Combinatorics, Probability and Computing (2019) 28, 423–464.
doi:10.1017/S0963548318000469

Sparse Highly Connected Spanning Subgraphs in Dense Directed Graphs

DONG YEAP KANG†

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu Daejeon, 34141 South Korea
(e-mail: dyk90@kaist.ac.kr)

Received 31 October 2017; revised 12 August 2018; first published online 5 November 2018

Mader proved that every strongly k-connected n-vertex digraph contains a strongly k-connected spanning subgraph with at most $2kn - 2k^2$ edges, where equality holds for the complete bipartite digraph $DK_{k,n-k}$. For dense strongly k-connected digraphs, this upper bound can be significantly improved. More precisely, we prove that every strongly k-connected n-vertex digraph D contains a strongly k-connected spanning subgraph with at most $kn + 800k(k + \overline{\Delta}(D))$ edges, where $\overline{\Delta}(D)$ denotes the maximum degree of the complement of the underlying undirected graph of a digraph D. Here, the additional term $800k(k + \overline{\Delta}(D))$ is tight up to multiplicative and additive constants. As a corollary, this implies that every strongly k-connected n-vertex semicomplete digraph contains a strongly k-connected spanning subgraph with at most $kn + 800k^2$ edges, which is essentially optimal since $800k^2$ cannot be reduced to the number less than $k(k-1)/2$.

We also prove an analogous result for strongly k-arc-connected directed multigraphs. Both proofs yield polynomial-time algorithms.

2010 *Mathematics subject classification*: Primary 05C20
Secondary 05C40

1. Introduction

Given a strongly connected digraph, what is the minimum number of edges of a strongly connected spanning subgraph? This minimum spanning strongly connected subgraph problem (or *MSSS*) is NP-hard, since it generalizes the Hamiltonian cycle problem. The problem is closely related to both extremal graph theory and combinatorial optimization from the perspective of studying the properties of extremal graphs and algorithmic aspects, and especially to industry, in order to build well-connected road systems with minimal cost. Even though the problem is NP-hard, it is known that the problem is polynomial-time

† Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (no. NRF-2017R1A2B4005020) and also by a TJ Park Science Fellowship of POSCO TJ Park Foundation.

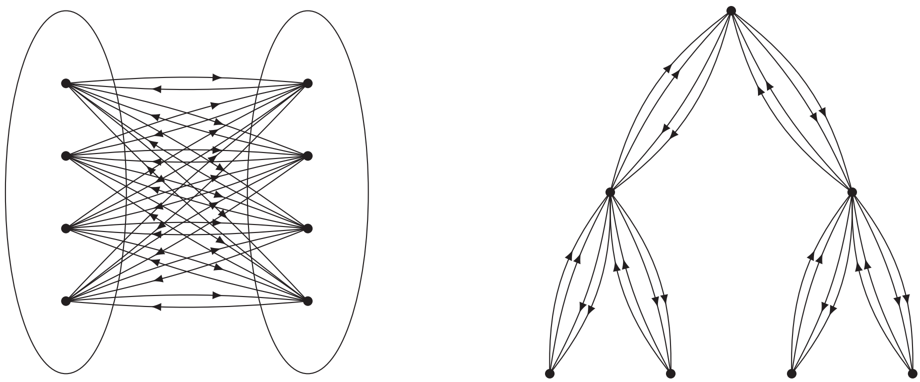

Figure 1. $DK_{4,4}$ and the directed multigraph obtained from the 7-vertex tree whose edges are replaced by two directed 2-cycles.

solvable for various classes of digraphs [4, 6], and there are algorithms that approximate the minimum number of edges of a strongly connected spanning subgraph [5, 22].

One of the natural generalizations of the MSSS problem is the problem of determining the minimum number of edges in a strongly k-connected (or k-arc-connected) spanning subgraph of a strongly k-connected (or k-arc-connected, respectively) digraph. Even though the problem is known to be NP-hard [11], there are algorithms that approximate the minimum number of edges of a strongly k-connected (or k-arc-connected) spanning subgraph [8]. For more on algorithmic aspects of both problems and their variants, the readers are referred to [1], [2, Chapter 12] and the recent survey [3] on tournaments and semicomplete digraphs.

We investigate an upper bound of the minimum number of edges in a strongly k-connected spanning subgraph and a strongly k-arc-connected spanning subgraph. The following are well-known results for general digraphs and directed multigraphs.

(1) (Mader [17]) For integers $k \geqslant 1$ and $n \geqslant 4k+3$, every strongly k-connected n-vertex digraph contains a strongly k-connected spanning subgraph with at most $2k(n-k)$ edges.

(2) (Dalmazzo [9]) For integers $k, n \geqslant 1$, every strongly k-arc-connected n-vertex directed multigraph contains a strongly k-arc-connected spanning subgraph with at most $2k(n-1)$ edges.

(3) (Berg and Jordán [7]) There exists a function $h(k)$ such that for integers $k \geqslant 1$ and $n \geqslant h(k)$, every strongly k-arc-connected n-vertex digraph contains a strongly k-arc-connected spanning subgraph with at most $2k(n-k)$ edges.

The upper bounds for these three cases are best possible; the digraph $DK_{k,n-k}$ obtained from $K_{k,n-k}$[1] by replacing each edge with a directed 2-cycle shows that the upper bounds given in (1) and (3) are tight, and a directed multigraph obtained from an n-vertex tree by replacing each edge with k directed 2-cycles shows that (2) cannot be improved (see Figure 1).

[1] An undirected graph $K_{k,n-k}$ is a complete bipartite graph with two independent sets of size k and size $n-k$, respectively.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Nevertheless, one may ask whether those upper bounds given in (1)–(3) can be improved for dense digraphs, because all of these extremal examples are sparse. As a starting point, Bang-Jensen, Huang and Yeo [5] proved the following result, which improves the result of Berg and Jordán for tournaments.

Theorem 1.1 (Bang-Jensen, Huang and Yeo [5]). *For all integers $k, n \geqslant 1$, every strongly k-arc-connected n-vertex tournament contains a strongly k-arc-connected spanning subgraph with at most $kn + 136k^2$ edges.*

They also proved that the number $136k^2$ of additional edges cannot be reduced to the number less than $(k(k-1))/2$, so the result is essentially best possible. In 2009, Bang-Jensen [1] asked whether there is a function $g(k)$ such that every strongly k-connected n-vertex tournament contains a strongly k-connected spanning subgraph with at most $kn + g(k)$ edges. Recently, Kim, Kim, Suh and the author [13] answered the question affirmatively.

Theorem 1.2 (Kang, Kim, Kim and Suh [13]). *For all integers $k, n \geqslant 1$, every strongly k-connected n-vertex tournament contains a strongly k-connected spanning subgraph with at most $kn + 750k^2 \log_2(k+1)$ edges.*

In particular, they answered the question of Bang-Jensen with $g(k) = 750k^2 \log_2(k+1)$. Since an example of Bang-Jensen, Huang and Yeo [5] shows that $g(k) \geqslant (k(k-1))/2$, there is a gap between the lower bound $(k(k-1))/2$ and the upper bound $750k^2 \log_2(k+1)$ of $g(k)$. We close this gap by showing that $g(k) = \Theta(k^2)$ and generalize both Theorems 1.1 and 1.2 to a larger class of directed digraphs and directed multigraphs, respectively.

Before stating the results, let us begin with some terminology. Let $UG(D)$ be an *underlying graph* of a directed multigraph D, a simple undirected graph obtained from D by removing orientations of edges and multiple edges. Let $\overline{\Delta}(D)$ be the maximum degree of the complement of $UG(D)$, which is equal to $\max_{v \in V(D)} |\{w \in V(D) \setminus \{v\} : (v,w),(w,v) \notin E(D)\}|$. A directed multigraph D is *semicomplete* if $\overline{\Delta}(D) = 0$.

Bang-Jensen, Huang and Yeo [5, Theorem 8.3] proved that every strongly connected digraph D contains a strongly connected spanning subgraph with at most $n + \overline{\Delta}(D)$ edges. We generalize this to strongly k-connected digraphs and strongly k-arc-connected directed multigraphs as follows.

Theorem 1.3. *For integers $k, n \geqslant 1$, the following hold.*

(1) *Every strongly k-connected n-vertex* digraph *D contains a strongly k-connected spanning subgraph with at most $kn + 800k\overline{\Delta}(D) + 800k^2$ edges.*
(2) *Every strongly k-arc-connected n-vertex* directed multigraph *D contains a strongly k-arc-connected spanning subgraph with at most $kn + 670k\overline{\Delta}(D) + 670k^2$ edges.*

Remark.

(1) Theorem 1.3 gives the better result for 'dense' digraphs and directed multigraphs. Given any $0 < \varepsilon < 1$, part (1) of Theorem 1.3 implies that any strongly k-connected

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

n-vertex digraph D with $\overline{\Delta}(D) < (1-\varepsilon)n/800$ has a strongly k-connected spanning subgraph of D with at most $(2-\varepsilon)kn + 800k^2$ edges, improving the result of Mader [17] for these dense digraphs. Similarly, the result of Dalmazzo [9] is also improved for strongly k-arc-connected n-vertex directed multigraphs with $\overline{\Delta}(D) < (1-\varepsilon)n/670$.

(2) Both additional terms $800k(k+\overline{\Delta}(D))$ and $670k(k+\overline{\Delta}(D))$ are optimal up to multiplicative and additive constants. In Section 3, it is proved that for all integers $k \geqslant 1$, $\overline{\Delta} \geqslant 0$ and $n \geqslant \max(5k+2, 4k+\overline{\Delta}+3)$, there is a strongly k-connected n-vertex oriented graph G with $\overline{\Delta}(G) \leqslant \overline{\Delta}$ such that every spanning subgraph D with $\delta^+(D), \delta^-(D) \geqslant k$ contains at least $kn + \max((k(k-1))/2, k\overline{\Delta})$ edges.

Note that the class of tournaments is a subclass of the class of semicomplete digraphs. Theorem 1.3 proves that $g(k) = O(k^2)$ suffices, which improves Theorem 1.2 and provides a function that is asymptotically sharp for the question of Bang-Jensen. Moreover, Theorem 1.3 extends Theorems 1.1 and 1.2 to semicomplete directed multigraphs.

Corollary 1.4. *For all integers $k, n \geqslant 1$, the following hold.*

(1) *Every strongly k-connected n-vertex semicomplete digraph D contains a strongly k-connected spanning subgraph with at most $kn + 800k^2$ edges.*

(2) *Every strongly k-arc-connected n-vertex semicomplete directed multigraph D contains a strongly k-arc-connected spanning subgraph with at most $kn + 670k^2$ edges.*

One of the main ideas of the proof is the use of transitive subtournaments that dominate almost all vertices in order to link the vertices, which builds on the recent methods (see [13, 14, 15, 16, 19, 20]). Another main idea of the proof is called a *sparse linkage structure*, which is introduced in [13] and will be discussed in Section 2. With some new ingredients, both ideas are extensively used in the proof of Theorem 1.3.

The proof of Theorem 1.3 is constructive so that there is a polynomial-time algorithm which, given a strongly k-connected digraph (strongly k-arc-connected directed multigraph) D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$, outputs a strongly k-connected (strongly k-arc-connected, respectively) spanning subgraph with at most $kn + 800k\overline{\Delta} + 800k^2$ ($kn + 670k\overline{\Delta} + 670k^2$, respectively) edges. Since every strongly k-arc-connected n-vertex directed multigraph has at least kn edges, the algorithm approximates the minimum number of edges of a strongly k-connected (or strongly k-arc-connected) spanning subgraph of G within an additive error $O(k(k+\overline{\Delta}))$.

Organization of the paper. We introduce terminology and tools used in the proof in Section 2. We discuss a lower bound on the minimum number of edges in a strongly k-connected subgraph and a strongly k-arc-connected subgraph in Section 3. We briefly sketch the proof of the main theorems in Section 4. Before the proof of the main results, we introduce some basic objects and notions for the construction of sparse highly connected subgraphs in Section 5. The main theorems are proved in Section 6, and we discuss questions related to the main results in Section 7.

2. Preliminaries

2.1. Basic notions and lemmas

We begin with some basic definitions.

Sets and orderings. For any integer $N \geqslant 0$, let $[N]$ denote the set $\{1,\dots,N\}$ if $N \geqslant 1$, $\emptyset$ otherwise. For any m-element finite set $S = \{s_1,\dots,s_m\}$, a *linear ordering* $\sigma = (s_1,\dots,s_m)$ is a map from $[m]$ to S such that $\sigma(i) := s_i$ for $1 \leqslant i \leqslant m$. For two integers p and q, $\sigma(p,q) := \sigma(\{p,\dots,q\} \cap [m])$ if $p \leqslant q$, and $\emptyset$ otherwise.

Directed graphs, directed multigraphs, oriented graphs. A *directed graph* (or *digraph*) D is a pair (V,E) with a finite set V of vertices and a set of E edges in $(V \times V) \setminus \{(v,v) : v \in V\}$. A *directed multigraph* D is a pair (V,E) with a finite set V of vertices and a multiset E of edges in $(V \times V) \setminus \{(v,v) : v \in V\}$. For simplicity, uv denotes a 2-tuple $(u,v) \in V(D) \times V(D)$ for $u,v \in V(D)$. For two directed multigraphs $D_1 = (V_1,E_1)$ and $D_2 = (V_2,E_2)$, its union $D_1 \cup D_2$ is a directed multigraph $(V_1 \cup V_2, E_1 \cup E_2)$. For a set $S \subseteq V(D)$, $D[S]$ denotes the subgraph of D induced by S. An *underlying graph* $UG(D)$ of a directed multigraph D is a simple undirected graph obtained from D by removing its orientation and multiple edges.

An *oriented graph* is a digraph obtained from an undirected graph by orienting each edge. An oriented graph G is *transitive* if $uv, vw \in E(G)$ then $uv \in E(G)$. For a vertex $v \in D$, a set $N^+_D(v)$ is the set of *out-neighbours* of v, and $N^-_D(v)$ is the set of *in-neighbours* of v. A set $\delta^+_D(v)$ is the multiset of edges out of v, and $\delta^-_D(v)$ is the multiset of edges into v. Let $d^+_D(v) := |\delta^+_D(v)|$ and $d^-_D(v) := |\delta^-_D(v)|$ be the *out-degree* and *in-degree* of v, respectively. Let $\delta^+(D)$ and $\delta^-(D)$ be the *minimum out-degree* and *minimum in-degree* of any vertex in D, respectively. For two sets $X, Y \subseteq V(D)$, let $E_D(X,Y)$ be the multiset of edges from X to Y, and $e_D(X,Y) := |E_D(X,Y)|$. A vertex $v \in V(D)$ is a *source* if the in-degree of v is 0, and a vertex v is a *sink* if the out-degree of v is 0. A vertex w is a *non-neighbour* of v if w is neither v nor an in-neighbour of v nor an out-neighbour of v. Let $\overline{\Delta}(D)$ be the maximum number of non-neighbours of any vertex in D, equivalently, the maximum degree of the complement of $UG(D)$. A digraph or a directed multigraph D is *semicomplete* if $\overline{\Delta}(D) = 0$, and a semicomplete oriented graph is called a *tournament*. We frequently use the following fact that $\overline{\Delta}(D') \leqslant \overline{\Delta}(D)$ for every induced subgraph D' of a multigraph D.

For any integer $k \geqslant 1$, a directed multigraph D is *k-regular* if, for every $v \in V(D)$, $d^+_D(v) = d^-_D(v) = k$. A set $A \subseteq V(D)$ *indominates* a vertex $v \in V(D)$ if $v \in A$ or there exists $a \in A$ with $va \in E(D)$. A set $B \subseteq V(D)$ *outdominates* a vertex $u \in V(D)$ if $u \in B$ or there exists $b \in B$ with $bu \in E(D)$.

Paths and fans. A *path* $P = (v_1,\dots,v_s)$ is a digraph with the set $V(P) := \{v_1,\dots,v_s\}$ of s distinct vertices and the set $E(P) := \{v_iv_{i+1} : 1 \leqslant i \leqslant s-1\}$ of edges. The set of *endvertices* of P is $\{v_1,v_s\}$, and the set $\mathrm{Int}(P)$ of *internal vertices* is $V(P) \setminus \{v_1,v_s\}$. A path $P = (v_1,\dots,v_s)$ in a directed multigraph D is *minimal* if $v_iv_j \notin E(D)$ for $2 \leqslant i+1 < j \leqslant s$.

Let $k \geqslant 1$ be an integer and $S \subseteq V(D)$. For a vertex $v \in V(D) \setminus S$, a *k-fan* from v to S (from S to v) is a collection of k paths from v to vertices in S (from vertices in S to v, respectively) such that each of them contains exactly one vertex in S, and any two of them have only the vertex v in common. A *k-arc-fan* from v to S (from S to v) is a collection of k paths from v to vertices in S (from vertices in S to v, respectively) such that each of them contains exactly one vertex in S, and any two of them have no edge in common.

Connectivity. A directed multigraph D is *strongly connected* if, for every $u, v \in V(D)$, there is a path from u to v. For any integer $k \geqslant 1$, a directed graph D is *strongly k-connected* if $|V(D)| \geqslant k+1$ and for every $S \subseteq V(D)$ of $|S| \leqslant k-1$, the directed multigraph $D - S$ is strongly connected. A directed multigraph D is *strongly k-arc-connected* if, for every $T \subseteq E(D)$ with $|T| \leqslant k-1$, the directed multigraph $D - T$ remains strongly connected. A directed multigraph D is *minimally strongly k-connected* (*minimally strongly k-arc-connected*) if D is strongly k-connected (strongly k-arc-connected, respectively) and $D - \{e\}$ is not strongly k-connected (strongly k-arc-connected, respectively) for every $e \in E(D)$.

We often use the following well-known facts easily deduced from Menger's theorem.

Proposition 2.1. *Let $k \geqslant 1$ be an integer, and let D be a directed multigraph and $\emptyset \neq S \subseteq V(D)$.*

(1) *If D is strongly k-connected and $|S| \geqslant k$, then for every $v \in V(D) \setminus S$, there are a k-fan from v to S and a k-fan from S to v.*
(2) *If D is strongly k-arc-connected, then for every $v \in V(D) \setminus S$, there are a k-arc-fan from v to S and a k-arc-fan from S to v.*
(3) *If D is strongly k-connected and $a_1, \ldots, a_k, b_1, \ldots, b_k \in V(D)$ are $2k$ distinct vertices of D, then there are k vertex-disjoint paths $P_1, \ldots, P_k$ such that there is a permutation $\sigma : [k] \rightarrow [k]$ and for $i \in [k]$, P_i is a path from a_i to $b_{\sigma(i)}$.*
(4) *If D is strongly k-arc-connected and $a_1, \ldots, a_k, b_1, \ldots, b_k \in V(D)$ are $2k$ distinct vertices of D, then there are k edge-disjoint paths $P_1, \ldots, P_k$ such that there is a permutation $\sigma : [k] \rightarrow [k]$ and for $i \in [k]$, P_i is a path from a_i to $b_{\sigma(i)}$.*

Now we prove the following elementary lemma, which extends [13, Lemma 2.1] to dense directed multigraphs.

Lemma 2.2. *For integers $k \geqslant 1$, $n \geqslant 2$, $\overline{\Delta} \geqslant 0$ with $n \geqslant k$, let D be an n-vertex directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. Then D has k vertices having at least $(n-k-\overline{\Delta})/2$ in-neighbours in D and k vertices having at least $(n-k-\overline{\Delta})/2$ out-neighbours in D.*

Proof. Let $x_1, \ldots, x_k$ be k vertices such that $|N_D^-(x_1)| \geqslant \cdots \geqslant |N_D^-(x_k)|$ and $|N_D^-(x_i)| \geqslant |N_D^-(v)|$ for every $v \in V(D) \setminus \{x_1, \ldots, x_k\}$ and $1 \leqslant i \leqslant k$. Since $D' = D - \{x_1, \ldots, x_{k-1}\}$ contains $n-k+1$ vertices and $\overline{\Delta}(D') \leqslant \overline{\Delta}$,

$$\sum_{x \in V(D')} |N_{D'}^-(x)| = |E(D')| \geqslant |E(UG(D'))| \geqslant \frac{1}{2}|V(D')|(n-k-\overline{\Delta}),$$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

and there is $x \in V(D')$ such that $|N^-_{D'}(x)| \geqslant (n-k-\overline{\Delta})/2$ since $|V(D')| \geqslant 1$. Therefore, for $1 \leqslant i \leqslant k$,

$$|N^-_D(x_i)| \geqslant |N^-_D(x_k)| \geqslant |N^-_D(x)| \geqslant |N^-_{D'}(x)| \geqslant \frac{n-k-\overline{\Delta}}{2}.$$

Similarly, there are k vertices having at least $(n-k-\overline{\Delta})/2$ out-neighbours in D. □

2.2. Sparse linkage structures

We need some notions introduced in [13, Section 3]. For any n-vertex digraph D and a linear ordering $\sigma = (v_1, \ldots, v_n)$ of $V(D)$, a digraph D is (σ, k, t)-*good* for positive integers k and t, if the following hold.

