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On identifying the true sources of aerodynamic sound

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A space–time filtering approach is used to divide an unbounded turbulent flow into its radiating and non-radiating components. The result is used to investigate the possibility of identifying the true sources of the sound generated by this flow.

1. Introduction

Lighthill's (1952) acoustic analogy approach and its extensions (Phillips 1960; Lilley 1974), remain the principal tools for predicting the noise from high-speed air jets. In the most general sense, they amount to rearranging the Navier–Stokes equations into a form that separates out the linear terms and associates them with propagation effects that can then be determined as part of the solution. The nonlinear terms are treated as 'known' source functions to be determined by modelling and, in more recent approaches, with some or all of the model parameters being determined from a steady RANS calculation. (Alternative approaches that attempt to represent the sound sources as linear instability waves include Crow's line antenna model; Crow & Champagne 1971.)

The major drawback with these approaches is that the unsteady effects, which actually generate the sound, must be included as part of the model. This clearly puts severe demands on the modelling aspects of the prediction, which usually amount to assuming a functional form for the two-point time-delayed velocity correlation spectra. The present paper describes an approach in which the acoustic sources can be determined as part of the base flow computation and, therefore, do not have to be modelled.

Goldstein (2003) shows that the Navier–Stokes equations can be recast into the convective form of the linearized Navier–Stokes (LNS) equations but with modified dependent variables, with the viscous stress perturbation replaced by a certain generalized Reynolds stress and with the heat flux perturbation replaced by a generalized stagnation enthalpy flux. The 'base flow' about which the equations are linearized can be any solution to a very general class of inhomogeneous Navier–Stokes equations with arbitrarily specified source strengths. The LNS equations are, of course, still nonlinear but the nonlinearity is effectively contained in the generalized Reynolds stresses and enthalpy flux which also contain contributions from the base flow sources. The acoustic analogy methods and their extensions correspond to taking the base flow to be a (steady) approximation to the mean flow field in the jet and treating the generalized stresses and enthalpy flux as known source terms that can be estimated or modelled as in the original Lighthill analysis. The current view is that these are only apparent sound sources and that the acoustic analogy approach cannot be used to identify the 'true sources of sound'. Fortunately, it is only the correlation of these sources and not their instantaneous values, that need to be modelled, but this is still a difficult task that requires a great deal of empiricism.

The so-called hybrid approaches were introduced in an attempt to minimize this requirement. The base flow is taken to be the large-eddy simulation (LES) equations in these approaches, i.e. the filtered Navier–Stokes equations with a purely spatial filter whose width is of the order of or larger than the mesh size. But computer storage limitations usually require that the latter be very coarse and the resulting computations are not able to adequately account for the, auditorially important, high-frequency component of the spectrum. The missing component is then calculated from the residual equations whose source terms still have to be modelled (Bodony & Lele 2002). Unfortunately, it is now necessary to model their instantaneous values and not their correlations, which is an extremely difficult task.

The present paper shows that the base flow filter can be chosen to pass almost all of the non-radiating components of the motion and none of the radiating components. The residual flow, which primarily consists of the radiating components of the motion, should therefore be much smaller than the base flow (since it is known that only a small fraction of the flow energy is radiated as sound) and the largest constituent of the residual equation source terms should then come from the base flow contribution, which can be determined as part of the base flow computation. The residual flow, which can now be identified with the acoustic part of the motion, is then governed by linear equations and almost entirely generated by known sources. The hope is that the latter can be identified with the highly elusive 'true sources of sound' (Fedorchenko 2001), since they generate only radiating components of the motion.

Most of the jet noise reduction achieved over the past fifty years can be attributed to reductions in the jet exhaust velocity. Noise suppression devices produce only relatively small reductions and require considerable (primarily empirical) development to achieve even these modest results. This is usually attributed to the almost complete lack of theoretical guidance. The hope is that computations of the non-radiating base flows developed herein will provide the needed insights.

