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ENERGY CONCENTRATION PROPERTIES OF A p-GINZBURG–LANDAU MODEL

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Abstract. This paper is concerned with the p-Ginzburg–Landau (p-GL) type model with $p \neq 2$. First, we obtain global energy estimates and energy concentration properties by the singularity analysis. Next, we give a decay rate of $1-|u_\varepsilon|$ in the domain away from the singularities when $\varepsilon \to 0$, where u_ε is a minimizer of p-GL functional with $p \in (1,2)$. Finally, we obtain a Liouville theorem for the finite energy solutions of the p-GL equation on $\mathbb{R}^2$.

§1. Introduction

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G, and g be a smooth map from ∂G to S^1 satisfying $d := \deg(g, \partial G) \neq 0$. Without loss of generality, we assume $d > 0$. Bethuel et al. [3] and Struwe [21] well studied the asymptotic behavior of the Ginzburg–Landau functional

$$E(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2$$

as $\varepsilon \to 0^+$, and established the C^k-convergence relation between the minimizer and some harmonic map u_* on G. Namely, if u_ε is the minimizer of $E(u)$ in $H^1_g(G,\mathbb{R}^2)$, then there exists a subsequence u_{ε_k} such that when $k \to \infty$,

$$u_{\varepsilon_k} \to u_*, \quad in\ C^l_{loc}(G \setminus \{a_j\}_{j=1}^d)$$

for any $l \geq 1$. Here, $a_1, a_2, \ldots, a_d$ are the singularities of u_* in G. In particular, [3, Theorem VII.2] shows the following property of the energy concentration

$$\lim_{\varepsilon_k \to 0} \frac{(1-|u_{\varepsilon_k}|^2)^2}{4\varepsilon_k^2} = \frac{\pi}{2}\sum_{j=1}^d \delta_{a_j}, \quad \text{in the weak star topology of} \quad C(\overline{G}). \tag{1.1}$$

These results come into play when studying the location of the vortices in phase transition problems occurring in superconductivity and superfluids. In addition, these results are also helpful to understand the regularity of harmonic maps and the distribution of the singularities.

In addition, 19 open problems were posed in [3]. Comte and Mironescu gave positive answers to the 7th problem (cf. [8], [9], and [20]) on the global analysis of the Ginzburg–Landau energy. In particular, there exists $C > 0$ such that as $\varepsilon \to 0$,

$$\int_G (1-|u_\varepsilon|^2)^\alpha |\nabla u_\varepsilon|^2 \leq C\alpha^{-1}, \quad \forall \alpha > 0, \tag{1.2}$$

$$\int_G |\det(\nabla u_\varepsilon)| \leq C. \tag{1.3}$$

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In this paper, we are concerned with the asymptotic behavior of the minimizer u_ε of the p-Ginzburg-Landau-type functional

$$E_\varepsilon(u,G)=\frac{1}{p}\int_G|\nabla u|^p+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2 \quad (p>1 \text{ and } p\neq 2)$$

in the space $W=\{v\in W^{1,p}(G,\mathbb{R}^2)\cap L^4(G,\mathbb{R}^2); v|_{\partial G}=g\}$.

Functional $E_\varepsilon(u,G)$ can be used to study the partial regularity of p-harmonic maps and the location of the singularities when $p\in(1,2)$ and p approaches the dimension 2 (cf. [14], [24]). Different from the case $p\in(1,2)$, $W_g^{1,p}(G,S^1)=\emptyset$ when $p>2$ and $d\neq 0$. Therefore, there does not exist p-harmonic map from G to S^1. Thus, functional $E_\varepsilon(u,G)$ is naturally used by the idea of penalization which is analogous to researching harmonic maps in [3]. In fact, the same idea had been used in studying flow of p-harmonic maps (cf. [6]).

In this paper, we investigate the global properties of functional $E_\varepsilon(u,G)$. In addition, the singularity properties of the functional is also interesting. When $p=2$, the results (1.1)–(1.3) describe the singularity and the global properties. We expect to generalize them to the case of $p\neq 2$.

By the direct methods, we know the existence of the minimizer u_ε of $E_\varepsilon(u,B)$ in the space W. Clearly, the minimizer is a weak solution to the following system

$$-div(|\nabla u|^{p-2}\nabla u)=\frac{1}{\varepsilon^p}u(1-|u|^2), \quad in \quad G. \tag{1.4}$$

By the regularity theory (cf. [22]), $u_\varepsilon\in C^\alpha(\overline{G})\cap C^{1,\alpha}_{loc}(G)$ for each ε. In addition, by [15, analogous proof of Theorem 2.2], we also have

$$|u_\varepsilon|\leq 1 \quad in\ \overline{G}. \tag{1.5}$$

There may be several minimizers of $E_\varepsilon(u,G)$. One of them, denoted by $\tilde{u}_\varepsilon$, can be obtained as the limit of a subsequence $u_\varepsilon^{\eta_k}$ of the minimizers u_ε^η of the regularized functional (if we follow Uhlenbeck's idea in [23])

$$E_\varepsilon^\eta(u,G):=\frac{1}{p}\int_G(|\nabla u|^2+\eta)^{\frac{p}{2}}+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2 \quad (\eta>0,\ \varepsilon>0)$$

in W (see also [13] and [15]). Namely, there exists a subsequence η_k of η such that

$$\lim_{\eta_k\to 0}u_\varepsilon^{\eta_k}=\tilde{u}_\varepsilon, \quad in \quad W^{1,p}(G), \tag{1.6}$$

where $\tilde{u}_\varepsilon$ is also a minimizer of $E_\varepsilon(u,G)$ in W. Here, $\tilde{u}_\varepsilon$ is called the regularized minimizer.

When $p>2$, [17, Theorem 6.3] shows a convergence result of the minimizer u_ε of $E_\varepsilon(u,G)$ in W. Namely, there exists a subsequence u_{ε_k} of u_ε, such that when $k\to\infty$,

$$\lim_{k\to\infty}u_{\varepsilon_k}=u_p, \quad in\ C^{1,\alpha}_{loc}(G\setminus\{a_j\}_{j=1}^N), \quad \forall\alpha\in(0,1), \tag{1.7}$$

where u_p is a p-harmonic map on $G\setminus\{a_1,a_2,\cdots,a_N\}$. In addition, [18, Theorem 1.1] shows that

$$\lim_{\varepsilon_k\to 0}\frac{(1-|\tilde{u}_{\varepsilon_k}|^2)^2}{4\varepsilon_k^2}=\sum_{j=1}^N m_j\delta_{a_j}, \quad \text{in the weak star topology of} \quad C(\overline{G}), \tag{1.8}$$

where $m_j>0$ for $j=1,2,\ldots,N$.

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The following theorem presents results similar to (1.2) and (1.3) when $p>2$, which will be proved in Section 3.

THEOREM 1.1. *Assume $p>2$, u_ε is a minimizer of $E_\varepsilon(u,G)$ in W. Then we can find a subsequence ε_k of ε, and $P_j\geq 0$ which is independent of ε_k $(j=1,2,\ldots,N)$, such that*

$$\lim_{\varepsilon_k\to 0}\varepsilon_k^{p-2}|\det(\nabla u_{\varepsilon_k})|^{p/2}=\sum_{j=1}^N P_j\delta_{a_j},\ \textit{weakly* in } C(\overline{G}), \tag{1.9}$$

where δ_{a_j} is the Dirac mass at a_j. In addition, for any $\alpha\geq 2-\frac{4}{p}$, there exists $L_j>0$ which is independent of ε_k $(j=1,2,\ldots,N)$, such that

$$\lim_{\varepsilon_k\to 0}(1-|u_{\varepsilon_k}|^2)^\alpha|\nabla u_{\varepsilon_k}|^2=\sum_{j=1}^N L_j\delta_{a_j},\ \textit{weakly* in } C(\overline{G}). \tag{1.10}$$

When $p\in(1,2)$, [16, Theorem 1.3] implies a convergence result of the regularized minimizer. Namely, $\tilde{u}_\varepsilon$ is a regularized minimizer. When $p\in(2-t,2)$ for some $t\in(0,1/2)$, there is a subsequence $\tilde{u}_k$ of $\tilde{u}_\varepsilon$ such that

$$\lim_{k\to\infty}\tilde{u}_k=u_p,\quad in\quad C^{1,\alpha}_{loc}(G\setminus\{a_j\}_{j=1}^N),\quad \forall\alpha\in(0,1), \tag{1.11}$$

where u_p is a p-harmonic map on G and $a_1,a_2,\ldots,a_N\in\overline{G}$ are singularities of u_p.

The following theorem shows a result similar to (1.1)–(1.3) in the case of $p\in(1,2)$, which will be proved in Section 5.

