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IMPLEMENTING THE BEST STEADY STATE WITH SAVINGS IN UNBACKED RISKY ASSETS

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This paper shows, in an overlapping-generations economy à la Diamond [*American Economic Review* 55, 1126–1150 (1965)], that when savings in an unbacked asset (e.g., fiat money) bear some risk of becoming suddenly worthless, the market does not implement the best steady state attainable with that asset. Nonetheless, in the absence of absolutely riskless fiat money and excluding resorting to redistributive fiscal policies that would make it possible to attain the first-best steady state, this *best monetary steady state* can be implemented as a competitive equilibrium with the adequate policy of taxes on returns to capital, subsidies to returns to monetary savings, and lump-sum transfers. The policy is, at the steady state, balanced every period and nonredistributive.

Keywords: Taxation of Savings, Overlapping Generations, Asset Bubble

1. INTRODUCTION

The first-best steady state of the overlapping-generations economy in Diamond (1965) is typically not a competitive equilibrium unless the agents can save in terms of a fiat money (or another unbacked asset such as public debt) on top of physical capital. But even with fiat money, for the first-best steady state to be a competitive outcome, money has to be absolutely riskless. In effect, as soon as this asset risks becoming worthless—because of not being accepted by the next generation if money, or being reneged upon if debt—then there is no hope of attaining the first-best steady state as a laissez-faire competitive equilibrium.[1] But for anyone expecting that the market does its best anyway given the circumstances (i.e., given the unavoidable riskiness, no matter how small, of unbacked assets),

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news gets even worse: not even the best steady state *implementable with the unbacked risky asset* is a laissez-faire competitive-equilibrium outcome. In other words, a competitive-equilibrium steady state fails to provide the representative agent with the highest utility not only among all feasible steady states (whether dynamically efficient or not), but even among only those steady states that are *implementable by saving in the risky asset*. The good news is nonetheless that if, for some reason, the intergenerational transfers required to implement the first-best are not possible[2] and unbacked assets such as money are unavoidably risky to some extent, at least the best steady state implementable by a risky money (the *best monetary steady state* henceforth) is a competitive-equilibrium outcome under the right fiscal policy. This paper tells what this policy is.

Specifically, in Diamond (1965), the agents' only endowment is their ability to work when young. Output can be reproduced each period using the labor the agents supply and the amount of previously produced output that has not been consumed or used up in the production process yet (thought of as the aggregate level of capital). In such a setup the best possible steady state—i.e., the steady state that maximizes the utility of the representative agent—requires, first, that the aggregate level of capital be such that the output net of capital replacement is maximized in each period; and, second, that this net production be split between young and old agents in such a way that the marginal rate of substitution between consumption when young and old equals the rate at which consumption can be redistributed from young to old at any given period. These two conditions amount to making both the marginal rate of substitution between present and future consumption and the marginal productivity of capital equal to the factor by which the population grows each period. Typically, this requires *not* remunerating the factors of production by their marginal productivities or, alternatively, requires making intergenerational redistributions of income, should the factors be remunerated in this way. Either of these two ways to implement the first-best steady state is clearly at odds with what characterizes a laissez-faire competitive equilibrium, because the latter both remunerates the factors by their marginal productivities and does not allow for redistributions of income among agents. Nonetheless, if the agents can save part of their labor income in terms of an unbacked and hence intrinsically worthless asset, e.g., a fiat money [an asset bubble in Tirole's (1985) terms],[3] then there is a specific portfolio of money and capital that, if chosen by the agents for their savings, implements the first-best steady state as a competitive-equilibrium outcome. Nevertheless, it is crucial for this result to hold true that every agent believe that money will not have, for sure, zero exchange value next period.

In effect, in a competitive-equilibrium steady state of the Diamond (1965) economy, the agents, in the absence of money or any other mechanism making it possible to implement intergenerational transfers of resources in a decentralized way, may end up dumping too much capital into the production process with their saving decisions, compared to the level that maximizes net output. To convince them to withdraw part of these savings from the production process[4]—and to devote them instead to increasing the consumption of their parents—they need to be reassured that they will be treated in the same way by the next generation. That is, they must believe that the mechanism in place today allowing intergenerational transfers will still be there tomorrow when their turn comes to receive from it, instead of contributing to it. Whether this mechanism is fiat money, rolled-over public debt, or a pay-as-you-go pensions system, the fact is that it amounts to just promises, and thus the essential element making any such social contrivance work is trust. Now, trusted promises risk not being honored. As a matter of fact, although these are rare events, it is nonetheless a fact of life that every now and then states do dissolve, wars are waged, revolutions topple governments, and as a result public debts of previous governments are repudiated, money issued by former regimes becomes worthless, and pension claims are not honored.[5] Financial crises in which banking and credit institutions disappear do happen and claimants lose their savings as a result. And, nevertheless, *some trust is put recurrently in similar promises*, institutions, or social compacts almost immediately after such crises take place. Thus it seems to be inherent to intergenerational financial arrangements based on trust that there is some probability, no matter how small, that they collapse, only to be restarted a little after. Weil (1987) established conditions for the existence of competitive equilibria in a Diamond (1965) economy with a money that risks losing value completely at any time, and the result was that existence obtained as long as this risk was small enough. In Weil (1987), the economy was assumed to revert to a nonmonetary equilibrium once the bubble burst, which happens in finite time with probability 1. Unfortunately, this is a counterfactual feature of the stochastic asset bubbles considered there, because asset bubbles are clearly recurrent, and fiat money in particular, as a bubble, is immediately replaced by another money should it lose value completely. Thus, I consider instead a steady state in which new money is issued (a new bubble starts) right after the dismissal of the current money, in case that event happens.[6]