2. The LNS equations

The Navier–Stokes equations can be written as

$$\frac{\partial}{\partial t}\Lambda_\nu + \frac{\partial}{\partial x_j}\Gamma_{\nu j}, \tag{2.1}$$

where the summation convention is being used, but with the Greek indices ranging from 1 to 5, while the Latin indices i, j are restricted to the range 1, 2, 3, and $\{\Lambda_\nu\} = \{\rho v_i, \rho h_o - p, \rho\}$, $\{\Gamma_{\nu j}\} = \{\rho v_i v_j + \delta_{ij} p - \sigma_{ij}, \rho v_j h_o + q_j - v_i \sigma_{ij}, \rho v_j\}$,

$$h_o \equiv h + \tfrac{1}{2} v^2, \tag{2.2}$$

denotes the stagnation enthalpy, h denotes the enthalpy, t denotes the time, $\boldsymbol{x} = \{x_1, x_2, x_3\}$ are Cartesian coordinates, p denotes the pressure, ρ denotes the density, $\boldsymbol{v} = \{v_1, v_2, v_3\}$ is the fluid velocity, σ_{ij} is the viscous stress tensor, q_i is the heat flux vector and the dependent variables are assumed to satisfy the ideal gas law:

$$p = \rho R T, \tag{2.3}$$

with $R = c_p - c_v$ being the gas constant, c_p and c_v are the specific heats at constant pressure and volume, respectively, and T the absolute temperature. Goldstein (2003) showed that can be recast into the form of the linearized Navier–Stokes equations by dividing the dependent variables

$$\rho = \overline{\rho} + \rho', \quad p = \overline{p} + p', \quad v_i = \widetilde{v}_i + v_i', \tag{2.4}$$

as well as the viscous stress σ_{ij} and heat flux q_i, into their 'base flow' components $\overline{\rho}$, $\overline{p}$, $\widetilde{h}$, $\widetilde{v}_i$, $\overline{\sigma}_{ij}$, $\overline{q}_i$ and into their 'residual' components ρ', p', h', σ'_{ij}, q'_i and requiring that the former satisfy the inhomogeneous Navier–Stokes equations (where the source terms are incorporated into the dependent-variable vectors $\widetilde{\Lambda}_\nu$ and $\widetilde{\Gamma}_{\nu j}$, rather then being placed on the right-hand side as is usually done)

$$\frac{\partial}{\partial t}\widetilde{\Lambda}_\nu + \frac{\partial}{\partial x_j}\widetilde{\Gamma}_{\nu j}, \tag{2.5}$$

along with an ideal-gas-law equation of state,

$$\widetilde{h} = c_p \widetilde{T} = \frac{c_p}{R}\frac{\overline{p}}{\overline{\rho}} \tag{2.6}$$

where $\{\widetilde{\Lambda}_\nu\} = \{\overline{\rho}\,\widetilde{v}_i, \overline{\rho}\widetilde{h}_o - \overline{p} - \widetilde{H}_o, \overline{\rho}\}$, $\{\widetilde{\Gamma}_{\nu j}\} = \{\overline{\rho}\widetilde{v}_i\widetilde{v}_j + \delta_{ij}\overline{p} - \overline{\sigma}_{ij} - \widetilde{T}_{ij}, \overline{\rho}\widetilde{v}_j\widetilde{h}_o + \overline{q}_j - \widetilde{v}_i\overline{\sigma}_{ij} - \widetilde{H}_j - \widetilde{v}_j\widetilde{H}_o, \overline{\rho}\widetilde{v}_j\}$,

$$\widetilde{h}_o = \widetilde{h} + \tfrac{1}{2}\widetilde{v}^2 \tag{2.7}$$

is the base-flow stagnation enthalpy, and the 'sources strengths' $\widetilde{T}_{ij}$, $\widetilde{H}_o$ and $\widetilde{H}_j$ which are assumed to be localized, can otherwise be specified arbitrarily. The reason for using both overbars and tildes to define the base-flow variables will become clear when equations (3.1) and (3.2)are introduced below.

The residual variables are governed by the (convective form of the) LNS equations

$$L_{\mu\nu}u_\nu = s_\mu, \tag{2.8}$$

where

$$\{u_\nu\} = \{\overline{\rho}u_i', p_o', \rho'\} \tag{2.9}$$

with

$$\bar{\rho}u_i' \equiv \rho v_i' \tag{2.10}$$

and

$$p_o' \equiv p' + (\gamma - 1)\left(\tfrac{1}{2}\rho v'^2 + \tilde{H}_o\right) \tag{2.11}$$

is a five-dimensional (non-linear) dependent-variable vector, $L_{\mu\nu}$ is the five-dimensional linearized Euler operator defined in the Appendix, the five-dimensional source vector s_μ is given by

$$s_\mu \equiv \frac{\partial}{\partial x_j}e'_{j\mu} + \delta_{\mu 4}(\gamma - 1)\,e'_{ij}\frac{\partial \tilde{v}_i}{\partial x_j}, \tag{2.12}$$