THEOREM 1.2. *Assume that $p\in(1,2)$ approaches the dimension 2, and u_ε is a minimizer in W. Then we can find a subsequence ε_k of ε, and $Q_j\geq 0$ which is independent of ε $(j=1,2,\ldots,N)$, such that*

$$\lim_{\varepsilon_k\to 0}|\det(\nabla u_{\varepsilon_k})|^{p/2}=\sum_{j=1}^N Q_j\delta_{a_j},\ \textit{weakly* in } C(\overline{G}). \tag{1.12}$$

Moreover, if $\tilde{u}_\varepsilon$ is a regularized minimizer, then we can find $K_j>0$ which is independent of ε $(j=1,2,\ldots,N)$, such that

$$\lim_{\varepsilon_k\to 0}\frac{(1-|\tilde{u}_{\varepsilon_k}|^2)^2}{4\varepsilon_k^2}=\sum_{j=1}^N K_j\delta_{a_j},\ \textit{weakly* in } C(\overline{G}). \tag{1.13}$$

In addition, for any $\alpha\geq 2-p$, there exist $M_j\geq 0$ which are independent of ε $(j=1,2,\ldots,N)$, such that

$$\lim_{\varepsilon_k\to 0}(1-|\tilde{u}_{\varepsilon_k}|^2)^\alpha|\nabla\tilde{u}_{\varepsilon_k}|^2=\sum_{j=1}^N M_j\delta_{a_j},\ \textit{weakly* in } C(\overline{G}). \tag{1.14}$$

Here, $M_j>0$ when $a_j\in G$, and $M_j=0$ when $a_j\in\partial G$.

REMARK 1.1. When $p\neq 2$, if u_ε has radial structure, that is, $G=B_1(0)$ and $u_\varepsilon(x)\in\{u\in W; u(x)=f(r)(\cos d\theta,\sin d\theta), r=|x|, x=(\cos\theta,\sin\theta)\}$, then $N=1$. According to [19, Theorems 1.1 and 1.2], P_1 in Theorem 1.1 and Q_1 in Theorem 1.2 are positive. For a general minimizer u_ε (i.e., u_ε without the radial structure), it seems difficult to determine the values of P_j and Q_j even if $p=2$ (cf. [9]).

REMARK 1.2. (i) Equations (1.8)–(1.10) and (1.12)–(1.14) generalize (1.1)–(1.3) in the cases of $p>2$ and $p\in(1,2)$, respectively. Different from the case of $p=2$, the conformal invariant of the functional with $p\neq 2$ is lost. So the results does not look concise. Namely, we need balance the energy functional by some proper weights to ensure the new energy integrals are globally bounded.

(ii) When $p>2$, (1.8) and (1.9) show the concentration properties of $\varepsilon^{p-2}E_\varepsilon(u_\varepsilon,G)$. For the regularized minimizer $\tilde{u}_\varepsilon$, [18, Theorem 1.1] shows that the first and the second terms of $\varepsilon^{p-2}E_\varepsilon(u_\varepsilon,G)$ have the same convergence orders, and $\varepsilon_k^{p-2}E_\varepsilon(\tilde{u}_{\varepsilon_k},G)\to\frac{p}{p-2}\Sigma_{j=1}^N m_j\delta_{a_j}$ when $k\to\infty$, where m_j is the positive coefficients in (1.8).

(iii) When $p\in(1,2)$, (1.13) shows the concentration property of the second term of $\varepsilon^{p-2}E_\varepsilon(\tilde{u}_\varepsilon,G)$. Moreover, when $p\in((\sqrt{17}-1)/2,2)$, the first term of $\varepsilon^{p-2}E_\varepsilon(\tilde{u}_\varepsilon,G)$ is blow-up if $\tilde{u}_\varepsilon$ is radial (cf. [19, (1.20)]). So the convergence orders of the first and the second terms of $\varepsilon^{p-2}E_\varepsilon(\tilde{u}_\varepsilon,G)$ are different.

To prove (1.12), we need the decay result of the gradient of $|u_\varepsilon|-1$ in arbitrary compact subset of $G\setminus(\cup_j\{a_j\})$. For the regularized minimizer $\tilde{u}_\varepsilon$, it is clear in view of (1.11). For the general minimizer u_ε it is not easy to be obtained.

In Section 4, we will prove the following decay result, which does not only come into play to deduce (1.12), but also has its own meaning of independence.

THEOREM 1.3. *Let $p\in(1,2)$. Assume u_ε is a minimizer in W. Then for any compact subset $K\subset G\setminus\{a_j\}_{j=1}^N$, there holds*

$$\int_K[|\nabla(1-|u_\varepsilon|)|^p+\frac{1}{\varepsilon^p}(1-|u_\varepsilon|)^2]\to 0,\quad \text{when } \varepsilon\to 0. \tag{1.15}$$

Finally, we consider the finite energy solutions of

$$-div(|\nabla u|^{p-2}\nabla u)=u(1-|u|^2),\quad in\quad \mathbb{R}^2. \tag{1.16}$$

When $p=2$, Brezis et al. [5] gave a Liouville theorem by an idea of Cazenave. Namely, if $\nabla u\in L^2(\mathbb{R}^2)$, then either $u(x)\equiv 0$, or $u(x)\equiv C$ with $|C|=1$. When $p\neq 2$, we have the following result which will be proved in Section 6.

THEOREM 1.4. *Let u be a classical solution of (1.16) with $p>1$ and $p\neq 2$. If*

$$\nabla u\in L^p(\mathbb{R}^2), \tag{1.17}$$

then

$$u(x)\equiv C, \tag{1.18}$$

where C is a constant vector satisfying $|C|\in\{0,1\}$.

REMARK 1.3. The Liouville theorem of the critical points of p-Ginzburg–Landau energy on the Riemannian manifold can be seen in [7]. In addition, the p-Ginzburg–Laudan functional on $\mathbb{R}^2$ was well studied in [1] and [2].

§2. Preliminaries

Besides (1.7), (1.8), and (1.11), we recall several results of the case $p \neq 2$.

Assume u_ε is a minimizer of $E_\varepsilon(u,G)$ in W. Denote the disc with x as center and r as radius by $B(x,r)$ or $B_r(x)$. Let λ,μ be two positive constants which are independent of ε. If

$$\int_{G\cap B(x^\varepsilon,2\lambda\varepsilon)}(1-|u_\varepsilon|^2)^2 \le \mu\varepsilon^2,$$

then $B(x^\varepsilon,\lambda\varepsilon)$ is called a *good disc*. Otherwise $B(x^\varepsilon,\lambda\varepsilon)$ is called a *bad disc*. When $p>2$, [17, Remark 2.7] shows that the zeros of u_ε are included in finite nonintersecting *bad* discs $B(x_i^\varepsilon,h\varepsilon)$ $(i=1,2,...,N_1)$, where N_1 and $h>0$ are independent of ε. As $\varepsilon\to 0$, there exists a subsequence $x_i^{\varepsilon_k}$ of the center x_i^ε of bad discs such that $x_i^{\varepsilon_k}\to a_i\in\overline{G}(i=1,2,...,N_1)$. Since there may be at least two subsequences that converge to the same point, we denote the limit points by $a_1,a_2,...,a_N(N\le N_1)$. Write $\Lambda_j := \{i; x_i^{\varepsilon_k}\to a_j \text{ when } k\to\infty\}$ for each j. We can choose $\sigma>0$ suitably small such that $B_\sigma(a_j)\subset G$ when $a_j\in G$, and $B_\sigma(a_j)\cap B_\sigma(a_m)=\emptyset$ for $j\neq m$. When $p\in(1,2)$, [16, Section 3] also presents the same conclusions above. Clearly, these results still hold for the regularized minimizers.

2.1 Case of $p>2$

We now introduce several results in [17] and [18] which will be used later.

Proposition 2.1. [17, Proposition 2.1] *Let u_ε be a minimizer of $E_\varepsilon(u,G)$ in W. Then, there exists a constant $C>0$ which is independent of $\varepsilon\in(0,1)$, such that $E_\varepsilon(u_\varepsilon,G)\le C\varepsilon^{2-p}$.*

Proposition 2.2. [17, Proposition 2.2] *There exists a constant $C_0>0$ which is independent of $\varepsilon\in(0,1)$, such that for any $x,y\in\overline{G}$,*

$$|u_\varepsilon(x)-u_\varepsilon(y)|\le C_0\varepsilon^{(2-p)/p}|x-y|^{(p-2)/p}.$$

Proposition 2.3. [17, Theorem 2.6] *All the zeros of u_ε are contained in finite, disjointed bad discs $\{B(x_j^\varepsilon,h\varepsilon); j=1,2,\dots,N_1\}$. In addition,*

$$|u_\varepsilon(x)|\ge\frac{1}{2},\quad \forall x\in\overline{G}\setminus\cup_{j=1}^{N_1}B(x_j^\varepsilon,h\varepsilon).$$

Proposition 2.4. [18, Proposition 2.4] *Write $d_i:=\deg(u_\varepsilon,\partial B(x_i^\varepsilon,h\varepsilon))$. Then for each i, there exists a subsequence ε_k of ε such that d_i is independent of ε_k as long as k is sufficiently large.*

Proposition 2.5. [17, Theorem 3.1] *Let u_ε be a minimizer of $E_\varepsilon(u,G)$ in W. Then for any compact subset K of $G\setminus\{a_1,a_2,...,a_N\}$, there exists a constant $C>0$ which does not depend on ε, such that $E_\varepsilon(u_\varepsilon,K)\le C$.*

2.2 Case of $p\in(1,2)$

In [16], the following free energy functional was studied

$$E_{\varepsilon,\varrho}(u)=\frac{1}{2}\int_{B_1}|\nabla u|^2+\frac{1}{4\varepsilon^2}\int_{B_1\setminus B_\varrho}(1-|u|^2)^2+\frac{1}{2\varepsilon^2}\int_{B_\varrho}|u|^2,$$

where $B_r=\{x\in\mathbb{R}^2;|x|<r\}$, ε and ϱ are small positive parameters. It is associated with the model of superconductivity with normal impurity inclusion such as superconducting-normal

junctions. There are two major differences between $E_{\varepsilon,\varrho}(u)$ and $E_\varepsilon(u,G)$. The former models an heterogenous superconductor and the latter models the homogenous case. In addition, the domain considered in [16] is the unit disk.