(a) If $v_i v_j \in E(D)$ for $1 \leqslant i, j \leqslant n$, then $i < j$.
(b) Every vertex v_j for $1 \leqslant j \leqslant n-t$ has out-degree at least k in D.
(c) Every vertex v_j for $t+1 \leqslant j \leqslant n$ has in-degree at least k in D.

The following lemma easily follows from the definition of (σ, k, t)-good digraphs. Note that (1) of the lemma follows by [13, Claim 3.1], and (2) is easily deduced from (1).

Lemma 2.3. *For integers $n \geqslant 1$, $t \geqslant k \geqslant 1$ and a (σ, k, t)-good n-vertex digraph D, the following hold.*

(1) *Let $S \subseteq V(D)$ be a set of at most $k-1$ vertices. For every $u \in V(D) \setminus S$, there are vertices $v \in \sigma(1, t) \setminus S$ and $w \in \sigma(n-t+1, n) \setminus S$ such that $D - S$ contains a path from v to u and a path from u to w.*
(2) *Let $F \subseteq E(D)$ be a set of at most $k-1$ edges. For every $u \in V(D)$, there are vertices $v \in \sigma(1, t)$ and $w \in \sigma(n-t+1, n)$ such that $D - F$ contains a path from v to u and a path from u to w.*

The following proposition, the heart of the proof of Theorem 1.3, asserts that if D is dense, then we can always find a sparse linkage structure (see [13, Lemma 3.4]).

Proposition 2.4 (Kang, Kim, Kim and Suh [13]). *For integers $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, let D be an n-vertex directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. There is a linear ordering σ of $V(D)$ and a $(\sigma, k, 2k + \overline{\Delta} - 1)$-good digraph D', where D' is a spanning subgraph of D with at most $kn - k + k\overline{\Delta}$ edges.*

Indeed, the proof of [13, Lemma 3.4] yields a polynomial-time algorithm that outputs D' in time $O(n^3 + kn^{2.5})$ using the algorithm of Hopcroft and Karp [12] that finds a maximum matching in a bipartite graph.

We also need the following applications of Lemma 2.3 and Proposition 2.4.

Lemma 2.5. *For integers $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, let D be a digraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. Let U be a non-empty subset of $V(D)$. Then there are a spanning subgraph D' of $D[U]$, and subsets $U_i, U_o \subseteq U$ satisfying the following.*

(1) $|E(D')| \leqslant k|U| - k + k\overline{\Delta}$.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

(2) $|U_i|, |U_o| \leqslant 2k + \overline{\Delta} - 1$.
(3) *For every* $S \subseteq V(D)$ *with* $|S| \leqslant k-1$ *and for every* $u, v \in U \setminus S$*, the digraph* $D' - S$ *has a path from* u *to a vertex in* $U_o \setminus S$*, and a path from a vertex in* $U_i \setminus S$ *to* v.

Proof. The proof is immediate from Lemma 2.3 and Proposition 2.4. □

Lemma 2.6. *For integers* $k, n \geqslant 1$ *and* $\overline{\Delta} \geqslant 0$*, let* D *be a digraph with* $\overline{\Delta}(D) \leqslant \overline{\Delta}$*, and* $\{P_1, \ldots, P_k\}$ *be a collection of* k *vertex-disjoint minimal paths in* D *such that* P_i *is a path from* $a_i \in V(D)$ *to* $b_i \in V(D)$.

For every non-empty $U \subseteq \bigcup_{i=1}^{k} \mathrm{Int}(P_i)$*, there are a spanning subgraph* D' *of* $D[U] - \bigcup_{i=1}^{k} E(P_i)$*, and subsets* $U_i, U_o \subseteq U$ *satisfying the following.*

(1) $|E(D')| \leqslant (k-1)|U| + (k-1)(\overline{\Delta}+1)$.
(2) $|U_i|, |U_o| \leqslant 2k + \overline{\Delta} - 1$.
(3) *For every* $S \subseteq V(D)$ *with* $|S| \leqslant k-1$ *and for every* $u, v \in U \setminus S$*, the subgraph* $D - S$ *has a path from* u *to a vertex in* $(U_o \cup \{b_1, \ldots, b_k\}) \setminus S$ *using only edges in* $E(D') \cup \bigcup_{i=1}^{k} E(P_i)$*, and a path from a vertex in* $(U_i \cup \{a_1, \ldots, a_k\}) \setminus S$ *to* v *only using edges in* $E(D') \cup \bigcup_{i=1}^{k} E(P_i)$.

Proof. Let $E_{\mathrm{path}} := \bigcup_{i=1}^{k} E(P_i)$. Since $\overline{\Delta}(D) \leqslant \overline{\Delta}$ and every vertex intersects at most one path in $\{P_1, \ldots, P_k\}$, we have $\overline{\Delta}(D[U] - E_{\mathrm{path}}) \leqslant \overline{\Delta} + 2$. By Proposition 2.4, there are a linear ordering σ of U and a $(\sigma, k-1, 2k+\overline{\Delta}-1)$-good spanning subgraph D' of $D[U] - E_{\mathrm{path}}$ that satisfies (1). Let $U_i := \sigma(1, 2k + \overline{\Delta} - 1)$ and $U_o := \sigma(|U| - 2k - \overline{\Delta} + 2, |U|)$. Then $|U_i|, |U_o| \leqslant 2k + \overline{\Delta} - 1$, satisfying (2).

Now it remains to prove (3). Let $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $u \in U \setminus S$. We aim to prove that there is a path P in $D - S$ from u to a vertex in $(U_o \cup \{b_1, \ldots, b_k\}) \setminus S$ with $E(P) \subseteq E(D') \cup E_{\mathrm{path}}$. Let us write $\sigma = (v_1, \ldots, v_{|U|})$ and let i be the maximum index such that u can reach v_i by a directed path in $D' - S$.

If $i \geqslant |U| - 2k - \overline{\Delta} + 2$, then $v_i \in U_o$. Let P be a directed path in $D' - S$ from u to v_i and we are done. We may assume that $i \leqslant |U| - 2k - \overline{\Delta} + 1$. By the maximality of i, we have $S = N^+_{D'}(v_i)$ since every vertex in $\sigma(1, |U| - 2k - \overline{\Delta} + 1)$ has out-degree at least $k-1$ in D' and $|S| \leqslant k-1$. From the definition of U, there is $t \in [k]$ such that $v_i \in V^{\mathrm{int}}(P_t)$, where P_t is a minimal path in D from a_t to b_t. Let Q be the subpath of P_t from v_i to b_t, and w_i be the out-neighbour of v_i in Q. Since P_t is a minimal path in D, we have $V(Q) \cap N^+_D(v_i) = \{w_i\}$. Hence it follows that $V(Q) \cap N^+_{D'}(v_i) = V(Q) \cap S = \emptyset$ since $E(D') \cap E(Q) \subseteq E(D') \cap E_{\mathrm{path}} = \emptyset$. Therefore, there is a path P in $D - S$ from u to b_t with $E(P) \subseteq E(D') \cup E_{\mathrm{path}}$, as desired. Similarly, for every $v \in U \setminus S$, there is a path P' in $D - S$ from a vertex in $U_i \cup \{a_1, \ldots, a_k\}$ to v with $E(P') \subseteq E(D') \cup E_{\mathrm{path}}$. □

Both Lemmas 2.5 and 2.6 have the following variations with the identical proofs. When applying Proposition 2.4, we may assume that D is a digraph by removing multiple edges.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Lemma 2.7. *For integers $k,n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, let D be a directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. Let U be a non-empty subset of $V(D)$. Then there are a spanning subgraph D' of $D[U]$, and subsets $U_i, U_o \subseteq U$ satisfying the following.*

(1) $|E(D')| \leqslant k|U| - k + k\overline{\Delta}$.
(2) $|U_i|, |U_o| \leqslant 2k + \overline{\Delta} - 1$.
(3) *For every $F \subseteq E(D)$ with $|F| \leqslant k-1$ and for every $u,v \in U$, the digraph $D' - F$ has a path from u to a vertex in U_o, and a path from a vertex in U_i to v.*

Lemma 2.8. *For integers $k,n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, let D be a directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ and $\{P_1,\dots,P_k\}$ be a collection of k edge-disjoint paths in D such that for $i \in [k]$, P_i is a path from $a_i \in V(D)$ to $b_i \in V(D)$.*

For every non-empty $U \subseteq \bigcup_{i=1}^k \mathrm{Int}(P_i)$, there are a spanning subgraph D' of $D[U] - \bigcup_{i=1}^k E(P_i)$, and subsets $U_i, U_o \subseteq U$ satisfying the following.

(1) $|E(D')| \leqslant (k-1)|U| + (k-1)(\overline{\Delta} + 2k - 1)$.
(2) $|U_i|, |U_o| \leqslant 4k + \overline{\Delta} - 3$.
(3) *For every $F \subseteq E(D)$ with $|F| \leqslant k-1$ and for every $u,v \in U$, a subgraph $D - F$ has a path from u to a vertex in $U_o \cup \{b_1,\dots,b_k\}$ using only edges in $E(D') \cup \bigcup_{i=1}^k E(P_i)$, and a path from a vertex in $U_i \cup \{a_1,\dots,a_k\}$ to v using only edges in $E(D') \cup \bigcup_{i=1}^k E(P_i)$.*

Proof. Let $E_{\text{path}} := \bigcup_{i=1}^k E(P_i)$. Since P_i intersects every vertex at most once for $i \in [k]$ and $\overline{\Delta}(D) \leqslant \overline{\Delta}$, we have $\overline{\Delta}(D[U] - E_{\text{path}}) \leqslant \overline{\Delta} + 2k$. By Proposition 2.4, there are a linear ordering σ of U and a $(\sigma, k-1, 4k + \overline{\Delta} - 3)$-good digraph D', where D' is a spanning subgraph of $D[U] - E_{\text{path}}$ that satisfies (1). Let $U_i := \sigma(1, 4k + \overline{\Delta} - 3)$ and $U_o := \sigma(|U| - 4k - \overline{\Delta} + 4, |U|)$. Then $|U_i|, |U_o| \leqslant 4k + \overline{\Delta} - 3$, satisfying (2).

Now it remains to prove (3). Let $F \subseteq E(D)$ with $|F| \leqslant k-1$ and $u \in U$. We aim to prove that there is a path P in $D - F$ from u to a vertex in $U_o \cup \{b_1,\dots,b_k\}$ with $E(P) \subseteq E(D') \cup E_{\text{path}}$. Let us write $\sigma = (v_1,\dots,v_{|U|})$ and let i be the maximum index such that u can reach v_i by a directed path in $D' - F$.

If $i \geqslant |U| - 4k - \overline{\Delta} + 4$, then $v_i \in U_o$. Let P be a directed path in $D' - F$ from u to v_i and we are done. We may assume that $i \leqslant |U| - 4k - \overline{\Delta} + 3$. By the maximality of i, we have $F = \delta^+_{D'}(v_i)$ since every vertex in $\sigma(1, |U| - 4k - \overline{\Delta} + 4)$ has out-degree at least $k-1$ in D' and $|F| \leqslant k-1$. From the definition of U, there is $t \in [k]$ such that $v_i \in V^{\text{int}}(P_t)$, where P_t is a path in D from a_t to b_t. Let Q be a subpath of P_t from v_i to b_t. Since $E(D') \cap E_{\text{path}} = \emptyset$, it follows that $E(Q) \cap F = \emptyset$. Therefore, there is a path P in $D - F$ from u to b_t with $E(P) \subseteq E(D') \cup E_{\text{path}}$, as desired. Similarly, for every $v \in U$, there is a path P' in $D - F$ from a vertex in $U_i \cup \{a_1,\dots,a_k\}$ to v with $E(P') \subseteq E(D') \cup E_{\text{path}}$. □

2.3. Minimally strongly k-connected digraphs

For any undirected graph G, a subgraph $C = (v_1,\dots,v_t)$ is a *circuit* in G if $v_1,\dots,v_t \in V(G)$ and $v_iv_{i+1} \in E(G)$ for $1 \leqslant i \leqslant t$, where we define $v_{t+1} = v_1$ and these t edges are distinct. Note that the vertices $v_1,\dots,v_t$ are not necessarily distinct, and we regard a circuit C as a subgraph of G, such that $V(C) := \{v_1,\dots,v_t\}$ and $E(C) := \{v_iv_{i+1} : 1 \leqslant i \leqslant t\}$.

For a digraph D, a subgraph $C = (v_1, \ldots, v_{2m})$ is an *anti-directed trail* in D if $v_1, \ldots, v_{2m} \in V(D)$, $v_{2i-1}v_{2i} \in E(D)$ and $v_{2i+1}v_{2i} \in E(D)$ for $1 \leqslant i \leqslant m$, where we define $v_{2m+1} = v_1$ and these $2m$ edges are distinct. Note that the vertices $v_1, \ldots, v_{2m}$ are not necessarily distinct, and we regard an anti-directed trail C as a subgraph of D, such that $V(C) := \{v_1, \ldots, v_{2m}\}$ and $E(C) := \bigcup_{i=1}^{m} \{v_{2i-1}v_{2i}, v_{2i+1}v_{2i}\}$.

For a digraph $D = (V, E)$, let $V' := \{v' : v \in V\}$ and $V'' := \{v'' : v \in V\}$ be two disjoint copies of V. A *bipartite representation* $BG(D)$ of D is an undirected bipartite graph with $V(BG(D)) := V' \cup V''$ and $E(BG(D)) := \{\{x', y''\} : (x, y) \in E(D)\}$.

It is easy to see that a subgraph D' of D is an anti-directed trail if and only if its bipartite representation $BG(D')$ is a circuit in $BG(D)$. Therefore, if D has no anti-directed trail then

$$|E(D)| = |E(BG(D))| \leqslant |V(BG(D))| - 1 = 2|V(D)| - 1,$$

since $BG(D)$ is a forest. This proves the following proposition (see [17, Lemma 2]), which characterizes digraphs without anti-directed trails.

Proposition 2.9. *A digraph D does not contain an anti-directed trail if and only if $BG(D)$ is a forest. In particular, $|E(D)| \leqslant 2|V(D)| - 1$ if D has no anti-directed trail.*

For a directed multigraph $D = (V, E)$ and a vertex $u \in V$, a spanning subgraph T is an *out-branching* (*in-branching*) of D rooted at u if T is an oriented graph obtained from a tree by orienting edges and u is the only vertex with in-degree (out-degree, respectively) zero in T. We make the use of the following theorem (see [10] or [2, Theorem 9.3.1]).

Theorem 2.10 (Edmonds [10]). *Let $D = (V, E)$ be a directed multigraph with a vertex $u \in V(D)$. Then the following hold.*

(1) *D contains k edge-disjoint out-branchings rooted at u if and only if, for every $\emptyset \neq S \subseteq V(D) \setminus \{u\}$, $e_D(V(D) \setminus S, S) \geqslant k$.*
(2) *D contains k edge-disjoint in-branchings rooted at u if and only if, for every $\emptyset \neq S \subseteq V(D) \setminus \{u\}$, $e_D(S, V(D) \setminus S) \geqslant k$.*

Theorem 2.10 has the following corollary, which extends the result of Dalmazzo [9] that every strongly k-arc-connected n-vertex directed multigraph contains a strongly k-arc-connected subgraph with at most $2k(n-1)$ edges (see [2, Theorem 5.6.1]).

Corollary 2.11. *Let $k \geqslant 1$ be an integer and let D be a minimally strongly k-arc-connected directed multigraph and $\emptyset \neq U \subseteq V(D)$. Then $|E(D[U])| \leqslant 2k(|U| - 1)$.*

Proof. Fix $u \in U$. By Theorem 2.10, there are k edge-disjoint out-branchings $T_1^+, \ldots, T_k^+$ rooted at u, and k edge-disjoint in-branchings $T_1^-, \ldots, T_k^-$ rooted at u. Since $\bigcup_{i=1}^{k} T_i^+ \cup \bigcup_{i=1}^{k} T_i^-$ is a strongly k-arc-connected spanning subgraph of D, we have $D = \bigcup_{i=1}^{k} T_i^+ \cup$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

$\bigcup_{i=1}^{k} T_i^-$. As $|E(T_i^+[U])| \leqslant |U|-1$ and $|E(T_i^-[U])| \leqslant |U|-1$ for every $i \in [k]$, we have

$$|E(D[U])| \leqslant \sum_{i=1}^{k} |E(T_i^+[U])| + \sum_{i=1}^{k} |E(T_i^-[U])| \leqslant 2k(|U|-1),$$

as desired. □

We use the following theorem by Mader (see [18] or [2, Corollary 5.6.20]).

Theorem 2.12 (Mader [18]). *For any integer $k \geqslant 2$ and a minimally strongly k-connected digraph $D=(V,E)$, let $D'=(V,E')$ be a strongly $(k-1)$-connected spanning subgraph of D. Then the digraph $(V, E\setminus E')$ contains no anti-directed trail.*

The following proposition proves that, if a digraph D is minimally strongly k-connected, then for any $U \subseteq V(D)$, the induced subgraph $D[U]$ contains only a few edges. This also proves that every strongly k-connected digraph D contains a strongly k-connected spanning subgraph with at most $2k|V(D)|$ edges, which is slightly weaker than the result of Mader [17].

Proposition 2.13. *For any integer $k \geqslant 1$, let D be a minimally strongly k-connected digraph and $\emptyset \neq U \subseteq V(D)$. Then $|E(D[U])| \leqslant 2k|U|-k-1$.*

Proof. We prove by induction on k. If $k=1$, the proposition follows from Corollary 2.11, as D is minimally strongly 1-arc-connected. Now we may assume that $k \geqslant 2$. Let D' be a minimally strongly $(k-1)$-connected spanning subgraph of D. By the induction hypothesis, $|E(D'[U])| \leqslant 2(k-1)|U|-k$.

By Theorem 2.12, the digraph $D'' := (V, E\setminus E')$ has no anti-directed trail by Theorem 2.12. As its induced subgraph $D''[U]$ also has no anti-directed trail, it has at most $2|U|-1$ edges by Proposition 2.9. Hence

$$|E(D[U])| = |E(D'[U])| + |E(D''[U])| \leqslant (2(k-1)|U|-k) + (2|U|-1) \leqslant 2k|U|-k-1,$$

as desired. □

3. Lower bounds

Inspired by the construction of $\mathcal{T}_{n,k}$ in [5, Section 2], we define a strongly k-connected $(n_1+n_2+\overline{\Delta}+1)$-vertex oriented graph $G_{n_1,n_2,k,\overline{\Delta}}$ for integers $n_1, n_2 \geqslant 2k+1$ as follows (see Figure 2). Let G_1 be an $(\overline{\Delta}+1)$-vertex digraph with no edges. Let T_2 be an n_1-vertex tournament obtained from an $\lfloor (n_1-1)/2 \rfloor$th power[2] of a directed cycle of length n_1 by adding arbitrary edges to ensure that T_2 is a tournament. Since $\lfloor (n_1-1)/2 \rfloor \geqslant k$, the tournament T_2 is strongly k-connected and $\delta^+(T_2), \delta^-(T_2) \geqslant \lfloor (n_1-1)/2 \rfloor$. Similarly, let T_3

[2] A kth power of a digraph D is a digraph that has the vertex-set $V(D)$ and (u,v) is an edge when the distance from u to v is at most k in D.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

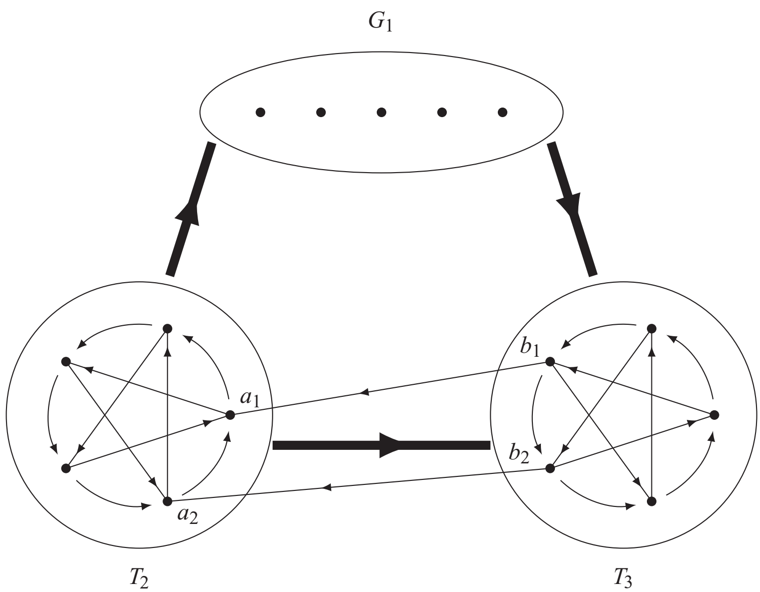

Figure 2. The oriented graph $G_{5,5,2,4}$.

be an n_2-vertex tournament obtained from an $\lfloor (n_2-1)/2 \rfloor$th power of a directed cycle of length n_2 by adding arbitrary edges. Since $\lfloor (n_2-1)/2 \rfloor \geqslant k$, the tournament T_3 is strongly k-connected and $\delta^+(T_3), \delta^-(T_3) \geqslant \lfloor (n_2-1)/2 \rfloor$. We may assume that $V(G_1), V(T_2)$, and $V(T_3)$ are disjoint. Let $a_1, \ldots, a_k \in V(T_2)$ and $b_1, \ldots, b_k \in V(T_3)$ be $2k$ distinct vertices and define

$$\begin{aligned} V(G_{n_1,n_2,k,\overline{\Delta}}) &:= V(G_1) \cup V(T_2) \cup V(T_3), \\ E(G_{n_1,n_2,k,\overline{\Delta}}) &:= (V(G_1) \times V(T_3)) \cup (V(T_2) \times V(G_1)) \\ &\qquad \cup ((V(T_2) \times V(T_3)) \setminus \{a_i b_i : 1 \leqslant i \leqslant k\}) \cup \{b_i a_i : 1 \leqslant i \leqslant k\}. \end{aligned}$$

Note that $G_{n_1,n_2,k,\overline{\Delta}}$ has the following properties.