$\gamma \equiv c_p/c_v$ is the specific heat ratio, the source strengths $e'_{i\nu}$ are given by

$$e'_{i\nu} \equiv -\rho v_i' v_\nu' - \tilde{T}_{i\nu} + \delta_{i\nu}(\gamma - 1)\left(\tfrac{1}{2}\rho v'^2 + \tilde{H}_o\right) + \sigma'_{i\nu}, \tag{2.13}$$

for $\nu = 1, 2, ..4$ and zero otherwise and we have put

$$v_4' \equiv (\gamma - 1)\left(h' + \tfrac{1}{2}v'^2\right) = c^{2\prime} + \frac{(\gamma - 1)}{2}v'^2, \tag{2.14}$$

$$\tilde{T}_{i4} = (\gamma - 1)(\tilde{H}_i - \tilde{T}_{ij}\tilde{v}_j), \tag{2.15}$$

and

$$\sigma'_{j4} = -(\gamma-1)\big(q'_j - \sigma_{jl} v'_l\big). \tag{2.16}$$

Equation (2.8) is an exact consequence of the original Navier Stokes equations, but has been rearranged so that its left-hand side is the same as the equation that would have been obtained by linearizing the convective form of the Euler equations about a 'fictitious' base flow. In other words, it is the linearized inhomogeneous Euler (LIE) equation but with different (nonlinear) dependent variables, u_ν, which causes no particular difficulty because, for the present purpose, they can be treated as five linear equations in five unknowns that satisfy linear far-field boundary conditions, since the dependent variables all become linear there. The right-hand side corresponds to the sources that would be obtained by imposing the external stress perturbation e'_{ij} and external energy flux perturbation e'_{i4} on the 'fictitious' flow. In other words, *the fundamental equation (2.8) is just the Navier–Stokes equations linearized about a fictitious base flow (or the LNS equations), but with modified dependent variables and with the viscous stress perturbation replaced by the generalized Reynolds stress and the heat flux perturbation replaced by the generalized stagnation enthalpy flux.*

In this paper we assume, as is usually done in jet acoustics, that solid boundaries have little or no effect on the sound generation process. Then equation (2.8) can be solved in terms of the free-space vector Green's function (Morse & Feshbach 1953, pp. 878–886) $g_{\nu\sigma}(\boldsymbol{x},t|\boldsymbol{x}',t')$, which satisfies

$$L_{\mu\nu}\, g_{\nu\sigma} = \delta_{\mu\sigma}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{2.17}$$

to obtain

$$u_\nu(\boldsymbol{x},t) = \int_{-\infty}^{\infty}\int_V g_{\nu\mu}(\boldsymbol{x},t|\boldsymbol{x}',t') s_\mu(\boldsymbol{x}',t')\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}t', \tag{2.18}$$

where the symbol V denotes integration over all space. This then becomes

$$u_\nu(\boldsymbol{x},t) = -\int_{-\infty}^{\infty}\int_V \tilde{\gamma}_{\nu j\mu}(\boldsymbol{x},t|\boldsymbol{x}',t') e'_{\mu j}(\boldsymbol{x}',t')\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t', \tag{2.19}$$

where

$$\tilde{\gamma}_{\nu j\mu}(\boldsymbol{x},t|\boldsymbol{x}',t') \equiv \frac{\partial}{\partial x'_j} g_{\nu\mu} - (\gamma-1)\frac{\partial \tilde{v}_\mu}{\partial x'_j} g_{\nu 4} \tag{2.20}$$

when the derivatives acting on the source strengths $e'_{j\mu}$ in (2.12) are transferred to the Green's function.

3. The acoustic analogy approaches

As noted in the Introduction, the base flow is taken to be the actual mean flow field in the jet, or some approximation to that flow, in the acoustic analogy approaches. The overbar on the dependent base-flow variables would then denote the time average

$$\bar{\bullet} \equiv \lim_{T\to\infty}\int_{-T}^{T} \bullet(\boldsymbol{x},t)\,\mathrm{d}t, \tag{3.1}$$

where the dot is a place holder for ρ, v_i, p, and h, and

$$\tilde{\bullet} \equiv (\overline{\rho\bullet})/\bar{\rho} \tag{3.2}$$

denotes a Favre-averaged quantity (Lele 1994) for all variables except $\widetilde{h}_o$, which is defined by (2.7). Notice that equation (2.6) is completely consistent with the ideal gas law $p=\rho RT$ when the tilde is defined in this fashion.

