Proceedings of the Royal Society of Edinburgh, **149**, 831–847, *2019*
DOI:10.1017/prm.2018.26

Positivity and continued fractions from the binomial transformation

Bao-Xuan Zhu
School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, PR China (bxzhu@jsnu.edu.cn)

(MS received 27 October 2016; accepted 2 January 2017)

Given a sequence of polynomials $\{x_k(q)\}_{k\geqslant 0}$, define the transformation

$$y_n(q) = a^n \sum_{i=0}^{n} \binom{n}{i} b^{n-i} x_i(q)$$

for $n \geqslant 0$. In this paper, we obtain the relation between the Jacobi continued fraction of the ordinary generating function of $y_n(q)$ and that of $x_n(q)$. We also prove that the transformation preserves q-TP$_{r+1}$ (q-TP) property of the Hankel matrix $[x_{i+j}(q)]_{i,j\geqslant 0}$, in particular for $r = 2, 3$, implying the r-q-log-convexity of the sequence $\{y_n(q)\}_{n\geqslant 0}$. As applications, we can give the continued fraction expressions of Eulerian polynomials of types A and B, derangement polynomials types A and B, general Eulerian polynomials, Dowling polynomials and Tanny-geometric polynomials. In addition, we also prove the strong q-log-convexity of derangement polynomials type B, Dowling polynomials and Tanny-geometric polynomials and 3-q-log-convexity of general Eulerian polynomials, Dowling polynomials and Tanny-geometric polynomials. We also present a new proof of the result of Pólya and Szegö about the binomial convolution preserving the Stieltjes moment property and a new proof of the result of Zhu and Sun on the binomial transformation preserving strong q-log-convexity.

Keywords: log-convexity; strong q-log-convexity; k-q-log-convexity; total positivity; hankel matrix; continued fraction

2010 *Mathematics subject classification:* 05A10; 05A20; 05A30; 11A55; 11B83

1. Introduction

Given a sequence $\{x_n\}_{n\geqslant 0}$, define the binomial transformation

$$y_n = \sum_{i=0}^{n} \binom{n}{i} x_i$$

for $n \geqslant 0$, which often arises in combinatorics. In fact, it has the general combinatorial interpretation from the famous sieve method or inclusion-exclusion-principle [**19**, Chaper IV]. It is very useful in studying the log-concavity and log-convexity. For instance, it is well known that the binomial transformation preserves the log-concavity property and log-convexity property (see Karlin [**23**] for instance).

Zhu and Sun [42] also proved that the binomial transformation preserves the strong q-log-concavity property. More generally, the log-convexity property and log-concavity property are preserved under the binomial convolution and the ordinary multinomials convolution, see Davenport and Pólya [20], Wang and Yeh [36] and Ahmia and Belbachir [1], respectively. However, there is still a gap for linear transformations preserving higher order q-log-convexity. This is one motivation for this paper.

For the convenience of the readers and for the sake of clarity of exposition, recall some definitions. Let $\{a_k\}_{k\geqslant 0}$ be a sequence of nonnegative numbers. The sequence is called *log-convex* (*log-concave*, resp.) if $a_k a_{k+2} \geqslant a_{k+1}^2$ ($a_k a_{k+2} \leqslant a_{k+1}^2$, resp.) for all $k \geqslant 0$. The log-convex and log-concave sequences arise often in combinatorics and have been extensively investigated. We refer the reader to [10, 32] for the log-concavity and [25, 39] for the log-convexity. For two polynomials with real coefficients $f(q)$ and $g(q)$, denote $f(q) \geqslant_q g(q)$ if the difference $f(q) - g(q)$ has only nonnegative coefficients. For a polynomial sequence $\{f_n(q)\}_{n\geqslant 0}$, it is called *$q$-log-concave* suggested by Stanley if

$$f_n(q)^2 \geqslant_q f_{n+1}(q) f_{n-1}(q),$$

for $n \geqslant 1$ and is called *q-log-convex* introduced by Liu and Wang if

$$f_n(q)^2 \leqslant_q f_{n+1}(q) f_{n-1}(q),$$

for any $n \geqslant 1$. It is also called *strongly q-log-convex* defined by Chen *et al.* if

$$f_{n+1}(q) f_{m-1}(q) - f_n(q) f_m(q) \geqslant_q 0$$

for any $n \geqslant m \geqslant 1$. Obviously, the q-log-concavity (q-log-convexity) reduces to the log-concavity (log-convexity) for $q = 0$. The q-log-concavity and q-log-convexity of polynomials have been extensively studied, see Butler [12], Leroux [24], Sagan [31], and Su, Wang and Yeh [33] for q-log-concavity, and refer to Chen *et al.* [15–17], Liu and Wang [25], Zhu [39, 40], and Zhu and Sun [42] for q-log-convexity.

Motivated by the notion of infinite log-concavity [26], Chen [13] defined the r-q-log-convexity as follows. Define the operator $\mathcal{L}$ which maps a polynomial sequence $\{f_i(q)\}_{i\geqslant 0}$ to a polynomial sequence $\{g_i(q)\}_{i\geqslant 0}$ given by

$$g_i(q) := f_{i-1}(q) f_{i+1}(q) - f_i(q)^2.$$

Then the q-log-convexity of the polynomial sequence $\{f_i(q)\}_{i\geqslant 0}$ is equivalent to the q-positivity of $\mathcal{L}\{f_i(q)\}$, that is, the coefficients of $g_i(q)$ are nonnegative for all $i \geqslant 1$. If the polynomial sequence $\{g_i(q)\}_{i\geqslant 1}$ is q-log-convex, then $\{f_i(q)\}_{i\geqslant 0}$ is called *2-q-log-convex*. In general, $\{f_i(q)\}_{i\geqslant 0}$ is called *r-q-log-convex* if the coefficients of $\mathcal{L}^r\{f_i(q)\}$ are nonnegative for $i \geqslant r$. In general, it is much more difficult to show the r-q-log-convexity for $r \geqslant 2$.