- $G_{n_1,n_2,k,\overline{\Delta}}$ is strongly k-connected.
- $\overline{\Delta}(G_{n_1,n_2,k,\overline{\Delta}}) \leqslant \overline{\Delta}$.
- The minimum in-degree and the minimum out-degree are at least

$$\min\left(\left\lfloor \frac{n_1-1}{2} \right\rfloor, \left\lfloor \frac{n_2-1}{2} \right\rfloor\right).$$

If $n = n_1 + n_2 + \overline{\Delta} + 1$ and $|n_1 - n_2| \leqslant 1$, then

$$\min(n_1, n_2) \geqslant \frac{n - \overline{\Delta} - 2}{2} \quad \text{and} \quad \min\left(\left\lfloor \frac{n_1-1}{2} \right\rfloor, \left\lfloor \frac{n_2-1}{2} \right\rfloor\right) \geqslant \left\lfloor \frac{n-\overline{\Delta}}{4} \right\rfloor - 1.$$

Let D be a spanning subgraph of $G_{n_1,n_2,k,\overline{\Delta}}$ with $\delta^+(D), \delta^-(D) \geqslant k$. Since every vertex in G_1 has in-degree at least k in D,

$$\sum_{v \in V(T_2)} d_D^+(v) - \sum_{w \in V(T_2)} d_D^-(w) \geqslant e_D(V(T_2), V(G_1)) - e_D(V(T_3), V(T_2)) \geqslant k(\overline{\Delta}+1) - k$$

and thus

$$\sum_{v \in V(T_2)} d_D^+(v) \geqslant \sum_{w \in V(T_2)} d_D^-(w) + k\overline{\Delta} \geqslant kn_1 + k\overline{\Delta}.$$

Hence

$$\begin{aligned}|E(D)| &= \sum_{u \in V(G_1)} d_D^+(u) + \sum_{v \in V(T_2)} d_D^+(v) + \sum_{w \in V(T_3)} d_D^+(w) \\ &\geqslant k|V(G_1)| + \sum_{v \in V(T_2)} d_D^+(v) + k|V(T_3)| \geqslant k(n_1+n_2+\overline{\Delta}+1) + k\overline{\Delta}.\end{aligned}$$

Let us define $T_{n_1,n_2,k}$ to be an (n_1+n_2+k)-vertex tournament obtained from an (n_1+n_2+k)-vertex oriented graph $G_{n_1,n_2,k,k-1}$ by replacing G_1 with a k-vertex transitive tournament T_1. Note that $T_{n_1,n_2,k}$ has the following properties.

- $T_{n_1,n_2,k}$ is strongly k-connected.
- The minimum in-degree and the minimum out-degree are at least

$$\min\left(\left\lfloor \frac{n_1-1}{2} \right\rfloor, \left\lfloor \frac{n_2-1}{2} \right\rfloor\right).$$

Let D be a spanning subgraph of $T_{n_1,n_2,k}$ with $\delta^+(D), \delta^-(D) \geqslant k$. Let $\sigma = (v_1, \ldots, v_t)$ be a transitive ordering of the transitive tournament T_1. Since $d_D^-(v_i) \geqslant k$ for $1 \leqslant i \leqslant k$, we have $e_D(V(T_2), v_i) + e_D(V(T_1), v_i) \geqslant k$. In particular, $e_D(V(T_2), v_i) \geqslant k - i + 1$, and thus

$$e_D(V(T_2), V(T_1)) \geqslant \sum_{i=1}^{k} (k-i+1) = \frac{k(k+1)}{2}.$$

Hence

$$\sum_{v \in V(T_2)} d_D^+(v) - \sum_{w \in V(T_2)} d_D^-(w) \geqslant e_D(V(T_2), V(T_1)) - e_D(V(T_3), V(T_2)) \geqslant \frac{k(k+1)}{2} - k$$

and thus

$$\begin{aligned}|E(D)| &= \sum_{u \in V(G_1)} d_D^+(u) + \sum_{v \in V(T_2)} d_D^+(v) + \sum_{w \in V(T_3)} d_D^+(w) \\ &\geqslant k|V(G_1)| + \sum_{v \in V(T_2)} d_D^+(v) + k|V(T_3)| \geqslant k(n_1+n_2+k) + \frac{k(k-1)}{2}.\end{aligned}$$

If $n = n_1 + n_2 + k$ and $|n_1 - n_2| \leqslant 1$, then

$$\min(n_1, n_2) \geqslant \frac{n-k-1}{2} \quad \text{and} \quad \min\left(\left\lfloor \frac{n_1-1}{2} \right\rfloor, \left\lfloor \frac{n_2-1}{2} \right\rfloor\right) \geqslant \left\lfloor \frac{n-k-3}{4} \right\rfloor.$$

The construction above proves the following proposition.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Proposition 3.1. *Let $k \geqslant 1$ and $\overline{\Delta} \geqslant 0$ be integers.*

(1) *For any integer $n \geqslant 4k + \overline{\Delta} + 3$, there is a strongly k-connected n-vertex oriented graph G with $\overline{\Delta}(G) \leqslant \overline{\Delta}$ and $\delta^+(G), \delta^-(G) \geqslant \lfloor (n - \overline{\Delta})/4 \rfloor - 1$, such that every spanning subgraph D with $\delta^+(D), \delta^-(D) \geqslant k$ contains at least $kn + k\overline{\Delta}$ edges.*

(2) *For any integer $n \geqslant 5k + 2$, there is a strongly k-connected n-vertex tournament T with $\delta^+(T), \delta^-(T) \geqslant \lfloor (n - k - 3)/4 \rfloor$, such that each spanning subgraph D with $\delta^+(D)$, $\delta^-(D) \geqslant k$ contains at least $kn + (k(k-1))/2$ edges.*

4. Brief idea of the proof of Theorem 1.3

Before introducing tools used in the proof, we illustrate the brief idea of the proof of (1) of Theorem 1.3 for $\overline{\Delta} = 0$, where the given digraph D is semicomplete (see Figure 3).

In order to provide enough intuition, we assume the simplest case. First, let us assume that we have $3k$ disjoint sets $A_1, \ldots, A_{3k} \subseteq V(D)$ and $3k$ disjoint sets $B_1, \ldots, B_{3k} \subseteq V(D) \setminus \bigcup_{i=1}^{3k} A_i$ satisfying the following.

- $|A_i| = |B_i| = 5$ for $1 \leqslant i \leqslant 3k$.
- $D[A_i]$ contains a spanning transitive tournament $T[A_i]$ with a sink a_i and $D[B_i]$ contains a spanning transitive tournament $T[B_i]$ with a source b_i for $1 \leqslant i \leqslant 3k$.
- Every vertex
$$v \in V(D) \setminus \left(\bigcup_{i=1}^{3k} A_i \cup \bigcup_{i=1}^{3k} B_i \right)$$
is indominated by A_i and outdominated by B_i for $1 \leqslant i \leqslant 3k$.

We may assume that
$$d^-_{D[\{a_1,\ldots,a_{3k}\}]}(a_1) \geqslant \cdots \geqslant d^-_{D[\{a_1,\ldots,a_{3k}\}]}(a_{3k}) \quad \text{and} \quad d^+_{D[\{b_1,\ldots,b_{3k}\}]}(b_1) \geqslant \cdots \geqslant d^+_{D[\{b_1,\ldots,b_{3k}\}]}(b_{3k})$$
by permuting indices in $[3k]$. By Lemma 2.2, $d^-_{D[\{a_1,\ldots,a_{3k}\}]}(a_i) \geqslant k$ and $d^+_{D[\{b_1,\ldots,b_{3k}\}]}(b_i) \geqslant k$ for $1 \leqslant i \leqslant k$.

Since D is strongly k-connected, we can use Menger's theorem. There exists a permutation $\sigma : [k] \to [k]$ such that for $1 \leqslant i \leqslant k$, there exists a path P_i from a_i to $b_{\sigma(i)}$ in D. We may assume that σ is an identity map by permuting indices in $[k]$. As we only permute indices in $[k]$ here, it is still preserved that $d^-_{D[\{a_1,\ldots,a_{3k}\}]}(a_i) \geqslant k$ and $d^+_{D[\{b_1,\ldots,b_{3k}\}]}(b_i) \geqslant k$ for $1 \leqslant i \leqslant k$.

Let $A = \bigcup_{i=1}^{3k} A_i$ and $B = \bigcup_{i=1}^{3k} B_i$. Using *escapers* (see Lemma 5.8), there exist a set $E_{\text{escape}} \subseteq E(D)$ of edges and a set $V_{\text{out}} \subseteq V(D) \setminus (A \cup B)$ of vertices such that $|E_{\text{escape}}| = O(k^2)$ and $|V_{\text{out}}| = O(k^2)$, where they allow vertices in $A \cup B$, can easily escape from $A \cup B$ using these edges, in the following sense.

(A4.1) For any $S \subseteq V(D)$ with $|S| \leqslant k - 1$ and $u \in (A \cup B) \setminus S$, there is a path from u to a vertex in V_{out} in $D - S$ using only edges in E_{escape}.

(A4.2) For any $S \subseteq V(D)$ with $|S| \leqslant k - 1$ and $u \in (A \cup B) \setminus S$, there is a path from a vertex in V_{out} to u in $D - S$ using only edges in E_{escape}.

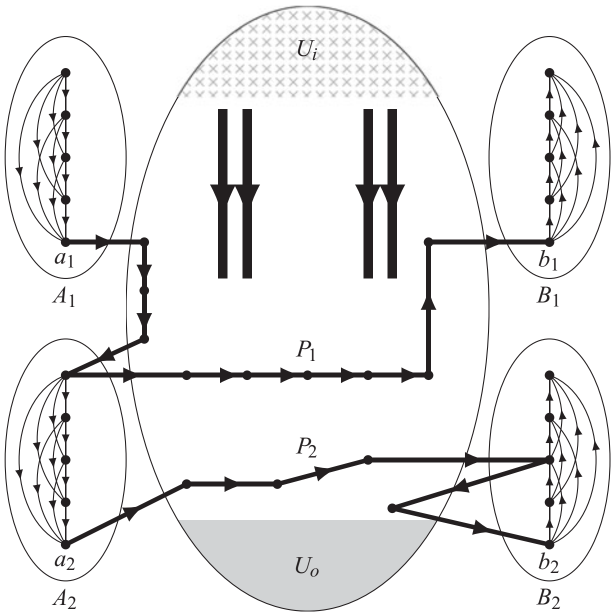

Figure 3. Indominating sets A_1, A_2 and outdominating sets B_1, B_2 with two paths P_1 and P_2 connecting pairs of vertices (a_1, b_1) and (a_2, b_2), respectively. The paths P_1 and P_2 may intersect other vertices in $A \cup B$. The thick lines depict that after removing one vertex in $V(D)$, each remaining vertex in $V(D) \setminus (A \cup B)$ can be reached from a vertex in U_i and can reach a vertex in U_o via sparse linkage structure.

Now we use the sparse linkage structure introduced in Section 2. Let us apply Lemma 2.5 to $D[V_{\text{out}}]$ and $D[V(D) \setminus (A \cup B \cup \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \cup V_{\text{out}})]$, where we get a spanning subgraph D' of $D[V_{\text{out}}]$, $U_i', U_o' \subseteq V_{\text{out}}$, a spanning subgraph D'' of

$$D\left[V(D) \setminus \left(A \cup B \cup \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \cup V_{\text{out}}\right)\right]$$

and

$$U_i'', U_o'' \subseteq V(D) \setminus \left(A \cup B \cup \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \cup V_{\text{out}}\right).$$

Similarly, let us apply Lemma 2.6 to

$$D\left[\bigcup_{i=1}^{k} V^{\text{int}}(P_i) \setminus (A \cup B \cup V_{\text{out}})\right],$$

where we get a spanning subgraph D''' of

$$D\left[\bigcup_{i=1}^{k} V^{\text{int}}(P_i) \setminus (A \cup B \cup V_{\text{out}})\right]$$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

and

$$U_i''', U_o''' \subseteq \bigcup_{i=1}^{k} V^{\mathrm{int}}(P_i) \setminus (A \cup B \cup V_{\mathrm{out}}).$$

Given any $S \subseteq V(D)$ with $|S| \leqslant k-1$, they satisfy the following.

- $|E(D')| \leqslant k|V_{\mathrm{out}}| - k$.
- $|E(D'')| \leqslant k|V(D) \setminus (A \cup B \cup \bigcup_{i=1}^{k} V^{\mathrm{int}}(P_i) \cup V_{\mathrm{out}})| - k$.
- $|E(D''')| \leqslant (k-1)|\bigcup_{i=1}^{k} V^{\mathrm{int}}(P_i) \setminus (A \cup B \cup V_{\mathrm{out}})| + (k-1)$.
- $|U_i'|, |U_o'|, |U_i''|, |U_o''|, |U_i'''|, |U_o'''| \leqslant 2k-1$.

(B4.1) For any vertex $w \in V(D') \setminus S$, there exist a path from w to a vertex in $U_o' \setminus S$ in $D' - S$ and a path from a vertex in $U_i' \setminus S$ to w in $D' - S$.

(B4.2) For any vertex $w \in V(D'') \setminus S$, there exist a path from w to a vertex in $U_o'' \setminus S$ in $D'' - S$ and a path from a vertex in $U_i'' \setminus S$ to w in $D'' - S$.

(B4.3) For any vertex $w \in V(D''') \setminus S$, there exist a path from w to a vertex in $(U_o''' \cup \{b_1, \ldots, b_k\}) \setminus S$ in $D''' - S$ and a path from a vertex in $(U_i''' \cup \{a_1, \ldots, a_k\}) \setminus S$ to w in $D''' - S$.

In the following section, an object *absorber* will be related to these properties above. Let

$$U_o := U_o' \cup U_o'' \cup U_o''', \quad U_i := U_i' \cup U_i'' \cup U_i'''.$$

For any $u \in U_o$ and $1 \leqslant i \leqslant 3k$, as $u \in V(D) \setminus (A \cup B)$, u is indominated by A_i and there exists a path $P_{u,i}$ of length at most two from u to a_i, since $D[A_i]$ contains a spanning transitive subtournament with a sink a_i. Similarly, for any $v \in U_i$ and $1 \leqslant i \leqslant k$, v is outdominated by B_i and there exists a path $Q_{v,i}$ of length at most two from b_i to v, since $D[B_i]$ contains a spanning transitive subtournament with a source b_i.

Let us define

$$E' := E(D[A \cup B]) \cup \bigcup_{u \in U_o} \bigcup_{i=1}^{3k} E(P_{u,i}) \cup \bigcup_{v \in U_i} \bigcup_{i=1}^{3k} E(Q_{v,i}).$$

Then $|E'| = O(k^2)$, as $|U_o| \leqslant 6k$ and $|U_i| \leqslant 6k$.

Let $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $u \in U_o \setminus S$. For any $1 \leqslant t \leqslant k$ with $a_t \notin S$, we claim that there exists a path from u to a_t in D only using edges in E'. Indeed, let $1 \leqslant i \leqslant 3k$ be an index with $a_i \in N_D^-(a_t)$ and $A_i \cap S = \emptyset$, which is guaranteed by $d^-_{D[\{a_1,\ldots,a_{3k}\}]}(a_t) \geqslant k$ and the disjointness of $A_1, \ldots, A_{3t}$. Since $A_i \cap S = \emptyset$ and $a_t \notin S$, the path $P^*_{u,t} := P_{u,i} \cup (a_i, a_t)$ does not intersect S and is from u to a_t only using edges in E'. Similarly, for any $v \in U_i \setminus S$ and $b_t \notin S$ with $1 \leqslant t \leqslant k$, there exists a path from b_t to v in D only using edges in E'. In summary:

(C4.1) For any $S \subseteq V(D)$ with $|S| \leqslant k-1$, $u \in U_o \setminus S$ and $a_t \notin S$ with $1 \leqslant t \leqslant k$, there exists a path from u to a_t in D only using edges in E'.

(C4.2) For any $S \subseteq V(D)$ with $|S| \leqslant k-1$, $v \in U_i \setminus S$ and $b_t \notin S$ with $1 \leqslant t \leqslant k$, there exists a path from b_t to v in D only using edges in E'

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

In the following section, an object *hub* will attain these properties above. Now, let D_{sparse} be a spanning subgraph of D with the edge set

$$\bigcup_{i=1}^{k} E(P_i) \cup E_{\text{escape}} \cup E(D') \cup E(D'') \cup E(D''') \cup E'.$$

Then it is straightforward to see that $|E(D_{\text{sparse}})| = k|V(D)| + O(k^2)$. Now we claim that D_{sparse} is strongly k-connected. Let $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $u, v \in V(D) \setminus S$. We aim to find a path from u to v in $D_{\text{sparse}} - S$. Let $i \in [k]$ be an index such that $V(P_i) \cap S = \emptyset$.

Now it suffices to find a path from u to $u^* \in U_o \setminus S$ in $D_{\text{sparse}} - S$ and a path from $v^* \in U_i \setminus S$ to v in $D_{\text{sparse}} - S$. Indeed, by (C4.1) and (C4.2) we have a path from u^* to a_i and a path from b_i to v^*. Together with the path P_i, there exists a path from u to v in $D_{\text{sparse}} - S$, as desired.

- If $u \in A \cup B$, then by (A4.1), there exists a path from u to $u' \in V_{\text{out}}$ in $D_{\text{sparse}} - S$. By (B4.1), there exists a path from u' to $u^* \in U_o \setminus S$ in $D_{\text{sparse}} - S$.
- If $u \in \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \setminus V_{\text{out}}$, then by (B4.3) there is a path P from u to a vertex $w \in U_o \cup \{b_1, \ldots, b_k\}$ in $D''' - S$. If $w \in U_o$, then let $u^* := w$. Otherwise, $w \in \{b_1, \ldots, b_k\} \subseteq A \cup B$, where this case has already been considered above.
- If $u \in V(D) \setminus (A \cup B \cup \bigcup_{i=1}^{k} V^{\text{int}}(P_i))$, then by (B4.1) and (B4.2) there is a path from u to a vertex $u^* \in U_o \setminus S$ in $D_{\text{sparse}} - S$.

Similarly, one can find a path from a vertex $v^* \in U_i \setminus S$ to v. This proves that D_{sparse} is strongly k-connected.

Note that this proof only works when for $1 \leqslant i \leqslant 3k$, every vertex in $V(D) \setminus (A \cup B)$ that is indominated by A_i is also outdominated by B_i, and A_i and B_i have small size. As we cannot guarantee the existence of these subsets of vertices, this ideal situation might not happen. Nevertheless, we are able to force all vertices in $V(D) \setminus (A \cup B)$ to satisfy the conditions close to the ideal one as follows.