The time derivatives now drop out of the base-flow equations (2.5), which do not, of course, form a closed system. The source strengths are given by

$$\widetilde{T}_{ij} = -\bar{\rho}(\widetilde{v_i v_j} - \tilde{v}_i \tilde{v}_j), \tag{3.3}$$

$$\tilde{H}_o = \tfrac{1}{2}\tilde{T}_{ii}, \tag{3.4}$$

and

$$\tilde{H}_j = -\bar{\rho}(\widetilde{h_o v_j} - \tilde{h}_o \tilde{v}_j) - \tilde{H}_o \tilde{v}_j. \tag{3.5}$$

The base flow equations are now the ordinary Reynolds-averaged Navier–Stokes (RANS) equations, which are usually closed by assuming some sort of model relating the source terms to the mean flow variables $\widetilde{v}_i$, $\overline{\rho}$, $\overline{p}$ and their derivatives, such as the Boussinesq model (Speziale 1991; Speziale & So 1998)

$$\tilde{T}_{ij} = \mu_T \left(\frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i} - \frac{2}{3} \frac{\partial \tilde{v}_k}{\partial x_k} \right) + \frac{2}{3} \tilde{H}_o \delta_{ij} \tag{3.6}$$

for the Reynolds stresses, with additional equations to determine the turbulent viscosity μ_T and a similar model for $\widetilde{H}_j$. The fundamental LNS equation (2.8) remains exact even when these approximations are introduced since, as already noted, the base-flow source strengths can be arbitrarily specified. But (3.1) would no longer hold and the base flow would be somewhat different from the actual mean flow in the jet.

However, when (3.1) does apply, the base-flow sources (3.3) to (3.5) can be written more compactly as

$$\tilde{T}_{ij} = -\bar{\rho}\widetilde{v_i' v_j'}, \tag{3.7}$$

$$\tilde{T}_{i4} = \tilde{H}_i - \tilde{T}_{ij}\tilde{v}_j = -\bar{\rho}\widetilde{h_o' v_i'}, \tag{3.8}$$

with $\widetilde{H}_o$ still given by (3.4). Then when the flow is inviscid, the LNS equation source strengths e'_{ij} and e'_{i4} are proportional to the differences between the fluctuating and Favre-averaged Reynolds stresses and enthalpy fluxes, which means that they are true fluctuating quantities with zero time averages which ensures that the residual variables will have this property as well.

The corresponding LNS equation sources must still be modelled. This is most easily accomplished by using the fourth (i.e. the pressure-like) component of the formal Greens function solution (2.19) to express the far-field pressure in terms of the near- field source distribution which, in view of (2.9), (2.13), (2.20), (3.4), (3.7), and (3.8), can be written as

$$p_o' = \int_{-\infty}^{\infty} \int_V \gamma_{j\mu}(\boldsymbol{x}, t | \boldsymbol{x}', t') \tau_{\mu j}(\boldsymbol{x}', t') \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t' \tag{3.9}$$

where

$$\gamma_{j\mu} \equiv -\frac{\partial}{\partial x_j'} g_{4\mu} + \frac{\gamma - 1}{2} \delta_{j\mu} \frac{\partial g_{4l}}{\partial x_l'} + (\gamma - 1) \left(\frac{\partial \tilde{v}_\mu}{\partial x_j'} - \frac{\gamma - 1}{2} \delta_{j\mu} \frac{\partial \tilde{v}_l}{\partial x_l'} \right) g_{44} \tag{3.10}$$

and

$$\tau_{\mu j} \equiv -(\rho v_j' v_\mu' - \bar{\rho}\widetilde{v_j' v_\mu'}) + \sigma_{j\mu}' \tag{3.11}$$

when the bulk viscosity is zero and the base flow is taken to be the actual mean flow in the jet. So in the inviscid limit, which is of primary interest here, $\tau_{\mu j}$ is just a generalized four-dimensional fluctuating Reynolds stress and equation (3.9) therefore provides a direct linear relation between this quantity and the far-field pressure fluctuation (recall that p_o' reduces to the latter in the far field).