Let $A = [a_{n,k}]_{n,k\geqslant 0}$ be a matrix of real numbers. It is called *totally positive* (*TP* for short) if all its minors are nonnegative and is called *TP$_r$* if all minors of order $\leqslant r$ are nonnegative. When each entry of A is a polynomial in q with nonnegative coefficients, then we have the similar concepts for q-TP (resp. q-TP$_r$)

if all its minors (resp. if all minors of order $\leqslant r$) are polynomials with nonnegative coefficients. Total positivity of matrices plays an important role in various branches of mathematics, statistics, probability, mechanics, economics, and computer science, see Karlin [23] and Pinkus [28] for instance. Theory of total positivity has successfully been applied to log-concavity and log-convexity problems in combinatorics, see Brenti [11] and Zhu [39, 41]. The total positivity of a kind of matrix called Hankel matrix is related to the continued fractions. In addition, the continued fractions often arise in combinatorics and have general combinatorial interpretations. Zhu [39, 41] also gave criterions for strong q-log-convexity and higher order q-log-convexity from the famous Jacobi continued fractions, respectively. For brevity, we let $\mathrm{JCF}[\mathrm{g_n(q)z, h_{n+1}(q)z^2}]$ denote the Jacobi continued fraction

$$\cfrac{1}{1-g_0(q)z-\cfrac{h_1(q)z^2}{1-g_1(q)z-\cfrac{h_2(q)z^2}{1-g_2(q)z-\dots}}},$$

where $\{g_n(q)\}_{n\geqslant 0}$ and $\{h_n(q)\}_{n\geqslant 1}$ are sequences of polynomials with nonnegative coefficients.

The following is the main result of this paper.

THEOREM 1.1. *Let $\{x_i(q)\}_{i\geqslant 0}$ be a sequence of polynomials. Assume the transformation*

$$y_n(q) = a^n \sum_{j=0}^{n} \binom{n}{j} b^{n-j} x_j(q),$$

for $n \geqslant 0$. Let r be a nonnegative integer.

(i) *If the generating function*

$$\sum_{k=0}^{\infty} x_k(q) z^k = \mathrm{JCF}[\mathrm{g_n(q)z, h_{n+1}(q)z^2}],$$

then we have

$$\sum_{k=0}^{\infty} y_k(q) z^k = \mathrm{JCF}[\mathrm{a(g_n(q)+b)z, a^2 h_{n+1}(q)z^2}].$$

(ii) *Let a and b be positive. If the Hankel matrix $[x_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP_{r+1}, then so is $[y_{i+j}(q)]_{i,j\geqslant 0}$. In particular, $\{y_i(q)\}_{i\geqslant 0}$ is r-q-log-convex for $1\leqslant r\leqslant 3$.*

(iii) *Let a and b be positive. If the Hankel matrix $[x_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP, then so is the Hankel matrix $[y_{i+j}(q)]_{i,j\geqslant 0}$.*

Barry also obtained (i) of theorem 1.1 for $a = 1$ using a different method [3]. Note that a sequence of polynomials $\{x_n(q)\}_{n\geqslant 0}$ is strongly q-log-convex if and only if the Hankel matrix $[x_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP$_2$. Thus, for $a = b = r = 1$, the (ii) of theorem 1.1 gives another proof of the following result [42].

Corollary 1.2. *[42] If the sequence of polynomials $\{x_i(q)\}_{i\geqslant 0}$ is strongly q-log-convex, then so is the binomial transformation*

$$y_n(q)=\sum_{j=0}^{n}\binom{n}{j}x_j(q),$$

for $n\geqslant 0$.

In particular, the following result is immediate from (ii) and (iii) of theorem 1.1 and lemma 2.2.

Theorem 1.3. *Let $\{x_i\}_{i\geqslant 0}$ be a sequence of nonnegative real numbers. Assume the binomial transformation*

$$y_n=\sum_{j=0}^{n}\binom{n}{j}x_j q^j,$$

for $n\geqslant 0$. Let r be a nonnegative integer.

(i) *If the Hankel matrix $[x_{i+j}]_{i,j\geqslant 0}$ is TP_{r+1}, then $\{y_i\}_{i\geqslant 0}$ is r-log-convex for $1\leqslant r\leqslant 3$ and $q=1$.*

(ii) *If $[x_{i+j}]_{i,j\geqslant 0}$ is TP, then $[y_{i+j}]_{i,j\geqslant 0}$ is q-TP.*

(iii) *If $\{x_i\}_{i\geqslant 0}$ is a Stieltjes moment sequence, then so is $\{y_i\}_{i\geqslant 0}$ for $q=1$.*

A sequence $\{a_k\}_{k\geqslant 0}$ is called a *Stieltjes moment* (*SM* for short) sequence if its Hankel matrix $[a_{i+j}]_{i,j\geqslant 0}$ is TP. It is well known that it is a Stieltjes moment sequence if and only if it has the form

$$a_k=\int_0^{+\infty}x^k\,\mathrm{d}\mu(x), \tag{1.1}$$

where μ is a non-negative measure on $[0,+\infty)$, see [**28**, theorem 4.4] for instance. Stieltjes moment problem is one of the classical moment problems and arises naturally in many branches of mathematics [**35**]. Note that Pólya and Szegö [**29**, Part VII, theorem 42] proved the following result.

Theorem 1.4. *If both $\{x_n\}_{n\geqslant 0}$ and $\{y_n\}_{n\geqslant 0}$ are Stieltjes moment sequences, then so is their binomial convolution*

$$z_n=\sum_{k=0}^{n}\binom{n}{k}x_k y_{n-k},\quad n=0,1,2,\ldots.$$

Remark 1.5. Wang and Zhu [**37**] gave a new proof for theorem 1.4. It is known [**6**, theorem 6] that if a triangular linear transformation preserves the Stieltjes moment property, then the same goes for its convolution. Thus by (iii) of theorem 1.3, we have another proof for theorem 1.4.

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

The organization of this paper is as follows. In § 2, we present the proof of theorem 1.1. In § 3, as applications of theorem 1.1, we give the continued fraction expressions of Eulerian polynomials of types A and B, derangement polynomials types A and B, general Eulerian polynomials, Dowling polynomials and Tanny-geometric polynomials. In addition, we also prove the strong q-log-convexity of derangement polynomials type B, Dowling polynomials and Tanny-geometric polynomials and 3-q-log-convexity of general Eulerian polynomials, Dowling polynomials and Tanny-geometric polynomials.

2. Proof of theorem 1.1

In this paper, the total positivity of matrices plays an important role in our proof. In order to present our proof, we need the following some basic results from total positivity of matrices. The first of the following lemmas follows from the classic Cauchy-Binet formula and the second can be obtained from the properties of determinants.

Lemma 2.1. *If two matrices are q-TP_r, then so is their product.*

Lemma 2.2. *If* $[x_{i+j}(q)]_{i,j\geqslant 0}$ *is q-TP_r, then so is* $[(a+bq)^{i+j}x_{i+j}(q)]_{i,j\geqslant 0}$ *for nonnegative real numbers a and b.*

Using the total positivity of matrices, Zhu [41] proved the following criterion for higher order q-log-convexity. In order to make the proof self-contained, we will state the proof here.