In fact, the domain considered in [16] can be replaced by the more general G and the purpose of using B_1 there is for convenience (see also [10]). In addition, if $\varrho = 0$, then $E_{\varepsilon,\varrho}(u)$ becomes $E_\varepsilon(u,G)$, and we can see the results from the corresponding conclusions of Case I (i.e., $\varrho = O(\varepsilon)$) in [16], as long as we replace B_1 by G and p approaches the dimension 2.

First, by [16, (1–6) in Theorem 1.2], we have

PROPOSITION 2.6. *Let $p \in (1,2)$, u_ε be a minimizer of $E_\varepsilon(u,G)$ in W. Then there exists a constant $C > 0$ which is independent of ε, such that $E_\varepsilon(u_\varepsilon,G) \le C$.*

Next, by [16, Theorem 3.1 and Proposition 2.2], we have

PROPOSITION 2.7. *Assume that $p \in (1,2)$, $u_\varepsilon \in W$ satisfies the (1.4) in the weak sense. Then there is a constant $\rho_0 > 0$, such that for $\rho \in (0,\rho_0)$,*

$$|u_\varepsilon(x)| \ge \frac{1}{2}, \quad as \quad x \in G \setminus G^{2\rho\varepsilon}. \tag{2.1}$$

Here, $G^{\rho\varepsilon} := \{x \in G; dist(x,\partial G) > \rho\varepsilon\}$. In addition, for any $\rho > 0$, there exists a positive constant C_1 which is independent of ε, such that

$$\|\nabla u_\varepsilon(x)\|_{L^\infty(B(x,\rho\varepsilon))} \le C_1\varepsilon^{-1}, \quad as \quad x \in G^{\rho\varepsilon}. \tag{2.2}$$

By [16, (3-1) in Proposition 3.2], we have

PROPOSITION 2.8. *Let $p \in (1,2)$, u_ε be a minimizer of $E_\varepsilon(u,G)$ in W. Then, there exists a constant $C > 0$ which is independent of $\varepsilon \in (0,\varepsilon_0)$ with ε_0 sufficiently small, such that,*

$$\frac{1}{\varepsilon^2}\int_G (1-|u_\varepsilon|^2)^2 \le C.$$

By [16, (3-10)], we have

PROPOSITION 2.9. *Let $p \in (1,2)$, u_ε be a minimizer of $E_\varepsilon(u,G)$ in W. For any given $\sigma > 0$, and $0 < \varepsilon \ll \sigma$, there holds*

$$|u_\varepsilon(x)| \ge \frac{1}{2}, \quad \forall x \in \overline{G} \setminus (\cup_{j=1}^{CardJ} B(a_j,\sigma)).$$

The next proposition shows the reversed Hölder inequality, which is crucial in the proofs of (1.14) and (1.15). We now use the same ideas in [11, Chapter 5] to prove this reversed Hölder inequality.

PROPOSITION 2.10. *Assume $p \in [3/2,2)$, u_ε is a minimizer of $E_\varepsilon(u,G)$ in W. Then there exists a constant $R_0 \in (0,1/2)$ which is independent of ε, such that for any $B_R \subset G(2R < R_0)$, we have*

$$\left(\int_{B_R} |\nabla u_\varepsilon|^{\tilde q}\right)^{1/\tilde q} \le C\left(\int_{B_{2R}} (|\nabla u_\varepsilon|^2+1)^{p/2}\right)^{1/p}, \quad \forall \tilde q \in [p,p(1+t)),$$

where $C > 0$ depends on $R_0, p, \tilde q$, and $t \in (0,1/2)$ only depends on R_0.

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Proof. *Step 1.* Let $Q\subset\mathbb{R}^2$ be a square and $m\in[3/2,2)$. Suppose

$$\frac{1}{|Q_R|}\int_{Q_R(x_0)} g^m \le b\left(\frac{1}{|Q_{2R}|}\int_{Q_{2R}(x_0)} g\right)^m \tag{2.3}$$

for each $x_0\in Q$ and each $0<R<\frac{1}{3}\min\{dist(x_0,\partial Q),R_0\}$, where $b>1$ and $R_0>0$ are absolute constants. By the same argument of [11, proof of Theorem 1.2], we claim that $g\in L^q_{loc}(Q)$ for $q\in[m,m(1+t))$ and

$$\left(\frac{1}{|Q_R|}\int_{Q_R} g^q\right)^{\frac{1}{q}} \le C\left(\frac{1}{|Q_{2R}|}\int_{Q_{2R}} g^m\right)^{\frac{1}{m}} \tag{2.4}$$

for $Q_{2R}\subset Q$, $0<R<R_0$, where t is a positive constant only depending on b, and $C>0$ depends on b,m,q. In particular, C is blowing up when q approaches $m(1+t)$.

In fact, when we check [12, Proposition 5.1], (2.3) implies $\theta=0$ and $f(x)\equiv 0$. Set

$$E(h,s)=\{x\in Q; h(x)>s\},\quad \alpha_k=(3^2\cdot 4^k)^{1/m},\quad G(x)=g(x)\|g\|^{-1}_{L^m(Q)},$$

$$\Phi(x)=\alpha_k^{-1}G(x)\ in\ C_k:=\{x\in Q; 2^{-k}<dist(x,\partial Q)\le 2^{-k+1}\}.$$

As in [12, proof of (5.4)], by the Calderon–Zygmund subdivision argument and an iteration, we still obtain

$$\int_{E(\Phi,T)}\Phi^m \le aT^{m-1}\int_{E(\Phi,T)}\Phi \tag{2.5}$$

for $T\ge 1$. According to the result of [12] (see line 7 in page 167), a in (2.5) can be chosen as

$$a=b\left(\frac{5m}{m-1}\right)^{m-1}(30^2\cdot 5m+2^2).$$

In view of $m\in[3/2,2)$, a can be bounded by an absolute constant $\tilde a$ (which is independent of m). Now, (2.5) with $a=\tilde a$ is (1.6) in Chapter 5 of [11], which implies that the conditions of [11, Lemma 1.2 in Chapter 5] are satisfied if we write $h(T)=\int_{E(\Phi,T)}\Phi$ and $H(T)\equiv 0$. By applying Lemma 1.2 in Chapter 5 of [11], we can deduce (2.4) for $q\in[m,\frac{\tilde a}{\tilde a-1}m)$. Set $\frac{\tilde a}{\tilde a-1}=1+t$, then

$$t=\frac{1}{\tilde a-1}.$$

This implies that t is independent of m. In addition, C in (2.4) depends on $m[\tilde a m-(\tilde a-1)q]^{-1}$, which implies that C is blowing up when q approaches $m(1+t)$.

Step 2. Let $y=x\varepsilon^{-1}$ in $E_\varepsilon(u,G)$ and denote $v_\varepsilon(y)=u_\varepsilon(x)$, $G_\varepsilon=\{y=x\varepsilon^{-1}; x\in G\}$. Then

$$\begin{aligned} E_\varepsilon(u_\varepsilon,G) &= \varepsilon^{2-p}\left(\frac{1}{p}\int_{G_\varepsilon}|\nabla v_\varepsilon|^p dy+\frac{1}{4}\int_{G_\varepsilon}(1-|v_\varepsilon|^2)^2 dy\right)\\ &:= \varepsilon^{2-p}E(v_\varepsilon,G_\varepsilon). \end{aligned}$$

It is clear that v_ε is also a minimizer of $E(v,G_\varepsilon)$.