This result can be time averaged to obtain the expression

$$\overline{p^2}(\boldsymbol{x}, t_o) = \int_{-\infty}^{\infty} \int_V \int_V \bar{\gamma}_{j\sigma l\mu}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{\eta}, t_o + \tau)\bar{\tau}_{\sigma j\mu l}(\boldsymbol{y}; \boldsymbol{\eta}, \tau)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\tau \tag{3.12}$$

for the pressure autocovariance (Pope 2000)

$$\overline{p^2}(\boldsymbol{x}, t_o) \equiv \frac{1}{2T}\int_{-T}^{T} p_o'(\boldsymbol{x}, t)p_o'(\boldsymbol{x}, t + t_o)\,\mathrm{d}t, \tag{3.13}$$

where T denotes some large but finite time interval,

$$\bar{\gamma}_{j\sigma\mu l} \equiv \int_{-\infty}^{\infty} \gamma_{j\sigma}(\boldsymbol{x}|\boldsymbol{y}, t_1 + t_o + \tau)\gamma_{\mu l}(\boldsymbol{x}|\boldsymbol{y} + \boldsymbol{\eta}, t_1)\,\mathrm{d}t_1 \tag{3.14}$$

accounts for the acoustic propagation and mean flow interaction effects and

$$\bar{\tau}_{\sigma i\mu j}(\boldsymbol{y}; \boldsymbol{\eta}, \tau) \equiv \frac{1}{2T}\int_{-T}^{T} \tau_{\sigma i}(\boldsymbol{y}, t')\tau_{\mu j}(\boldsymbol{y} + \boldsymbol{\eta}, t' + \tau)\,\mathrm{d}t' \tag{3.15}$$

is the density-weighted, fourth-order, two-point, time-delayed fluctuating velocity/ stagnation enthalpy correlation.

This result can be used to relate the mean-square pressure in the far field to the source correlation function, which means that it is only necessary to model this latter quantity and not the instantaneous sources themselves. Unfortunately, this is still a difficult task that requires a great deal of empiricism.

4. The hybrid methods

The so-called hybrid formulations (Bodony & Lele 2002) were introduced in an attempt to minimize this requirement. They correspond to taking the base flow to be a large-eddy simulation (LES), which may include the large-scale coherent structures in the jet. The base-flow equations then correspond to the filtered Navier–Stokes equations (Goldstein 2000, 2002), which amounts to interpreting the overbars in equation (2.5) as the filtered variables

$$\overline{f} \equiv \int_{-\infty}^{\infty} \int_V g(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)f(\boldsymbol{\xi}, \tau)\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\tau, \tag{4.1}$$

where f can be any function and g denotes a generalized filter in both space and time (Aldama 1990). We have, for reasons that will become apparent, written (4.1) as a spatial-temporal filter, even though pure spatial filters are used in virtually all large-eddy simulations. The tilde is defined by equations (3.2) and (4.1), the source terms $\widetilde{T}_{ij}$, $\widetilde{H}_o$ and $\widetilde{H}_j$ are still given by (3.3) to (3.5), and the residual variables, v_i', p' and h', which in the present context might best be referred to as the 'unresolved components' of the flow, are still determined by (2.8) to (2.16).

The base-flow sources must again be modelled in order to close the system and the most commonly used model is still of the Boussinesq type (3.6) (Rogallo & Moin 1984). Unfortunately, as noted in the Introduction, the calculations must be performed on relatively coarse grids and are, therefore, not able to adequately resolve the high-frequency component of the sound field. This missing component is then calculated from the residual, or LNS, equations and the complete sound field can, at least in principle, be determined by adding the latter to the base-flow sound field.

The governing equations are, in principle, exact, but the LNS equation sources still have to be modelled and, since the base flow is unsteady, it is no longer possible to use the procedure described in the acoustic analogy context to relate the far-field pressure correlation to the correlation of the source function. It is therefore necessary to model the instantaneous source strengths rather than their correlations, which is much easier said than done. It may be possible to use some sort of stochastic source model such as the ones used by Bailly, Lafon & Candel (1995) and Bodony & Lele (2002). But this requires modelling the time history of the sources which is much more difficult than modelling their statistics as in the RANS approach. This difficulty would, however, be overcome if the residual equation source strengths could be determined from the base flow computation, i.e. if e'_{iv} were dominated by the base flow contribution, which would be the case if the residual component of the motion could be treated as a small perturbation about the base flow component.