Proposition 2.3. *Let* $\{a_n(q)\}_{n\geqslant 0}$ *be a sequence of polynomials with nonnegative coefficients. If the Hankel matrix* $[a_{i+j}(q)]_{i,j\geqslant 0}$ *is q-TP_{r+1}, then* $\{a_n(q)\}_{n\geqslant 0}$ *is r-q-log-convex for* $1\leqslant r\leqslant 3$.

Proof. For brevity, we write a_k for $a_k(q)$. Note for $r=1$ that

$$\mathcal{L}(a_k)=a_{k+1}a_{k-1}-a_k^2=\begin{vmatrix} a_{k-1} & a_k \\ a_k & a_{k+1}\end{vmatrix}.$$

Thus it is obvious that $\{a_n(q)\}_{n\geqslant 0}$ is q-log-convex if the Hankel matrix $[a_{i+j}]_{i,j\geqslant 0}$ is q-TP$_2$. Furthermore, for $r=2$, we get that

$$\begin{aligned}\mathcal{L}^2(a_k)&=\mathcal{L}(a_{k-1})\mathcal{L}(a_{k+1})-[\mathcal{L}(a_k)]^2\\&=\left(a_{k+2}a_k-a_{k+1}^2\right)\left(a_ka_{k-2}-a_{k-1}^2\right)-\left(a_{k+1}a_{k-1}-a_k^2\right)^2\\&=a_k\left(2a_{k-1}a_ka_{k+1}+a_ka_{k+2}a_{k-2}-a_k^3-a_{k+1}^2a_{k-2}-a_{k-1}^2a_{k+2}\right)\\&=a_k\begin{vmatrix} a_{k-2} & a_{k-1} & a_k\\ a_{k-1} & a_k & a_{k+1}\\ a_k & a_{k+1} & a_{k+2}\end{vmatrix},\end{aligned} \tag{2.1}$$

which implies that if the Hankel matrix $[a_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP$_3$ then $\{a_n(q)\}_{n\geqslant 0}$ is 2-q-log-convex.

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

In the following, we proceed to consider the case for $r = 3$. By (2.1), we have

$$\begin{aligned}
\mathcal{L}^3(a_k) &= \mathcal{L}^2(a_{k-1})\mathcal{L}^2(a_{k+1}) - \left[\mathcal{L}^2(a_k)\right]^2 \\
&= a_{k+1}a_{k-1}\begin{vmatrix} a_{k-1} & a_k & a_{k+1} \\ a_k & a_{k+1} & a_{k+2} \\ a_{k+1} & a_{k+2} & a_{k+3} \end{vmatrix}\begin{vmatrix} a_{k-3} & a_{k-2} & a_{k-1} \\ a_{k-2} & a_{k-1} & a_k \\ a_{k-1} & a_k & a_{k+1} \end{vmatrix} \\
&\quad - a_k^2\begin{vmatrix} a_{k-2} & a_{k-1} & a_k \\ a_{k-1} & a_k & a_{k+1} \\ a_k & a_{k+1} & a_{k+2} \end{vmatrix}^2 \\
&= (a_{k+1}a_{k-1} - a_k^2) \\
&\quad \times \left(a_k^2\begin{vmatrix} a_{k-3} & a_{k-2} & a_{k-1} & a_k \\ a_{k-2} & a_{k-1} & a_k & a_{k+1} \\ a_{k-1} & a_k & a_{k+1} & a_{k+2} \\ a_k & a_{k+1} & a_{k+2} & a_{k+3} \end{vmatrix} + \begin{vmatrix} a_{k-3} & a_{k-2} & a_{k-1} \\ a_{k-2} & a_{k-1} & a_k \\ a_{k-1} & a_k & a_{k+1} \end{vmatrix}\begin{vmatrix} a_{k-1} & a_k & a_{k+1} \\ a_k & a_{k+1} & a_{k+2} \\ a_{k+1} & a_{k+2} & a_{k+3} \end{vmatrix} \right).
\end{aligned}$$

So if the Hankel matrix $[a_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP$_4$ then $\{a_n(q)\}_{n\geqslant 0}$ is 3-q-log-convex. This completes the proof. □

For a triangular array $A = [A_{n,k}]_{n,k\geqslant 0}$, define its row generating function

$$A_n(z) = \sum_{k=0}^{n} A_{n,k}z^k.$$

The following result will be used in the proof of theorem 1.1, which was proved in [8] for the case of real sequences.

Proposition 2.4. *Let A, B, C be three infinite matrices. For a sequence $\{x_n(q)\}_{n\geqslant 0}$, define a sequence $\{y_n(q)\}_{n\geqslant 0}$ by*

$$y_n(q) = \sum_{i=0}^{n} A_{n,i}x_i(q) \tag{2.2}$$

for $n \geqslant 0$. Let the infinite Hankel matrices $Y = [y_{i+j}(q)]_{i,j\geqslant 0}$ and $X = [x_{i+j}(q)]_{i,j\geqslant 0}$. If $A_{i+j}(z) = B_i(z)C_j(z)$, for all $i, j \geqslant 0$, then we have $Y = BXC'$.

Proof. By (2.2), we deduce for $i, j \geqslant 0$ that

$$\begin{aligned}
y_{i+j}(q) &= \sum_{k\geqslant 0} A_{i+j,k}x_k(q) \\
&= \sum_{k\geqslant 0}\sum_{r=0}^{k} B_{i,r}C_{j,k-r}x_k(q) \\
&= \sum_{r\geqslant 0}\left(\sum_{k\geqslant r} C_{j,k-r}x_k(q)\right)B_{i,r} \\
&= \sum_{r\geqslant 0}\sum_{k\geqslant 0} B_{i,r}C_{j,k}x_{k+r}(q),
\end{aligned}$$

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

which implies that $Y = BXC'$. □

Proof of theorem 1.1. (i) : Since

$$\sum_{k=0}^{\infty} x_k(q)z^k = \mathrm{JCF}[\mathrm{g_n(q)z, h_{n+1}(q)z^2}],$$

the sequence $\{x_n(q)\}_{n\geqslant 0}$ can be arranged as the first column of a triangular array $[A_{n,k}(q)]_{n,k\geqslant 0}$ satisfying the recurrence relation

$$\begin{aligned} A_{n,k}(q) &= A_{n-1,k-1}(q) + g_k(q)A_{n-1,k}(q) + h_{k+1}(q)A_{n-1,k+1}(q), \\ A_{n,0}(q) &= g_0(q)A_{n-1,0}(q) + h_1(q)A_{n-1,1}(q) \end{aligned}$$

for $n \geqslant 1$ and $k \geqslant 1$, where $A_{0,0}(q) = 1$, $A_{0,k}(q) = 0$ for $k > 0$, see [22] for instance. Let $b^{n-k}\binom{n}{k} = T_{n,k}$ for $n \geqslant k \geqslant 0$. It is clear that $[T_{n,k}]_{n,k\geqslant 0}$ is an array of nonnegative numbers satisfying the recurrence relation

$$T_{n,k} = bT_{n-1,k} + T_{n-1,k-1} \tag{2.3}$$

with $T_{n,k} = 0$ unless $0 \leqslant k \leqslant n$ and $T_{0,0} = 1$.

