, \ldots, n-1$, contains at least one zero of $P(z)$. We shall also determine regions which contain at least one zero of the polynomials $mP(z) + zP'(z)$ and $P'(z)$ under some appropriate assumptions. Finally some other results of similar nature will be obtained.

1. Introduction and statement of results

Let all the zeros of a polynomial $P(z)$ of degree n lie in the closed unit disk $|z| \leq 1$ and let $P(a) = 0$, then according to a conjecture of Sendov, better known as "Ilieff's conjecture" [*4*, Problem 4.5], [*6*, p. 795], the disk $|z-a| \leq 1$ contains at least one zero of $P'(z)$, the derivative of $P(z)$. The boundary case, that is when $|a| = 1$, has been proved by Rubinstein [*10*]. A conjecture stronger than that of Ilieff, in which the disk $|z-a| \leq 1$ is replaced by the disk $|z-(a/2)| \leq 1 - |a|/2$, is stated in [3] by Goodman, Rahman and Ratti and

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is proved there only for the boundary case. In full generality, these conjectured results have been proved [2], [3], [6], [7], [10], [12] only for polynomials of degree at most five.

Ilieff's conjecture might suggest that close to every zero of $P(z)$ there should always lie a zero of $P'(z)$. In this paper we shall first determine a neighbourhood of a zero w of $P'(z)$, which will always contain a zero of $P(z)$. We prove

THEOREM 1. *If $P(z)$ is a polynomial of degree n and w is a zero of $P'(z)$, then for every given real or complex number α, $P(z)$ has at least one zero in the region*

$$\left|w - \frac{\alpha+z}{2}\right| \le \left|\frac{\alpha-z}{2}\right| .$$

Taking $\alpha = 0$ in Theorem 1 and noting that $|w-(z/2)| \le |z/2|$ implies $|w-z| \le |z|$, we get the following interesting result.

COROLLARY 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \le 1$, and w is a zero of $P'(z)$, then $P(z)$ has at least one zero in both the circles*

$$\left|w - \frac{z}{2}\right| \le \tfrac{1}{2} \text{ and } |w-z| \le 1 .$$

Next we prove

THEOREM 2. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \le 1$ and $P(a) = 0$, $a \ne 0$, then for every positive integer m, the polynomial $F(z) = mP(z) + zP'(z)$ has at least one zero in the circle*

$$|z-a| \le 1 .$$

THEOREM 3. *If $P(z) = (z-a)Q(z)$ is a polynomial of degree n and if all the zeros of $Q(z)$ lie in the circle $|z+\alpha-a| \le |\alpha|$ for some real or complex number $\alpha \ne 0$, then at least one zero of $P'(z)$ lies in the circle*

$$\left|z - a + \frac{\alpha}{2}\right| \le \left|\frac{\alpha}{2}\right| .$$

For $a = 1 = \alpha$, this reduces to the result of Goodman, Rahman and Ratti [3].

An immediate consequence of Theorem 3 is

COROLLARY 2. *If all the zeros of a polynomial* $P(z) = (z-a)Q(z)$, $0 \le a \le 1$, *lie in the region* $S = \{|z| \le 1\} \cap \{|z+1-a| \le 1\}$, *then* $P'(z)$ *has at least one zero in both the circles*

$$|z-a+\tfrac{1}{2}| \le \tfrac{1}{2} \quad \text{and} \quad \left|z - \frac{a}{2}\right| \le 1 - \frac{a}{2} .$$

THEOREM 4. *Let* $P(z) = (z-a)Q(z)$ *be a polynomial of degree* n, *If*

$$\operatorname{Re} \frac{\alpha Q'(a)}{(n-1)Q(a)} \ge \tfrac{1}{2} ,$$

for some real or complex number α, *then* $P'(z)$ *has at least one zero in the circle*

$$\left|z - a + \frac{\alpha}{2}\right| \le \left|\frac{\alpha}{2}\right| .$$

The next corollary immediately follows from Theorem 4.