5. Non-radiating unsteady base flows

Since the radiated sound is typically four to five orders of magnitude smaller than the non-radiating component of the motion in virtually all high-speed air jets, as well as in most other high-speed flows† this effective linearization would be achieved if the base flow in the decomposition (2.4) were taken to be the entire (or nearly the entire) non-radiating component of the motion. To construct such a base flow it is first necessary to demonstrate that the filter g in equation (4.1) can be chosen to make the base flow non-radiating (Goldstein 2002). This can be done by applying Lighthill's (1952) analysis to the base flow equation (2.5) with source term (3.3) to obtain

$$\frac{\partial^2 \bar{\rho}}{\partial t^2} - c_o^2 \frac{\partial^2 \bar{\rho}}{\partial x_j \partial x_j} = \frac{\partial^2}{\partial x_i \partial x_j} \tilde{\theta}_{ij} \tag{5.1}$$

where

$$\theta_{ij} \equiv \rho v_i v_j + \delta_{ij}\left(p - c_o^2 \rho\right) \tag{5.2}$$

denotes the Lighthill stress tensor,

$$\tilde{\theta}_{ij} \equiv \bar{\rho}\widetilde{v_i v_j} + \delta_{ij}\left(\bar{p} - c_o^2 \bar{\rho}\right) \tag{5.3}$$

denotes the corresponding filtered tensor, and the viscous terms, which are believed to play an insignificant role in the sound generation process, have been omitted. This equation can be solved to obtain (recall that we are assuming the flow to be unbounded)

$$\bar{\rho} = \frac{1}{4\pi c_o^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{\tilde{\theta}_{ij}(\boldsymbol{y}, t - |\boldsymbol{x} - \boldsymbol{y}|/c_o)}{|\boldsymbol{x} - \boldsymbol{y}|} \mathrm{d}\boldsymbol{y} \tag{5.4}$$

which behaves like

$$c_o^2 \bar{\rho} \rightarrow \frac{x_i x_j}{4\pi c_o^2 |\,\boldsymbol{x}|^3} \frac{\partial^2}{\partial t^2} \int_V \tilde{\theta}_{ij}\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_o} + \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{|\boldsymbol{x}| c_o}\right) \mathrm{d}\boldsymbol{y} \quad \text{as } |\boldsymbol{x}| \rightarrow \infty. \tag{5.5}$$

† In fact, it is usually many orders of magnitude smaller than the errors incurred in computing the non-radiating part of the flow. A number of investigators have even suggested that the 'acoustic component' of the numerical solution to the full Navier–Stokes equations would be hopelessly corrupted by these errors or even by the computational noise itself (Crighton 1993).

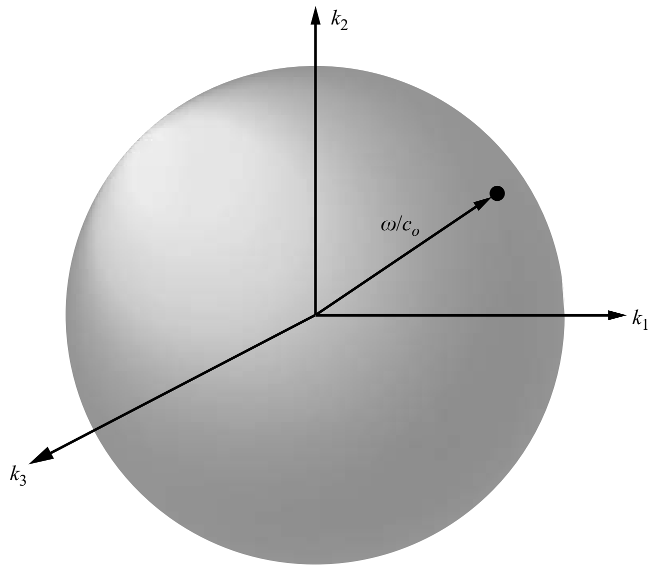

FIGURE 1. Surface of sound-producing wavenumbers.

Taking Fourier transforms yields

$$\bar{P} \to -\frac{2\pi^2\omega^2 x_i x_j \mathrm{e}^{\mathrm{i}\omega|\boldsymbol{x}|/c_o}}{c_o^2|\boldsymbol{x}|^3}\tilde{\Theta}_{ij}(\omega\boldsymbol{x}/|\boldsymbol{x}|c_o, \omega) \quad \text{as } |\boldsymbol{x}| \longrightarrow \infty, \tag{5.6}$$

where $k=\{k_1, k_2, k_3\}$, $\overline{P}$ denotes the Fourier transform with respect to time of $c_o^2\overline{\rho}$ and the remaining capital letters are used to denote the four-dimensional Fourier transforms

$$F(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_V \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} f(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{5.7}$$

of the corresponding lower-case symbols. Equation (5.6) shows that only the wavenumber components lying on the sphere $k^2=(\omega/c_o)^2$ can radiate to the far field and that all wavenumbers lying on this sphere will radiate at some angle (see figure 1). But the convolution theorem (Morse & Feshbach 1953) shows that