Using Lemma 5.3, we choose 5-indominators $A_1, \ldots, A_{5k}$ and 5-outdominators $B_1, \ldots, B_{5k}$ (see Definitions 5.1 and 5.2). Each of these 5-indominators A_i (5-outdominators B_i) would indominate (outdominate, respectively) all vertices in $V(D) \setminus (A \cup B)$ but a few exceptional vertices U_i^+ (U_i^-, respectively). As the size of U_i^+ or U_i^- could be $\Omega(n)$, we utilize the following two observations to reduce the size. First, we do not need to force all vertices to in/outdominated by all $5k$ 5-in/outdominators. Second, if the size of U_i^+ (U_i^-) is big enough, then all vertices in U_i^+ (U_i^-, respectively) have large out-degree (in-degree, respectively) so they can easily escape from U_i^+ (U_i^-, respectively).

Hence, we regard any vertex $v \in V(D) \setminus (A \cup B)$ as an exceptional vertex only when there are more than k indices $i \in [5k]$ such that v is not indominated by A_i (not outdominated by B_i) and $|U_i^+|$ ($|U_i^-|$, respectively) is not big enough. Let O^+ (O^-, respectively) be the set of all these vertices in $V(D) \setminus (A \cup B)$ and $O^* := O^+ \cup O^-$ be the set of exceptional vertices. In summary, it can be shown that 5-indominators $A_1, \ldots, A_{5k}$, 5-outdominators $B_1, \ldots, B_{5k}$ and O^* attain the following properties (see Lemma 5.5).

- $|O^*| = O(k)$.
- For any vertex $w \in V(D) \setminus (A \cup B \cup O^*)$, there exist a path of length at most two from w to a vertex in A_i for at least $4k$ indices $i \in [5k]$, and a path of length at most two from a vertex in B_i to w for at least $4k$ indices $i \in [5k]$.

Indeed, as every vertex in $V(D) \setminus (A \cup B)$ is not in/outdominated by all 5-in/out-dominators, we cannot simply follow the proof illustrated in this section and it is required to develop more ideas. In the following section, we introduce the objects according to the modification discussed as above.

5. Basic objects in the construction

As the proof of the main result consists of many technical parts, we divide the proof into statements constructing objects called *dominators*, *trios*, *escapers*, *hubs* and *absorbers*. Dominators are the most basic objects, very simple but useful in controlling the length of many disjoint paths. A collection of many dominators with many good properties are called a trio, which is our main interest when involving collections of many dominators. Based on trios, we construct hubs and absorbers, and combine them into a highly connected spanning subgraph with few edges to prove Theorem 1.3.

5.1. Dominators

In this subsection, we define indominators and outdominators in digraphs, which are the most basic objects in constructing a sparse highly connected spanning subgraph.

Definition 5.1. Let $t \geqslant 1$ be an integer. A *t-indominator* is a quadtuple (D, A, x, a) such that D is a directed multigraph, A is a subset of $V(D)$ with at most t vertices, and $x, a \in A$, satisfying the following.

(ID1) $D[A]$ contains a spanning transitive tournament with a source x and a sink a.
(ID2) x has at least $2^{t-1}|U^+|$ out-neighbours in D, where $U^+ := \bigcap_{v\in A} N_D^+(v) \setminus \bigcup_{v\in A} N_D^-(v)$.

Definition 5.2. Let $t \geqslant 1$ be an integer. A *t-outdominator* is a quadtuple (D, B, x', b) such that D is a directed multigraph, B is a subset of $V(D)$ with at most t vertices, and $x', b \in B$, satisfying the following.

(OD1) $D[B]$ contains a spanning transitive tournament with a source b and a sink x'.
(OD2) x' has at least $2^{t-1}|U^-|$ in-neighbours in D, where $U^- := \bigcap_{v\in B} N_D^-(v) \setminus \bigcup_{v\in B} N_D^+(v)$.

The following lemma guarantees the existence of a t-in/outdominator in directed multigraphs. This is a variation of [19, Lemma 2.3] proved for tournaments.

Lemma 5.3. *Let $t \geqslant 1$ be an integer. For each vertex x of a directed multigraph D, there exist $A \subseteq V(D)$ and $a \in A$ such that (D, A, x, a) is a t-indominator, and $B \subseteq V(D)$ and $b \in B$ such that (D, B, x, b) is a t-outdominator.*

Proof. We only prove that there exist $A \subseteq V(D)$ and $a \in A$ such that (D, A, x, a) is a t-indominator. The rest of the proof follows by reversing orientations of all edges.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Let G be an oriented graph obtained from D by removing multiple edges and exactly one edge from each directed 2-cycle. Let $V_1 := N_G^+(x)$ and $v_1 := x$. Let s be the maximum integer that satisfies $1 \leqslant s \leqslant t$ and $v_1, \ldots, v_s \in V(D)$ and $V_1, \ldots, V_s \subseteq V(D)$ satisfying the following properties.

(i) For $1 \leqslant i < j \leqslant s$, $v_j \in N_G^+(v_i)$.
(ii) For $1 \leqslant i \leqslant s$, $V_i := \bigcap_{k=1}^{i} N_G^+(v_k)$.
(iii) For $1 \leqslant i < s$, $|V_{i+1}| \leqslant \frac{1}{2}|V_i|$.

Note that such s exists as (i), (ii) and (iii) hold for $s = 1$. We claim that $V_s = \emptyset$ or $s = t$. Otherwise, let $v_{s+1} \in V_s$ with $d^+_{G[V_s]}(v_{s+1}) \leqslant |V_s|/2$. Indeed, since G is an oriented graph, $G[V_s]$ contains at most $(|V_s|(|V_s|-1))/2$ edges, proving that there is a vertex in V_s with out-degree at most $(|V_s|-1)/2$. Let us define $V_{s+1} := N^+_{G[V_s]}(v_{s+1}) = V_s \cap N_G^+(v_{s+1})$, then $|V_{s+1}| \leqslant |V_s|/2$, contradicting the maximality of s.

Therefore, $V_s = \emptyset$ or $s = t$. Let us define $A := \{v_1, \ldots, v_s\}$ with $a := v_s$. Then $G[A]$ is a transitive tournament with a source x and a sink a. Let $V^+ := V_s = \bigcap_{k=1}^{s} N_G^+(v_k)$. Since $|V^+| \leqslant 2^{-t+1}|V_1|$ by (iii), we deduce $|N_G^+(x)| = |V_1| \geqslant 2^{t-1}|V^+|$. Now we claim that

$$\bigcap_{v\in A} N_D^+(v) \setminus \bigcup_{v\in A} N_D^-(v) \subseteq \bigcap_{v\in A} N_G^+(v).$$

For every $w \in \bigcap_{v\in A} N_D^+(v) \setminus \bigcup_{v\in A} N_D^-(v)$ we have $w \in \bigcap_{v\in A} N_G^+(v)$, otherwise there exists $v \in A$ such that $wv, vw \in E(D)$, implying that $w \in \bigcup_{v\in A} N_D^-(v)$ and contradicting the assumption on w. Therefore, $|V^+| \geqslant |U^+|$ and we have

$$|N_D^+(x)| \geqslant |N_G^+(x)| \geqslant 2^{t-1}|V^+| \geqslant 2^{t-1}|U^+|,$$

where $U^+ := \bigcap_{v\in A} N_D^+(v) \setminus \bigcup_{v\in A} N_D^-(v)$. This proves that (D, A, x, a) is a t-indominator. □

Throughout the proof, it is worth noting that t will always be 5 when regarding t-indominators and t-outdominators.

5.2. Trios

In Section 4, we sketched the proof provided that every vertex in $V(D) \setminus (A \cup B)$ is indominated by $A_1, \ldots, A_{3k}$ and outdominated by $B_1, \ldots, B_{3k}$. However, we cannot guarantee these sets in/outdominate all other vertices, but they in/outdominate almost all other vertices by Lemma 5.3. In this subsection, we introduce the object called a *trio*, allowing that most of the vertices can reach many 5-indominators and can be reached from many 5-outdominators by paths of length at most two. The other subsections will introduce other objects to follow the sketched proof in Section 4 according to this modification.

Definition 5.4. Let $d, k, t_1, t_2 \geqslant 1$, $m \geqslant k$, $\overline{\Delta} \geqslant 0$ be integers, and let $u > 0$ be a real number. Let D be a directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. A 3-tuple $(\mathcal{A}, \mathcal{B}, O^*)$ is called a (t_1, t_2, d, m, u)-*trio* in D if $\mathcal{A}$ is a collection of m distinct 5-indominators $\{(D_i, A_i, x_i, a_i)\}_{i=1}^m$, and $\mathcal{B}$ is a collection of m distinct 5-outdominators $\{(D_i', B_i, x_i', b_i)\}_{i=1}^m$, and a subset $O^* \subseteq V(D)$ of vertices satisfying the following properties, where

$$U_i^+ := \bigcap_{w\in A_i} N^+_{D_i}(w) \setminus \bigcup_{w\in A_i} N^-_{D_i}(w) \quad \text{and} \quad U_i^- := \bigcap_{w\in B_i} N^-_{D_i'}(w) \setminus \bigcup_{w\in B_i} N^+_{D_i'}(w).$$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

(T1) For every $i \in [m]$, D_i is a subgraph of D, and contains $D - (\bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m B_i)$ as a subgraph.
(T2) For every $i \in [m]$, D'_i is a subgraph of $D - \bigcup_{i=1}^m A_i$, and contains $D - (\bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m B_i)$ as a subgraph.
(T3) $A_1, \dots, A_m$, $B_1, \dots, B_m$ are disjoint subsets.
(T4) For every $i \in [k]$,

$$|N^-_{D[\{a_1,\dots,a_m\}]}(a_i)| \geqslant \frac{m-k-\overline{\Delta}}{2} \quad \text{and} \quad |N^+_{D[\{b_1,\dots,b_m\}]}(b_i)| \geqslant \frac{m-k-\overline{\Delta}}{2}.$$

(T5) For every $v \in V(D) \setminus (\bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m B_i \cup O^*)$, there are at least $m - t_1 - t_2$ indices $i \in [m]$ such that either v is indominated by A_i, or v is in U_i^+ with $|U_i^+| \geqslant u$.
(T6) For every $v \in V(D) \setminus (\bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m B_i \cup O^*)$, there are at least $m - t_1 - t_2$ indices $i \in [m]$ such that either v is outdominated by B_i, or v is in U_i^- with $|U_i^-| \geqslant u$.
(T7) For every $u \in U_i^+$ with $i \in [m]$ and $|U_i^+| \geqslant u$, the vertex u has at least $d + |U_i^+|$ out-neighbours in D_i.
(T8) For every $u \in U_i^-$ with $i \in [m]$ and $|U_i^-| \geqslant u$, the vertex u has at least $d + |U_i^-|$ in-neighbours in D'_i.
(T9) $|O^*|$ is small enough; $|O^*|$ is at most

$$\frac{2mu}{t_1} + \frac{10\overline{\Delta}m}{t_2},$$

and if $t_2 \geqslant \overline{\Delta}$ then

$$|O^*| \leqslant \frac{2mu}{t_1}.$$

The following lemma guarantees a (t_1, t_2, d, m, u)-trio for dense digraphs.

Lemma 5.5. *Let $d, k, n, m, t_1, t_2 \geqslant 1$, $\overline{\Delta} \geqslant 0$ be integers with $m \geqslant k$, and let $u > 0$ be a real number. Let D be an n-vertex directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. If $n \geqslant 10m$ and $u \geqslant d/15$, then D contains a (t_1, t_2, d, m, u)-trio $(\mathcal{A}, \mathcal{B}, O^*)$.*

Proof. First of all, we construct m distinct 5-indominators satisfying some properties.

Claim 1. *There exist a collection $\mathcal{A}$ of m distinct 5-indominators $\{(D_i, A_i, x_i, a_i)\}_{i=1}^m$ satisfying the following. For every $1 \leqslant i \leqslant m$,*
(1) $D_i := D - \bigcup_{j=1}^{i-1} A_j$,
(2) x_i is a vertex in D_i with the smallest number of out-neighbours in $V(D_i)$,
(3) (D_i, A_i, x_i, a_i) is a 5-indominator.

Proof of Claim 1. As $|V(D)| = n \geqslant 5m$, the claim follows by successively applying Lemma 5.3. □

Let us define $A := \bigcup_{i=1}^m A_i$. Now we construct m distinct 5-outdominators satisfying some properties.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Claim 2. *There exist a collection $\mathcal{B}$ of m distinct 5-outdominators $\{(D'_i, B_i, x'_i, b_i)\}_{i=1}^m$ satisfying the following. For every $1 \leqslant i \leqslant m$,*
(1) $D'_i := D - (A \cup \bigcup_{j=1}^{i-1} B_j)$,
(2) x'_i *is a vertex in* D'_i *with the smallest number of in-neighbours in* $V(D'_i)$,
(3) (D'_i, B_i, x'_i, b_i) *is a 5-outdominator.*

Proof of Claim 2. Since $|V(D) \setminus A| \geqslant n - 5m \geqslant 5m$, the claim follows by successively applying Lemma 5.3. □

Let us define $B := \bigcup_{i=1}^m B_i$, and for every $i \in [m]$, let us define

$$U_i^+ := \bigcap_{v\in A_i} N^+_{D_i}(v) \setminus \bigcup_{v\in A_i} N^-_{D_i}(v), \quad U_i^- := \bigcap_{v\in B_i} N^-_{D'_i}(v) \setminus \bigcup_{v\in B_i} N^+_{D'_i}(v).$$

By (ID2) and (OD2), for every $i \in [m]$ we have

$$|N^+_{D_i}(x_i)| \geqslant 16|U_i^+|, \quad |N^-_{D'_i}(x_i)| \geqslant 16|U_i^-|. \tag{5.1}$$

Since both D_i and D'_i contain $D - (A \cup B)$ as a subgraph for $1 \leqslant i \leqslant m$, this proves (T1) and (T2) of Definition 5.4. From the construction of $\mathcal{A}$ and $\mathcal{B}$, (T3) is clear.

By Lemma 2.2 and permuting indices, we may assume that for every $i \in [k]$,

$$|N^-_{D[\{a_1,\dots,a_m\}]}(a_i)| \geqslant \frac{m-k-\overline{\Delta}}{2}, \quad |N^+_{D[\{b_1,\dots,b_m\}]}(b_i)| \geqslant \frac{m-k-\overline{\Delta}}{2},$$

which proves (T4) of Definition 5.4.

For $1 \leqslant i \leqslant m$, let

$$F_i^+ := V(D_i) \setminus \left(A_i \cup U_i^+ \cup \bigcup_{v\in A_i} N^-_{D_i}(v)\right), \quad F_i^- := V(D'_i) \setminus \left(B_i \cup U_i^- \cup \bigcup_{v\in B_i} N^+_{D'_i}(v)\right),$$

where F_i^+ is the set of vertices v in $V(D_i) \setminus A_i$ that are not indominated by A_i and are non-neighbours of some vertices in A_i, and F_i^- is the set of vertices v in $V(D'_i) \setminus B_i$ that are not outdominated by B_i and are non-neighbours of some vertices in B_i.

Since every vertex in D has at most $\overline{\Delta}$ other non-neighbour vertices and $|A_i|, |B_i| \leqslant 5$ for $i \in [m]$, it follows that

$$|F_i^+|, |F_i^-| \leqslant 5\overline{\Delta}. \tag{5.2}$$

It is easy to observe the following, from the definitions of U_i^+, F_i^+, U_i^- and F_i^-.

Observation 5.6. *For every vertex $v \in V(D) \setminus (A \cup B)$ and $i \in [m]$, the following hold.*

- *Either v is indominated by A_i, or v is in U_i^+, or v is in F_i^+.*
- *Either v is outdominated by B_i, or v is in U_i^-, or v is in F_i^-.*

Let us define

$$I^+ := \{i \in [m] : |U_i^+| < u\}, \quad I^- := \{i \in [m] : |U_i^-| < u\},$$
$$O^+ := \{v \in V(D) : |\{i \in I^+ : v \in U_i^+\}| > t_1\},$$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

$$\begin{aligned}
F^+ &:= \{v \in V(D) : |\{i \in [m] : v \in F_i^+\}| > t_2\},\\
O^- &:= \{v \in V(D) : |\{i \in I^- : v \in U_i^-\}| > t_1\},\\
F^- &:= \{v \in V(D) : |\{i \in [m] : v \in F_i^-\}| > t_2\},\\
O &:= O^+ \cup O^-,\\
F &:= F^+ \cup F^-.
\end{aligned}$$

Let $O^* := O \cup F$. By Observation 5.6 and the definition of O^*, both (T5) and (T6) of Definition 5.4 are satisfied.

Claim 3. *The following hold.*

(1) *For every $i \in [m] \setminus I^+$ and $v \in V(D_i) \setminus A_i$, $|N^+_{D_i}(v)| \geqslant d + |U_i^+|$.*

(2) *For every $i \in [m] \setminus I^-$ and $w \in V(D'_i) \setminus B_i$, $|N^-_{D'_i}(w)| \geqslant d + |U_i^-|$.*

(3) $|O| \leqslant (2mu)/t_1$.

(4) $|F| \leqslant (10\overline{\Delta}m)/t_2$. *Moreover, if $t_2 \geqslant \overline{\Delta}$, then $F = \emptyset$.*

Proof of Claim 3. For every $i \in [m]$, we have $|N^+_{D_i}(x_i)| \geqslant 16|U_i^+|$ and $|N^-_{D'_i}(x'_i)| \geqslant 16|U_i^-|$ by (5.1). From the definition of x_i and x'_i, it follows that for every $v \in V(D_i) \setminus A_i$ and $w \in V(D'_i) \setminus B_i$, we have

$$\begin{aligned}
|N^+_D(v)| \geqslant |N^+_{D_i}(x_i)| \geqslant 16|U_i^+|,\\
|N^-_{D-A}(w)| \geqslant |N^-_{D'_i}(x'_i)| \geqslant 16|U_i^-|,
\end{aligned}$$

by Claims 1 and 2.

For every $i \in [m] \setminus I^+$ and $v \in V(D_i) \setminus A_i$, since $|U_i^+| \geqslant u$ it follows that $|N^+_{D_i}(v)| \geqslant 16|U_i^+| \geqslant d + |U_i^+|$ since $u \geqslant d/15$. Similarly, for every $i \in [m] \setminus I^-$ and $w \in V(D'_i) \setminus B_i$, we have $|N^-_{D'_i}(w)| \geqslant d + |U_i^-|$. This proves (1) and (2).

Since every vertex in O^+ is in U_i^+ for more than t_1 indices $i \in I^+$,

$$t_1|O^+| \leqslant \sum_{i \in I^+} |U_i^+| \leqslant |I^+| \cdot u \leqslant m \cdot u$$

and $|O^+| \leqslant (m \cdot u)/t_1$. Similarly, $|O^-| \leqslant (m \cdot u)/t_1$, implying that $|O| \leqslant (2mu)/t_1$. This proves (3).

If $\overline{\Delta} = 0$, then (4) is trivial. We may assume that $\overline{\Delta} > 0$. Since every vertex in F^+ is in F_i^+ for more than t_2 indices $i \in [m]$ and by (5.2),

$$t_2|F^+| \leqslant \sum_{i \in [m]} |F_i^+| \leqslant m \cdot 5\overline{\Delta}$$

and $|F^+| \leqslant (5\overline{\Delta}m)/t_2$. Similarly, $|F^-| \leqslant (5\overline{\Delta}m)/t_2$, implying that $|F| \leqslant (10\overline{\Delta}m)/t_2$.

If $t_2 \geqslant \overline{\Delta}$, then for every $v \in F^+$, there are more than $\overline{\Delta}$ indices $i \in [m]$ such that $v \in F_i^+$ and there is $w \in A_i$ with $(v,w),(w,v) \notin E(D)$, implying that v has more than $\overline{\Delta}$ non-neighbours. Hence $F^+ = \emptyset$. Similarly, we have $F^- = \emptyset$. This proves (4). □

Since $O^* = O \cup F$,

$$|O^*| \leqslant |O| + |F| \leqslant \frac{2mu}{t_1} + \frac{10\overline{\Delta}m}{t_2}$$

by Claim 3. If $t_2 \geqslant \overline{\Delta}$, then $F = \emptyset$ and thus $|O^*| \leqslant |O| \leqslant (2mu)/t_1$. Hence $(\mathcal{A}, \mathcal{B}, O^*)$ is a (t_1, t_2, d, m, u)-trio since (T7)–(T9) hold by Claim 3. □

5.3. Escapers

In this subsection, we consider objects called escapers. Roughly speaking, given a directed multigraph D and a small set $U \subseteq V(D)$, a k-escaper is a set of edges such that every vertex in U can escape from U to $V(D) \setminus U$ by a path, after we remove fewer than k vertices of D. Finding k-escapers with few edges is one of the most crucial parts in constructing a sparse strongly k-connected subgraph of D.

