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GROUPS WITH NORMALITY CONDITIONS FOR UNCOUNTABLE SUBGROUPS

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Dedicated to Leonid A. Kurdachenko on the occasion of his 70th birthday

Abstract

This paper continues the investigation of the structure of uncountable groups whose large subgroups are normal. Moreover, we describe the behaviour of uncountable groups in which every large subgroup is close to normal with the only obstruction of a finite section.

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1. Introduction

In a series of recent papers, it has been shown that the structure of an uncountable group is strongly influenced by that of its large subgroups (see, for instance, [1, 3, 4]). In particular, the behaviour of uncountable groups, of cardinality $\aleph$ say, in which all subgroups of cardinality $\aleph$ are normal or subnormal was investigated in [1]. The aim of this paper is to make a further contribution to this topic, by studying uncountable groups whose subgroups of large cardinality satisfy other relevant embedding properties, generalizing normality.

The main obstacle in the study of groups of large cardinality $\aleph$ is the existence of infinite groups in which all proper subgroups have cardinality strictly smaller than $\aleph$ (the so-called *Jónsson groups*). Clearly, a countably infinite group is a Jónsson group if and only if all its proper subgroups are finite, so that Prüfer groups and Tarski groups (that is, infinite simple groups whose proper nontrivial subgroups have prime order) have the Jónsson property. On the other hand, it is known that if G is an uncountable Jónsson group, then $G/Z(G)$ is simple and $G' = G$ (see, for instance, [5, Corollary 2.6]). Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by Shelah [15] and Obraztsov [12].

Throughout this paper, $\aleph$ will denote an uncountable regular cardinal number, and if G is a group of cardinality $\aleph$, a subgroup X of G will be called *large* if it likewise has cardinality $\aleph$, and *small* otherwise. Section 2 is dedicated to the study of uncountable groups whose large subgroups are normal, with the purpose of improving the results obtained in [1].

Uncountable groups in which every large subgroup is close to normal, with the only obstruction of a finite section, are considered in Section 3. Recall that a subgroup X of a group G is called *almost normal* if it has only finitely many conjugates in G, or equivalently if its normalizer $N_G(X)$ of X has finite index in G, while X is said to be *nearly normal* in G if it has finite index in its normal closure X^G. In a famous paper of 1955, Neumann [11] proved that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index, and that groups admitting only nearly normal subgroups are precisely those with a finite commutator subgroup. The imposition of almost normality or near normality only on the members of a relevant system of subgroups also gives rise to strong restrictions on the group structure (see, for instance, [2, 8, 9]).

Most of our notation is standard and can be found in [13].

2. Normality of large subgroups

Our first result collects some known properties of groups in which all large subgroups are normal (see [1, Theorem 2.5 and Lemma 3.2], and [3, Theorem 2.5]).

LEMMA 2.1. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are normal. Then the following statements hold.*

(a) *If G has no Jónsson large subgroups, then G' has cardinality strictly smaller than $\aleph$ and G is nilpotent of class at most 2.*

(b) *If G contains an abelian subgroup of cardinality $\aleph$, then G is a Dedekind group.*

As a consequence of the above lemma, we have that if G is an uncountable group whose large subgroups are normal, then there are only few commutator subgroups.

COROLLARY 2.2. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are normal, and let X be a large subgroup of G. Then G/X' is a Dedekind group. Moreover, if the factor group G/X' is not abelian, then $Y' = X'$ for each large subgroup Y of X.*

PROOF. Of course, X' is a normal subgroup of G. Suppose first that also the commutator subgroup X' of X is large; then all subgroups of G containing X' are normal, and so G/X' is obviously a Dedekind group. On the other hand, if the subgroup X' is small, we have that X/X' is an abelian subgroup of cardinality $\aleph$ of G/X', and so G/X' is a Dedekind group by Lemma 2.1(b).

Suppose that G/X' is not abelian, and let Y be any large subgroup of X. Then Y' is normal in G, and G/Y' is a Dedekind nonabelian group by the first part of the statement, so that G'/Y' has order 2. As $Y' \leq X' < G'$, it follows that $Y' = X'$. □

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Our first main result is a refinement of the main theorem proved in [1] on uncountable groups whose large subgroups are normal.