COROLLARY 3. *If* $P(z) = (z-1)Q(z)$ *is a polynomial of degree* n *and* $P'(z)$ *does not vanish in the circle* $|z-\tfrac{1}{2}| \le \tfrac{1}{2}$, *then*

$$\operatorname{Re} \frac{Q'(1)}{Q(1)} < \frac{n-1}{2} .$$

We also prove

THEOREM 5. *If the polynomial* $P(z) = (z-1)Q(z)$ *of degree* n *has all its zeros in* $|z| \ge 1$, *then* $P'(z)$ *cannot have all its zeros in the disk*

$$|z-\tfrac{1}{2}| < \tfrac{1}{2} .$$

Finally we establish

THEOREM 6. *If* $P(z)$ *is a polynomial of degree* n *such that*

$$\max_{|z|=1} |P(z)| = \left|P\left(e^{i\theta}\right)\right| ,$$

then $P(z)$ *cannot have all its zeros in the disk*

$$\left|z - \frac{e^{i\theta}}{2}\right| < \tfrac{1}{2} .$$

2. Proofs

For the proofs of these theorems we need the following lemmas.

LEMMA 1. *If* $P(z)$ *is a polynomial of degree* n *such that* $P(a) = P(b)$, $a \neq b$, *then* $P'(z)$ *has at least one zero in each of the regions*

$$|z-a| \le |z-b| \quad \text{and} \quad |z-a| \ge |z-b| .$$

Proof of Lemma 1. Without loss of generality we suppose $P(a) = P(b) = 0$. Consider the polynomial

$$G(z) = P\left(\left(\frac{a-b}{2}\right)z + \frac{a+b}{2}\right) ,$$

then $G(1) = P(a) = 0$ and $G(-1) = P(b) = 0$. Now it follows from the proof of the Grace-Heawood theorem [5, p. 107] that $G'(z)$ is apolar to the polynomial

$$H(z) = \frac{(z-1)^n-(z+1)^n}{n} ,$$

whose zeros are $z_k = -i \cot(k\pi/n)$, $k = 1, 2, \ldots, n-1$. Since all the zeros of $H(z)$ lie in $\operatorname{Re} z \ge 0$, it follows by Grace's theorem [5, p. 61] that $G'(z)$ has at least one zero in $\operatorname{Re} z \ge 0$. That is, at least one zero of $G'(z)$ lie in $|z-1| \le |z+1|$. Replacing z by $\left(z-\left((a+b)/2\right)\right)\left(\left(2/(a-b)\right)\right)$, it follows that, at least one zero of $P'(z)$ lies in $|z-a| \le |z-b|$.

Since all the zeros of $H(z)$ lie also in $\operatorname{Re} z \le 0$, it follows by a similar argument as above that $P'(z)$ has at least one zero in the region $|z-a| \ge |z-b|$. This completes the proof of Lemma 1.

Let $P(z)$ be a polynomial of degree n . The first polar derivative of $P(z)$ with respect to the point α_1 is defined by

$$D_{\alpha_1} P(z) = nP(z) + \left(\alpha_1 - z\right)P'(z) .$$

Similarly the second polar derivative of $P(z)$ with respect to α_2 is defined by

$$D_{\alpha_1} D_{\alpha_2} P(z) = D_{\alpha_2}\left(D_{\alpha_1} P(z)\right) ,$$

and so on. For the proof of Theorem 4, we need

LEMMA 2. *If all the zeros of a polynomial* $P(z)$ *of degree* n *lie* *in a circular region* C *and if none of the points* $\alpha_1, \alpha_2, \ldots, \alpha_k$, $k \le n-1$, *lies in region* C , *then each of the polar derivatives*

$$D_{\alpha_1}P(z),\ D_{\alpha_1}D_{\alpha_2}P(z),\ \ldots,\ D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)$$

has all of its zeros in region C .

This lemma follows by repeated application of Laguerre's theorem [5, p. 49].

Finally for the proof of Theorem 6, we need the following lemma which is an immediate consequence of Berstein's theorem on the derivative of a trigonometric polynomial [11] (see also [1]).