Let $m=(p+2)/2$, then $m\in[3/2,2)$ when $p\in[3/2,2)$. Clearly, $(A+B)^m\le(2\max\{A,B\})^m\le 2^m(A^m+B^m)$ as long as A,B are positive. Checking the proof of

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Theorem 3.1 in Chapter 5 of [11], we find that $c(m)$ in lines 13–14 of page 160 satisfies $c(m) = 2^m \leq 4$ for $p \in [3/2, 2)$. Therefore, c_4 in (3.5) (see line 5 of page 161) is independent of p after an iteration (by Lemma 3.1). Next, the Sobolev–Poincare inequality shows that c_5 can be chosen as a suitably large absolute constant in view of $p \in [3/2, 2)$. Thus, we also derive a condition which satisfies (2.3) with $b = c_5$. Using the reversed Hölder inequality (2.4) with $g = (\varepsilon + |\nabla v_\varepsilon|)^{\frac{2p}{2+p}}$, we know that there exist constants $t, R_0 \in (0, 1/2)$ and $C > 0$, such that for any $B_R \subset B_{R_0/2} \subset G$ and $q \in [m, m(1+t))$, the inequality

$$\begin{aligned}
&\left(\frac{1}{|B_{R/\varepsilon}|}\int_{B_{R/\varepsilon}} |\nabla v_\varepsilon|^{\frac{2pq}{2+p}}\,dy\right)^{1/q} \\
&\quad\leq \left(\frac{1}{|B_{R/\varepsilon}|}\int_{B_{R/\varepsilon}} (|\nabla v_\varepsilon| + \varepsilon)^{\frac{2pq}{2+p}}\,dy\right)^{1/q} \\
&\quad\leq C\left(\frac{1}{|B_{2R/\varepsilon}|}\int_{B_{2R/\varepsilon}} (|\nabla v_\varepsilon| + \varepsilon)^{p}\,dy\right)^{\frac{2}{p+2}}
\end{aligned}$$

holds, where $B_{R/\varepsilon} = \{y = x\varepsilon^{-1}; x \in B_R\}$. Letting $x = y\varepsilon$ and multiplying by $\varepsilon^{-\frac{2p}{p+2}}$, we obtain

$$\left(\int_{B_R} |\nabla u_\varepsilon|^{\frac{2pq}{2+p}}\,dx\right)^{\frac{p+2}{2pq}} \leq C\left(\int_{B_{2R}} (|\nabla u_\varepsilon|^2 + 1)^p\,dx\right)^{1/p}.$$

Let $\tilde{q} = \frac{2pq}{p+2}$. Noticing $q \in [m, m(1+t))$, we can see $\tilde{q} \in [p, p(1+t))$. And hence the proposition holds. ∎

Clearly, the results above (Propositions 2.6–2.10) still hold for the regularized minimizers. By [16, Proposition 5.3], we have

PROPOSITION 2.11. *Assume $p \in (1,2)$ approaches the dimension 2, and $\tilde{u}_\varepsilon$ is a regularized minimizer. Then, for any compact subset $K \subset G \setminus (\cup_{j=1}^N \{a_j\})$, there exists a positive constant $C = C(K)$ which is independent of ε, such that*

$$\|\frac{1}{\varepsilon^p}(1 - |\tilde{u}_\varepsilon|^2)\|_{L^\infty(K)} \leq C.$$

§3. Proof of Theorem 1.1

In this section, we assume $p > 2$, and u_ε is the minimizer of $E_\varepsilon(u, G)$ in W.

Proof of (1.10).

Noting $\alpha \geq 2 - \frac{4}{p}$, using (1.5), the Hölder inequality and Proposition 2.1, we deduce that

$$\begin{aligned}
&\int_G |\nabla u_\varepsilon|^2 (1 - |u_\varepsilon|^2)^\alpha \\
&\quad\leq C\int_G |\nabla u_\varepsilon|^2 (1 - |u_\varepsilon|^2)^{2-\frac{4}{p}} \\
&\quad\leq C\left(\int_G (1 - |u_\varepsilon|^2)^2\right)^{1-\frac{2}{p}} \left(\int_G |\nabla u_\varepsilon|^p\right)^{\frac{2}{p}} \\
&\quad\leq C\varepsilon^{2(1-\frac{2}{p}) + \frac{2}{p}(2-p)} = C.
\end{aligned}$$

Namely, $|\nabla u_\varepsilon|^2(1-|u_\varepsilon|^2)^\alpha$ is bounded in $L^1(G)$. Thus, there exists a Radon measure ω_1 such that

$$\lim_{\varepsilon_k\to 0}|\nabla u_{\varepsilon_k}|^2(1-|u_{\varepsilon_k}|^2)^\alpha=\omega_1, \quad \text{weakly star in} \quad C(\overline{G}).$$

Next, using the Hölder inequality and Proposition 2.5, we see that as $\varepsilon\to 0$,

$$\begin{aligned}&\int_{G\setminus\cup_{j=1}^N B(a_j,\sigma)}|\nabla u_\varepsilon|^2(1-|u_\varepsilon|^2)^\alpha\\ &\le\left[\int_{G\setminus\cup_j B(a_j,\sigma)}|\nabla u_\varepsilon|^p\right]^{\frac{2}{p}}\\ &\cdot\left[\int_{G\setminus\cup_j B(a_j,\sigma)}(1-|u_\varepsilon|^2)^{\frac{p\alpha}{p-2}}\right]^{\frac{p-2}{p}}\to 0.\end{aligned}\tag{3.1}$$

Thus, $supp(\omega_1)\subset\{a_j\}_{j=1}^N$. Therefore, we can find $L_j\ge 0$ such that

$$\omega_1=\sum_{j=1}^N L_j\delta_{a_j}.$$

We claim $L_j>0$ for each j (and hence $supp(\omega_1)=\{a_j\}_{j=1}^N$). For convenience, we here drop k from ε_k.

Noting $B_{h\varepsilon}(x_i)$ contains zeros of u_ε, by Propositions 2.3 we have

$$\frac{1}{2}\le|u_\varepsilon|\le\frac{3}{4}, \quad on\ \partial B_{h\varepsilon}(x_i)$$

as long as $h\varepsilon$ is suitably small. For each $x_1\in\partial B_{h\varepsilon}(x_i)$, by Proposition 2.2, we get

$$\frac{3}{8}\le|u_\varepsilon(y)|\le\frac{7}{8}, \quad \forall y\in\overline{B_{\hat{l}\varepsilon}(x_1)}\cap B_{h\varepsilon}(x_i).$$

Here, $\hat{l}:=\min\{(8C_0)^{\frac{p}{2-p}},h\}$. Therefore,

$$\frac{3}{8}\le|u_\varepsilon(y)|\le\frac{7}{8}, \quad \forall y\in B_{h\varepsilon}(x_i)\setminus B_{(h-\hat{l}/2)\varepsilon}(x_i).\tag{3.2}$$

As in [15, Theorem 3.9], by (3.2) we can write

$$u_\varepsilon(x)=|u_\varepsilon(x)|\phi(r,\tau),\ r=|x|,\ \tau=\frac{x}{|x|},$$

and hence

$$|\nabla u_\varepsilon|^2\ge|u_\varepsilon|^2r^{-2}|\nabla_\tau\phi(r,\tau)|^2.$$

Thus, by the Hölder inequality, there holds

$$\begin{aligned}&\int_{B_{h\varepsilon}(x_i)\setminus B_{(h-\hat{l}/2)\varepsilon}(x_i)}(1-|u_\varepsilon|^2)^\alpha|\nabla u_\varepsilon|^2\\ &\ge\left(1-\frac{7}{8}\right)^\alpha\int_{B_{h\varepsilon}(x_i)\setminus B_{(h-\hat{l}/2)\varepsilon}(x_i)}|u_\varepsilon|^2r^{-2}|\nabla_\tau\phi(r,\tau)|^2\end{aligned}$$

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$$\geq \left(\frac{1}{8}\right)^{\alpha}\left(\frac{3}{8}\right)^{2}\int_{(h-\hat{l}/2)\varepsilon}^{h\varepsilon}\left(\int_{S^1}|\nabla_\tau\phi|^2 ds\right)\frac{dr}{r}$$
$$\geq \left(\frac{1}{8}\right)^{\alpha}\left(\frac{3}{8}\right)^{2}\int_{(h-\hat{l}/2)\varepsilon}^{h\varepsilon}\frac{1}{2\pi}\left(\int_{S^1}|\nabla_\tau\phi| ds\right)^2\frac{dr}{r}.$$

Theorem 8.2 in [4] shows that

$$\int_{S^1}|\nabla_\tau\phi|ds \geq 2\pi|d_i|,$$

where $d_i = \deg(u_\varepsilon, \partial B_{h\varepsilon}(x_i))$ is independent of ε (see Proposition 2.4). Therefore,

$$\int_{B_{h\varepsilon}(x_i)\setminus B_{(h-\hat{l}/2)\varepsilon}(x_i)}(1-|u_\varepsilon|^2)^\alpha|\nabla u_\varepsilon|^2 \geq \left(\frac{1}{8}\right)^{\alpha}\left(\frac{3}{8}\right)^{2}(2\pi d_i^2)\cdot\left(\log\frac{h}{h-\hat{l}/2}\right).$$

This implies $L_j > 0$, and hence (1.10) is proved.

Proof of (1.9).