THEOREM 2.3. *Let G be an uncountable group of cardinality $\aleph$ which has no Jónsson large subgroups. Then all large subgroups of G are normal if and only if one of the following conditions holds:*

(a) *G is a Dedekind group;*
(b) *$X' = G'$ for each large subgroup X of G;*
(c) *$G = P \times Q$, where P is a nilpotent p-group of cardinality $\aleph$ for some prime p, Q is a periodic Dedekind p'-group, and either $p > 2$, P satisfies condition* (b) *and Q is nonabelian or $p = 2$ and P contains a nonabelian subgroup N of finite index satisfying* (b) *and such that P/N' is a Dedekind nonabelian group.*

PROOF. Assume that all large subgroups of G are normal, so that G is nilpotent by Lemma 2.1. Suppose that G neither is a Dedekind group nor satisfies condition (b), and let Y be a large subgroup of G such that $Y' \neq G'$. It follows from Corollary 2.2 that G/Y' is a Dedekind nonabelian group and so there exists a normal subgroup K of G such that G/K is isomorphic to the quaternion group of order 8. Since G/Y' is periodic and G is nilpotent, we have that G itself is periodic. Then it follows from [1, Theorem 3.5] that there exists a prime number p such that $G = P \times Q$, where P is a p-group of cardinality $\aleph$ and Q is a Dedekind group of cardinality strictly smaller than $\aleph$ which has no elements of order p.

Assume first that $p > 2$, so that P is contained in K and hence G/P is not abelian. It follows from Lemma 2.2 that $X' = P'$ for each large subgroup X of P, and so P satisfies condition (b) of the statement. Suppose now that $p = 2$, so that Q is an abelian subgroup contained in K. Put $N = K \cap P$. Thus P/N is isomorphic to the quaternion group of order 8; in particular, N has cardinality $\aleph$ and so it cannot be abelian. Moreover, P/N' is a Dedekind nonabelian group by Corollary 2.2, while it follows again from Lemma 2.2 that $X' = N'$ for every large subgroup X of N.

Conversely, let $G = P \times Q$ be a group satisfying condition (c) of the statement, and let X be any large subgroup of G. Then

$$X = (X \cap P) \times (X \cap Q),$$

where $X \cap P$ has cardinality $\aleph$ and $X \cap Q$ is a normal subgroup of G. Moreover, we have $(X \cap P)' = P'$ if $p > 2$ and $(X \cap P)' = N'$ when $p = 2$, so that $X \cap P$ is normal in P, and so also in G. Therefore the subgroup X is normal in G, and the proof is complete. □

The consideration of the direct product of a quaternion group of order 8 by a Prüfer 2-group shows that the statement of Theorem 2.3 does not hold if $\aleph = \aleph_0$.

Our aim is now to describe the behaviour of uncountable groups in which all large subgroups have the same commutator subgroup. Notice that these groups may have an infinite commutator subgroup; in fact, Ehrenfeucht and Faber [6] exhibited a nilpotent group G of class 2 and cardinality $\aleph_1$ whose commutator subgroup is

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isomorphic to the additive group of rational numbers, but $X' = G'$ for each uncountable subgroup X. In the same paper, the authors constructed (for a suitable prime number p) an extraspecial p-group G of cardinality $\aleph_1$ whose abelian subgroups are countable, so that also in this case G' is the commutator subgroup of any large subgroup.

Let G be a group, and let X be a subgroup of G. Recall that the *isolator* of X in G is the set $I_G(X)$ of all elements g of G such that g^k belongs to X for some positive integer k. It is known that in a locally nilpotent group the isolator of any subgroup is likewise a subgroup. Moreover, if G is a torsion-free locally nilpotent group and X is a subgroup of G which is nilpotent of class c, then $I_G(X)$ is also nilpotent of class c (see [10, Section 2.3]).