Let $T = [T_{n,k}]_{n,k\geqslant 0}$, $A = [A_{n,k}(q)]_{n,k\geqslant 0}$ and $B = TA = [B_{n,k}]_{n,k\geqslant 0}$. We claim the following. □

Claim 2.5. The triangular array $[B_{n,k}]_{n,k\geqslant 0}$ satisfies the recurrence relation

$$\begin{aligned} B_{n,k} &= B_{n-1,k-1} + [g_k(q) + b]B_{n-1,k} + h_{k+1}(q)B_{n-1,k+1}, \\ B_{n,0} &= [g_0(q) + b]B_{n-1,0} + h_1(q)B_{n-1,1} \end{aligned} \tag{2.4}$$

for $n \geqslant 1$ and $k \geqslant 0$, where $B_{0,0} = 1$, $B_{0,k} = 0$ for $k > 0$.

Proof. We will complete the proof by induction on n. It is obvious for $n = 0$. For $n \geqslant 1$, it follows from $B = TA$ that

$$B_{n,k} = \sum_{i=0}^{n} T_{n,i}A_{i,k}(q). \tag{2.5}$$

Thus, by the recurrence relation of $T_{n,i}$, we have

$$\begin{aligned} B_{n,k} &= \sum_i T_{n,i}A_{i,k}(q) \\ &= \sum_i [T_{n-1,i-1} + bT_{n-1,i}]A_{i,k}(q) \\ &= \sum_i T_{n-1,i}A_{i+1,k}(q) + \sum_{i=0}^{n-1} bT_{n-1,i}A_{i,k}(q) \\ &= \sum_i T_{n-1,i}[A_{i,k-1}(q) + (g_k(q) + b)A_{i,k}(q) + h_{k+1}(q)A_{i,k+1}(q)] \\ &= B_{n-1,k} + [g_k(q) + b]B_{i,k} + h_{k+1}(q)B_{i,k+1}, \end{aligned}$$

as desired. This proves the claim. □

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

By (2.5), we get

$$B_{n,0} = \sum_{i=0}^{n} T_{n,i} A_{i,0}(q) = \sum_{i=0}^{n} T_{n,i} x_i(q) = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} x_i(q). \tag{2.6}$$

Hence by claim 2.5, we have

$$\sum_{k=0}^{\infty} B_{k,0}(q) z^k = \mathrm{JCF}[(\mathrm{g_n(q)} + \mathrm{b})\mathrm{z}, \mathrm{h_{n+1}(q) z^2}].$$

Thus

$$\sum_{k=0}^{\infty} y_k(q) z^k = \sum_{k=0}^{\infty} a^k B_{k,0} z^k = \mathrm{JCF}[\mathrm{a}(\mathrm{g_n(q)} + \mathrm{b})\mathrm{z}, \mathrm{a^2 h_{n+1}(q) z^2}].$$

In the following, we will present the proofs of (ii) and (iii). Let $L = (L_{i,j})_{i,j\geqslant 0}$ be the infinite Pascal matrix, where $L_{i,j} = \binom{i}{j}$ is binomial coefficients. Note that

$$L_{i+j}(z) = L_i(z) L_j(z) = (1+z)^{i+j}.$$

Thus, if the infinite Hankel matrices $Z = [z_{i+j}(q)]_{i,j\geqslant 0}$ and $X^* = [b^{-i-j} x_{i+j}(q)]_{i,j\geqslant 0}$, where $z_n = \sum_{k=0}^{n} \binom{n}{k} b^{-k} x_k(q)$, then we get

$$Z = L X^* L'$$

from proposition 2.4 by taking $A = B = C = L$. Note that L is a TP matrix (see [**23**, p. 132]), so is L'. Thus the classical Cauchy–Binet theorem implies that Z is q-TP$_{r+1}$ since the Hankel matrix X^* is q-TP$_{r+1}$ by lemma 2.2. It follows from lemma 2.2 that

$$[y_{i+j}(q)]_{i,j\geqslant 0} = [(ab)^{i+j} z_{i+j}(q)]_{i,j\geqslant 0}$$

is q-TP$_{r+1}$. So $\{y_i(q)\}_{i\geqslant 0}$ is r-q-log-convex by proposition 2.3 for $1 \leqslant r \leqslant 3$. This proves that (ii) holds. Similarly, the q-TP property of X also implies that of Y by the classical Cauchy–Binet theorem. Thus (iii) holds. We complete the proof. □

3. Applications

In this section, we give some applications of the results obtained in § 1.

3.1. The classical Eulerian polynomials

Let $\pi = a_1 a_2 \cdots a_n$ be a permutation of $[n]$. An element $i \in [n-1]$ is called a descent of π if $a_i > a_{i+1}$. The Eulerian number $A(n,k)$ is defined as the number of permutations of $[n]$ having $k-1$ descents. Moreover, the Eulerian numbers satisfy

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

the recurrence

$$A(n,k) = kA(n-1,k) + (n-k+1)A(n-1,k-1),$$

whose generating function, that is, the classical Eulerian polynomials, is denoted by $A_n(q)$. It is known that the exponential generating function of $\{A_n(q)\}_{n\geqslant 0}$ [**19**] is

$$\sum_{n\geqslant 0} A_n(q)\frac{x^n}{n!} = \frac{(1-q)}{1-q\mathrm{e}^{x(1-q)}}, \tag{3.1}$$

and the ordinary generating function of $\{A_n(q)\}_{n\geqslant 0}$ is

$$\begin{aligned}\sum_{k\geqslant 0} A_k(q)x^k &= \mathrm{JCF}[[\mathrm{n}+(\mathrm{n}+1)\mathrm{q}]\,\mathrm{x}, (\mathrm{n}+1)^2\mathrm{qx}^2] \\ &= \cfrac{1}{1-qx-\cfrac{qx^2}{1-(1+2q)x-\cfrac{4qx^2}{1-(2+3q)x-\cdots}}}\end{aligned}$$

(see [**22**] for instance).