Denote u_ε by u. Clearly, by the Cauchy inequality,

$$|\det(\nabla u)| = |u_{1x_1}u_{2x_2} - u_{1x_2}u_{2x_1}| \leq \frac{1}{2}|\nabla u|^2. \tag{3.3}$$

By Proposition 2.1, we obtain that for each j,

$$\varepsilon^{p-2}\int_{G\cap B(a_j,\sigma)}|\det(\nabla u)|^{p/2} \leq \frac{\varepsilon^{p-2}}{2}\int_G|\nabla u|^p \leq C. \tag{3.4}$$

Using Proposition 2.5, we get

$$\varepsilon^{p-2}\int_{G\setminus\cup_j B(a_j,\sigma)}|\det(\nabla u)|^{p/2} \leq \frac{\varepsilon^{p-2}}{2}\int_{G\setminus\cup_j B(a_j,\sigma)}|\nabla u|^p \leq C\varepsilon^{p-2}. \tag{3.5}$$

Combining (3.4) and (3.5), we see that $\varepsilon^{p-2}|\det(\nabla u)|^{p/2}$ is bounded in $L^1(G)$. Thus, there exists a Radon measure ω_2 such that

$$\lim_{\varepsilon_k\to 0}\varepsilon_k^{p-2}|\det(\nabla u_{\varepsilon_k})|^{p/2} = \omega_2, \quad \text{weakly star in} \quad C(\overline{G}).$$

By virtue of (3.5), $supp(\omega_2) \subset \{a_j\}_{j=1}^N$. Thus, we can find $P_j \geq 0$ such that

$$\omega_2 = \sum_{j=1}^N P_j\delta_{a_j}.$$

Equation (1.9) is proved.

§4. Convergence rate of $|u_\varepsilon| - 1$ when $p \in (1,2)$

In this section, we consider the case of $p \in (1,2)$. Proposition 2.8 implies that $|u_\varepsilon| - 1$ tends to zero and the decay rate is presented. Here, we will give a decay rate of the gradient of $|u_\varepsilon| - 1$.

Proof of Theorem 1.3

Proof. *Step 1.* Let $R>0$ be a small constant such that $B(x,2R)\subset\subset G\setminus\cup_{i=1}^{N}\{a_i\}$. Applying (1.5) and Proposition 2.9, we have

$$\frac{1}{2}\leq|u_\varepsilon|\leq 1 \quad in\ B(x,2R). \tag{4.1}$$

By the integral mean value theorem and Proposition 2.6, there is $r\in[R,2R]$ such that

$$\int_{\partial B(x,r)}|\nabla|u_\varepsilon||^p d\xi+\frac{1}{\varepsilon^p}\int_{\partial B(x,r)}(1-|u_\varepsilon|^2)^2 d\xi\leq C \tag{4.2}$$

with $C=C(r)>0$ independent of ε. Denote $B(x,r)$ by B. If ρ_1 is a minimizer of the functional

$$E(\rho,B)=\frac{1}{p}\int_B(|\nabla\rho|^2+1)^{p/2}+\frac{1}{2\varepsilon^p}\int_B(1-\rho)^2$$

in $W^{1,p}_{|u_\varepsilon|}(B,\mathbb{R}^+\cup\{0\})$, then it solves

$$-div[(|\nabla\rho|^2+1)^{(p-2)/2}\nabla\rho]=\frac{1}{\varepsilon^p}(1-\rho), \tag{4.3}$$

$$\rho|_{\partial B}=|u_\varepsilon|, \tag{4.4}$$

By the maximum principle, it follows from (4.1) that

$$\frac{1}{2}\leq\rho_1\leq 1 \quad on\ \overline{B}. \tag{4.5}$$

Applying Proposition 2.6 we see easily that

$$E(\rho_1,B)\leq E(|u_\varepsilon|,B)\leq C(E_\varepsilon(u_\varepsilon,B)+1)\leq C. \tag{4.6}$$

Step 2. Multiplying (4.3) with $\rho=\rho_1$ by $(\nu\cdot\nabla\rho_1)$, we have

$$\begin{aligned}-\int_{\partial B}v^{(p-2)/2}|\partial_\nu\rho_1|^2 d\xi\quad&+\int_B v^{(p-2)/2}\nabla\rho_1\cdot\nabla(\nu\cdot\nabla\rho_1)\\&=\frac{1}{\varepsilon^p}\int_B(1-\rho_1)(\nu\cdot\nabla\rho_1),\end{aligned} \tag{4.7}$$

where ν denotes the unit outside norm vector on ∂B and $v=|\nabla\rho_1|^2+1$. Integrating by parts yields

$$\begin{aligned}&\int_B v^{(p-2)/2}\nabla\rho_1\cdot\nabla(\nu\cdot\nabla\rho_1)\\&=\int_B v^{(p-2)/2}|\nabla\rho_1|^2+\frac{1}{p}\int_B\nu\cdot\nabla(v^{p/2})\\&=\int_B v^{(p-2)/2}|\nabla\rho_1|^2+\frac{1}{p}\int_{\partial B}v^{p/2}d\xi-\frac{1}{p}\int_B v^{p/2}(\mathrm{div}\nu),\end{aligned} \tag{4.8}$$

and

$$\frac{1}{\varepsilon^p}\int_B(1-\rho_1)(\nu\cdot\nabla\rho_1)=\frac{1}{2\varepsilon^p}\int_B(1-\rho_1)^2(\mathrm{div}\nu)-\frac{1}{2\varepsilon^p}\int_{\partial B}(1-\rho_1)^2 d\xi.$$

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Substitute this result and (4.8) into (4.7). Noting $\text{div}\nu = r^{-1} > 0$, we can use (4.5), (4.6), (4.4), and (4.2) to obtain

$$\int_{\partial B} v^{(p-2)/2}|\partial_\nu \rho_1|^2 d\xi \le C + \frac{1}{p}\int_{\partial B} v^{p/2} d\xi. \tag{4.9}$$

Step 3. By the Jensen inequality $(1+a^2+b^2)^{1/2} \le 1+|a|+|b|$ and (4.4), there holds

$$\int_{\partial B} v^{p/2} d\xi \le \int_{\partial B} v^{(p-1)/2}[(1+|\partial_\tau|u_\varepsilon||)+|\partial_\nu \rho_1|]d\xi,$$

where τ denotes the unit tangent vector on ∂B. Using the Hölder inequality and (4.2), we deduce from the result above that

$$\int_{\partial B} v^{p/2} d\xi \le C\left(\int_{\partial B} v^{\frac{p}{2}} d\xi\right)^{\frac{p-1}{p}} + \left(\int_{\partial B} v^{\frac{p-2}{2}}|\partial_\nu \rho_1|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{\partial B} v^{\frac{p}{2}} d\xi\right)^{\frac{1}{2}}.$$

Inserting (4.9) into this result and using the Young inequality, we obtain that for any $\delta \in (0,1)$,

$$\int_{\partial B} v^{p/2} d\xi \le C(\delta) + \left(\delta + \frac{1}{2p} + \frac{1}{2}\right)\int_{\partial B} v^{p/2} d\xi,$$

Therefore, by choosing $\delta > 0$ sufficiently small we get

$$\int_{\partial B} (|\nabla \rho_1|^2+1)^{p/2} d\xi \le C. \tag{4.10}$$

Multiply (4.3) by $(1-\rho_1)$. In view of (4.4), applying the Hölder inequality, and using (4.2) and (4.10), we get

$$\begin{aligned} &\int_B [(|\nabla \rho_1|^2+1)^{(p-2)/2}|\nabla \rho_1|^2 + \frac{1}{\varepsilon^p}(1-\rho_1)^2] \\ &\le \left|\int_{\partial B} (1-\rho_1)(|\nabla \rho_1|^2+1)^{(p-2)/2}(\nu\cdot\nabla \rho_1) d\xi\right| \le C\varepsilon. \end{aligned} \tag{4.11}$$

Step 4. Set $U = \rho_1 w$ on B; $U = u_\varepsilon$ on $G\setminus B$, where $w = u_\varepsilon/|u_\varepsilon|$. Then $U \in W$. Since u_ε is a minimizer of $E_\varepsilon(u,G)$, we have

$$E_\varepsilon(u_\varepsilon, G) \le E_\varepsilon(U,G) = E_\varepsilon(\rho_1 w, B) + E_\varepsilon(u_\varepsilon, G\setminus B). \tag{4.12}$$

In view of $p \in (1,2)$, we get

$$\begin{aligned} &\int_B (|\nabla \rho_1|^2 + \rho_1^2|\nabla w|^2)^{p/2} - \int_B (\rho_1^2|\nabla w|^2)^{p/2} \\ &= \frac{p}{2}\int_B\int_0^1 (s|\nabla \rho_1|^2 + \rho_1^2|\nabla w|^2)^{(p-2)/2}|\nabla \rho_1|^2 ds dx \\ &\le \frac{p}{2}\int_B |\nabla \rho_1|^p dx \cdot \int_0^1 s^{(p-2)/2} ds = \int_B |\nabla \rho_1|^p. \end{aligned}$$

Here, we use the fact $\int_0^1 s^{(p-2)/2} ds = 2/p$. Combining this with (4.12) leads to

$$E_\varepsilon(u_\varepsilon, B) \le E_\varepsilon(\rho_1 w, B) \le \frac{1}{p}\int_B (\rho_1^2|\nabla w|^2)^{p/2} + E_\varepsilon(\rho_1, B).$$

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Using (4.5) and (4.11), we can see from the result above that

$$E_\varepsilon(u_\varepsilon, B) \le \frac{1}{p}\int_B |\nabla w|^p + C\varepsilon. \tag{4.13}$$

Step 5. Here, we denote $|u_\varepsilon|$ by h_ε.