Lemma 2.4. *Let G be a torsion-free locally nilpotent group whose commutator subgroup G' is divisible abelian. Then the factor group G/G' is torsion-free.*

Proof. The isolator $I_G(G')$ of G' in G is an abelian normal subgroup of G, and of course the factor group $G/I_G(G')$ is torsion-free. On the other hand, $I_G(G')$ splits over its divisible subgroup G', and so $I_G(G') = G'$ because $I_G(G')/G'$ is periodic. Therefore the group G/G' is torsion-free. □

Theorem 2.5. *Let G be an uncountable group of cardinality $\aleph$ which has no Jónsson large subgroups. If $X' = G'$ for each large subgroup X of G, then G is nilpotent of class at most 2. Moreover, either the subgroup T consisting of all elements of finite order of G is large and G' has prime exponent or T is small and G' is divisible.*

Proof. All large subgroups of G are obviously normal, so that it follows from Lemma 2.1 that G is nilpotent of class at most 2 and G' has cardinality strictly smaller than $\aleph$.

Assume first that T is a large subgroup of G, so that $T' = G'$. Of course, it follows from the hypotheses that all finite homomorphic images of T are abelian, and hence in this case G' has prime exponent (see [1, Theorem 3.5]).

Suppose now that T has cardinality strictly smaller than $\aleph$ and assume for a contradiction that $(G')^k \neq G'$ for some positive integer k. As $\overline{G} = G/T$ is a torsion-free group of cardinality $\aleph$ in which all large subgroups are normal, its commutator subgroup $\overline{G}\,'$ is divisible (see [1, Theorem 3.6]) and hence $\overline{G}/\overline{G}\,'$ is likewise torsion-free by Lemma 2.4. It follows that $(\overline{G}/\overline{G}\,')^k$ has cardinality $\aleph$, and so also G^k has cardinality $\aleph$. Moreover, $[G^k, G] \leq (G')^k$ because G has class 2, so that $G^k/(G')^k$ is contained in the centre of G/G^k and hence $Z(G/(G')^k)$ has cardinality $\aleph$. Thus $G/(G')^k$ is a Dedekind group by Lemma 2.1. On the other hand, $G/(G')^k$ cannot be abelian, and so G contains a normal subgroup N such that G/N is isomorphic to the quaternion group of order 8, which is impossible, since in this case N has cardinality $\aleph$ and $N' \neq G'$. Therefore G' is a divisible group. □

The above quoted examples due to Ehrenfeucht and Faber show that both situations described in Theorem 2.5 may actually occur.

Our next statement shows that the situation is much easier in the countable case, at least within the universe of locally graded groups. Recall here that a group G is

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said to be *locally graded* if every finitely generated nontrivial subgroup of G has a proper subgroup of finite index. Locally graded groups form a wide class, containing in particular all locally (soluble-by-finite) groups.

Proposition 2.6. *Let G be an infinite locally graded group. If $X' = G'$ for each infinite subgroup X of G, then G is abelian.*

Proof. Assume for a contradiction that G is not abelian. Then G has no infinite abelian subgroups, so that it is periodic but not locally finite (see [13, Part 1, Theorem 3.43]). In particular, the commutator subgroup G' of G must be infinite. Moreover, G' is contained in the intersection N of all infinite subgroups of G, and hence $G' = N$. It follows that $G' = G''$ is perfect and all its proper subgroups are finite. As G' is not locally finite, it is finitely generated, which is impossible, because G is locally graded. This contradiction proves the statement. □

3. Further normality conditions for large subgroups

For our purposes we need the following lemma, which has been proved in [1].

Lemma 3.1. *Let $\aleph$ be a regular uncountable cardinal number, and let R be a principal ideal domain of cardinality strictly smaller than $\aleph$. If M is an R-module of cardinality $\aleph$, then M contains an R-submodule which is the direct sum of a collection of cardinality $\aleph$ of nontrivial R-submodules.*

The above result can of course be specialized by choosing as R the ring $\mathbb{Z}$ of integers. In this case, we obtain the following statement, which shows in particular that any uncountable abelian group A of cardinality $\aleph$ contains a subgroup which can be decomposed into the direct product of two subgroups of cardinality $\aleph$.