$$\tilde{\Theta}_{ij}(\boldsymbol{k}, \omega) = (2\pi)^4 G(\boldsymbol{k}, \omega)\Theta_{ij}\boldsymbol{k}, \omega), \tag{5.8}$$

which means that the base flow will be non-radiating if the filter is chosen so that its Fourier transform G vanishes when $k=\pm\omega/c_o$. An appropriate choice would be

$$G(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^4(1+10\pi)}\left\{1+\exp(2\omega/\Delta c_o)^2 - \exp\left[-\left(k-\frac{\omega}{c_o}\right)^2\Big/\Delta^2\right]\right.$$
$$\left. - \exp\left[-\left(k+\frac{\omega}{c_o}\right)^2\Big/\Delta^2\right]\right\}, \tag{5.9}$$

where Δ is a suitably small constant. Other choices are, of course, possible. The filter can now be constructed by taking the inverse transform of (5.9). Since the source and nonlinear terms are expected to vanish at large distances from the flow, the base-flow equations should reduce to the homogeneous linear acoustic equations there, which ensures that the remaining flow variables will also be non-radiating.

It is also possible to construct Fourier transform filters G that vanish on only a portion of the radiating sphere $k^2=(\omega/c_o)^2$ corresponding to a range of streamwise wavenumbers, say,

$$\cos\theta_1 < (c_o k_1/\omega) < \cos\theta_2. \tag{5.10}$$

This would then restrict the acoustic radiation to the range of polar angles $\theta_1 < \theta < \theta_2$ which could be chosen to, say, minimize the environmental impact of the radiated noise. In higher-Mach-number supersonic flows, it may be desirable to choose the range of angles so that propagating disturbances that remain within the jet boundaries are retained as part of the base flow.

It is, of course, still necessary to model the filtered stresses (3.3) to (3.5), which now account for the effect of the radiating component of the flow on the non-radiating component. The Boussinesq model is almost certainly inappropriate here, but the base-flow equations can be closed by replacing v_ν by $\widetilde{v}_\nu$ (i.e. neglecting the extremely small contribution from the radiating part v'_ν) in the $\widetilde{T}_{ij}$ and the $\widetilde{H}_j$ components of the source function so that

$$\tilde{T}_{ij} \approx -\bar{\rho}(\widetilde{\tilde{v}_i\tilde{v}_j} - \tilde{v}_i\tilde{v}_j) \tag{5.11}$$

and

$$\tilde{H}_j \approx -\bar{\rho}(\widetilde{\tilde{h}_o\tilde{v}_j} - \tilde{h}_o\tilde{v}_j) - \tilde{H}_o\tilde{v}_j. \tag{5.12}$$

The result is that the original differential equation is replaced by an integro-differential equation, which could be difficult to solve numerically. It may be easier to compute the base flow by using a Fourier–Spectral method with the radiating spectral base functions eliminated from the computation.

Notice that the first terms in the base-flow source components (3.3) to (3.5), or (5.11) and (5.12), are non-radiating disturbances but the second terms, which involve quadratic interactions between the non-radiating components, can generate radiating wavenumbers. The expectation is that the difference between these two terms will be much smaller than either of them individually. The complete residual equation sources $e'_{i\nu}$ involve both the base flow sources and quadratic residual components. The latter, which can either represent true sound sources or nonlinear propagation effects, are likely to be small at subsonic and moderate supersonic speeds, since, as noted above, only a very small fraction of the flow energy can radiate, which means that the base-flow component sources produce most of the sound. The relative importance of these terms, of course, should shift when the Mach number becomes very large, causing the radiation field to exhibit a bimodal structure.

Lighthill (1952) argued that the strength of his quadrupole source could be obtained to a good approximation by calculating its value for an equivalent flow devoid of sound. The present result provides an analytical basis for that idea. It is, of course, possible to move the residual stresses to the left side of the equations and calculate the sound from the full nonlinear equations, but this would make the present approach more complicated and computationally more expensive than solving the original Navier–Stokes equations. In fact, the main justification for using the present approach would be to ensure that the radiating component of the motion is uncontaminated by errors incurred in computing the non-radiating component, which is best accomplished by requiring that the residual variables be much smaller than the base-flow variables, which would, in turn, imply that the LNS equations can, in fact, be linearized. (Recall that the term linearized Navier–Stokes equation is somewhat of a misnomer in that they actually contain nonlinear terms that are embedded in the source functions.)