Definition 5.7. Let $k \geqslant 1$ be an integer and let D be a digraph. A *k-escaper* in D is a triple $(E_{\text{escape}}, U, U_{\text{out}})$ of a subset E_{escape} of $E(D)$ and subsets U and U_{out} of $V(D)$ satisfying the following.

- (E1) $U_{\text{out}} \subseteq V(D) \setminus U$.
- (E2) For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and any vertex $u \in U \setminus S$, a subgraph $D - S$ contains a path from u to a vertex in U_{out} only using edges in E_{escape}.
- (E3) For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and any vertex $v \in U \setminus S$, a subgraph $D - S$ contains a path from a vertex in U_{out} to v only using edges in E_{escape}.

The following lemma is the main lemma of this subsection, which allows us to find a sparse k-escaper of a set U of vertices.

Lemma 5.8. *Let $k, n \geqslant 1$ be integers. Let D be a strongly k-connected digraph, and $U \subseteq V(D)$. If $|U| \leqslant |V(D)| - k$, then there is a k-escaper $(E_{\text{escape}}, U, U_{\text{out}})$ in D such that $|E_{\text{escape}}| \leqslant 4k|U|$ and $|U_{\text{out}}| \leqslant 2k|U|$.*

Proof. Let D' be a minimally strongly k-connected spanning subgraph of D. Since $|V(D) \setminus U| \geqslant k$, we can apply Proposition 2.1 as follows. For every $u \in U$, there are a k-fan $\{P^+_{u,i}\}_{i=1}^k$ from u to $V(D) \setminus U$ and a k-fan $\{P^-_{u,i}\}_{i=1}^k$ from $V(D) \setminus U$ to u.

Let us define

$$E_{\text{escape}} := \bigcup_{u \in U} \left(\bigcup_{i=1}^{k} E(P^+_{u,i}) \cup \bigcup_{i=1}^{k} E(P^-_{u,i}) \right), \tag{5.3}$$

$$U_{\text{out}} := \bigcup_{u \in U} \left(\bigcup_{i=1}^{k} V(P^+_{u,i}) \cup \bigcup_{i=1}^{k} V(P^-_{u,i}) \right) \setminus U, \tag{5.4}$$

which proves (E1).

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

For every $u \in U$, it follows that

$$\left|U_{\text{out}} \cap \bigcup_{i=1}^{k} V(P^+_{u,i})\right| = k, \quad \left|U_{\text{out}} \cap \bigcup_{i=1}^{k} V(P^-_{u,i})\right| = k, \tag{5.5}$$

and thus $|U_{\text{out}}| \leqslant 2k|U|$ and $|E_{\text{escape}}| \leqslant |E(D'[U])| + |U_{\text{out}}| \leqslant 4k|U|$ by Proposition 2.13.

Since $E_{\text{escape}} \subseteq E(D')$, it is a subset of $E(D)$. Now we claim that $(E_{\text{escape}}, U, U_{\text{out}})$ is a k-escaper. For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $u \in U \setminus S$, there is $i \in [k]$ with $V(P^+_{u,i}) \cap S = \emptyset$. Since $E(P^+_{u,i}) \subseteq E_{\text{escape}}$ and by the definition of U_{out}, this proves (E2). Similarly (E3) holds by the same proof. □

We also define an edge version of escapers.

Definition 5.9. Let $k \geqslant 1$ be an integer and let D be a directed multigraph. A *k-arc-escaper* in D is a 3-tuple $(E_{\text{escape}}, U, U_{\text{out}})$ satisfying the following.

(E1′) $U_{\text{out}} \subseteq V(D) \setminus U$.
(E2′) For every $F \subseteq E(D)$ with $|F| \leqslant k-1$ and any vertex $u \in U$, a subgraph $D - F$ contains a path from u to a vertex in U_{out} only using edges in E_{escape}.
(E3′) For every $F \subseteq E(D)$ with $|F| \leqslant k-1$ and any vertex $v \in U$, a subgraph $D - F$ contains a path from a vertex in U_{out} to v only using edges in E_{escape}.

Replacing Proposition 2.13 by Corollary 2.11 in the proof of Lemma 5.8, the following lemma easily follows.

Lemma 5.10. *Let $k, n \geqslant 1$ be integers. Let D be an n-vertex strongly k-arc-connected directed multigraph, and $U \subsetneq V(D)$. Then there exists a k-arc-escaper $(E_{\text{escape}}, U, U_{\text{out}})$ in D such that $|E_{\text{escape}}| \leqslant 4k|U|$ and $|U_{\text{out}}| \leqslant 2k|U|$.*

5.4. Hubs

In this subsection, we consider objects called *hubs*, which allow us to connect a set of vertices with the vertices of dominators. Hubs are one of the main tools when constructing highly connected sparse spanning subgraphs of dense digraphs.

Definition 5.11. Let k be an integer and let D be a digraph. Then a *k-hub* $\mathcal{H}$ in D is a 5-tuple $(E_{\text{hub}}, A_0, B_0, U_o, U_i)$ that consists of a set $E_{\text{hub}} \subseteq E(D)$, two sets $A_0, B_0 \subseteq V(D)$ with $|A_0| = |B_0| = k$, and subsets $U_o, U_i \subseteq V(D)$ satisfying the following.

(H1) $A_0 =: \{a_1, \ldots, a_k\}$, $B_0 =: \{b_1, \ldots, b_k\}$ and $A_0 \cap B_0 = \emptyset$.
(H2) For every $t \in [k]$ and $S \subseteq V(D)$ with $|S| \leqslant k-1$, if $u \in U_o \setminus S$ and $a_t \notin S$, then $D - S$ contains a path from u to a_t only using edges in E_{hub}.
(H3) For every $t \in [k]$ and $S \subseteq V(D)$ with $|S| \leqslant k-1$, if $v \in U_i \setminus S$ and $b_t \notin S$, then $D - S$ contains a path from b_t to v only using edges in E_{hub}.

We also define an edge version of hubs.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Definition 5.12. Let k be an integer and let D be a directed multigraph. A *k-arc-hub* $\mathcal{H}$ in D is a 5-tuple $(E_{\text{hub}}, A_0, B_0, U_o, U_i)$ that consists of a set $E_{\text{hub}} \subseteq E(D)$, two sets $A_0, B_0 \subseteq V(D)$ with $|A_0| = |B_0| = k$, and subsets $U_o, U_i \subseteq V(D)$ satisfying the following.

(H1′) $A_0 =: \{a_1, \dots, a_k\}$, $B_0 =: \{b_1, \dots, b_k\}$ and $A_0 \cap B_0 = \emptyset$.

(H2′) For every $t \in [k]$, $u \in U_o$ and $F \subseteq E(D)$ with $|F| \leqslant k-1$, the subgraph $D-F$ contains a path from u to a_t only using edges in E_{hub}.

(H3′) For every $t \in [k]$, $v \in U_i$ and $F \subseteq E(D)$ with $|F| \leqslant k-1$, the subgraph $D-F$ contains a path from b_t to v only using edges in E_{hub}.

The following lemma guarantees the existence of a k-hub under some conditions for dense digraphs.

Lemma 5.13. *Let $d, k, m, t_1, t_2 \geqslant 1$, $\overline{\Delta}, w \geqslant 0$ be integers with $m \geqslant k$ $d \geqslant 6m + 5\overline{\Delta}$ and a real number $u \geqslant d/15$. Let D be a digraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ and at least $10m$ vertices. If D contains a (t_1, t_2, d, m, u)-trio $(\mathcal{A}, \mathcal{B}, O^*)$ such that*

- *$(\mathcal{A}, \mathcal{B}, O^*)$ satisfies the assumptions in Lemma 5.5,*
- *$\mathcal{A}$ consists of 5-indominators $\{(D_i, A_i, x_i, a_i)\}_{i=1}^m$, and*
- *$\mathcal{B}$ consists of 5-outdominators $\{(D_i', B_i, x_i', b_i)\}_{i=1}^m$.*

then for every $W_o, W_i \subseteq V(D) \setminus (\bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m B_i \cup O^*)$ with $|W_o|, |W_i| \leqslant w$, then D satisfies the following.

(1) *If $m \geqslant t_1 + t_2 + k$, then there is $E_{\text{conn}} \subseteq E(D)$ with $|E_{\text{conn}}| \leqslant 6w(m - t_1 - t_2)$ such that for every $S \subseteq V(D)$ with $|S| \leqslant k-1$, if $u \in W_o \setminus S$, then there is $t \in [m]$ such that $D - S$ contains a path from u to a_t only using edges in E_{conn}, and if $v \in W_i \setminus S$ then there is $t' \in [m]$ such that $D - S$ contains a path from $b_{t'}$ to v only using edges in E_{conn}.*

(2) *If $m > 2t_1 + 2t_2 + 3k + \overline{\Delta} - 2$, then D contains a k-hub*

$$\mathcal{H} := (E_{\text{hub}}, \{a_1, \dots, a_k\}, \{b_1, \dots, b_k\}, W_o, W_i)$$

with $|E_{\text{hub}}| \leqslant 2km + 6w(m - t_1 - t_2)$.

Proof. Since D is a digraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ and $|V(D)| \geqslant 10m$, there is $(\mathcal{A}, \mathcal{B}, O^*)$ such that

$$(\mathcal{A}, \mathcal{B}, O^*) \text{ is a } (t_1,\ t_2,\ d,\ m,\ u)\text{-trio in } D, \tag{5.6}$$

by Lemma 5.5, where $\mathcal{A}$ consists of m distinct 5-indominators $\{(D_i, A_i, x_i, a_i)\}_{i=1}^m$, $\mathcal{B}$ consists of m distinct 5-outdominators $\{(D_i', B_i, x_i', b_i)\}_{i=1}^m$, $|O^*| < (2mu)/t_1$ if $t_2 \geqslant \overline{\Delta}$ and otherwise

$$|O^*| \leqslant \frac{2mu}{t_1} + \frac{10\overline{\Delta}m}{t_2}.$$

Let $A := \bigcup_{i=1}^m A_i$ and $B := \bigcup_{i=1}^m B_i$. For $1 \leqslant i \leqslant m$, let

$$U_i^+ := \bigcap_{v \in A_i} N^+_{D_i}(v) \setminus \bigcup_{v \in A_i} N^-_{D_i}(v), \quad U_i^- := \bigcap_{v \in B_i} N^-_{D_i'}(v) \setminus \bigcup_{v \in B_i} N^+_{D_i'}(v).$$

For each $i \in [m]$, let $F_i^+ \subseteq V(D_i) \setminus A_i$ be the set of vertices in $V(D_i) \setminus A_i$ that are not indominated by A_i and not in U_i^+, and let $F_i^- \subseteq V(D_i') \setminus B_i$ be the set of vertices in

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

$V(D_i') \setminus B_i$ that are not outdominated by B_i and not in U_i^-. Since every vertex of D has at most $\overline{\Delta}$ non-neighbours and each $|A_i|, |B_i| \leqslant 5$ for $i \in [m]$, we have

$$|A|, |B| \leqslant 5m, \tag{5.7}$$

$$|F_i^+|, |F_i^-| \leqslant 5\overline{\Delta}. \tag{5.8}$$

Let W_o and W_i be any subsets of $V(D) \setminus (A \cup B \cup O^*)$ with $|W_o|, |W_i| \leqslant w$. For each $u \in W_o$, let $I_0^+(u)$ be the set of indices $i \in [m]$ such that A_i indominates u, and let $I_1^+(u) \subseteq [m] \setminus I_0^+(u)$ be the set of indices $i \in [m] \setminus I_0^+(u)$ such that $u \in U_i^+$ and $|U_i^+| \geqslant u$. Let $S^+(u) := \{a_i : i \in I_0^+(u) \cup I_1^+(u)\}$. By (T5), we have $|S^+(u)| \geqslant m - t_1 - t_2$. By removing some elements in $I_0^+(u)$ and $I_1^+(u)$, we may assume that

$$|S^+(u)| = m - t_1 - t_2. \tag{5.9}$$

Now we construct a $|S^+(u)|$-fan $\{P_{u,i}^+\}_{i \in I_0^+(u) \cup I_1^+(u)}$ from u to $S^+(u)$ as follows. For each $i \in I_0^+(u)$, since A_i indominates u we pick any vertex $u_i \in A_i \cap N_D^+(u)$. If $u_i \neq a_i$, then we can define $P_{u,i}^+$ to be the path (u, u_i, a_i) since $D[A_i]$ contains a spanning transitive tournament by (ID1) and (5.6). If $u_i = a_i$, then we define $P_{u,i}^+$ to be the path (u, a_i).

For each $i \in I_1^+(u)$, we have $d \geqslant 6m + 5\overline{\Delta}$ by the assumption of the lemma. By (T7), (5.7) and (5.8),

$$|N_{D_i}^+(u)| \geqslant d + |U_i^+| \geqslant 6m + 5\overline{\Delta} + |U_i^+| \geqslant m + |U_i^+| + |A| + |F_i^+|.$$

Thus we may choose $u_i \in N_{D_i}^+(u) \setminus (A \cup U_i^+ \cup F_i^+)$ for each $i \in I_1^+(u)$, so that $u_i \neq u_j$ for two distinct $i, j \in I_0^+(u) \cup I_1^+(u)$ as $|I_0^+(u) \cup I_1^+(u)| \leqslant m$.

For each $i \in I_1^+(u)$, $u_i \in V(D_i) \setminus (A_i \cup U_i^+ \cup F_i^+)$ by (T1). This shows that u_i is indominated by A_i in D_i and thus we can pick any $u_i' \in N_{D_i}^+(u_i) \cap A_i$. If $u_i' \neq a_i$, then we define $P_{u,i}^+$ to be the path (u, u_i, u_i', a_i), otherwise we define $P_{u,i}^+$ to be the path (u, u_i, a_i). Since $u_i \notin A$, $\{P_{u,i}^+\}_{i \in I_0^+(u) \cup I_1^+(u)}$ is an $(m - t_1 - t_2)$-fan from u to $S^+(u)$. Note that each path in the $|S^+(u)|$-fan is of length at most 3.

Similarly, for each $v \in W_i$, let $I_0^-(v)$ be the set of $i \in [m]$ such that B_i outdominates v, and let $I_1^-(v) := [m] \setminus I_0^-(v)$ be the set of indices $i \in [m] \setminus I_0^-(v)$ such that $v \in U_i^-$ and $|U_i^-| \geqslant u$. Let $S^-(v) := \{b_i : i \in I_0^-(v) \cup I_1^-(v)\}$. By (T6), we have $|S^-(v)| \geqslant m - t_1 - t_2$. By removing some elements in $I_0^-(v)$ and $I_1^-(v)$, we may assume that

$$|S^-(v)| = m - t_1 - t_2. \tag{5.10}$$

Now we construct a $|S^-(v)|$-fan $\{P_{v,i}^-\}_{i \in I_0^-(v) \cup I_1^-(v)}$ from $S^-(v)$ to v. For each $i \in I_0^-(v)$, since B_i outdominates u we pick any vertex $v_i \in B_i \cap N_D^-(u)$. If $v_i \neq b_i$, then we can define $P_{v,i}^-$ to be the path (b_i, v_i, u) since $D[B_i]$ contains a spanning transitive tournament by (OD1) and (5.6). If $v_i = b_i$, then we define $P_{v,i}^-$ to be the path (b_i, v).

For each $i \in I_1^-(v)$, we have $d \geqslant 6m + 5\overline{\Delta}$ by the assumption of the lemma. By (T8), (5.7) and (5.8),

$$|N_{D_i'}^-(v)| \geqslant d + |U_i^-| \geqslant 6m + 5\overline{\Delta} + |U_i^-| \geqslant m + |U_i^-| + |B| + |F_i^-|.$$

Thus we may choose $v_i \in N_{D_i'}^-(u) \setminus (B \cup U_i^- \cup F_i^-)$ for each $i \in I_1^-(v)$, so that $v_i \neq v_j$ for two distinct $i, j \in I_0^-(v) \cup I_1^-(v)$ as $|I_0^-(v) \cup I_1^-(v)| \leqslant m$.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

For each $i \in I_1^-(v)$, $v_i \in V(D_i') \setminus (B_i \cup U_i^- \cup F_i^-)$ by (T2). This shows that v_i is out-dominated by B_i in D_i' and thus we can pick any $v_i' \in N_{D_i'}^-(v_i) \cap B_i$. If $v_i' \neq b_i$, then we define $P_{v,i}^-$ to be the path (b_i, v_i', v_i, v), otherwise we define $P_{v,i}^-$ to be the path (b_i, v_i, v). Since $v_i \notin A \cup B$, $\{P_{v,i}^-\}_{i \in I_0^-(v) \cup I_1^-(v)}$ is an $(m - t_1 - t_2)$-fan from $S^-(v)$ to v. Note that each path in the $|S^-(v)|$-fan is of length at most 3.

Now we prove (1). For $m \geqslant t_1 + t_2 + k$, let us define

$$E_{\text{conn}} := \bigcup_{u \in W_o} \bigcup_{i \in S^+(u)} E(P_{u,i}^+) \cup \bigcup_{v \in W_i} \bigcup_{i \in S^-(v)} E(P_{v,i}^-).$$

By $|W_o|, |W_i| \leqslant w$, (5.9) and (5.10), we have

$$|E_{\text{conn}}| \leqslant 6w(m - t_1 - t_2). \tag{5.11}$$

For every $S \subseteq V(D)$ with $|S| \leqslant k - 1$, since for $u \in W_o$, $|S^+(u)| \geqslant m - t_1 - t_2 \geqslant k$ and for $v \in W_i$, $|S^-(v)| \geqslant m - t_1 - t_2 \geqslant k$, there are $t \in I_0^+(u) \cup I_1^+(u)$ with $V(P_{u,t}^+) \cap S = \emptyset$. Similarly, there is $t' \in I_0^-(v) \cup I_1^-(v)$ with $V(P_{v,t'}^-) \cap S = \emptyset$. This proves (1).

Now we prove (2). Let us assume that $m \geqslant 2t_1 + 2t_2 + 3k + \overline{\Delta} - 2$. Note that $m \geqslant t_1 + t_2 + k$ and thus (1) is satisfied. Let us define

$$E_{\text{hub}} := E_D(\{a_1, \ldots, a_k\}, \{a_1, \ldots, a_m\}) \cup E_D(\{b_1, \ldots, b_k\}, \{b_1, \ldots, b_m\}) \cup E_{\text{conn}}.$$

By (5.11), we have

$$|E_{\text{hub}}| \leqslant 2km + |E_{\text{conn}}| \leqslant 2km + 6w(m - t_1 - t_2).$$

We prove that $(E_{\text{hub}}, \{a_1, \ldots, a_k\}, \{b_1, \ldots, b_k\}, W_o, W_i)$ satisfies (H2). Let $S \subseteq V(D)$ be a set of at most $k - 1$ vertices. For $t \in [k]$ with $a_t \notin S$ and $u \in W_o \setminus S$, it follows that a_t has at least $(m - k - \overline{\Delta})/2$ in-neighbours in $D[\{a_1, \ldots, a_m\}]$ by (T4) and (5.6). There is a $|S^+(u)|$-fan from u to $S^+(u) \subseteq A_0$ and $|S^+(u)| = m - t_1 - t_2$ by (5.9); it follows that there are at least $m - t_1 - t_2 - k + 1$ i's with $i \in I_0^+(u) \cup I_1^+(u)$ and $V(P_{u,i}^+) \cap S = \emptyset$. Since $m > 2t_1 + 2t_2 + 3k + \overline{\Delta} - 2$ by the assumption of the lemma, we have

$$|N_{D[\{a_1,\ldots,a_m\}]}^-(a_t)| + |S^+(u)| - |S| \geqslant \frac{m - k - \overline{\Delta}}{2} + (m - t_1 - t_2) - (k - 1) > m$$

and by the pigeonhole principle, there is $i \in I_0^+(u) \cup I_1^+(u)$ with $V(P_{u,i}^+) \cap S = \emptyset$ and $a_i \in N_D^-(a_t)$. Then $P := P_{u,i}^+ \cup (a_i, a_t)$ is a path from u to a_t that does not intersect with S. Note that $E(P) \subseteq E_{\text{hub}}$, as $P_{u,i}^+ \subseteq E_{\text{hub}}$ and $a_i a_t \in E_{\text{hub}}$. The proof of (H3) is similar. □

The following lemma guarantees a k-arc-hub for dense digraphs under some conditions. Since the proof is almost identical to the proof of Lemma 5.13 except for a few parts, we only sketch the proof. The proof differs from the proof of Lemma 5.13 for two parts. For every $i \in I_1^+(u)$, we choose each $u_i \in N_{D_i}^+(u) \setminus (U_i^+ \cup F_i^+)$ which may be in A, since the paths in $|S^+(u)|$-fan are not necessarily vertex-disjoint. Similarly, for $i \in I_1^-(v)$, we choose $v_i \in N_{D_i'}^-(u) \setminus (U_i^- \cup F_i^-)$ which may be in B, since the paths in $|S^-(v)|$-fan are not necessarily vertex-disjoint. Therefore, we only need $d \geqslant m + 5\overline{\Delta}$ in the assumption. As the rest of the proof is identical, we omit the proof.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Lemma 5.14. *Let $d,k,m,t_1,t_2 \geqslant 1$, $\overline{\Delta}, w \geqslant 0$ be integers with $m \geqslant k$ $d \geqslant m + 5\overline{\Delta}$ and a real number $u \geqslant d/15$. Let D be a directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ and at least $10m$ vertices. If D contains a (t_1,t_2,d,m,u)-trio $(\mathcal{A},\mathcal{B},O^*)$ such that*

- *$(\mathcal{A},\mathcal{B},O^*)$ satisfies the assumptions in Lemma 5.5,*
- *$\mathcal{A}$ consists of 5-indominators $\{(D_i,A_i,x_i,a_i)\}_{i=1}^m$, and*
- *$\mathcal{B}$ consists of 5-outdominators $\{(D_i',B_i,x_i',b_i)\}_{i=1}^m$.*

then for any $W_o, W_i \subseteq V(D) \setminus (\bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m B_i \cup O^*)$ with $|W_o|,|W_i| \leqslant w$, then D satisfies the following.