Corollary 3.2. *Let A be an uncountable abelian group of cardinality $\aleph$. Then A contains a subgroup which is the direct product of a collection of cardinality $\aleph$ of nontrivial subgroups.*

The example of Ehrenfeucht and Faber proves also that statements similar to Neumann's theorem do not hold in the case of uncountable groups in which all large subgroups are almost or nearly normal. However, this phenomenon depends on the absence of large abelian subgroups.

Theorem 3.3. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are almost normal. If G contains a large abelian subgroup, then the centre $Z(G)$ has finite index in G.*

Proof. It follows from Corollary 3.2 that G contains an abelian subgroup of the form $A = A_1 \times A_2$, where both A_1 and A_2 have cardinality $\aleph$. Then the normalizer $N_G(A_i)$ has finite index in G for $i = 1, 2$. Moreover, all subgroups of the group $N_G(A_i)/A_i$ are almost normal, and so $N_G(A_i)/A_i$ is finite over its centre C_i/A_i. Thus the subgroups C_1 and C_2 have finite index in G, and so the core $C = (C_1 \cap C_2)_G$ is an abelian normal

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subgroup of finite index of G. If g is any element of G, by Lemma 3.1 we have that C contains a subgroup $C(g)$ which is the direct product of a collection of cardinality $\aleph$ of nontrivial $\langle g\rangle$-invariant subgroups.

Let X be any small subgroup of $\langle C, g\rangle$. Then there exist two $\langle g\rangle$-invariant large subgroups U_1 and U_2 of $C(g)$ such that

$$U_1 \cap U_2 = X \cap \langle U_1, U_2\rangle = \{1\}.$$

As the subgroups XU_1 and XU_2 are almost normal, their intersection

$$X = XU_1 \cap XU_2$$

is also almost normal. It follows that all subgroups of $\langle C, g\rangle$ are almost normal, so that the centre $Z(\langle C, g\rangle)$ has finite index in $\langle C, g\rangle$ and hence even in G. Consider now a transversal $\{g_1, \ldots, g_t\}$ to C in G. Then

$$G = \bigcup_{i=1}^{t} \langle C, g_i\rangle$$

and so the intersection

$$\bigcap_{i=1}^{t} Z(\langle C, g_i\rangle)$$

is a subgroup of finite index of G which is contained in $Z(G)$. Therefore the centre $Z(G)$ has finite index in G. □

A celebrated theorem of I. Schur states that any group which is finite over the centre has a finite commutator subgroup (see, for instance, [13, Part 1, Theorem 4.12]). Thus it follows from Neumann's theorems that if a group G has only almost normal subgroups, then all subgroups of G are also nearly normal. The following consequence of Theorem 3.3 shows in particular that a corresponding result holds when the condition of being almost normal is imposed only on large subgroups.

Corollary 3.4. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are almost normal. Then X^G/X_G is finite for each large subgroup X of G.*

Proof. Of course, it can be assumed that the normal subgroup X_G has cardinality strictly smaller than $\aleph$, since otherwise the centre of G/X_G has finite index. Since X is a large subgroup, its normalizer $N_G(X)$ has finite index in G and $N_G(X)/X$ is finite over its centre C/X. It follows that the core N of C has finite index in G. Moreover, $N' \leq C' \leq X$, so that N' is contained in X_G and hence it is a small subgroup of G. Therefore N/N' is an abelian subgroup of cardinality $\aleph$ of G/N', and so G/N' is finite over the centre by Theorem 3.3. As $N' \leq X_G$, we have that X^G/X_G is finite. □

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements admitting only finitely many conjugates, and G is an *FC-group* if it coincides with its FC-centre. Thus G is an FC-group if and only if the centralizer $C_G(g)$ has finite index in G for each element g of G. It was proved by Semenova [14] that if G is any infinite

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FC-group, then every subgroup of G which is maximal with respect to the condition of being nilpotent of class at most 2 has the same cardinality of G (see also [16, Theorem 8.5]). This result was later sharpened by Tomkinson in the following way (see [16, p. 155]).