6. Conclusions

A general set of linearized inhomogeneous Euler (LIE) equations is used to develop a non-acoustic analogy approach to aerodynamic noise prediction. The 'base flow' about which the equations are linearized can be any solution to a very general class of inhomogeneous Navier–Stokes equations with arbitrarily specified source strengths. The acoustic analogy methods and their extensions correspond to taking the base flow to be a (steady) approximation to the mean flow field in the jet and treating the source terms as known quantities that can be estimated or modelled as in the original Lighthill analysis. The more recently developed hybrid approaches amount to taking the base flow to be the LES equations, i.e. the filtered Navier–Stokes equations with a purely spatial filter (but see Bodony & Lele 2003).

Since the Fourier transform filter width, in (5.9), can be made arbitrarily small, the present result shows that the base-flow can be chosen so that it is nearly the entire non-radiating component of the motion. The residual component of the LNS equation source term should therefore be small compared to the base-flow component, which can be calculated as part of the base-flow computation. The residual flow, which consists almost entirely of the radiating components of the motion, therefore satisfies linear equations and is largely generated by known sources.

This decomposition has certain computational advantages in that the nonlinear but non-radiating (and therefore relatively localized) base flow can be calculated by using conventional well-established computational fluid dynamic (CFD) techniques. The sound field can then be determined from the linear residual equations (with radiation boundary conditions) by using methods designed to accurately capture the propagating wave motion. Even more importantly, however, it also has, as noted in the Introduction, certain theoretical significance in that it provides a rigorous mechanism for identifying the highly elusive 'true sources of sound'.

Ever since Kovasnay (1953) showed that a small-amplitude inviscid motion on a completely uniform flow could be decomposed into its acoustic and vortical components in the sense that the acoustic component carries all the pressure fluctuations but no vorticity, there have been numerous unsuccessful attempts to find similar decompositions for the small-amplitude motion on other base flows (e.g. Fedorchenko 2001). Part of the difficulty is that the term 'acoustic component' is usually not defined or, at best, only vaguely defined. Here we consider only relatively low-Mach-number unbounded flows and identify the 'acoustic component' with the radiating part of the motion. Unfortunately, the prevailing view seems to be that the acoustic field is just an unavoidable by-product of any compressible motion and that it is impossible to decompose an arbitrary flow into 'acoustic' and 'non-acoustic' components of this type. The present result is consistent with this idea in the sense that the filter width cannot be set to zero even though the radiating wavenumbers occupy zero volume in wavenumber space. The radiating part of the motion will therefore always contain some non-radiating component – it can, however, be made arbitrarily small. The implication is that an arbitrary motion can be decomposed into a non-radiating (i.e. non-acoustic) component and a nearly acoustic component – in the sense that it is almost completely, but not entirely, radiating.

Appendix

The five-dimensional linear Euler operator can be written as

$$L_{\mu\nu} \equiv \delta_{\mu\nu} D_o + \delta_{\nu 4}\partial_\mu + \partial_\nu(\widetilde{c^2}\delta_{\mu 4} + \delta_{\mu 5}) + K_{\mu\nu}, \tag{A 1}$$

with

$$K_{\mu\nu} \equiv \partial_\nu \tilde{v}_\mu - \frac{1}{\bar{\rho}} \frac{\partial \tilde{\tau}_{\mu j}}{\partial x_j} \delta_{\nu 5} + (\gamma - 1) \left(\frac{\partial \tilde{v}_j}{\partial x_j} \delta_{\nu 4} - \frac{1}{\bar{\rho}} \frac{\partial \tilde{\tau}_{\nu j}}{\partial x_j} \right) \delta_{\mu 4}, \tag{A 2}$$

$$\begin{aligned} \tilde{\tau}_{ij} &\equiv \delta_{ij} \bar{p} - \tilde{T}_{ij} - \bar{\sigma}_{ij}, \\ \partial_\mu &\equiv \frac{\partial}{\partial x_i}, \; i = \mu = 1, 2, 3, \end{aligned} \tag{A 3}$$

∂_μ, $\tilde{\tau}_\mu$ and $\tilde{\tau}_{\mu j}$ all equal to zero when $\mu > 3$ and D_o being the linear operator

$$\mathrm{D}_o \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j} \tilde{v}_j. \tag{A 4}$$

It is easy to see that the fifth component of the base-flow equation (2.5) can be used to put (2.8) into convective form.

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