(1) *If $m \geqslant t_1 + t_2 + k$, then there is $E_{\mathrm{conn}} \subseteq E(D)$ with $|E_{\mathrm{conn}}| \leqslant 6w(m - t_1 - t_2)$ such that for every $F \subseteq E(D)$ with $|F| \leqslant k-1$, if $u \in W_o$ then there is $t \in [m]$ such that $D - F$ contains a path from u to a_t only using edges in E_{conn}, and if $v \in W_i$ then there is $t' \in [m]$ such that $D - F$ contains a path from $b_{t'}$ to v only using edges in E_{conn}.*

(2) *If $m > 2t_1 + 2t_2 + 3k + \overline{\Delta} - 2$, then D contains a k-arc-hub*

$$\mathcal{H} := (E_{\mathrm{hub}}, \{a_1,\dots,a_k\}, \{b_1,\dots,b_k\}, W_o, W_i)$$

with $|E_{\mathrm{hub}}| \leqslant 2km + 6w(m - t_1 - t_2)$.

5.5. Absorbers

In this subsection, we consider objects called absorbers. Roughly speaking, even though we remove a few vertices from a digraph, we can connect vertices to a small set of vertices by a path in an absorber. This plays an important role in preserving the vertex-connectivity in a spanning subgraph, and finding sparse absorbers are directly related to finding highly connected sparse spanning subgraphs.

Definition 5.15. Let $k \geqslant 1$ be an integer and let D be a digraph. We say that a *k-absorber* is a 5-tuple $(E_{\mathrm{abs}}, V_{\mathrm{ex}}, \mathcal{P}, W_i, W_o)$ that consists of a set $E_{\mathrm{abs}} \subseteq E(D)$, a set $V_{\mathrm{ex}} \subseteq V(D)$, a collection $\mathcal{P} = \{P_i\}_{i=1}^k$ of k vertex-disjoint paths, and sets $W_i, W_o \subseteq V(D)$ satisfying the following.

(A1) For every $t \in [k]$, both endvertices of P_t are in V_{ex}.
(A2) $\bigcup_{t=1}^k E(P_t) \subseteq E_{\mathrm{abs}}$.
(A3) For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $u \in V(D) \setminus S$, the subgraph $D - S$ has a path from u to a vertex in $W_o \setminus S$ only using edges in E_{abs}.
(A4) For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $v \in V(D) \setminus S$, the subgraph $D - S$ has a path from a vertex in $W_i \setminus S$ to v only using edges in E_{abs}.

We also define an edge version of absorbers.

Definition 5.16. Let $k \geqslant 1$ be an integer and let D be a directed multigraph. A *k-arc-absorber* is a 5-tuple $(E_{\mathrm{abs}}, V_{\mathrm{ex}}, \mathcal{P}, W_i, W_o)$ that consists of a set $E_{\mathrm{abs}} \subseteq E(D)$, a set $V_{\mathrm{ex}} \subseteq V(D)$, a collection $\mathcal{P} = \{P_i\}_{i=1}^k$ of k edge-disjoint paths, and sets $W_i, W_o \subseteq V(D)$ satisfying the following.

(A1′) For each $t \in [t]$, both endvertices of P_t are in V_{ex}.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

(A2′) $\bigcup_{t=1}^{k} E(P_t) \subseteq E_{\text{abs}}$.
(A3′) For every $F \subseteq E(D)$ with $|F| \leqslant k-1$ and $u \in V(D)$, the subgraph $D - F$ has a path from u to a vertex in W_o using only edges in E_{abs}.
(A4′) For every $F \subseteq E(D)$ with $|F| \leqslant k-1$ and $v \in V(D)$, the subgraph $D - F$ has a path from a vertex in W_i to v using only edges in E_{abs}.

The following lemma guarantees the existence of a k-absorber that uses only few edges in dense digraphs.

Lemma 5.17. *Let $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$ be integers, and let D be a strongly k-connected n-vertex digraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. Let $V_{\text{ex}} \subseteq V(D)$ with $|V(D) \setminus V_{\text{ex}}| \geqslant 39k + 38\overline{\Delta}$, and $\mathcal{P}$ be a collection of k vertex-disjoint paths $\{P_1, \ldots, P_k\}$ such that P_i is a minimal path with endvertices in V_{ex} for every $i \in [k]$.*

Then D has a k-absorber $\mathcal{D} = (E_{\text{abs}}, V_{\text{ex}}, \mathcal{P}, W_i, W_o)$ satisfying the following.

(1) *$W_i, W_o \subseteq V(D) \setminus V_{\text{ex}}$ and $|W_i|, |W_o| = 3k$.*
(2) *$|E_{\text{abs}}| \leqslant kn + 226k(k + \overline{\Delta}) + 38(k + \overline{\Delta}) + (5k+1)|V_{\text{ex}}|$.*

Proof. For $t \in [k]$, let us define $E_{\text{path}} := \bigcup_{t=1}^{k} E(P_t)$ and $D' := D - V_{\text{ex}}$.

Since $|V(D')| \geqslant 39k + 38\overline{\Delta} \geqslant 10 \cdot 3k$, by applying Lemma 5.5 to D' we deduce that

$$\text{there is a } \left(k,\ k,\ 18k + 5\overline{\Delta},\ 3k,\ \frac{18k + 5\overline{\Delta}}{15}\right)\text{-trio } (\mathcal{A}', \mathcal{B}', S^*) \text{ in } D', \tag{5.12}$$

where $\mathcal{A}'$ consists of $3k$ distinct 5-indominators $\{(D_i, A_i', y_i, a_i')\}_{i=1}^{3k}$, $\mathcal{B}$ consists of $3k$ distinct 5-outdominators $\{(D_i', B_i', y_i', b_i')\}_{i=1}^{3k}$, and $|S^*| \leqslant 8k + 32\overline{\Delta}$.

Let us define

$$A' := \bigcup_{i=1}^{3k} A_i', \quad B' := \bigcup_{i=1}^{3k} B_i', \quad V_{\text{ex}}' := V_{\text{ex}} \cup A' \cup B' \cup S^*,$$

and

$$V_i^+ := \bigcap_{v \in A_i'} N_{D_i}^+(v) \setminus \bigcup_{v \in A_i'} N_{D_i}^-(v), \quad V_i^- := \bigcap_{v \in B_i'} N_{D_i'}^-(v) \setminus \bigcup_{v \in B_i'} N_{D_i'}^+(v)$$

for every $i \in [3k]$.

Since $|A'|, |B'| \leqslant 5 \cdot 3k$, it follows that

$$|A' \cup B'| \leqslant 30k, \tag{5.13}$$
$$|A' \cup B' \cup S^*| \leqslant 38(k + \overline{\Delta}), \tag{5.14}$$
$$|V_{\text{ex}}'| \leqslant |V_{\text{ex}}| + 38(k + \overline{\Delta}). \tag{5.15}$$

Since

$$\begin{aligned} |V(D) \setminus V_{\text{ex}}'| &\geqslant |V(D) \setminus V_{\text{ex}}| - |A' \cup B' \cup S^*| \\ &\geqslant 39k + 38\overline{\Delta} - 38(k + \overline{\Delta}) \geqslant k, \end{aligned}$$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

by applying Lemma 5.8 to a set V'_{ex}, there is a k-escaper $(E_{\text{escape}}, V'_{\text{ex}}, V_{\text{out}})$ with $V_{\text{out}} \subseteq V(D) \setminus V'_{\text{ex}}$ such that

$$|V_{\text{out}}| \leqslant 2k|V'_{\text{ex}}| \leqslant 2k|V_{\text{ex}}| + 76k(k + \overline{\Delta}), \tag{5.16}$$

$$|E_{\text{escape}}| \leqslant 4k|V'_{\text{ex}}| \leqslant 4k|V_{\text{ex}}| + 152k(k + \overline{\Delta}). \tag{5.17}$$

Let us define

$$X'_1 := \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \setminus (V'_{\text{ex}} \cup V_{\text{out}}), \tag{5.18}$$

$$X_1 := V(D) \setminus (V'_{\text{ex}} \cup V_{\text{out}} \cup X'_1). \tag{5.19}$$

Claim 4. *There exist sets $U_i^0, U_o^0 \subseteq V_{\text{out}}$, a set $E_0 \subseteq E(D)$, sets $U_i^1, U_o^1 \subseteq X_1$, a set $E_1 \subseteq E(D)$, sets $U_i'^1, U_o'^1 \subseteq X'_1$ and a set $E'_1 \subseteq E(D)$ satisfying the following.*

(1) $|E_0| \leqslant k|V_{\text{out}}| - k + k\overline{\Delta}$.

(2) *There are $U_i^0, U_o^0 \subseteq V_{\text{out}}$ such that $|U_i^0|, |U_o^0| \leqslant 2k + \overline{\Delta} - 1$, and for every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and for every $u, v \in V_{\text{out}} \setminus S$, the subgraph $D - S$ has a path from u to a vertex in $U_o^0 \setminus S$, and a path from a vertex in $U_i^0 \setminus S$ to v such that both paths only use edges in E_0.*

(3) $|E_1| \leqslant k|X_1| - k + k\overline{\Delta}$.

(4) *There are $U_i^1, U_o^1 \subseteq X_1$ such that $|U_i^1|, |U_o^1| \leqslant 2k + \overline{\Delta} - 1$, and for every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and for every $u, v \in X_1 \setminus S$, the subgraph $D - S$ has a path from u to a vertex in $U_o^1 \setminus S$, and a path from a vertex in $U_i^1 \setminus S$ to v such that both paths only use edges in E_1.*

(5) $|E'_1| \leqslant (k-1)|X'_1| + (\overline{\Delta} + 1)(k-1)$.

(6) *There are $U_i'^1, U_o'^1 \subseteq X'_1$ such that $|U_i'^1|, |U_o'^1| \leqslant 2k + \overline{\Delta} - 1$, and for every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and for every $u, v \in X'_1 \setminus S$, the subgraph $D - S$ has a path from u to a vertex in $(U_o'^1 \cup V_{\text{ex}}) \setminus S$, and a path from a vertex in $(U_i'^1 \cup V_{\text{ex}}) \setminus S$ to v such that both paths only use edges in $E_{\text{path}} \cup E'_1$.*

Proof of Claim 4. By applying Lemma 2.5 to $D[V_{\text{out}}]$ and $D[X_1]$, (1), (2), (3) and (4) follow. Similarly, applying Lemma 2.6 to $D[X'_1]$, (5) and (6) follow. □

Let us define

$$U_o := U_o^0 \cup U_o^1 \cup U_o'^1, \quad U_i := U_i^0 \cup U_i^1 \cup U_i'^1. \tag{5.20}$$

Then $|U_i|, |U_o| \leqslant 3(2k + \overline{\Delta})$.

Claim 5. *There is a set $E_{\text{conn}} \subseteq E(D')$ of edges satisfying the following.*

(1) $|E_{\text{conn}}| \leqslant 18k(2k + \overline{\Delta})$.

(2) *For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $u \in U_o \setminus S$, there is $t \in [3k]$ such that $D' - S$ contains a path from u to a'_t, only using edges in E_{conn}.*

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

(3) *For every $S \subseteq V(D)$ with $|S| \leqslant k-1$ and $v \in U_i \setminus S$, there is $t \in [3k]$ such that $D' - S$ contains a path from b'_t to v, only using edges in E_{conn}.*

Proof. Note that $U_o, U_i \subseteq V(D) \setminus V'_{\text{ex}} \subseteq V(D')$. By (5.12), $(\mathcal{A}', \mathcal{B}', S^*)$ satisfies the requirements of Lemma 5.13, hence the claim follows by (1) of Lemma 5.13. □

Now let us define

$$E_{\text{abs}} := E_{\text{path}} \cup E_{\text{escape}} \cup E_0 \cup E_1 \cup E'_1 \cup E_{\text{conn}}, \tag{5.21}$$
$$W_o := \{a'_1, \dots, a'_{3k}\}, \tag{5.22}$$
$$W_i := \{b'_1, \dots, b'_{3k}\}. \tag{5.23}$$

Then $W_o, W_i \subseteq V(D') = V(D) \setminus V_{\text{ex}}$. Since $\bigcup_{t=1}^{k} \text{Int}(P_t) \subseteq V'_{\text{ex}} \cup V_{\text{out}} \cup X'_1$, we have $|E_{\text{path}}| \leqslant |V'_{\text{ex}}| + |V_{\text{out}}| + |X'_1| + k$ by (5.15).

Note that $V(D) = V'_{\text{ex}} \cup V_{\text{out}} \cup X_1 \cup X'_1$ by (5.19). By (5.15), (5.16), (5.17), Claim 4, Claim 5, and $V(D) = V'_{\text{ex}} \cup V_{\text{out}} \cup X_1 \cup X'_1$ we have

$$\begin{aligned}
|E_{\text{abs}}| &\leqslant |E_{\text{escape}}| + |E_{\text{path}}| + |E_0| + |E_1| + |E'_1| + |E_{\text{conn}}| \\
&\leqslant 4k|V'_{\text{ex}}| + (|V'_{\text{ex}}| + |V_{\text{out}}| + |X'_1| + k) + (k|V_{\text{out}}| - k + k\overline{\Delta}) + (k|X_1| - k + k\overline{\Delta}) \\
&\quad + ((k-1)|X'_1| + k - 1 + k\overline{\Delta}) + 18k(2k + \overline{\Delta}) \\
&\leqslant k(|V'_{\text{ex}}| + |V_{\text{out}}| + |X_1| + |X'_1|) + (3k+1)|V'_{\text{ex}}| + |V_{\text{out}}| + 3k\overline{\Delta} + 18k(2k + \overline{\Delta}) \\
&\leqslant kn + (3k+1)|V_{\text{ex}}| + 114k(k + \overline{\Delta}) + 38(k + \overline{\Delta}) + |V_{\text{out}}| + 36k(k + \overline{\Delta}) \\
&\leqslant kn + 226k(k + \overline{\Delta}) + 38(k + \overline{\Delta}) + (5k+1)|V_{\text{ex}}|.
\end{aligned} \tag{5.24}$$

Let us define

$$\mathcal{D} := (E_{\text{abs}}, V_{\text{ex}}, \mathcal{P}, W_i, W_o).$$

Claim 6. *$\mathcal{D}$ is a k-absorber in D.*

Proof. Both (A1) and (A2) are clear. Let $S \subseteq V(D)$ with $|S| \leqslant k-1$, and $u, v \in V(D) \setminus S$ be two distinct vertices.

(a) If $u \in V'_{\text{ex}}$, then since $(E_{\text{escape}}, V'_{\text{ex}}, V_{\text{out}})$ is a k-escaper, there is a path from u to $u' \in V_{\text{out}}$ in $D - S$ using only edges in E_{escape}, and there is a path from u' to a vertex $u'' \in U_o$ in $D - S$ only using edges in E_0 by Claim 4. By Claim 5, there is a path from u'' to a vertex $u^+ \in W_o$ in $D - S$ only using edges in E_{conn}.

(b) If $u \in X'_1$, then there is a path from u to $u' \in U_o \cup V_{\text{ex}}$ in $D - S$ only using edges in $E_{\text{path}} \cup E'_1$ by Claim 4. If $u' \in U_o$, then there is a path from u' to a vertex $u^+ \in W_o$ in $D - S$ only using edges in E_{conn} by Claim 5. Otherwise if $u' \in V_{\text{ex}} \setminus S$, then there is a path from u' to a vertex $u^+ \in W_o$ in $D - S$ only using edges in E_{abs} by (a).

(c) If $u \in V_{\text{out}} \cup X_1$, then there is a path from u to a vertex $u' \in U_o$ in $D - S$ using only edges in $E_0 \cup E_1$ by Claim 4. By Claim 5, there is a path from u' to a vertex $u^+ \in W_o$ in $D - S$ only using edges in E_{conn}.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Hence there is a path in $D-S$ from u to $u^+ \in W_o$ only using edges in E_{abs}, proving (A3). Similarly, there is a path in $D-S$ from a vertex $v^+ \in W_i$ to v only using edges in E_{abs}, proving (A4). This proves the claim. □

By Claim 6 and (5.24), this completes the proof of the lemma. □

Similarly, the following lemma guarantees the existence of a k-arc-absorber that uses only a few edges in dense digraphs.

Lemma 5.18. *Let $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$ be integers, and let D be a strongly k-connected n-vertex directed multigraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. Let $V_{\text{ex}} \subseteq V(D)$ with $|V(D) \setminus V_{\text{ex}}| \geqslant 33k + 32\overline{\Delta}$, and let $\mathcal{P}$ be a collection of k edge-disjoint paths $\{P_1, \ldots, P_k\}$ such that P_i is a path with endvertices in V_{ex} for every $i \in [k]$.*

Then D has a k-arc-absorber $\mathcal{D} = (E_{\text{abs}}, V_{\text{ex}}, \mathcal{P}, W_i, W_o)$ satisfying the following.

(1) $W_i, W_o \subseteq V(D) \setminus V_{\text{ex}}$ *and* $|W_i|, |W_o| = 3k$.

(2) $|E_{\text{abs}}| \leqslant kn + 210k(k + \overline{\Delta}) + 32(k + \overline{\Delta}) + (5k+1)|V_{\text{ex}}|$.

Proof. For $t \in [k]$, let us define $E_{\text{path}} := \bigcup_{t=1}^{k} E(P_t)$ and $D' := D - V_{\text{ex}}$.

Since $|V(D')| \geqslant 33k + 32\overline{\Delta} \geqslant 10 \cdot 3k$, by applying Lemma 5.5 to D' we deduce that

$$\text{there is a } \left(k,\ k,\ 3k + 5\overline{\Delta},\ 3k,\ \frac{3k + 5\overline{\Delta}}{15}\right)\text{-trio } (\mathcal{A}', \mathcal{B}', S^*) \text{ in } D', \tag{5.25}$$

where $\mathcal{A}'$ consists of $3k$ distinct 5-indominators $\{(D_i, A_i', y_i, a_i')\}_{i=1}^{3k}$, $\mathcal{B}$ consists of $3k$ distinct 5-outdominators $\{(D_i', B_i', y_i', b_i')\}_{i=1}^{3k}$, and $|S^*| \leqslant 1.2k + 32\overline{\Delta}$.

Let us define

$$A' := \bigcup_{i=1}^{3k} A_i', \quad B' := \bigcup_{i=1}^{3k} B_i', \quad V_{\text{ex}}' := V_{\text{ex}} \cup A' \cup B' \cup S^*,$$

and

$$V_i^+ := \bigcap_{v \in A_i'} N_{D_i}^+(v) \setminus \bigcup_{v \in A_i'} N_{D_i}^-(v), \quad V_i^- := \bigcap_{v \in B_i'} N_{D_i'}^-(v) \setminus \bigcup_{v \in B_i'} N_{D_i'}^+(v)$$

for every $i \in [3k]$.