Clearly, $(sa^2+b^2)^{(p-2)/2} \ge (a^2+b^2)^{(p-2)/2}$ when $s\in(0,1)$ and $p\in(1,2)$. Therefore, according to the mean value theorem, it follows that

$$\begin{aligned}
&(|\nabla h_\varepsilon|^2 + h_\varepsilon^2|\nabla w|^2)^{p/2} - (h_\varepsilon|\nabla w|^2)^{p/2}\\
&= \frac{p}{2}\left(\int_0^1 [s(|\nabla h_\varepsilon|^2 + h_\varepsilon^2|\nabla w|^2) + (1-s)(h_\varepsilon^2|\nabla w|^2)]^{\frac{p-2}{2}} ds\right)\cdot |\nabla h_\varepsilon|^2\\
&\ge \frac{p}{2}|\nabla u_\varepsilon|^{p-2}|\nabla h_\varepsilon|^2.
\end{aligned}$$

This and (4.13) imply

$$\begin{aligned}
&\frac{1}{2}\int_B |\nabla u_\varepsilon|^{p-2}|\nabla h_\varepsilon|^2 + \frac{1}{p}\int_B (h_\varepsilon^p - 1)|\nabla w|^p + \frac{1}{4\varepsilon^p}\int_B (1-h_\varepsilon^2)^2\\
&\le E_\varepsilon(u_\varepsilon, B) - \frac{1}{p}\int_B |\nabla w|^p \le C\varepsilon.
\end{aligned} \tag{4.14}$$

In view of (4.1), there holds

$$\frac{1}{p}\int_B (1-h_\varepsilon^p)|\nabla w|^p \le \frac{2^p}{p}\int_B (1-h_\varepsilon^2)|\nabla u_\varepsilon|^p.$$

Applying the Hölder inequality and Propositions 2.10 and 2.6, we can obtain

$$\frac{1}{p}\int_B (1-h_\varepsilon^p)|\nabla w|^p \le C\varepsilon^{2t_0/(1+t_0)}$$

with $t_0 \in (0,t)$ is suitably small. Combining this with (4.14), we can see

$$\frac{1}{2}\int_B |\nabla u_\varepsilon|^{p-2}|\nabla h_\varepsilon|^2 + \frac{1}{4\varepsilon^p}\int_B (1-h_\varepsilon^2)^2 \le C\varepsilon^{2t_0/(1+t_0)}. \tag{4.15}$$

Using the Hölder inequality, by (4.15) and Proposition 2.6, we see that

$$\int_B |\nabla h|^p \le \left(\int_B |\nabla u_\varepsilon|^{p-2}|\nabla h_\varepsilon|^2\right)^{\frac{p}{2}} \left(\int_B |\nabla u_\varepsilon|^p\right)^{\frac{2-p}{2}} \le C\varepsilon^{pt_0/(1+t_0)}.$$

Combining with (4.15) and by an argument of finite coverings, we can see (1.15). ∎

§5. Proof of Theorem 1.2

Proof of (1.12).

By (3.3) and Proposition 2.6, we get

$$\int_G |\det(\nabla u_\varepsilon)|^{p/2} \le \int_G |\nabla u_\varepsilon|^p \le C. \tag{5.1}$$

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Therefore, we can find a Radon measure ω_3 and a subsequence ε_k of ε such that

$$\lim_{k\to\infty} |\det(\nabla u_{\varepsilon_k})|^{p/2} = \omega_3, \quad \text{weakly star in} \quad C(\overline{G}). \tag{5.2}$$

Proposition 2.9 implies that there exists ϕ_ε such that when $x \in G \setminus \cup_j B_\sigma(a_j)$,

$$u_\varepsilon(x) = h_\varepsilon(x)(\cos\phi_\varepsilon(x), \sin\phi_\varepsilon(x)),$$

where $h_\varepsilon(x) = |u_\varepsilon(x)|$. Thus, on $G \setminus \cup_j B_\sigma(a_j)$,

$$|\det(\nabla u_\varepsilon)| = h_\varepsilon |\partial_\nu h_\varepsilon \partial_\tau \phi_\varepsilon - \partial_\tau h_\varepsilon \partial_\nu \phi_\varepsilon|.$$

Therefore, by the Hölder inequality, there holds

$$\begin{aligned} &\int_{G\setminus\cup_j B_\sigma(a_j)} |\det(\nabla u_\varepsilon)|^{p/2} \\ &\le \left(\int_{G\setminus\cup_j B_\sigma(a_j)} |\nabla h_\varepsilon|^p\right)^{\frac{1}{2}} \left(\int_{G\setminus\cup_j B_\sigma(a_j)} |h_\varepsilon \nabla\phi_\varepsilon|^p\right)^{\frac{1}{2}}. \end{aligned} \tag{5.3}$$

In view of Proposition 2.6 and (1.15),

$$\int_{G\setminus\cup_j B_\sigma(a_j)} |\det(\nabla u_\varepsilon)|^{p/2} \le C\left(\int_{G\setminus\cup_j B_\sigma(a_j)} |\nabla h_\varepsilon|^p\right)^{\frac{1}{2}} \to 0$$

when $\varepsilon \to 0$. Therefore, $supp(\omega_3) \subset \{a_j\}_{j=1}^N$. Thus, there exists constants $Q_j \ge 0$ such that

$$\omega_3 = \sum_{j=1}^N Q_j \delta_{a_j}.$$

Equation (1.12) is proved.

Proof of (1.14).

First we observe that $2 - p < pt$ since p is sufficiently close to 2, where t is the constant in Proposition 2.10 (which implies that t is independent of p). Thus we can choose $\gamma \in (2-p, pt)$. By (1.5), the Hölder inequality and Propositions 2.10 and 2.6, we have

$$\int_{G\setminus G^{2\rho\varepsilon}} (1-|\tilde{u}_\varepsilon|^2)^\alpha |\nabla\tilde{u}_\varepsilon|^2 \le \left(\int_{G\setminus G^{2\rho\varepsilon}} |\nabla\tilde{u}_\varepsilon|^{p+\gamma}\right)^{\frac{2}{p+\gamma}} |G \setminus G^{2\rho\varepsilon}|^{1-\frac{2}{p+\gamma}} \to 0 \tag{5.4}$$

when $\varepsilon \to 0$.

In view of (1.5), the right hand side of (1.4) with $u = \tilde{u}_\varepsilon$ is bounded by ε^{-p}. Thus, checking the proof of Proposition 5.1 in [22], and applying Proposition 2.6 we get

$$\|\nabla\tilde{u}_\varepsilon\|^p_{L^\infty(G^{\rho\varepsilon}\cap B_\sigma(a_j))} \le C\sigma^{-2}\varepsilon^{-p} \int_G (1 + |\nabla\tilde{u}_\varepsilon|^p) \le C\varepsilon^{-p}.$$

Therefore, by the Hölder inequality, we obtain that for any $\gamma > 0$,

$$\begin{aligned}
&\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} (1-|\tilde{u}_\varepsilon|^2)^\alpha |\nabla \tilde{u}_\varepsilon|^2 \\
&\le C\varepsilon^{p-2} \int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} (1-|\tilde{u}_\varepsilon|^2)^\alpha |\nabla \tilde{u}_\varepsilon|^p \\
&\le C\varepsilon^{p-2} \left(\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} |\nabla \tilde{u}_\varepsilon|^{p+\gamma}\right)^{\frac{p}{p+\gamma}} \\
&\quad \cdot \left(\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} (1-|\tilde{u}_\varepsilon|^2)^{(p+\gamma)\alpha/\gamma}\right)^{\frac{\gamma}{p+\gamma}}.
\end{aligned} \tag{5.5}$$

We set

$$\gamma_0 = 2-p.$$

Then for any $\alpha \ge 2-p$,

$$\frac{(p+\gamma_0)\alpha}{\gamma_0} \ge 2. \tag{5.6}$$

In addition, $\gamma_0 \in (0, pt)$ since p is sufficiently close to 2. Here, t is the constant in Proposition 2.10.