LEMMA 3.5. *Let G be an uncountable FC-group of cardinality $\aleph$. Then G contains a large subgroup X which is nilpotent of class at most 2 and has a finite commutator subgroup.*

THEOREM 3.6. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are nearly normal. If G contains a large FC-subgroup, then the commutator subgroup G' of G is finite.*

PROOF. Suppose first that G contains a large abelian subgroup. In this case it follows from Corollary 3.2 that there exists in G an abelian subgroup of the form $A = A_1 \times A_2$, where both A_1 and A_2 have cardinality $\aleph$. Then all subgroups of the factor group G/A_i^G are nearly normal, and hence the commutator subgroup $G'A_i^G/A_i^G$ of G/A_i^G is finite, for $i = 1, 2$. On the other hand, as the indices $|A_1^G : A_1|$ and $|A_2^G : A_2|$ are finite, the intersection $A_1^G \cap A_2^G$ is also finite, and so G' is likewise finite.

Assume now that all abelian subgroups of G are small. It follows from Lemma 3.5 that G contains a large subgroup U whose commutator subgroup U' is finite. By hypothesis U has finite index in its normal closure $V = U^G$, and so also the core $W = U_V$ has finite index in V. Then W/W' is a large abelian subgroup of V/W', and hence V/W' has finite commutator subgroup by the first part of the proof. It follows that V' is finite, so that V/V' is a large abelian subgroup of G/V', and again we obtain that the commutator subgroup G'/V' of G/V' is finite. Therefore G' is finite. □

In the last part of the paper we look for further information on the structure of uncountable groups whose large subgroups are nearly normal.

Recall first that every group G contains a maximum locally nilpotent normal subgroup $H(G)$, the so-called *Hirsch–Plotkin radical*, and that the *upper Hirsch–Plotkin series*

$$\{1\} = H_0(G) \leq H_1(G) \leq \cdots \leq H_\alpha(G) \leq H_{\alpha+1}(G) \leq \cdots$$

is defined recursively by putting

$$H_{\alpha+1}(G)/H_\alpha(G) = H(G/H_\alpha(G))$$

for each ordinal α and

$$H_\lambda(G) = \bigcup_{\alpha<\lambda} H_\alpha(G)$$

if λ is a limit ordinal. A group G is said to be *radical* if it admits an ascending (normal) series whose factors are locally nilpotent, or equivalently if its upper Hirsch–Plotkin series terminates with G. It is easy to show that in any group G the subgroup $R(G)$ generated by all radical normal subgroups is likewise radical and coincides with the last term of the upper Hirsch–Plotkin series of G.

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Our next statement should be seen in relation to the classical and easy result of Fedorov [7] which proves that any infinite group, in which all nontrivial subgroups have finite index, must be cyclic.

LEMMA 3.7. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups have finite index. Then the normal subgroup $R(G)$ is small, and so the factor group $G/R(G)$ has cardinality $\aleph$.*

PROOF. Assume for a contradiction that $R = R(G)$ is a large subgroup of G. Let

$$\{1\} = H_0 < H_1 < \cdots < H_\alpha < H_{\alpha+1} < \cdots < H_\tau = R$$

be the upper Hirsch–Plotkin series of G, and consider the smallest ordinal number $\lambda \leq \tau$ such that H_λ has cardinality $\aleph$. Let X be any small G-invariant subgroup of H_λ. If x is an element of X, the conjugacy class of x in G is contained in X, so that $|G : C_G(x)| < \aleph$ and hence the centralizer $C_G(x)$ has cardinality $\aleph$. It follows that $C_G(x)$ has finite index in G, and so x belongs to the FC-centre F of G. Therefore the join K of all small G-invariant subgroups of H_λ lies in F. Moreover, K cannot be properly contained in any G-invariant subgroup of infinite index of H_λ, and so in particular the factor group H_λ/K satisfies the maximal condition on G-invariant subgroups. Since H_λ has finite index in G, it follows that H_λ/K also satisfies the maximal condition on normal subgroups (see [17]). On the other hand, H_α is contained in K for each ordinal $\alpha < \lambda$, so that H_λ/K is locally nilpotent, and hence it is a finitely generated nilpotent group (see [13, Part 1, Theorem 5.37]). Thus K is an FC-group of cardinality $\aleph$, and so by Lemma 3.5 it contains a large nilpotent subgroup U whose commutator subgroup U' is finite. As the abelian group U/U' likewise has cardinality $\aleph$, it follows from Corollary 3.2 that U contains a large subgroup V such that $|U : V| = \aleph$. This contradiction proves that the cardinality of R is strictly smaller than $\aleph$. □