LEMMA 3. *Let* $P(z)$ *be a polynomial of degree* $n \ge 1$, *then*

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| .$$

3. Proofs of the theorems

Proof of Theorem 1. Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$ and let w be a zero of $P'(z)$. If $w = \alpha$ or $w = z_j$ for some $j = 1, 2, \ldots, n$, then the result follows and we have nothing to prove. Hence we suppose that $w \ne \alpha$ and $w \ne z_j$ for any $j = 1, 2, \ldots, n$. Since w is a zero of $P'(z)$ and $P(w) \ne 0$, we have

$$\sum_{j=1}^{n} \frac{1}{w-z_j} = \frac{P'(w)}{P(w)} = 0 .$$

This gives

$$\sum_{j=1}^{n} \frac{(w-z_j)-(\alpha-z_j)}{w-z_j} = \sum_{j=1}^{n} \frac{w-\alpha}{w-z_j} = 0 ,$$

and therefore

$$\sum_{j=1}^{n} \frac{\alpha-z_j}{w-z_j} = n .$$

This implies

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$$n = \sum_{j=1}^{n} \text{Re}\, \frac{\alpha-z_j}{w-z_j} \le n \max_{1\le j\le n} \text{Re}\, \frac{\alpha-z_j}{w-z_j}$$

which shows that for at least one $j = 1, 2, \ldots, n$,

$$\text{Re}\, \frac{\alpha-z_j}{w-z_j} \ge 1 .$$

Thus for at least one $j = 1, 2, \ldots, n$ we have

$$\left|1 - \frac{\alpha-z_j}{2(w-z_j)}\right| \le \left|\frac{\alpha-z_j}{2(w-z_j)}\right| .$$

This gives

$$\left|w - \frac{\alpha+z_j}{2}\right| \le \left|\frac{\alpha-z_j}{2}\right|$$

for at least one $j = 1, 2, \ldots, n$, which is equivalent to the desired result.

Proof of Theorem 2. Consider the polynomial

$$G(z) = z^m P(z) ,$$

where m is a positive integer greater or equal to 1, then

$$G'(z) = z^{m-1}\big(mP(z)+zP'(z)\big) = z^{m-1}F(z) .$$

By hypothesis, $P(a) = 0$, $a \ne 0$; therefore $G(a) = 0 = G(0)$. Hence by using Lemma 1, with $b = 0$, it follows that the polynomial $G'(z)$ has at least one zero in the region

(1) $$|z-a| \le |z| .$$

As $a \ne 0$, this zero cannot be $z = 0$. Therefore this zero must be a zero of $F(z)$. Since all the zeros of $P(z)$ lie in $|z| \le 1$, it follows by the Gauss-Lucas theorem that all the zeros of $G'(z)$ lie in $|z| \le 1$ and hence all the zeros of $F(z)$ also lie in $|z| \le 1$. Thus from (1) we conclude that at least one zero of $F(z)$ lie in the circle $|z-a| \le 1$ and this completes the proof of Theorem 2.

Proof of Theorem 3. We have

(2) $$P'(z) = (z-a)Q'(z) + Q(z)$$

and

$$(3) \qquad P''(z) = (z-a)Q''(z) + 2Q'(z) \ .$$

If $z = a$ is a multiple zero of $P(z)$, then $z = a$ is also a zero of $P'(z)$ and since $z = a$ lies in the circle $|z-a+(\alpha/2)| \le |\alpha/2|$, the assertion is true in this case. Henceforth we assume that $z = a$ is a simple zero of $P(z)$, so that $P'(a) \ne 0$. Now from (2) and (3) we get

$$(4) \qquad \frac{P''(a)}{P'(a)} = \frac{2Q'(a)}{Q(a)} \ .$$

If $z_1, z_2, \ldots, z_{n-1}$ are the zeros of $Q(z)$ and $w_1, w_2, \ldots, w_{n-1}$ are those of $P'(z)$, then from (4) we have

$$\sum_{j=1}^{n-1} \frac{1}{a-w_j} = 2 \sum_{j=1}^{n-1} \frac{1}{a-z_j} \ .$$

Multiplying the two sides of this equation by $\alpha \ne 0$ and then taking the real parts on both sides, we obtain

$$(5) \qquad \sum_{j=1}^{n-1} \operatorname{Re} \frac{\alpha}{a-w_j} = 2 \sum_{j=1}^{n-1} \operatorname{Re} \frac{\alpha}{\alpha-(\alpha-a+z_j)} \ .$$