According to Proposition 2.10, and by Proposition 2.6, it follows that

$$\left(\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} |\nabla \tilde{u}_\varepsilon|^{p+\gamma_0}\right)^{\frac{p}{p+\gamma_0}} \le C\int_G (|\nabla \tilde{u}_\varepsilon|^2+1)^{p/2} \le C.$$

Inserting this result into (5.5) with $\gamma = \gamma_0$ yields

$$\begin{aligned}
&\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} (1-|u_\varepsilon|^2)^\alpha |\nabla \tilde{u}_\varepsilon|^2 \\
&\le C\varepsilon^{p-2} \left(\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} (1-|\tilde{u}_\varepsilon|^2)^{(p+\gamma_0)\alpha/\gamma_0}\right)^{\frac{\gamma_0}{p+\gamma_0}}.
\end{aligned} \tag{5.7}$$

Using (5.6), Proposition 2.8, we can deduce from (5.7) that

$$\int_{G^{\rho\varepsilon}\cap B(a_j,\sigma)} (1-|\tilde{u}_\varepsilon|^2)^\alpha |\nabla \tilde{u}_\varepsilon|^2 \le C\varepsilon^{p-2+\frac{2\gamma_0}{p+\gamma_0}} = C. \tag{5.8}$$

Next, using

$$\sup_{G\setminus\cup_j B(a_j,\sigma)} |\nabla \tilde{u}_\varepsilon| \le C$$

and

$$\sup_{G\setminus\cup_j B(a_j,\sigma)} (1-|\tilde{u}_\varepsilon|^2) \to 0$$

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which are implied by (1.11), we derive that when $\varepsilon \to 0$,

$$\begin{aligned}&\int_{G^{\rho\varepsilon}\setminus\cup_j B(a_j,\sigma)}(1-|\tilde{u}_\varepsilon|^2)^\alpha|\nabla\tilde{u}_\varepsilon|^2\\ &\le\left(\sup_{G\setminus\cup_j B(a_j,\sigma)}(1-|\tilde{u}_\varepsilon|^2)^\alpha\right)\left(\sup_{G\setminus\cup_j B(a_j,\sigma)}|\nabla\tilde{u}_\varepsilon|^2\right)|G\setminus\cup_j B_\sigma(a_j)|\\ &\to 0.\end{aligned} \tag{5.9}$$

Combining (5.4), (5.8) with (5.9), we obtain that $(1-|\tilde{u}_\varepsilon|^2)^\alpha|\nabla\tilde{u}_\varepsilon|^2$ is bounded in $L^1(G)$, and hence

$$\lim_{\varepsilon_k\to 0}(1-|\tilde{u}_{\varepsilon_k}|^2)^\alpha|\nabla\tilde{u}_{\varepsilon_k}|^2=\omega_4, \quad \text{weakly star in} \quad C(\overline{G}),$$

where ω_4 is a Radon measure. In addition, (5.9) implies $supp(\omega_4)\subset\{a_j\}_{j=1}^N$. Thus, there exists constants $M_j\ge 0$ such that

$$\omega_4=\sum_{j=1}^N M_j\delta_{a_j}.$$

By (5.4), we see that $M_j=0$ when $a_j\in\partial G$.

For the other a_j which are contained in G, using (2.2) instead of Proposition 2.2 in the proof of $L_j>0$, we also get $M_j>0$. Equation (1.14) is proved.

Proof of (1.13).

By Proposition 2.8, $\frac{(1-|\tilde{u}_\varepsilon|^2)^2}{\varepsilon^2}$ is bounded in $L^1(G)$. Thus, there exist a subsequence ε_k of ε and a Radon measure ω_5, such that

$$\lim_{k\to 0}\frac{(1-|\tilde{u}_{\varepsilon_k}|^2)^2}{\varepsilon_k^2}=\omega_5, \quad \text{weakly star in} \quad C(\overline{G}).$$

In addition, according to Proposition 2.11,

$$\frac{1}{\varepsilon_k^2}\int_{G\setminus\cup_j B_\sigma(a_j)}(1-|\tilde{u}_{\varepsilon_k}|^2)^2\le C\varepsilon_k^{2(p-1)}\to 0$$

when $\varepsilon_k\to 0$. Therefore, we can see that $supp(\omega_5)\subset\{a_j\}_{j=1}^N$, and hence

$$\omega_5=\sum_{j=1}^N K_j\delta_{a_j},$$

where $K_j\ge 0$.

We claim $K_j>0$ for each j. In fact, there exists x_j^ε which is the center of the bad disc tends to a_j when $\varepsilon\to 0$. Recalling the definition of the bad disc in Section 2, we have

$$\frac{1}{\varepsilon^2}\int_{B(x_j^\varepsilon,h\varepsilon)}(1-|\tilde{u}_\varepsilon|^2)^2\ge\mu>0.$$

for each j. This implies $K_j>0$ (and hence $supp(\omega_5)=\{a_j\}_{j=1}^N$). Equation (1.13) is proved.

§6. Proof of Theorem 1.4

Step 1. First we claim

$$|u| \le 1 \quad a.e.\ on\ \mathbb{R}^2. \tag{6.1}$$

For convenience, sometimes we denote $B_R(0)$ by B_R.

From (1.17), it follows that

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{B_{2R}\setminus B_R} |\nabla u|^p dx \\ &= \lim_{R\to\infty} \int_R^{2R} \left[r \int_{\partial B_r} |\nabla u|^p d\xi \right] \frac{dr}{r} \\ &\ge \lim_{R\to\infty} \inf_{r\in[R,2R]} \left[r \int_{\partial B_r} |\nabla u|^p d\xi \right] \cdot (\log 2). \end{aligned}$$

Therefore, we can find a subsequence R_k of R such that

$$\lim_{R_k\to\infty} R_k \int_{\partial B_{R_k}(0)} |\nabla u|^p d\xi = 0. \tag{6.2}$$

When $p \in (1,2)$, the Sobolev inequality implies $u \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$. By the same proof of (6.2), there also holds that

$$\lim_{R_k\to\infty} R_k \int_{\partial B_{R_k}(0)} |u|^{\frac{2p}{2-p}} d\xi = 0. \tag{6.3}$$

Here, R_k is also a subsequence of R.

Set $\Phi = u - u\min\{1,|u|\}/|u|$ and $B_+ = \{x \in \mathbb{R}^2; |u(x)| > 1\}$, then

$$\begin{cases} \nabla\Phi = \nabla u - |u|^{-1}\nabla u + (u\cdot\nabla u)u/|u|^3, & on\ B_+; \\ \nabla\Phi = 0 & on\ \mathbb{R}^2 \setminus B_+. \end{cases}$$

Multiplying (1.16) by Φ and integrating on B_{R_k} and then letting $R_k \to \infty$, we get

$$\begin{aligned} &- \lim_{R_k\to\infty} \int_{\partial B_{R_k}(0)} u|\nabla u|^{p-2}\partial_\nu u d\xi \\ &+ \lim_{R_k\to\infty} \int_{\partial B_{R_k}(0)} \frac{u}{|u|} \min\{1,|u|\}|\nabla u|^{p-2}\partial_\nu u d\xi \\ &+ \int_{B_+} (1-1/|u|)|\nabla u|^p + \int_{B_+} |\nabla u|^{p-2}(u\cdot\nabla u)^2/|u|^3 \\ &+ \int_{B_+} |u|(|u|+1)(|u|-1)^2 = 0. \end{aligned} \tag{6.4}$$

When $R_k \to \infty$, by the Hölder inequality and (6.2) and (6.3), there holds

$$\begin{aligned} &\int_{\partial B_{R_k}} |u||\nabla u|^{p-1} d\xi \\ &\le R_k^{-\frac{1}{2}} \left(R_k \int_{\partial B_{R_k}} |\nabla u|^p d\xi \right)^{1-\frac{1}{p}} \left(R_k \int_{\partial B_{R_k}} |u|^{\frac{2p}{2-p}} d\xi \right)^{\frac{2-p}{2p}} |\partial B_{R_k}|^{\frac{1}{2}} \\ &\to 0, \end{aligned} \tag{6.5}$$

which implies both the first and the second terms of the left hand side of (6.4) are equal to zero. This shows $|B_+| = 0$ and hence (6.1) is proved when $p \in (1.2)$.

When $p > 2$, set $\Phi := (|u|-1)_+$. Then,

$$\begin{cases} \nabla\Phi = 0, & on \quad \mathbb{R}^2 \setminus B_+; \\ \nabla\Phi = \frac{u\cdot\nabla u}{|u|}, & on \quad B_+. \end{cases}$$

Obviously, (1.17) implies

$$\|\nabla\Phi\|_{L^p(\mathbb{R}^2)} < \infty. \tag{6.6}$$

Let $\zeta \in C^\infty(\mathbb{R}^2, [0,1])$ be a cut-off function satisfying $\zeta(y) = 1$ for $|y| \le 1$, and $\zeta(y) = 0$ for $|y| \ge 2$. Set $\zeta_t(y) = \zeta(\frac{y}{t})$. Multiply (1.16) by ξ, where

$$\begin{cases} \xi = 0, & on \quad \mathbb{R}^2 \setminus B_+; \\ \xi = \frac{u}{|u|}\zeta_t, & on \quad B_+. \end{cases}$$

Then,

$$\int_{B_+} |\nabla u|^{p-2}\nabla u \nabla(\frac{u}{|u|}\zeta_t) = -\int_{B_+} |u|(1+|u|)\Phi\zeta_t.$$

By calculating the left hand side, we can obtain that

$$\begin{aligned} &\int_{B_+} |u|^{-1}|\nabla u|^p\zeta_t - \int_{B_+} |u|^{-3}|\nabla u|^{p-2}(u\cdot\nabla u)^2\zeta_t \\ &+ \int_{B_+} |u|(1+|u|)\Phi\zeta_t + \int_{B_+} |\nabla u|^{p-2}\nabla\Phi\nabla\zeta_t = 0. \end{aligned} \tag{6.7}$$

In view of $|\nabla u|^2 \ge (u\cdot\nabla u)^2/|u|^2$, the first and the second terms in the left hand side of (6.7) is nonnegative. Therefore, using (1.17) and (6.6), we can deduce that

$$\begin{aligned} \int_{B_+} |u|(1+|u|)\Phi\zeta_t &\le |\int_{B_+} |\nabla u|^{p-2}\nabla\Phi\nabla\zeta_t| \\ &\le \frac{1}{t}\int_{B_+\cap\{y; t\le|y|\le 2t\}} |\nabla u|^{p-2}|\nabla\Phi| \le \frac{1}{t}\|\nabla u\|_p^{p-2}\|\nabla\Phi\|_p t^{2/p}. \end{aligned}$$

When $t \to \infty$, the right hand side converges to zero by virtue of $p > 2$. And hence so is the left hand side. This means $|B_+| = 0$ or $\Phi = 0$ a.e. on $\mathbb{R}^2$. Thus, (6.1) is proved when $p > 2$.