3.2. The Eulerian polynomials of type *A*

Given a finite Coxeter group W, define the Eulerian polynomials of W by

$$P_n(W,q) = \sum_{\pi\in W} q^{d_W(\pi)},$$

where $d_W(\pi)$ is the number of W-descents of π. We refer the reader to Björner [**8**] for relevant definitions.

For Coxeter groups of type A, it is known that $P_n(A,q) = A_n(q)/q$, the shifted Eulerian polynomials. Note that the exponential generating function is

$$\sum_{n\geqslant 0} P_n(A,q)\frac{x^n}{n!} = \frac{(1-q)e^{x(1-q)}}{1-qe^{x(1-q)}}. \tag{3.2}$$

So combining (3.1) and (3.2), we have

$$P_n(A,q) = \sum_{k=0}^{n} \binom{n}{k}(1-q)^{n-k}A_k(q). \tag{3.3}$$

Thus, by (3.3) and theorem 1.1(i) for $a=1$ and $b=1-q$, we have

$$\sum_{k\geqslant 0} P_k(A,q)x^k = \mathrm{JCF}[(\mathrm{nq}+\mathrm{n}+1)\mathrm{x}, (\mathrm{n}+1)^2\mathrm{qx}^2] \tag{3.4}$$

$n\geqslant 0$, that is, the next corollary.

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

Corollary 3.1. *[4]*

$$\sum_{k\geqslant 0} P_k(A,q)x^k = \cfrac{1}{1-x-\cfrac{qx^2}{1-(q+2)x-\cfrac{4qx^2}{1-(2q+3)x-\cfrac{9qx^2}{1-(3q+4)x-\cdots}}}}.$$

3.3. The derangement polynomials

A bijection $\pi : T \mapsto T$, with $T \subseteq \mathbb{Z}$ is a derangement if $\pi(i) \neq i$ for all $i \in T$. The set of all derangements on $[1, n]$ is denoted by D_n. Brenti [11] defined the derangement polynomials (to type A) by $d_0(q) = 1$ and

$$d_n(q) := \sum_{\sigma\in D_n} q^{\mathrm{exc}(\sigma)}$$

for $n \geqslant 1$. The following formula is given in [14, theorem 1.1] and it is derived from [11]. For $n \geqslant 0$,

$$d_n(q) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} P_k(A,q). \tag{3.5}$$

Thus, combining (3.4) and (3.5), we have the next result for derangement polynomials by theorem 1.1(i) for $a = 1$ and $b = -1$.

Proposition 3.2. *For $n \geqslant 0$,*

$$\sum_{k\geqslant 0} d_k(q)x^k = \mathrm{JCF}[(\mathrm{nq}+\mathrm{n})\,\mathrm{x}, (\mathrm{n}+1)^2\mathrm{qx}^2]$$

$$= \cfrac{1}{1-\cfrac{qx^2}{1-(q+1)x-\cfrac{4qx^2}{1-(2q+2)x-\cfrac{9qx^2}{1-(3q+3)x-\cdots}}}}.$$

3.4. The Eulerian polynomials of type *B*

Denote by B_n the group of all bijections σ in $S([-n, n] \setminus 0)$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in [-n, n] \setminus 0$, with composition as the group operation. This group is usually known as the group of signed permutations on $[1, n]$, or as the hyperoctahedral group of rank n.

Bagno and Garber [2] introduced a definition of excedance on the set of signed permutations, called colored excedance, $exc^{Clr(\sigma)} := 2exc_A(\sigma) + neg(\sigma)$. Mongelli

[27] defined the generating function of the colored descent statistic on B_n by

$$P_n(B,q) = \sum_{\sigma\in B_n} q^{exc^{Clr}(\sigma)}$$

and proved

$$P_n(B,q) = (1+q)^n P_n(A,q). \tag{3.6}$$

Let $D_n(B)$ denote the set of all derangements on B_n and $d_n(B,q)$ be the generating function of the colored excedances on the set $D_n(B)$, that is

$$d_n(B,q) := \sum_{\sigma\in D_n(B)} q^{\mathrm{exc}^{\mathrm{Clr}}(\sigma)}.$$

The derangement polynomial $d_n(B,q)$ shares most of the main properties of $d_n(q)$, for instance,

$$d_n(B,q) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} P_k(B,q). \tag{3.7}$$

Thus applying theorem 1.1 to (3.6) and (3.7), by corollary 3.1 we have the next result.

Proposition 3.3. *For $n \geqslant 0$,*

$$\sum_{k\geqslant 0} P_k(B,q)x^k = \mathrm{JCF}[(\mathrm{nq+n+1})(\mathrm{q+1})\mathrm{x}, (\mathrm{n+1})^2\mathrm{q}(\mathrm{q+1})^2\mathrm{x}^2]$$

$$= \cfrac{1}{1-(q+1)x-\cfrac{q(q+1)^2x^2}{1-(q+2)(q+1)x-\cfrac{4q(q+1)^2x^2}{1-(2q+3)(q+1)x-\cdots}}}$$

and

$$\sum_{k\geqslant 0} d_k(B,q)x^k = \mathrm{JCF}[[\mathrm{n}(\mathrm{q+1})^2+\mathrm{q}]\mathrm{x}, (\mathrm{n+1})^2\mathrm{q}(\mathrm{q+1})^2\mathrm{x}^2]$$

$$= \cfrac{1}{1-qx-\cfrac{q(q+1)^2x^2}{1-[(q+1)^2+q]x-\cfrac{4q(q+1)^2x^2}{1-[2(q+1)^2+q]x-\cdots}}}.$$

Proposition 3.4. *The sequences $\{P_k(B,q)\}_{k\geqslant 0}$ and $\{d_k(B,q)\}_{k\geqslant 0}$ are both strongly q-log-convex. In addition, $\{P_k(B,q)\}_{k\geqslant 0}$ is 3-q-log-convex.*

Proof. It follows from $P_k(B,q) = (1+q)^k P_k(A,q)$ that

$$P_k(B,q) = (1+q)^k A_k(q)/q.$$

Thus the q-total positivity of $[A_{i+j}(q)]_{i,j\geqslant 0}$ [37] implies that of $[P_{i+j}(B,q)]_{i,j\geqslant 0}$. So $\{P_k(B,q)\}_{k\geqslant 0}$ is strongly q-log-convex and 3-q-log-convex by proposition 2.3.