The following two consequences of Lemma 3.7 apply in particular to the case of radical groups.

COROLLARY 3.8. *Let G be an uncountable group of cardinality $\aleph$. If the factor group $G/R(G)$ is countable, then G contains a large subgroup X such that the index $|G : X|$ is infinite.*

COROLLARY 3.9. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are nearly normal. If the factor group $G/R(G)$ is countable, then the commutator subgroup G' of G has cardinality strictly smaller than $\aleph$.*

PROOF. Assume for a contradiction that G' is a large subgroup of G, so that by Corollary 3.8 it contains a large subgroup X such that the index $|G' : X|$ is infinite. Then all subgroups of G/X^G are nearly normal, and hence the commutator subgroup $G'X^G/X^G$ is finite. On the other hand, X has finite index in X^G and so also in G', and this contradiction proves the statement. □

LEMMA 3.10. *Let G be a group, and let g be an element of G whose centralizer $C_G(g)$ is nearly normal. Then the normal closure $\langle g\rangle^G$ is an FC-group.*

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Proof. Put $C = C_G(g)$. As the index $|C^G : C|$ is finite, the element g has only finitely many conjugates in C^G. Then $\langle g\rangle^G$ is contained in the FC-centre of C^G, and in particular it is an FC-group. □

Theorem 3.11. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups of G are nearly normal. If the commutator subgroup G' has cardinality strictly smaller than $\aleph$, then the elements of finite order of G form a subgroup T and the factor group G/T is nilpotent of class at most* 2*.*

Proof. Let g be any element of G. As the normal closure $\langle g\rangle^G$ is contained in $\langle g, G'\rangle$, it is a small subgroup of G. Then the index $|G : C_G(g)|$ is strictly smaller than $\aleph$, and hence the centralizer $C_G(g)$ is a large subgroup. It follows that $C_G(g)$ is nearly normal in G, and so $\langle g\rangle^G$ is an FC-group by Lemma 3.10. In particular, the normal closure of any element of finite order of G is periodic, and so the set T of all elements of finite order of G is a subgroup. Moreover, $\langle g\rangle^G T/T$ is abelian for each element g of G, so that the torsion-free group G/T is 2-Engel and hence also nilpotent of class at most 2 (see [13, Part 2, Theorem 7.14]). □

Corollary 3.12. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are nearly normal. If the factor group $G/R(G)$ is countable, then the elements of finite order of G form a subgroup T and the factor group G/T is nilpotent of class at most* 2*.*

Corollary 3.13. *Let G be a torsion-free uncountable group of cardinality $\aleph$ in which all large subgroups are nearly normal. If the commutator subgroup G' of G is small, then the index $|G' : (G')^k|$ is finite for each positive integer k.*

Proof. Application of Theorem 3.11 yields that G is nilpotent of class at most 2, and hence $G/Z(G)$ is a torsion-free abelian group. Of course, it can be assumed that G is not abelian, so that its centre $Z(G)$ is a small subgroup by Theorem 3.6. If k is any positive integer, it follows that

$$G^k Z(G)/Z(G) \simeq G/Z(G)$$

has cardinality $\aleph$, and hence G^k is a large subgroup of G. Moreover,

$$[G^k, G] = (G')^k,$$

so that $G^k/(G')^k$ is a subgroup of cardinality $\aleph$ of $Z(G/(G')^k)$ and hence the commutator subgroup $G'/(G')^k$ is finite by Theorem 3.6. □

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