Step 2. Multiplying (1.16) by u and integrating on $B_R(0)$ yield

$$\int_{B_R(0)} |u|^2(1-|u|^2) = \int_{B_R(0)} |\nabla u|^p - \int_{\partial B_R(0)} u|\nabla u|^{p-2}\partial_\nu u d\xi. \tag{6.8}$$

When $p \in (1,2)$, we use (6.5) and (1.17) to get

$$\int_{\mathbb{R}^2} |u|^2(1-|u|^2) = \int_{\mathbb{R}^2} |\nabla u|^p < \infty. \tag{6.9}$$

When $p > 2$, by (6.1) and (6.2), there holds

$$\left|\int_{\partial B_{R_k}} u|\nabla u|^{p-2}\partial_\nu u d\xi\right| \le CR_k^{\frac{1}{p}-1}\left(R_k\int_{\partial B_{R_k}} |\nabla u|^p d\xi\right)^{1-\frac{1}{p}} |\partial B_{R_k}|^{\frac{1}{p}} \to 0$$

when $R_k \to \infty$. From (6.8) with $R = R_k$, we also deduce (6.9).

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Multiply (1.16) by $(x\cdot\nabla u)$ and integrate on $B_R(0)$. Integrating by parts, we can see the left hand side

$$\begin{aligned}
&-\int_{B_R}\operatorname{div}(|\nabla u|^{p-2}\nabla u)(x\cdot\nabla u)\\
&\quad=-R\int_{\partial B_R}|\nabla u|^{p-2}|\partial_\nu u|^2d\xi+\int_{B_R}|\nabla u|^p+\frac{1}{p}\int_{B_R}x\cdot\nabla(|\nabla u|^p)\\
&\quad=-R\int_{\partial B_R}|\nabla u|^{p-2}|\partial_\nu u|^2d\xi+(1-\frac{2}{p})\int_{B_R}|\nabla u|^p+\frac{R}{p}\int_{\partial B_R}|\nabla u|^pd\xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\frac{2}{p}-1\right)\int_{B_R}|\nabla u|^p+\frac{1}{2}\int_{B_R}(1-|u|^2)^2\\
&\quad=\frac{R}{4}\int_{\partial B_R}(1-|u|^2)^2d\xi-R\int_{\partial B_R}|\nabla u|^{p-2}|\partial_\nu u|^2d\xi\\
&\qquad+\frac{R}{p}\int_{\partial B_R}|\nabla u|^pd\xi,
\end{aligned}\tag{6.10}$$

and

$$\begin{aligned}
&\left(1-\frac{2}{p}\right)\int_{B_R}|\nabla u|^p+\int_{B_R}(|u|^2-\frac{1}{2}|u|^4)\\
&\quad=\frac{R}{2}\int_{\partial B_R}(|u|^2-\frac{1}{2}|u|^4)d\xi+R\int_{\partial B_R}|\nabla u|^{p-2}|\partial_\nu u|^2d\xi\\
&\qquad-\frac{R}{p}\int_{\partial B_R}|\nabla u|^pd\xi.
\end{aligned}\tag{6.11}$$

Step 3. We claim that there exists $R_0>0$ such that either $|u|<\frac{1}{4}$ or $|u|>T$ on $\mathbb{R}^2\setminus B_{R_0}$, where

$$T:=\begin{cases}3/4, & \textit{when } p\in(1,2);\\ \sqrt{p/(3p-4)}, & \textit{when } p>2.\end{cases}$$

In fact, the following set is bounded

$$S:=\left\{x\in\mathbb{R}^2;\frac{1}{4}\le|u(x)|\le T\right\}.$$

Otherwise, there exists a sequence $x_m\to\infty$ (when $m\to\infty$) such that $\frac{1}{4}\le|u(x_m)|\le T$. In view of (1.17) and (6.1), by the Tolksdorf theorem (cf. [22]), we have $|\nabla u(x)|\le C$ for each $x\in\mathbb{R}^2$. Therefore, for $\sigma\in(0,1-T)$, we can find $\delta\in(0,1)$ which is independent of m such that

$$\frac{1}{8}\le|u(x)|\le T+\sigma,\quad \textit{when } x\in B(x_m,\delta).$$

Therefore,

$$\int_{B(x_m,\delta)}|u|^2(1-|u|^2)\ge\left(\frac{1}{8}\right)^2(1-(T+\sigma)^2)\pi\delta^2:=M_*.\tag{6.12}$$

Clearly, M_* is independent of m. On the other hand, by (6.9), we can find $R_*>0$ such that

$$\int_{|x|>R_*}|u|^2(1-|u|^2)<M_*. \tag{6.13}$$

Noting $B(x_m,\delta)\subset\mathbb{R}^2\setminus B_{R_*}$ for sufficiently large m, (6.13) contradicts with (6.12). This implies S is bounded. Namely, there exists $R_0>0$ such that $S\subset B_{R_0}$. Since $\mathbb{R}^2\setminus B_{R_0}$ is connected and u is continuous, then either $|u|<\frac{1}{4}$ or $|u|>T$ on $\mathbb{R}^2\setminus B_{R_0}$ in view of the definition of S.

Step 4. When $p\in(1,2)$, we will prove Theorem 1.4.

When $|u|<\frac{1}{4}$ on $\mathbb{R}^2\setminus B_{R_0}$, (6.9) and (6.1) lead to

$$u\in L^2(\mathbb{R}^2)\cap L^4(\mathbb{R}^2). \tag{6.14}$$

This implies

$$\frac{R_k}{2}\int_{\partial B_{R_k}}(|u|^2-\frac{1}{2}|u|^4)d\xi\to 0$$

when $R_k\to\infty$. Inserting this and (6.2) into (6.11) with $R=R_k$, we get

$$\left(1-\frac{2}{p}\right)\int_{\mathbb{R}^2}|\nabla u|^p+\int_{\mathbb{R}^2}(|u|^2-\frac{1}{2}|u|^4)=0. \tag{6.15}$$

Inserting (6.9) into (6.15) yields

$$\left(\frac{3}{2}-\frac{2}{p}\right)\int_{\mathbb{R}^2}|u|^4=\left(2-\frac{2}{p}\right)\int_{\mathbb{R}^2}|u|^2\geq\left(2-\frac{2}{p}\right)\int_{\mathbb{R}^2}|u|^4.$$

This implies $|u|\equiv 0$.

When $|u|>3/4$ on $\mathbb{R}^2\setminus B_{R_0}$, (6.9) and (6.1) lead to

$$1-|u|^2\in L^2(\mathbb{R}^2). \tag{6.16}$$

This implies

$$\frac{R_k}{4}\int_{\partial B_{R_k}}(1-|u|^2)^2d\xi\to 0$$

when $R_k\to\infty$. Inserting this and (6.2) into (6.10), we get

$$\left(\frac{2}{p}-1\right)\int_{\mathbb{R}^2}|\nabla u|^p+\frac{1}{2}\int_{\mathbb{R}^2}(1-|u|^2)^2=0. \tag{6.17}$$

In view of $p\in(1,2)$, (6.17) implies $|\nabla u|=1-|u|^2=0$ on $\mathbb{R}^2$. Therefore, $u\equiv C$ with $|C|=1$.

Step 5. When $p>2$, we will prove Theorem 1.4.

When $|u|<\frac{1}{4}$ on $\mathbb{R}^2\setminus B_{R_0}$, by (6.1) and (6.15) we get $u\equiv 0$.

When $|u|>T$ on $\mathbb{R}^2\setminus B_{R_0}$, (6.17) still holds. Combining with (6.9) leads to

$$\left(2-\frac{4}{p}\right)\int_{\mathbb{R}^2}|u|^2(1-|u|^2)=\int_{\mathbb{R}^2}(1-|u|^2)^2.$$

Namely,

$$\int_{\mathbb{R}^2}(1-|u|^2)[(3-\frac{4}{p})|u|^2-1]=0.$$

By (6.1) and $|u|>T=\sqrt{p/(3p-4)}$, we have $|u|\equiv 1$ on $\mathbb{R}^2$. Inserting this into (6.9) we see that $u\equiv C$ with $|C|=1$.

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