Since $|A'|, |B'| \leqslant 5 \cdot 3k$, it follows that

$$|A' \cup B'| \leqslant 30k, \tag{5.26}$$

$$|A' \cup B' \cup S^*| \leqslant 32(k + \overline{\Delta}), \tag{5.27}$$

$$|V_{\text{ex}}'| \leqslant |V_{\text{ex}}| + 32(k + \overline{\Delta}). \tag{5.28}$$

The rest of the proof is almost identical to the proof of Lemma 5.17, except for a few parts. We use Lemma 5.10 for k-arc-escapers instead of Lemma 5.8 for k-escapers. Since the paths in $\{P_1, \ldots, P_k\}$ are edge-disjoint and each P_i is not necessarily minimal, we use Lemmas 2.7 and 2.8 instead of Lemmas 2.5 and 2.6 respectively. Note that Lemma 2.8 has a slightly worse bound than Lemma 2.6. Finally, we replace Lemma 5.13 by Lemma 5.14 in the proof of Claim 5.

Since

$$|V(D) \setminus V'_{\text{ex}}| \geqslant |V(D) \setminus V_{\text{ex}}| - |A' \cup B' \cup S^*|$$
$$\geqslant 33k + 32\overline{\Delta} - 32(k + \overline{\Delta}) \geqslant k,$$

by applying Lemma 5.10 to a set V'_{ex}, there is a k-arc-escaper $(E_{\text{escape}}, V'_{\text{ex}}, V_{\text{out}})$ with $V_{\text{out}} \subseteq V(D) \setminus V'_{\text{ex}}$ such that

$$|V_{\text{out}}| \leqslant 2k|V'_{\text{ex}}| \leqslant 2k|V_{\text{ex}}| + 64k(k + \overline{\Delta}), \tag{5.29}$$
$$|E_{\text{escape}}| \leqslant 4k|V'_{\text{ex}}| \leqslant 4k|V_{\text{ex}}| + 128k(k + \overline{\Delta}). \tag{5.30}$$

Let us define

$$X'_1 := \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \setminus (V'_{\text{ex}} \cup V_{\text{out}}), \tag{5.31}$$
$$X_1 := V(D) \setminus (V'_{\text{ex}} \cup V_{\text{out}} \cup X'_1). \tag{5.32}$$

Claim 7. *There exist sets $U_i^0, U_o^0 \subseteq V_{\text{out}}$, a set $E_0 \subseteq E(D)$, sets $U_i^1, U_o^1 \subseteq X_1$, a set $E_1 \subseteq E(D)$, sets $U_i'^1, U_o'^1 \subseteq X'_1$ and a set $E'_1 \subseteq E(D)$ satisfying the following.*

(1) $|E_0| \leqslant k|V_{\text{out}}| - k + k\overline{\Delta}$.

(2) *There are $U_i^0, U_o^0 \subseteq V_{\text{out}}$ such that $|U_i^0|, |U_o^0| \leqslant 2k + \overline{\Delta} - 1$ and for every $F \subseteq E(D)$ with $|S| \leqslant k - 1$ and for every $u, v \in V_{\text{out}}$, the subgraph $D - F$ has a path from u to a vertex in U_o^0, and a path from a vertex in U_i^0 to v such that both paths only use edges in E_0.*

(3) $|E_1| \leqslant k|X_1| - k + k\overline{\Delta}$.

(4) *There are $U_i^1, U_o^1 \subseteq X_1$ such that $|U_i^1|, |U_o^1| \leqslant 2k + \overline{\Delta} - 1$ and for every $F \subseteq E(D)$ with $|F| \leqslant k - 1$ and for every $u, v \in X_1$, the subgraph $D - F$ has a path from u to a vertex in U_o^1, and a path from a vertex in U_i^1 to v such that both paths only use edges in E_1.*

(5) $|E'_1| \leqslant (k - 1)|X'_1| + (k - 1)(\overline{\Delta} + 2k - 1)$.

(6) *There are $U_i'^1, U_o'^1 \subseteq X'_1$ such that $|U_i'^1|, |U_o'^1| \leqslant 4k + \overline{\Delta} - 3$ and for every $F \subseteq E(D)$ with $|F| \leqslant k - 1$ and for every $u, v \in X'_1 \setminus S$, the subgraph $D - F$ has a path from u to a vertex in $U_o'^1 \cup V_{\text{ex}}$, and a path from a vertex in $U_i'^1 \cup V_{\text{ex}}$ to v such that both paths only use edges in $E_{\text{path}} \cup E'_1$.*

Proof of Claim 7. By applying Lemma 2.7 to $D[V_{\text{out}}]$ and $D[X_1]$, (1), (2), (3) and (4) follow. Similarly, applying Lemma 2.8 to $D[X'_1]$, (5) and (6) follow. □

Let us define

$$U_o := U_o^0 \cup U_o^1 \cup U_o'^1, \quad U_i := U_i^0 \cup U_i^1 \cup U_i'^1. \tag{5.33}$$

Then $|U_i|, |U_o| \leqslant 8k + 3\overline{\Delta}$.

Claim 8. *There is a set $E_{\text{conn}} \subseteq E(D')$ of edges satisfying the following.*

(1) $|E_{\text{conn}}| \leqslant 6k(8k + 3\overline{\Delta})$.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

(2) *For every* $S \subseteq V(D)$ *with* $|S| \leqslant k-1$ *and* $u \in U_o \setminus S$, *there is* $t \in [3k]$ *such that* $D' - S$ *contains a path from* u *to* a'_t, *only using edges in* E_{conn}.

(3) *For every* $S \subseteq V(D)$ *with* $|S| \leqslant k-1$ *and* $v \in U_i \setminus S$, *there is* $t \in [3k]$ *such that* $D' - S$ *contains a path from* b'_t *to* v, *only using edges in* E_{conn}.

Proof. Note that $U_o, U_i \subseteq V(D) \setminus V'_{\text{ex}} \subseteq V(D')$. By (5.25), $(\mathcal{A}', \mathcal{B}', S^*)$ satisfies the requirements of Lemma 5.14, hence the claim follows by (1) of Lemma 5.14. □

Now let us define

$$E_{\text{abs}} := E_{\text{path}} \cup E_{\text{escape}} \cup E_0 \cup E_1 \cup E'_1 \cup E_{\text{conn}}, \tag{5.34}$$

$$W_o := \{a'_1, \ldots, a'_{3k}\}, \tag{5.35}$$

$$W_i := \{b'_1, \ldots, b'_{3k}\}. \tag{5.36}$$

Then $W_o, W_i \subseteq V(D') = V(D) \setminus V_{\text{ex}}$. Since $\bigcup_{t=1}^{k} \text{Int}(P_t) \subseteq V'_{\text{ex}} \cup V_{\text{out}} \cup X'_1$, we have $|E_{\text{path}}| \leqslant |V'_{\text{ex}}| + |V_{\text{out}}| + |X'_1| + k$ by (5.28).

Note that $V(D) = V'_{\text{ex}} \cup V_{\text{out}} \cup X_1 \cup X'_1$ by (5.32). By (5.28), (5.29), (5.30), Claim 7, Claim 8 and $V(D) = V'_{\text{ex}} \cup V_{\text{out}} \cup X_1 \cup X'_1$, we have

$$\begin{aligned}
|E_{\text{abs}}| &\leqslant |E_{\text{escape}}| + |E_{\text{path}}| + |E_0| + |E_1| + |E'_1| + |E_{\text{conn}}| \\
&\leqslant 4k|V'_{\text{ex}}| + (|V'_{\text{ex}}| + |V_{\text{out}}| + |X'_1| + k) + (k|V_{\text{out}}| - k + k\overline{\Delta}) + (k|X_1| - k + k\overline{\Delta}) \\
&\qquad + ((k-1)|X'_1| + (k-1)\overline{\Delta} + 2k^2 - 3k + 1) + 6k(8k + 3\overline{\Delta}) \\
&\leqslant k(|V'_{\text{ex}}| + |V_{\text{out}}| + |X_1| + |X'_1|) + (3k+1)|V'_{\text{ex}}| + |V_{\text{out}}| + 50k^2 + 21k\overline{\Delta} \\
&\leqslant kn + (3k+1)|V_{\text{ex}}| + 96k(k + \overline{\Delta}) + 32(k + \overline{\Delta}) + |V_{\text{out}}| + 50k^2 + 21k\overline{\Delta} \\
&\leqslant kn + (210k^2 + 181k\overline{\Delta}) + 32(k + \overline{\Delta}) + (5k+1)|V_{\text{ex}}|.
\end{aligned} \tag{5.37}$$

Let us define

$$\mathcal{D} := (E_{\text{abs}}, V_{\text{ex}}, \mathcal{P}, W_i, W_o).$$

Claim 9. $\mathcal{D}$ *is a* k*-arc-absorber in* D.

Proof. Both (A1′) and (A2′) are clear. Let $F \subseteq E(D)$ with $|F| \leqslant k-1$, and $u, v \in V(D)$ be two distinct vertices.

(a) If $u \in V'_{\text{ex}}$, then since $(E_{\text{escape}}, V'_{\text{ex}}, V_{\text{out}})$ is a k-arc-escaper, there is a path from u to $u' \in V_{\text{out}}$ in $D - F$ using only edges in E_{escape}, and there is a path from u' to a vertex $u'' \in U_o$ in $D - F$ only using edges in E_0 by Claim 7. By Claim 8, there is a path from u'' to a vertex $u^+ \in W_o$ in $D - F$ only using edges in E_{conn}.

(b) If $u \in X'_1$, then there is a path from u to $u' \in U_o \cup V_{\text{ex}}$ in $D - F$ only using edges in $E_{\text{path}} \cup E'_1$ by Claim 7. If $u' \in U_o$, then there is a path from u' to a vertex $u^+ \in W_o$ in $D - F$ only using edges in E_{conn} by Claim 8. Otherwise, if $u' \in V_{\text{ex}}$, then there is a path from u' to a vertex $u^+ \in W_o$ in $D - F$ only using edges in E_{abs} by (a).

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

(c) If $u \in V_{\mathrm{out}} \cup X_1$, then there is a path from u to a vertex $u' \in U_o$ in $D - F$ using only edges in $E_0 \cup E_1$ by Claim 7. By Claim 8, there is a path from u' to a vertex $u^+ \in W_o$ in $D - F$ only using edges in E_{conn}.

Hence there is a path in $D - F$ from u to $u^+ \in W_o$ only using edges in E_{abs}, proving (A3′). Similarly, there is a path in $D - F$ from a vertex $v^+ \in W_i$ to v only using edges in E_{abs}, proving (A4′). □

By Claim 9 and (5.37), this completes the proof of the lemma. □

6. Proof of the main result

We divide Theorem 1.3 into two parts as follows. First of all, the following theorem establishes the upper bound of the minimum number of edges in a strongly k-connected spanning subgraph.

Theorem 6.1. *For all integers $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, every strongly k-connected n-vertex digraph D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ contains a strongly k-connected spanning subgraph with at most $kn + 790k\overline{\Delta} + 790k^2$ edges.*

Secondly, the following theorem establishes the upper bound of the minimum number of edges in a strongly k-arc-connected spanning subgraph.

Theorem 6.2. *For all integers $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, every strongly k-arc-connected n-vertex directed multigraph D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ contains a strongly k-arc-connected spanning subgraph with at most $kn + 666k\overline{\Delta} + 666k^2$ edges.*

Both Theorems 6.1 and 6.2 prove Theorem 1.3. Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let D be a strongly k-connected n-vertex digraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. For $n < 4k + 3$, we have

$$|E(D)| \leqslant 2\binom{n}{2} < 16k^2 + 20k + 6 \leqslant 790k(k + \overline{\Delta}).$$

For $4k + 3 \leqslant n < 200(k + \overline{\Delta})$, let D' be a minimally strongly k-connected spanning subgraph of D. By the result of Mader [17], we have $|E(D')| \leqslant 2kn \leqslant 400k(k + \overline{\Delta}) \leqslant 790k(k + \overline{\Delta})$.

We may assume that $n \geqslant 200(k + \overline{\Delta})$. By Lemma 5.5, D contains a 3-tuple $(\mathcal{A}, \mathcal{B}, O^*)$ such that

$$(\mathcal{A}, \mathcal{B}, O^*) \text{ is a } \left(k + \overline{\Delta},\ \overline{\Delta},\ 30k + 35\overline{\Delta},\ 5(k + \overline{\Delta}),\ \frac{7(k + \overline{\Delta})}{3}\right)\text{-trio,} \tag{6.1}$$

where $\mathcal{A}$ consists of $5(k + \overline{\Delta})$ distinct 5-indominators $\{(D_i, A_i, x_i, a_i)\}_{i=1}^{5(k+\overline{\Delta})}$, $\mathcal{B}$ consists of $5(k + \overline{\Delta})$ distinct 5-outdominators $\{(D'_i, B_i, x'_i, b_i)\}_{i=1}^{5(k+\overline{\Delta})}$, and $|O^*| \leqslant 24(k + \overline{\Delta})$.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Let $A := \bigcup_{i=1}^{5(k+\overline{\Delta})} A_i$ and $B := \bigcup_{i=1}^{5(k+\overline{\Delta})} B_i$. For $i \in [5(k+\overline{\Delta})]$, let

$$U_i^+ := \bigcap_{v\in A_i} N_{D_i}^+(v) \setminus \bigcup_{v\in A_i} N_{D_i}^-(v) \quad \text{and} \quad U_i^- := \bigcap_{v\in B_i} N_{D_i'}^-(v) \setminus \bigcup_{v\in B_i} N_{D_i'}^+(v).$$

Since $|A|, |B| \leqslant 5\cdot 5(k+\overline{\Delta})$ and $|O^*| < 24(k+\overline{\Delta})$, it follows that

$$|A\cup B\cup O^*| \leqslant 74(k+\overline{\Delta}).$$

By Menger's theorem, let $P_1,\dots,P_k$ be k vertex-disjoint paths from $\{a_1,\dots,a_k\}$ to $\{b_1,\dots,b_k\}$ such that there is a permutation $\sigma : [k] \to [k]$, and for $i\in[k]$, P_i is a path from a_i to $b_{\sigma(i)}$. Without loss of generality, we may assume that P_i is a minimal path from a_i to $b_{\sigma(i)}$ for $i\in[k]$. Let $\mathcal{P} := \{P_1,\dots,P_k\}$.

Since $|V(D)| - |A\cup B\cup O^*| \geqslant 200(k+\overline{\Delta}) - 74(k+\overline{\Delta}) \geqslant 39k + 38\overline{\Delta}$, we apply Lemma 5.17 so that D contains a k-absorber

$$\mathcal{D} := (E_{\text{abs}}, A\cup B\cup O^*, \mathcal{P}, W_i, W_o)$$

with $W_i, W_o \subseteq V(D)\setminus(A\cup B\cup O^*)$, $|W_i|, |W_o| = 3k$, and

$$\begin{aligned}|E_{\text{abs}}| &\leqslant kn + 226k(k+\overline{\Delta}) + 38(k+\overline{\Delta}) + (5k+1)|A\cup B\cup O^*| \\ &\leqslant kn + 596k(k+\overline{\Delta}) + 112(k+\overline{\Delta}),\end{aligned} \tag{6.2}$$

since $|A\cup B\cup O^*| \leqslant 74(k+\overline{\Delta})$.

Since $W_i, W_o \subseteq V(D)\setminus(A\cup B\cup O^*)$ with $|W_i|, |W_o| = 3k$ and (6.1), we apply Lemma 5.13 with $3k$ playing the role of w. By (2) of Lemma 5.13, D has a k-hub

$$\mathcal{H} := (E_{\text{hub}}, \{a_1,\dots,a_k\}, \{b_1,\dots,b_k\}, W_o, W_i)$$

such that

$$|E_{\text{hub}}| \leqslant 82k(k+\overline{\Delta}). \tag{6.3}$$

Let $E_L := E_{\text{abs}} \cup E_{\text{hub}}$. By (6.2) and (6.3),

$$\begin{aligned}|E_L| &\leqslant |E_{\text{abs}}| + |E_{\text{hub}}| \\ &\leqslant kn + 596k(k+\overline{\Delta}) + 82k(k+\overline{\Delta}) + 112(k+\overline{\Delta}) \\ &\leqslant kn + 678k(k+\overline{\Delta}) + 112(k+\overline{\Delta}) \\ &\leqslant kn + 790k(k+\overline{\Delta}).\end{aligned}$$

Let $D' := (V(D), E_L)$ be a spanning subgraph of D. Now it remains to prove that D' is strongly k-connected. Let $S\subseteq V(D')$ with $|S|\leqslant k-1$ and $u,v\in V(D')\setminus S$. Let $i\in[k]$ be an index such that $V(P_i)\cap S = \emptyset$. If $u\in W_o$, then $u' := u$. Otherwise, $D'-S$ contains a path from u to a vertex $u'\in W_o\setminus S$ since $\mathcal{D}$ is a k-absorber in D. Since $\mathcal{H}$ is a k-hub, $D'-S$ contains a path from u' to a_i, showing that $D'-S$ contains a path from u to a_i. Similarly, $D'-S$ contains a path from $b_{\sigma(i)}$ to v. Connecting from a_i to $b_{\sigma(i)}$ by P_i, we deduce that $D'-S$ contains a path from u to v, as desired. □

Now we prove Theorem 6.2, and the proof is analogous to the proof of Theorem 6.1.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Proof of Theorem 6.2. Let D be a strongly k-arc-connected n-vertex digraph with $\overline{\Delta}(D) \leqslant \overline{\Delta}$. For $n < 100(k+\overline{\Delta})$, let D' be a minimally strongly k-arc-connected spanning subgraph of D. By the result of Dalmazzo [9], we have $|E(D')| \leqslant 2kn \leqslant 200k(k+\overline{\Delta}) \leqslant 666k(k+\overline{\Delta})$.

We may assume that $n \geqslant 100(k+\overline{\Delta})$. By Lemma 5.5, D contains a 3-tuple $(\mathcal{A}, \mathcal{B}, O^*)$ such that

$$(\mathcal{A}, \mathcal{B}, O^*) \text{ is a } \left(k+\overline{\Delta},\ \overline{\Delta},\ 5k+10\overline{\Delta},\ 5(k+\overline{\Delta}),\ \frac{k+2\overline{\Delta}}{3}\right)\text{-trio,} \tag{6.4}$$

where $\mathcal{A}$ consists of $5(k+\overline{\Delta})$ distinct 5-indominators $\{(D_i, A_i, x_i, a_i)\}_{i=1}^{5(k+\overline{\Delta})}$, $\mathcal{B}$ consists of $5(k+\overline{\Delta})$ distinct 5-outdominators $\{(D'_i, B_i, x'_i, b_i)\}_{i=1}^{5(k+\overline{\Delta})}$, and $|O^*| \leqslant (10k+20\overline{\Delta})/3 \leqslant 4k+7\overline{\Delta}$.

Let $A := \bigcup_{i=1}^{5(k+\overline{\Delta})} A_i$ and $B := \bigcup_{i=1}^{5(k+\overline{\Delta})} B_i$. For $i \in [5(k+\overline{\Delta})]$, let

$$U_i^+ := \bigcap_{v \in A_i} N^+_{D_i}(v) \setminus \bigcup_{v \in A_i} N^-_{D_i}(v) \quad \text{and} \quad U_i^- := \bigcap_{v \in B_i} N^-_{D'_i}(v) \setminus \bigcup_{v \in B_i} N^+_{D'_i}(v).$$

Since $|A|, |B| \leqslant 5 \cdot 5(k+\overline{\Delta})$ and $|O^*| < 4k+7\overline{\Delta}$, it follows that

$$|A \cup B \cup O^*| \leqslant 57(k+\overline{\Delta}).$$

By Menger's theorem, let $P_1, \dots, P_k$ be k edge-disjoint paths from $\{a_1, \dots, a_k\}$ to $\{b_1, \dots, b_k\}$ such that there is a permutation $\sigma : [k] \to [k]$ where for $i \in [k]$, P_i is a path from a_i to $b_{\sigma(i)}$. Let $\mathcal{P} := \{P_1, \dots, P_k\}$.