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

Note that Zhu [39, proposition 3.13] proved that for the continued fraction expansion

$$\sum_{k\geqslant 0} x_k(q)z^k = \mathrm{JCF}[\mathrm{g_n(q)z, h_{n+1}(q)z^2}]$$

with $g_n(q) \geqslant_q 0$ and $h_{n+1}(q) \geqslant_q 0$ for $n \geqslant 0$, the sequence $\{x_n(q)\}_{n\geqslant 0}$ is strongly q-log-convex if $g_n(q)g_{n+1}(q) \geqslant_q h_{n+1}(q)$. So by proposition 3.3 the continued fraction expansion

$$\sum_{k\geqslant 0} d_k(B,q)x^k = \mathrm{JCF}[[\mathrm{n(q+1)^2+q]x, (n+1)^2q(q+1)^2x^2}],$$

we have $g_n = n(q+1)^2 + q$ and $h_n = n^2q(q+1)^2$. Thus

$$\begin{aligned} & g_n g_{n+1} - h_{n+1} \\ & \quad = n + n^2 + (4n+3n^2)q + (1+6n+4n^2)q^2 + (4n+3n^2)q^3 + (n+n^2)q^4 \\ & \quad \geqslant_q 0, \end{aligned}$$

which implies that $\{d_k(B,q)\}_{k\geqslant 0}$ is strongly q-log-convex. □

3.5. The general Eulerian polynomials

Recently, Xiong, Tsao and Hall [38] defined the general Eulerian numbers $A_{n,k}(a,d)$ associated with an arithmetic progression $\{a, a+d, a+2d, a+3d, \ldots\}$ as

$$A_{n,k}(a,d) = (-a+(k+2)d)A_{n-1,k}(a,d) + (a+(n-k-1)d)A_{n-1,k-1}(a,d),$$

where $A_{0,-1} = 1$ and $A_{n,k} = 0$ for $k \geqslant n$ or $k \leqslant -2$. In particular, when $a = d = 1$, $A_{n,k}(1,1) = A_{n,k}$, the classical Eulerian numbers which enumerate the number of A_n with $k-1$ descents. Similarly, the general Eulerian polynomials associated with an arithmetic progression $\{a, a+d, a+2d, a+3d, \ldots\}$ can be defined as

$$P_n(q,a,d) = \sum_{k=-1}^{n-1} A_{n,k}(a,d)q^{k+1}.$$

It was proved that the exponential generating function of $\{P_n(q,a,d)\}_{n\geqslant 0}$ has the following expression

$$\sum_{n\geqslant 0} P_n(q,a,d)\frac{x^n}{n!} = \frac{(1-q)e^{ax(1-q)}}{1-qe^{dx(1-q)}} \tag{3.8}$$

and

$$P_n(q,a,d) = \sum_{k=0}^{n} \binom{n}{k} d^k P_k(A,q)(aq-a)^{n-k}. \tag{3.9}$$

Using the exponential Riordan Arrays, Barry [5] proved the next result. We will give a new proof by theorem 1.1(i).

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

COROLLARY 3.5. *[5]*

$$\sum_{n\geqslant 0} P_n(q,a,d)x^n = \cfrac{1}{1-s_0(q)x-\cfrac{t_1(q)x^2}{1-s_1(q)x-\cfrac{t_2(q)x^2}{1-s_2(q)x-\cfrac{t_3(q)x^2}{1-s_3(q)x-\cdots}}}}$$

with $s_i(q) = (di+a)q + (di+d-a)$ *and* $t_{i+1}(q) = (d(i+1))^2 q$ *for* $i \geqslant 0$.

Proof. Because

$$\sum_{k\geqslant 0} P_k(A,q)x^k = \mathrm{JCF}[(\mathrm{nq}+\mathrm{n}+1)\mathrm{x}, (\mathrm{n}+1)^2\mathrm{qx}^2],$$

which implies

$$\sum_{k\geqslant 0} P_k(A,q)d^k x^k = \mathrm{JCF}[\mathrm{d}(\mathrm{nq}+\mathrm{n}+1)\mathrm{x}, (\mathrm{n}+1)^2\mathrm{qd}^2\mathrm{x}^2].$$

Thus, by (3.9) and theorem 1.1(i), we have

$$\begin{aligned}\sum_{k\geqslant 0} P_k(q,a,d)x^k &= \sum_{k\geqslant 0} \frac{P_k(q,a,d)}{[a(q-1)]^k}[a(q-1)x]^k \\ &= \mathrm{JCF}[[\mathrm{d}(\mathrm{nq}+\mathrm{n}+1)+\mathrm{aq}-\mathrm{a}]\mathrm{x}, (\mathrm{n}+1)^2\mathrm{qd}^2\mathrm{x}^2] \\ &= \mathrm{JCF}[[(\mathrm{dn}+\mathrm{a})\mathrm{q}+\mathrm{dn}+\mathrm{d}-\mathrm{a}]\mathrm{x}, (\mathrm{n}+1)^2\mathrm{d}^2\mathrm{qx}^2].\end{aligned}$$

□

PROPOSITION 3.6. *The sequence* $\{P_k(q,a,d)\}_{k\geqslant 0}$ *is 3-q-log-convex for* $d \geqslant a \geqslant 0$.

Proof. Note that Zhu [41] proved that for the continued fraction expansion

$$\sum_{k\geqslant 0} x_k(q)z^k = \mathrm{JCF}[\mathrm{g_n(q)z}, \mathrm{h_{n+1}(q)z^2}]$$

with $g_n(q) \geqslant_q 0$ and $h_{n+1}(q) \geqslant_q 0$ for $n \geqslant 0$, the sequence $\{x_n(q)\}_{n\geqslant 0}$ is 3-q-log-convex if $g_k(q)g_{k+1}(q) \geqslant_q h_{k+1}(q)$ and

$$g_k(q)g_{k+1}(q)g_{k+2}(q)g_{k+3}(q) - g_{k+2}(q)g_{k+3}(q)h_{k+1}(q) - g_k(q)g_{k+3}(q)h_{k+2}(q) - g_k(q)g_{k+1}(q)h_{k+3}(q) + h_{k+1}(q)h_{k+3}(q) \geqslant_q 0$$

for $k \geqslant 0$. Thus, by corollary 3.5, using Mathematic, it is easy to check these inequalities for the case of $\{P_k(q,a,d)\}_{k\geqslant 0}$, whose details are omitted for brevity. So $\{P_k(q,a,d)\}_{k\geqslant 0}$ is 3-q-log-convex for $d \geqslant a \geqslant 0$. □

3.6. The Dowling polynomials

In [34], Tanny introduced the geometric polynomials $F_n(q) = \sum_{k=0}^{n} k!S_{n,k}q^k$. In addition, it was shown

$$F_n(q) = \sum_{k=1}^{n} A(n,k)q^{n-k+1}(q+1)^{k-1} = q^n P_n\left(A, \frac{q+1}{q}\right), \tag{3.10}$$

where $A(n, k)$ is the Eulerian numbers.