Since by hypothesis

$$\left| \frac{\alpha-a+z_j}{\alpha} \right| \le 1 \quad \text{for all} \quad j = 1, 2, \ldots, n-1 \ ,$$

therefore,

$$\operatorname{Re} \frac{\alpha}{\alpha-(\alpha-a+z_j)} \ge \tfrac{1}{2} \ ,$$

for all $j = 1, 2, \ldots, n-1$. Hence from (5) we get

$$\sum_{j=1}^{n-1} \operatorname{Re} \frac{\alpha}{a-w_j} \ge 2 \sum_{j=1}^{n-1} \tfrac{1}{2} = n-1 \ .$$

This shows that

$$\operatorname{Re} \frac{\alpha}{a-w_j} \ge 1 \ , \text{ for al least one } j = 1, 2, \ldots, n-1 \ ,$$

from which it follows that

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$$\left|w_j - a + \frac{\alpha}{2}\right| \le \left|\frac{\alpha}{2}\right| \quad \text{for at least one } j = 1, 2, \ldots, n-1 .$$

This is equivalent to the desired result and Theorem 3 is proved.

Proof of Theorem 4. If $z = a$ is a multiple zero of $P(z)$, then the result follows as in the proof of Theorem 3. Hence we assume that $z = a$ is a simple zero of $P(z)$, so that $P'(a) \ne 0$. Since $P(z) = (z-a)Q(z)$, therefore, $P'(a) = Q(a) \ne 0$, $P''(a) = 2Q'(a)$. Also it follows by hypothesis that $Q'(a) \ne 0$ and $\alpha \ne 0$. We have to show that $P'(z)$ has at least one zero in the circle $|z-a+(\alpha/2)| \le |\alpha/2|$. Assume the contrary. That is, assume that all the zeros of $P'(z)$ lie in $|z-a+(\alpha/2)| > |\alpha/2|$. Since the point a does not lie in $|z-a+(\alpha/2)| > |\alpha/2|$, it follows from Lemma 2 that all the zeros of the $(n-2)$th polar derivative

$$D_a^{n-2}P'(z) = D_aD_a \ldots D_aP'(z)$$

of $P'(z)$ lie in $|z-a+(\alpha/2)| > |\alpha/2|$. But $D_a^{n-2}P'(z)$ is a polynomial of degree one and its only zero is (see [9, p. 235, Problem V 137]) given by

$$z = a - \frac{(n-1)P'(a)}{P''(a)} = a - \frac{(n-1)Q(a)}{2Q'(a)} ,$$

so that

$$\frac{1}{a-z} = \frac{2Q'(a)}{(n-1)Q(a)} .$$

This gives with the help of the hypothesis

$$\text{Re}\,\frac{\alpha}{a-z} = 2\,\text{Re}\,\frac{\alpha Q'(a)}{(n-1)Q(a)} \ge 1 ,$$

which implies

$$\left|1 - \frac{\alpha}{2(a-z)}\right| \le \left|\frac{\alpha}{2(a-z)}\right| .$$

This shows that the only zero of $D_a^{n-2}P'(z)$ lies in $|z-a+(\alpha/2)| \le |\alpha/2|$, which is a contradiction and therefore the desired result follows.

Proof of Theorem 5. Here we have $P(z) = (z-1)Q(z)$, so that $P'(1) = Q(1)$ and $P''(1) = 2Q'(1)$. Since $Q(z)$ has all its zeros in

$|z| \geq 1$, therefore, if $z_1, z_2, \ldots, z_{n-1}$ are the zeros of $Q(z)$, then $|z_j| \geq 1$, $j = 1, 2, \ldots, n-1$, and

$$\frac{zQ'(z)}{Q(z)} = \sum_{j=1}^{n-1} \frac{z}{z-z_j} .$$

Now for points $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $Q(z)$, we have

$$\text{Re } \frac{e^{i\theta}Q'(e^{i\theta})}{Q(e^{i\theta})} = \sum_{j=1}^{n-1} \frac{e^{i\theta}}{e^{i\theta}-z_j} \leq \sum_{j=1}^{n-1} \tfrac{1}{2} = \frac{n-1}{2} .$$