The rest of the proof is analogous to the proof of Theorem 6.1. As $|V(D)| - |A \cup B \cup O^*| \geqslant 100(k+\overline{\Delta}) - 57(k+\overline{\Delta}) \geqslant 33k + 32\overline{\Delta}$, we apply Lemma 5.18 so that D contains a k-arc-absorber

$$\mathcal{D}_{\mathrm{arc}} := (E_{\mathrm{abs}}, A \cup B \cup O^*, \mathcal{P}, W_i, W_o)$$

with $W_i, W_o \subseteq V(D) \setminus (A \cup B \cup O^*)$, $|W_i|, |W_o| = 3k$, and

$$\begin{aligned} |E_{\mathrm{abs}}| &\leqslant kn + 210k(k+\overline{\Delta}) + 32(k+\overline{\Delta}) + (5k+1)|A \cup B \cup O^*| \\ &\leqslant kn + 495k(k+\overline{\Delta}) + 89(k+\overline{\Delta}), \end{aligned} \tag{6.5}$$

since $|A \cup B \cup O^*| \leqslant 57(k+\overline{\Delta})$.

Since $W_i, W_o \subseteq V(D) \setminus (A \cup B \cup O^*)$ with $|W_i|, |W_o| = 3k$ and (6.4), we apply Lemma 5.14 with $3k$ playing the role of w. By (2) of Lemma 5.14, D has a k-arc-hub

$$\mathcal{H}_{\mathrm{arc}} := (E_{\mathrm{hub}}, \{a_1, \dots, a_k\}, \{b_1, \dots, b_k\}, W_o, W_i)$$

such that

$$|E_{\mathrm{hub}}| \leqslant 82k(k+\overline{\Delta}). \tag{6.6}$$

Let $E_L := E_{\mathrm{abs}} \cup E_{\mathrm{hub}}$. By (6.5) and (6.6),

$$\begin{aligned} |E_L| &\leqslant |E_{\mathrm{abs}}| + |E_{\mathrm{hub}}| \\ &\leqslant kn + 495k(k+\overline{\Delta}) + 82k(k+\overline{\Delta}) + 89(k+\overline{\Delta}) \\ &\leqslant kn + 577k(k+\overline{\Delta}) + 89(k+\overline{\Delta}) \\ &\leqslant kn + 666k(k+\overline{\Delta}). \end{aligned}$$

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Let $D' := (V(D), E_L)$ be a spanning subgraph of D. Now it remains to prove that D' is strongly k-arc-connected. Let $F \subseteq E(D')$ with $|F| \leqslant k-1$ and $u, v \in V(D')$. Let $i \in [k]$ be an index such that $E(P_i) \cap F = \emptyset$. If $u \in W_o$, then $u' := u$. Otherwise, $D' - F$ contains a path from u to a vertex $u' \in W_o$ since $\mathcal{D}_{\mathrm{arc}}$ is a k-arc-absorber in D. Since $\mathcal{H}_{\mathrm{arc}}$ is a k-arc-hub, $D' - F$ contains a path from u' to a_i, showing that $D' - F$ contains a path from u to a_i. Similarly, $D' - F$ contains a path from $b_{\sigma(i)}$ to v. Connecting from a_i to $b_{\sigma(i)}$ by P_i, we deduce that $D' - F$ contains a path from u to v, as desired. □

7. Concluding remarks

7.1. Improving the upper bound

For any integer $k \geqslant 1$ and a digraph D, let $h(k, D)$ be the minimum number of edges in a spanning subgraph D' of D with $\delta^+(D'), \delta^-(D') \geqslant k$. Bang-Jensen, Huang and Yeo [5] proved that $h(k, T) \leqslant k|V(T)| + (k(k+1))/2$ for every tournament T with $\delta^+(T), \delta^-(T) \geqslant k$, and $h(k, T) \leqslant k|V(T)| + (k(k-1))/2$ if the tournament T is strongly k-arc-connected (see [5, Proposition 2.1]). They also conjectured that $h(k, T)$ is equal to the minimum number of edges in a strongly k-arc-connected spanning subgraph of T, for every strongly k-arc-connected tournament T. Using the ideas of the proof of [5, Proposition 2.1], we prove the following.

Proposition 7.1. *For integers $k, n \geqslant 1$ and an integer $\overline{\Delta} \geqslant 2k-1$, $h(k, D) \leqslant kn + k\overline{\Delta}$ for every strongly k-arc-connected n-vertex digraph D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$.*

Proof. Let $V_1 := \{v_1 : v \in V(D)\}$ and $V_2 := \{v_2 : v \in V(D)\}$ be two disjoint copies of $V(D)$. Let $\mathcal{N}$ be a network with a vertex-set $\{s, t\} \cup V_1 \cup V_2$ and an edge-set

$$\{sv_1 : v \in V(D)\} \cup \{v_2 t : v \in V(D)\} \cup \{u_1 v_2 : uv \in E(D)\}.$$

We may assume that $s, t \notin V_1 \cup V_2$. Let $\ell : E(\mathcal{N}) \to \mathbb{R}_{\geqslant 0}$ be a lower bound function such that $\ell(sv_1) = \ell(v_2 t) = k$ for every $v \in V(D)$, and $\ell(e) = 0$ for the other edges $e \in E(\mathcal{N})$. Let $c : E(\mathcal{N}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a capacity function such that $c(sv_1) = c(v_2 t) = \infty$ for every $v \in V(D)$ and $c(u_1 v_2) = 1$ for every $uv \in E(D)$. One can easily check that the minimum (s, t)-flow of $\mathcal{N}$ is equal to $h(k, D)$. By Min-Flow Max-Demand Theorem (see [2, Theorem 4.9.1]), the minimum (s, t)-flow is equal to the maximum of $\ell(S, T) - c(T, S)$, where $\{S, T\}$ is a partition of $V(\mathcal{N})$ with $s \in S$ and $t \in T$.

Let $\{S, T\}$ be a partition of $V(\mathcal{N})$ with $s \in S$ and $t \in T$. For $A, B \in \{S, T\}$, let $V_{A,B} := \{v \in V(D) : v_1 \in A,\ v_2 \in B\}$. Then

$$\begin{aligned} \ell(S, T) &= k(V_{T,S} + V_{T,T}) + k(V_{S,S} + V_{T,S}) = k|V(D)| + k|V_{T,S}| - k|V_{S,T}|, \\ c(T, S) &= e_D(V_{T,S} \cup V_{T,T}, V_{S,S} \cup V_{T,S}) = |E(D[V_{T,S}])| + e_D(V_{T,S}, V_{S,S}) \\ &\quad + e_D(V_{T,T}, V_{S,S} \cup V_{T,S}). \end{aligned}$$

Now we aim to prove $\ell(S, T) - c(T, S) \leqslant kn + k\overline{\Delta}$. If there are at least three empty sets in $\{V_{S,S}, V_{S,T}, V_{T,S}, V_{T,T}\}$, then it is easy to check that $\ell(S, T) - c(T, S) \leqslant kn$. Hence we may assume that there are at least two non-empty sets in $\{V_{S,S}, V_{S,T}, V_{T,S}, V_{T,T}\}$. We claim that

$$\ell(S,T) - c(T,S) \leqslant kn + k|V_{T,S}| - |E(D[V_{T,S}])| - k \leqslant kn + k\overline{\Delta}.$$

If $V_{S,T} = V_{T,T} = \emptyset$ then $e_D(V_{T,T}, V_{S,S} \cup V_{T,S}) = 0$ and $e_D(V_{T,S}, V_{S,S}) \geqslant k$, implying $\ell(S,T) - c(T,S) \leqslant kn + k|V_{T,S}| - |E(D[V_{T,S}])| - k$. If $V_{S,T} = \emptyset$ and $V_{T,T} \neq \emptyset$ then $e_D(V_{T,T}, V_{S,S} \cup V_{T,S}) \geqslant k$ since D is strongly k-arc-connected. Therefore, either $|V_{S,T}| \geqslant 1$ or $e_D(V_{T,T}, V_{S,S} \cup V_{T,S}) \geqslant k$. In either case, it follows that $\ell(S,T) - c(T,S) \leqslant kn + k|V_{T,S}| - |E(D[V_{T,S}])| - k$.

Since $|E(D[V_{T,S}])| \geqslant \max(0, |V_{T,S}|(|V_{T,S}| - 1 - \overline{\Delta})/2)$, we have

$$k|V_{T,S}| - |E(D[V_{T,S}])| \leqslant \begin{cases} k|V_{T,S}| & \text{if } |V_{T,S}| < \overline{\Delta} + 1, \\ k|V_{T,S}| - \dfrac{|V_{T,S}|}{2}(|V_{T,S}| - \overline{\Delta} - 1) & \text{otherwise.} \end{cases}$$

If $|V_{T,S}| < \overline{\Delta} + 1$, then $k|V_{T,S}| - |E(D[V_{T,S}])| < k\overline{\Delta} + k$ and thus $\ell(S,T) - c(T,S) < kn + k\overline{\Delta}$. Let us assume that $|V_{T,S}| \geqslant \overline{\Delta} + 1$. Since the function $f(x) = (x(2k + \overline{\Delta} + 1 - x))/2$ is a decreasing function for $x \geqslant k + (\overline{\Delta} + 1)/2$ and $|V_{T,S}| \geqslant \overline{\Delta} + 1 \geqslant k + (\overline{\Delta} + 1)/2$, we have

$$\begin{aligned} k|V_{T,S}| - |E(D[V_{T,S}])| &\leqslant k|V_{T,S}| - \frac{|V_{T,S}|}{2}(|V_{T,S}| - \overline{\Delta} - 1) = f(|V_{T,S}|) \\ &\leqslant f(\overline{\Delta} + 1) = k\overline{\Delta} + k. \end{aligned}$$

and thus

$$\ell(S,T) - c(T,S) \leqslant kn + k|V_{T,S}| - |E(D[V_{T,S}])| - k \leqslant kn + k\overline{\Delta}.$$

This completes the proof. □

As the oriented graph $G_{n_1,n_2,k,\overline{\Delta}}$ in Section 3 with $n = n_1 + n_2 + \overline{\Delta} + 1$ satisfies $h(k, G_{n_1,n_2,k,\overline{\Delta}}) \geqslant kn + k\overline{\Delta}$ if $\overline{\Delta} \geqslant 2k - 1$, Proposition 7.1 implies that $h(k, G_{n_1,n_2,k,\overline{\Delta}}) = kn + k\overline{\Delta}$ when $\overline{\Delta} \geqslant 2k - 1$.

For $k = 1$, Bang-Jensen, Huang and Yeo [5, Theorem 8.3] proved that every strongly connected n-vertex digraph D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ contains a spanning strongly connected subgraph with at most $n + \overline{\Delta}$ edges. We conjecture that the multiplicative constant of $k\overline{\Delta}$ of Theorem 1.3 can be improved to 1, which is best possible.

Conjecture 7.2.

(1) *There is $C > 0$ such that for integers $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, every strongly k-connected n-vertex digraph D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ contains a strongly k-connected spanning subgraph with at most $kn + k\overline{\Delta} + Ck^2$ edges.*

(2) *There is $C' > 0$ such that for integers $k, n \geqslant 1$ and $\overline{\Delta} \geqslant 0$, every strongly k-arc-connected n-vertex directed multigraph D with $\overline{\Delta}(D) \leqslant \overline{\Delta}$ contains a strongly k-arc-connected spanning subgraph with at most $kn + k\overline{\Delta} + C'k^2$ edges.*

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

Since Mader [17] proved that every strongly k-connected n-vertex digraph contains a strongly k-connected spanning subgraph with at most $2kn - k(k+1)$ edges, Conjecture 7.2 is true for $\overline{\Delta} \geqslant n-k-1$.

7.2. Almost-regular spanning subgraphs

There are many studies regarding finding spanning regular subgraphs in tournaments. One of the typical examples of spanning regular subgraphs is a union of edge-disjoint Hamiltonian cycles, and there are some results relating edge-disjoint Hamiltonian cycles and the vertex-connectivity of tournaments. Thomassen [21] conjectured that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every strongly $f(k)$-connected tournament contains k edge-disjoint Hamiltonian cycles, and Kühn, Lapinskas, Osthus and Patel [15] proved that $f(k) = O(k^2(\log k)^2)$ suffices and constructed a strongly $((k-1)^2)/4$-connected tournament with no k edge-disjoint Hamiltonian cycles. Recently, Pokrovskiy [20] proved that $f(k) = O(k^2)$ suffices, which is asymptotically sharp.

As a variation of the problem, one may ask the minimum $m = m(k)$ such that every strongly mk-connected tournament T contains a spanning k-regular subgraph. The next lemma proves that $m \geqslant (k+1)/2$, and the result of Pokrovskiy [20] is asymptotically best possible even if we relax the condition of existence of k edge-disjoint Hamiltonian cycles to the existence of spanning k-regular subgraph. Recall that $T_{n_1,n_2,k}$ is a strongly k-connected (n_1+n_2+k)-vertex tournament defined in Section 3. We remark that an almost identical construction can be found in [15, Proposition 5.1].

Lemma 7.3. *Let $m, k \geqslant 1$ be integers. For a $(5mk+2)$-vertex tournament $T_{2mk+1,2mk+1,mk}$, every spanning subgraph D of $T_{2mk+1,2mk+1,mk}$ satisfying $\delta^+(D), \delta^-(D) \geqslant k$ contains at least $(k-2m+1)/5m$ vertices of either in-degree or out-degree more than k in D.*

Proof. Let $T_{2mk+1,2mk+1,mk}$ be the tournament with subtournaments T_1, T_2 and T_3 defined in Section 3.

Let D be any spanning subgraph of T such that $\delta^+(D), \delta^-(D) \geqslant k$. Let $S^+ \subseteq V(T_2)$ be the set of vertices v in $V(T_2)$ such that $d_D^+(v) > k$.

Since $d_T^+(v) \leqslant 5mk+1$ for any $v \in V(T_2)$ and every vertex in $V(T_1)$ has in-degree at least k in D, it follows that $(5mk+1)|S^+| + k(2mk+1-|S^+|) \geqslant \sum_{v \in V(T_2)} d_D^+(v)$ and $e_D(V(T_2), V(T_1))$ is at least $(k(k+1))/2$. Hence

$$\begin{aligned}(5mk+1)|S^+| + k(2mk+1-|S^+|) &\geqslant \sum_{v \in V(T_2)} d_D^+(v) \\ &\geqslant e_D(V(T_2), V(T_1)) - e_D(V(T_3), V(T_2)) + \sum_{w \in V(T_2)} d_D^-(w) \\ &\geqslant \frac{k(k+1)}{2} - mk + k(2mk+1),\end{aligned}$$

implying that

$$|S^+| \geqslant \frac{k(k+1-2m)}{2(5mk-k+1)} \geqslant \frac{k+1-2m}{10m}.$$

Let $S^- \subseteq V(T_3)$ be the set of vertices v in $V(T_3)$ such that $d_D^-(v) > k$. Similarly,

$$|S^-| \geqslant \frac{k+1-2m}{10m},$$

and it follows that D contains at least $(k-2m+1)/5m$ vertices with either in-degree or out-degree more than k in D. □

Rather than finding spanning regular subgraphs in semicomplete digraphs, we may consider finding *almost* regular spanning subgraph (all vertices except a few vertices have the same in/out-degrees) in semicomplete digraphs. Corollary 1.4 implies that every strongly k-connected semicomplete digraph contains a strongly k-connected spanning subgraph such that all vertices except for $O(k^2)$ vertices have both in-degree and out-degree exactly k. We conjecture the following.

Conjecture 7.4.

(1) *For integers $k,n \geqslant 1$ and given a strongly k-connected semicomplete digraph D, there exists a set $S \subseteq V(D)$ with $|S| = O(k)$ such that there is a strongly k-connected spanning subgraph D' of D with $d_{D'}^+(v) = d_{D'}^-(v) = k$ for every $v \in V(D) \setminus S$, and $d_{D'}^+(w) = d_{D'}^-(w) = O(k)$ for every $w \in V(D)$.*

(2) *For integers $k,n \geqslant 1$ and given a strongly k-arc-connected semicomplete directed multigraph D, there exists a set $S \subseteq V(D)$ with $|S| = O(k)$ such that there is a strongly k-arc-connected spanning subgraph D' of D with $d_{D'}^+(v) = d_{D'}^-(v) = k$ for every $v \in V(D) \setminus S$, and $d_{D'}^+(w) = d_{D'}^-(w) = O(k)$ for every $w \in V(D)$.*

Note that the statements in Conjecture 7.4 imply that $|E(D')| \leqslant k|V(D)| + O(k^2)$, strengthening Corollary 1.4. By Lemma 7.3, we remark that the size $O(k)$ of S cannot be improved further, since every spanning subgraph D of a tournament $T_{2k+1,2k+1,k}$ with $\delta^+(D), \delta^-(D) \geqslant k$ contains at least $(k-1)/4$ vertices of either in-degree or out-degree more than k in D.

Acknowledgements

The author would like to thank Sang-il Oum and Jaehoon Kim for their valuable comments and suggestions. The author would also like to thank the anonymous referees for their careful reading and suggestions.

References

[1] Bang-Jensen, J. (2009) Problems and conjectures concerning connectivity, paths, trees and cycles in tournament-like digraphs. *Discrete Math.* **309** 5655–5667.

[2] Bang-Jensen, J. and Gutin, G. (2009) *Digraphs: Theory, Algorithms and Applications*, second edition, Springer Monographs in Mathematics, Springer.

[3] Bang-Jensen, J. and Havet, F. (2018) *Tournaments and Semicomplete Digraphs*, Springer, pp. 35–124.

[4] Bang-Jensen, J., Huang, J. and Yeo, A. (2003) Strongly connected spanning subdigraphs with the minimum number of arcs in quasi-transitive digraphs. *SIAM J. Discrete Math.* **16** 335–343.

https://doi.org/10.1017/S0963548318000469 Published online by Cambridge University Press

[5] Bang-Jensen, J., Huang, J. and Yeo, A. (2004) Spanning k-arc-strong subdigraphs with few arcs in k-arc-strong tournaments. *J. Graph Theory* **46** 265–284.
[6] Bang-Jensen, J. and Yeo, A. (2001) The minimum spanning strong subdigraph problem for extended semicomplete digraphs and semicomplete bipartite digraphs. *J. Algorithms* **41** 1–19.
[7] Berg, A. R. and Jordán, T. (2005) Minimally k-edge-connected directed graphs of maximal size. *Graphs Combin.* **21** 39–50.
[8] Cheriyan, J. and Thurimella, R. (2000) Approximating minimum-size k-connected spanning subgraphs via matching. *SIAM J. Comput.* **30** 528–560.
[9] Dalmazzo, M. (1977) Nombre d'arcs dans les graphes k-arc-fortement connexes minimaux. *CR Acad. Sci. Paris Sér. A–B* **285** A341–A344.
[10] Edmonds, J. (1973) Edge-disjoint branchings. In *Combinatorial Algorithms* (B. Rustin, ed.), Academic Press, pp. 91–96.
[11] Garey, M. R. and Johnson, D. S. (1979) *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman.
[12] Hopcroft, J. E. and Karp, R. M. (1973) An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput.* **2** 225–231.
[13] Kang, D. Y., Kim, J., Kim, Y. and Suh, G. (2017) Sparse spanning k-connected subgraphs in tournaments. *SIAM J. Discrete Math.* **31** 2206–2227.
[14] Kim, J., Kühn, D. and Osthus, D. (2016) Bipartitions of highly connected tournaments. *SIAM J. Discrete Math.* **30** 895–911.
[15] Kühn, D., Lapinskas, J., Osthus, D. and Patel, V. (2014) Proof of a conjecture of Thomassen on Hamilton cycles in highly connected tournaments. *Proc. Lond. Math. Soc.* (3) **109** 733–762.
[16] Kühn, D., Osthus, D. and Townsend, T. (2016) Proof of a tournament partition conjecture and an application to 1-factors with prescribed cycle lengths. *Combinatorica* **36** 451–469.
[17] Mader, W. (1985) Minimal n-fach zusammenhängende Digraphen. *J. Combin. Theory Ser. B* **38** 102–117.
[18] Mader, W. (1996) On vertices of degree n in minimally n-connected graphs and digraphs. In *Combinatorics: Paul Erdős is Eighty*, (D. Miklós *et al.*, eds), Vol. 2, János Bolyai Mathematical Society, pp. 423–449.
[19] Pokrovskiy, A. (2015) Highly linked tournaments. *J. Combin. Theory Ser. B* **115** 339–347.
[20] Pokrovskiy, A. (2017) Edge disjoint Hamiltonian cycles in highly connected tournaments. *Int. Math. Res. Not.* **2017** 429–467.
[21] Thomassen, C. (1982) Edge-disjoint Hamiltonian paths and cycles in tournaments. *Proc. London Math. Soc.* (3) **45** 151–168.
[22] Vetta, A. (2001) Approximating the minimum strongly connected subgraph via a matching lower bound. In *Proceedings of the Twelfth Annual ACM–SIAM Symposium on Discrete Algorithms*, SIAM, pp. 417–426.