The Dowling lattice $Q_n(G)$ is a geometric lattice of rank n over a finite group G of order m and has many remarkable properties, see [7, 21]. When $m = 1$, that is, G is the trivial group, $Q_n(G)$ is the lattice $\prod_{n+1}$ of partitions of an $(n+1)$-element set. So the Dowling lattices can be viewed as group-theoretic analogs of the partition lattices. Let $W_m(n, k)$ be the Whitney numbers of the second kind, which satisfy the recurrence relation

$$W_m(n,k) = (mk+1)W_m(n-1,k) + W_m(n-1,k-1).$$

Benoumhani [7] defined the Dowling polynomials $D_m(n, q) = \sum_{k=0}^{n} W_m(n, k)q^k$ and the Tanny-geometric polynomials $F_m(n, q) = \sum_{k=0}^{n} k!W_m(n, k)q^k$. It was proved that

$$D_m(n,q) = \sum_{k=0}^{n} \binom{n}{k} m^k B_k\left(\frac{q}{m}\right), \tag{3.11}$$

$$F_m(n,q) = \sum_{k=0}^{n} \binom{n}{k} m^k F_k\left(\frac{q}{m}\right), \tag{3.12}$$

see [30] for instance. For a generalization of Whitney numbers of the second kind, the r-Whitney numbers of the second kind, denoted by $W_{m,r}(n, k)$, satisfy the recurrence relation

$$W_{m,r}(n,k) = (mk+r)W_{m,r}(n-1,k) + W_{m,r}(n-1,k-1),$$

whose generating function $D_{m,r}(n, q) = \sum_{k=0}^{n} W_{m,r}(n, k)q^k$ is called the r-Dowling polynomial [18]. In [18], it was proved

$$D_{m,r}(n,q) = \sum_{k=0}^{n} \binom{n}{k} (r-1)^{n-k} D_m(k,q). \tag{3.13}$$

Proposition 3.7. *For any positive integers m and r, we have the following results.*

(i) *The generating function of r-Dowling polynomials*

$$\sum_{k=0}^{\infty} D_{m,r}(k,q)x^k = \mathrm{JCF}[(\mathrm{q}+\mathrm{mn}+\mathrm{r})\mathrm{x}, (\mathrm{n}+1)\mathrm{mqx}^2].$$

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

(ii) *The generating function of Tanny-geometric polynomials*

$$\sum_{k=0}^{\infty} F_m(k,q)x^k = \mathrm{JCF}[[(2n+1)q+mn+1]x, (n+1)^2 q(q+m)x^2].$$

(iii) *Both sequences* $\{D_{m,r}(n,q)\}_{n\geqslant 0}$ *and* $\{F_m(n,q)\}_{n\geqslant 0}$ *are strongly q-log-convex and 3-q-log-convex.*

Proof. Since the continued fraction expression of the generating function of Bell polynomials

$$\sum_{k=0}^{\infty} B_k(q)x^k = \mathrm{JCF}[(q+n)x, (n+1)qx^2],$$

we deduce that

$$\begin{aligned}\sum_{k=0}^{\infty} B_k\left(\frac{q}{m}\right) m^k x^k &= \mathrm{JCF}\left[\left(\frac{q}{m}+n\right) mx, \frac{(n+1)qm^2x^2}{m}\right]\\ &= \mathrm{JCF}[(q+mn)x, (n+1)mqx^2].\end{aligned}$$

It follows that

$$\sum_{k=0}^{\infty} D_m(k,q)x^k = \mathrm{JCF}[(q+mn+1)x, (n+1)mqx^2] \tag{3.14}$$

by (3.11) and theorem 1.1(i). Thus we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_{m,r}(n,q)x^n &= \mathrm{JCF}[(q+mn+1+r-1)x, (n+1)mqx^2]\\ &= \mathrm{JCF}[(q+mn+r)x, (n+1)mqx^2]\end{aligned}$$

by (3.13) and theorem 1.1(i). Therefore, we prove that (i) holds.

(ii) Because the continued fraction expression of the generating function of the Eulerian polynomials of Type A

$$\sum_{k\geqslant 0} P_k(A,q)x^k = \mathrm{JCF}\left[(nq+n+1)x, (n+1)^2 qx^2\right],$$

it follows from (3.10) that

$$\begin{aligned}\sum_{k=0}^{\infty} F_k(q)x^k &= \mathrm{JCF}\left[\left(n\frac{q+1}{q}+n+1\right) qx, n^2\frac{q+1}{q}(qx)^2\right]\\ &= \mathrm{JCF}\left[[(2n+1)q+n]x, n^2 q(q+1)x^2\right].\end{aligned}$$

So

$$\begin{aligned}\sum_{k=0}^{\infty} m^k F_k\left(\frac{q}{m}\right) x^k &= \mathrm{JCF}\left[\left[(2n+1)\frac{q}{m}+n\right] mx, n^2\frac{q}{m}(\frac{q}{m}+1)m^2x^2\right]\\ &= \mathrm{JCF}[[(2n+1)q+mn]x, n^2q(q+m)x^2].\end{aligned}$$

https://doi.org/10.1017/prm.2018.26 Published online by Cambridge University Press

Thus we get

$$\sum_{k=0}^{\infty} F_m(k,q)x^k = \mathrm{JCF}[[(2n+1)q+mn+1]x, n^2q(q+m)x^2]$$

by (3.10) and theorem 1.1(i).

(iii) We first show that 3-q-log-convexity of the sequence $\{D_{m,r}(n,q)\}_{n\geqslant 0}$ as follows. Since the Hankel matrix $[B_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP [**37**], so is $[m^{i+j}B_{i+j}(q/m)]_{i,j\geqslant 0}$ by lemma 2.2. Thus by (3.11) and theorem 1.1(iii), we get that $[D_m(i+j,q)]_{i,j\geqslant 0}$ is q-TP. It follows from (3.13) that $[D_{m,r}(i+j,q)]_{i,j\geqslant 0}$ is q-TP by lemma 2.2 and theorem 1.1 (iii). Thus the sequence $\{D_{m,r}(n,q)\}_{n\geqslant 0}$ is strongly q-log-convex and 3-q-log-convex by proposition 2.3.

Since the Hankel matrix $[A_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP [**37**], so is $[P_{i+j}(A,q)]_{i,j\geqslant 0}$ since $P_n(A,q) = A_n(q)/q$. Hence we deduce that the Hankel matrix $[F_{i+j}(q)]_{i,j\geqslant 0}$ is q-TP from (3.10). Thus, applying theorem 1.1(iii) to (3.12), we get that $[F_m(i+j,q)]_{i,j\geqslant 0}$ is q-TP by lemma 2.2, which implies strong q-log-convexity and 3-q-log-convexity of $\{F_m(n,q)\}_{n\geqslant 0}$ by proposition 2.3. The proof is complete. □

Acknowledgements

Supported partially by the National Natural Science Foundation of China (No. 11571150).