This implies

$$|e^{i\theta}Q'(e^{i\theta})| \leq |(n-1)Q(e^{i\theta})-e^{i\theta}Q'(e^{i\theta})|$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $Q(z)$. Since this inequality obviously holds for points $e^{i\theta}$ which are the zeros of $Q(z)$, therefore, it follows that

$$(6) \qquad |Q'(z)| \leq |(n-1)Q(z)-zQ'(z)| \quad \text{for} \quad |z| = 1 .$$

If $P''(1) = 0$, then $P'(z)$ has at least one zero in $|z-\frac{1}{2}| \geq \frac{1}{2}$. Because if all the zeros of $P'(z)$ lie in $|z-\frac{1}{2}| < \frac{1}{2}$, then by the Gauss-Lucas theorem, all the zeros of $P''(z)$ also lie in $|z-\frac{1}{2}| < \frac{1}{2}$. Since $P''(1) = 0$ and 1 does not lie in $|z-\frac{1}{2}| < \frac{1}{2}$, we get a contradiction.

We now suppose that $P''(1) \neq 0$, so that $Q'(1) \neq 0$. We have to show that $P'(z)$ cannot have all its zeros in the disk $|z-\frac{1}{2}| < \frac{1}{2}$. Assume that all the zeros of $P'(z)$ lie in $|z-\frac{1}{2}| < \frac{1}{2}$. Since 1 does not lie in $|z-\frac{1}{2}| < \frac{1}{2}$, it follows by Lemma 2 that all the zeros of $(n-2)$th polar derivative

$$D_1^{n-2}P'(z) = D_1D_1 \ldots D_1P'(z)$$

of $P'(z)$ lie in $|z-\frac{1}{2}| < \frac{1}{2}$. But the only zero of the polynomial $D_1^{n-2}P'(z)$, which is of the first degree, is given by (see [9, p. 235])

$$z = 1 - \frac{(n-1)P'(1)}{P''(1)} = 1 - \frac{(n-1)Q(1)}{2Q'(1)} .$$

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This gives, with the help of (6),

$$|z-\tfrac{1}{2}| = \tfrac{1}{2}\left|\frac{Q'(1)-(n-1)Q(1)}{Q'(1)}\right| \geq \tfrac{1}{2} .$$

This shows that the only zero of the polynomial $D_1^{n-2}P'(z)$ lies in $|z-\tfrac{1}{2}| \geq \tfrac{1}{2}$, which is a contradiction and therefore the result follows.

Proof of Theorem 6. Since $|P(z)|$ takes its maximum at $z = e^{i\theta}$ on $|z| = 1$, it follows that (see [8, p. 132, Problem III 144]) $e^{i\theta}P'(e^{i\theta})/P(e^{i\theta})$ is real and positive and therefore $P'(e^{i\theta}) \neq 0$. We have to show that $P(z)$ has at least one zero in $|z-(e^{i\theta}/2)| \geq \tfrac{1}{2}$. Suppose that all the zeros of $P(z)$ lie in $|z-(e^{i\theta}/2)| < \tfrac{1}{2}$. Since $e^{i\theta}$ does not lie in $|z-(e^{i\theta}/2)| < \tfrac{1}{2}$, it follows from Lemma 2 that all the zeros of the $(n-1)$th polar derivative

$$D^{n-1}_{e^{i\theta}}P(z) = D_{e^{i\theta}}D_{e^{i\theta}} \cdots D_{e^{i\theta}}P(z)$$

of $P(z)$ lie in $|z-(e^{i\theta}/2)| < \tfrac{1}{2}$. But $D^{n-1}_{e^{i\theta}}P(z)$ is a polynomial of degree 1 and its only zero (see [9, p. 235, Problem V 137]) is given by

$$z = e^{i\theta} - \frac{nP(e^{i\theta})}{P'(e^{i\theta})} .$$

With the help of Lemma 3, this zero lies in

$$\left|z - \frac{e^{i\theta}}{2}\right| = \left|\frac{e^{i\theta}}{2} - \frac{nP(e^{i\theta})}{P'(e^{i\theta})}\right| \geq \frac{n|P(e^{i\theta})|}{|P'(e^{i\theta})|} - \tfrac{1}{2}$$
$$\geq 1 - \tfrac{1}{2} = \tfrac{1}{2} ,$$

which is a contradiction and Theorem 6 is established.

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