Having thus introduced some probability for the money bubble bursting, one can consider (as when money was assumed to be valued for sure) which is the best steady state that can be implemented by *saving in such a "risky" money*, on top of in terms of capital. Of course, this will depend on the specific probability of money losing value, and as a first approach (admittedly unsatisfactory) I will consider that probability to be exogenously determined, as in Weil (1987). Thus I characterize later the best steady state that a risky money can buy, i.e., the best monetary steady state, for a given probability of money losing value completely. The best monetary steady state turns out not to be, unfortunately, a competitive outcome *under laissez-faire*. In other words, free markets are unable to reach the best steady state allocation of resources that is implementable with any given intergenerational transfers mechanism whenever (quite realistically) the latter may collapse at some point, no matter how small this risk is. This is bad news. The good news is that the best monetary steady state is nonetheless a competitive-equilibrium outcome under a well-defined policy of taxes and transfers *not requiring any intergenerational redistribution*. This is interesting because, given the obvious difficulties in building a consensus on any kind of redistribution, in the hypothetical extreme case in which the government is barred from using redistributive taxes (so that money still fulfills an essential role), the first-best steady state cannot be reached in a decentralized way as long as the risk of the unbacked savings becoming worthless (or almost) exists. And still the policy makes it possible to implement (without redistribution) the unanimously preferred best monetary steady state.[7] More specifically, in the case in which the best monetary steady state overaccumulates capital with respect to the first-best steady state, this fiscal policy consists of (i) taxing linearly the returns to capital, (ii) subsidizing monetary savings returns linearly, and (iii) making second-period lump-sum transfers (which at the steady state equilibrium will be equal to the taxes net of subsidies raised from the same generation).

In case it seems awkward that the implementation of the best monetary steady state may require the taxation of productive savings (in capital) and the subsidizing of unproductive ones (in money), one should recall that the dynamic inefficiency—besides the inefficiency generated by the risk that prevents the money to implement the first-best steady state—comes from the agents dumping too much capital into the productive process, and hence the need to disincentivize that kind of savings. At the same time, unproductive (in a direct sense only) monetary savings[8] work instead in the direction of unclogging the production process in this case, from which the need to not to disincentivize them follows. This result may challenge the widespread view that values directly productive investments above supposedly unproductive or "speculative" ones, and hence may provide some food for thought about what is the real role of each kind of investments.

The rest of the paper is organized as follows. Section 2 provides (mainly to fix notation and for the sake of completeness) the well-known characterization of the unique first-best steady state of the Diamond (1965) overlapping-generations economy with production. In Section 3 I allow the probability of money losing value completely to be positive at any time, and I characterize the laissez-faire competitive equilibrium steady state in that case. In Section 4 I show that, as a consequence of money being risky, the laissez-faire competitive steady state is not the best monetary steady state. Section 5 establishes that the best monetary steady state can nonetheless be made into a competitive outcome with an adequate policy of taxes and transfers, which I characterize there. A concluding Section 6 closes the paper.

2. THE FIRST-BEST STEADY STATE OF THE DIAMOND OVERLAPPING-GENERATIONS ECONOMY

In the Diamond (1965) overlapping-generations economy with production, each of the two-period-lived identical members of a population of overlapping generations (growing by a factor of $1 + n$ every period) is endowed with, say, 1 unit of labor when young and nothing when old. The consumption good can be produced out of their labor and of the previously produced and not consumed good by means of a constant-returns technology.[9] Utility from the consumption profile (c_0, c_1) is given by $u(c_0) + v(c_1)$, with u and v being as usual differentiable, strictly increasing, strictly concave on nonnegative consumptions, and satisfying $\lim_{c_0 \to 0^+} u'(c_0) = +\infty = \lim_{c_1 \to 0^+} v'(c_1)$. Without loss of generality, and for the sake of notational simplicity, capital is assumed to depreciate completely every period.