References

1 M. Ahmia and H. Belbachir. Preserving log-convexity for generalized Pascal triangles. *Electron. J. Combin.* **19** (2012), Research Paper 16, 6 pp.

2 E. Bagno and D. Garber. On the excedance number of colored permutation groups. *Sém. Lothar. Combin.* **53** (2004/2006), Article B53f, 17 pp.

3 P. Barry. Continued fractions and transformations of integer sequences. *J. Integer Seq.* **12** (2009), Article 09.7.6.

4 P. Barry. Eulerian polynomials as moments, via exponential Riordan arrays **14** (2011), Article 11.9.5, 14 pp.

5 P. Barry. General Eulerian polynomials as moments using exponential Riordan array. *J. Integer Seq.* **16** (2013), Article 13.9.6, 15 pp.

6 G. Bennett. Hausdorff means and moment sequences. *Positivity* **15** (2011), 17–48.

7 M. Benoumhani. On some numbers related to Whitney numbers of Dowling lattices. *Adv. Appl. Math.* **19** (1997), 106–116.

8 A. Björner and F. Brenti. Combinatorics of Coxeter Groups. *Grad. Texts in Math.*, vol. 231(New York: Springer-Verlag, 2005).

9 F. Brenti. Unimodal, log-concave, and Pólya frequency sequences in combinatorics. Thesis (Ph.D.)–Massachusetts Institute of Technology, 1988.

10 F. Brenti. Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update. *Contemp. Math.* **178** (1994), 71–89.

11 F. Brenti. Combinatorics and total positivity. *J. Combin. Theory Ser. A* **71** (1995), 175–218.

12 L. M. Butler. The q-log concavity of q-binomial coeffcients. *J. Combin. Theory Ser. A* **54** (1990), 54–63.

13 W. Y. C. Chen. Log-concavity and q-Log-convexity Conjectures on the Longest Increasing Subsequences of Permutations, arXiv: 0806.3392v2, 2008.

14 W. Y. C. Chen, R. L. Tang and A. F. Y. Zhao. Derangement polynomials and excedances of type B. *Electron. J. Combin.* **16** (2009), R15.

15 W. Y. C. Chen, R. L. Tang, L. X. W. Wang and A. L. B. Yang. The q-log-convexity of the Narayana polynomials of type B. *Adv. in Appl. Math.* **44** (2010), 85–110.

16 W. Y. C. Chen, L. X. W. Wang and A. L. B. Yang. Schur positivity and the q-log-convexity of the Narayana polynomials. *J. Algebraic Combin.* **32** (2010), 303–338.

17 W. Y. C. Chen, L. X. W. Wang and A. L. B. Yang. Recurrence relations for strongly q-log-convex polynomials. *Canad. Math. Bull.* **54** (2011), 217–229.

18 G.-S. Cheon and J.-H. Jung. The r-Whitney numbers of Dowling lattices. *Discrete Math.* **312** (2012), 2337–2348.

19 L. Comtet. *Advanced combinatorics.* (Dordrecht: D. Reidel Publishing Co., 1974).

20 H. Davenport and G. Pólya. On the product of two power series. *Canadian J. Math.* **1** (1949), 1–5.

21 T. A. Dowling. A class of geometric lattices based on finite groups. *J. Combin. Theory Ser. B* **14** (1973), 61–86; Erratum, *J. Combin. Theory Ser.* B **15** (1973), 211.

22 P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.* **32** (1980), 125–161.

23 S. Karlin. *Total Positivity*, vol. I, (Stanford: Stanford University Press, 1968).

24 P. Leroux. Reduced matrices and q-log concavity properties of q-Stirling numbers. *J. Combin. Theory Ser. A* **54** (1990), 64–84.

25 L. L. Liu and Y. Wang. On the log-convexity of combinatorial sequences. *Adv. in. Appl. Math.* **39** (2007), 453–476.

26 V. H. Moll. The evaluation of integrals: A personal story. *Notices Amer. Math. Soc.* **49** (2002), 311–317.

27 P. Mongelli. Excedances in classical and affine Weyl groups. *Journal of Combinatorial Theory, Series A* **120** (2013), 1216–1234.

28 A. Pinkus. *Totally Positive Matrices.* (Cambridge: Cambridge University Press, 2010).

29 G. Pólya and G. Szegö. *Problems and Theorems in Analysis*, vol. II, 3rd ed. (New York: Springer-Verlag, 1964).

30 M. Rahmani. Some results on Whitney numbers of Dowling lattices. *Arab J. Math. Sci.* **20** (2014), 11–27.

31 B. E. Sagan. Log concave sequences of symmetric functions and analogs of the Jacobi-Trudi determinants. *Trans. Amer. Math. Soc.* **329** (1992), 795–811.

32 R. P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. *Ann. New York Acad. Sci.* **576** (1989), 500–534.

33 X.-T. Su, Y. Wang and Y.-N. Yeh. Unimodality Problems of Multinomial Coefficients and Symmetric Functions. *Electron. J. Combin.* **18** (2011), Research Paper 73.

34 S. Tanny. On some numbers related to the Bell numbers. *Canad. Math. Bull.* **17** (1975), 733–738.

35 H. S. Wall. *Analytic Theory of Continued Fractions.* (Princeton, NJ: Van Nostrand, 1948).

36 Y. Wang and Y.-N. Yeh. Log-concavity and LC-positivity. *J. Combin. Theory Ser. A* **114** (2007), 195–210.

37 Y. Wang and B.-X. Zhu. Log-convex and Stieltjes moment sequences. *Adv. in Appl. Math.* **81** (2016), 115–127.

38 T. Xiong, H.-P. Tsao and J. I. Hall. General Eulerian numbers and the Eulerian polynomials. *J. Math.* (2013), Article 629132, 19 pp.

39 B.-X. Zhu. Log-convexity and strong q-log-convexity for some triangular arrays. *Adv. in Appl. Math.* **50** (2013), 595–606.

40 B.-X. Zhu. Some positivities in certain triangular arrays. *Proc. Amer. Math. Soc.* **142** (2014), 2943–2952.

41 B.-X. Zhu. Positivity of iterated sequences of polynomials. *SIAM J. Discrete Math.* **32** (2018), 1993–2010.

42 B.-X. Zhu and H. Sun. Linear transformations preserving the strong q-log-convexity of polynomials. *Electron. J. Combin.* **22** (2015), P3.27.