In a steady state feasible allocation, all agents consume the same profile and get, because they have identical preferences, the same utility. Steady states providing the highest possible utility to all agents are thus characterized by being solutions to the problem

$$\max_{0 \leq c_0, c_1, k} u(c_0) + v(c_1), \tag{1}$$

$$c_0 + \frac{c_1}{1+n} + k \leq F\left(\frac{k}{1+n}, 1\right),$$

where k is the output saved per worker each period (and used as capital the next period) and the feasibility constraint is hence written in per-worker terms.[10] Under the assumptions made on u and v, a solution to (1) is completely characterized by the equations

$$\frac{u'(c_0)}{v'(c_1)} = 1 + n = F_K\left(\frac{k}{1+n}, 1\right), \tag{2}$$

$$c_0 + \frac{c_1}{1+n} + k = F\left(\frac{k}{1+n}, 1\right).$$

In effect, such a level k of per-worker capital savings maximizes net output in any period, whereas the latter is distributed between the young and old alive in that period in such a way that the marginal rate of substitution between consumption when young c_0 and consumption when old c_1 equals always the rate at which they can be transformed into each other, i.e., the growth factor of the population $1 + n$.

It easily follows from the assumptions on u and v that there is only one solution (c_0^*, c_1^*, k^*) to problem (1) and hence there exists a unique first-best steady state.

PROPOSITION 1. *The Diamond* (1965) *overlapping-generations economy with production has a unique first-best steady state, i.e., a unique feasible allocation such that*

(1) *it provides the same consumption profile to all generations*
(2) *for no other allocation providing the same consumption profile to all generations do the agents get a higher utility.*

Proof. Assume that both (c_0, c_1, k) and (c_0', c_1', k') solve (1). Then, necessarily,

$$F_K\left(\frac{k}{1+n}, 1\right) = 1 + n = F_K\left(\frac{k'}{1+n}, 1\right), \tag{3}$$

so that $k = \bar{k} = k'$ for some $\bar{k}$, and because

$$\max_{0 \leq c_0, c_1} u(c_0) + v(c_1), \qquad (4)$$
$$c_0 + \frac{c_1}{1+n} \leq F\left(\frac{\bar{k}}{1+n}, 1\right) - \bar{k}$$

has a unique interior solution because of v and u being strictly concave and because of their behavior at the boundary $\lim_{c \to 0^+} u(c) = +\infty = \lim_{c \to 0^+} v(c)$, $(c_0, c_1) = (c'_0, c'_1)$ as well. ■

From the statement of problem (1), it is clear that its only constraint, the feasibility constraint, makes it possible to distribute the output of each period freely among the contemporaneous young and old agents in order to maximize the representative agent's utility. As a consequence, the agents need not receive in the first-best steady state (c^*_0, c^*_1, k^*) the marginal productivity of the factors they contribute to the production of output in case this was a private property economy in which young agents only have their labor endowment and old agents only the return to their saved labor income, i.e., the return to capital. In effect, this would only be the case if it happened to hold that

$$c^*_0 + k^* = F_L\left(\frac{k^*}{1+n}, 1\right), \qquad (5)$$
$$c^*_1 = F_K\left(\frac{k^*}{1+n}, 1\right),$$

which is not guaranteed by the conditions (2) characterizing the first-best steady state (c^*_0, c^*_1, k^*). In other words, the first-best steady state need not be (and typically will not be) a competitive-equilibrium outcome in the absence of some mechanism making it possible to implement intergenerational transfers. Nonetheless, if the first-best steady state (c^*_0, c^*_1, k^*) solution to (2) is such that

$$\frac{c^*_1}{1+n} - k^* \geq 0, \qquad (6)$$

then, as it is well known, it can be attained as a competitive equilibrium of such a private property economy by introducing an unbacked (and hence intrinsically worthless) asset such as fiat money that the agents can trade for the good, and in terms of which they can therefore save as well, *conditional on the probability of this money being accepted next period being* 1. In effect, given the solution (c^*_0, c^*_1, k^*) satisfying (2), there exists an $m^* = \frac{c^*_1}{1+n} - k^* \geq 0$ (and hence m^* equals $F_L(\frac{k^*}{1+n}) - k^* - c^*_0$ as well, from the feasibility condition in (2) and the

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homogeneity of degree 1 of F) such that

$$c_0^* + k^* + m^* = F_L\left(\frac{k^*}{1+n}, 1\right), \tag{7}$$

$$c_1^* = F_K\left(\frac{k^*}{1+n}, 1\right) k^* + (1+n)m^*,$$

so that if *every period* the young agents buy the fiat money from the old agents in exchange for an amount m^* of the good, thus getting a return $1 + n$ on it next period, then the first-best steady state (c_0^*, c_1^*, k^*) obtains as a competitive-equilibrium steady state.[11]

The condition just stated—namely, that the probability that money is not accepted next period is zero—is crucial for the decentralization of the first-best as a competitive outcome in this way.[12] In effect, in the next section I show the consequence of money risking becoming worthless at any period:[13] the new unique[14] competitive-equilibrium steady state supported by such a stochastic bubble asset turns out to be distinct not only from the unique first-best steady state, but even from the best monetary steady state.[15] The latter can, however, be implemented as a competitive outcome under the fiscal policy detailed further in Section 5.[16]

3. LAISSEZ-FAIRE COMPETITIVE EQUILIBRIA WITH "RISKY" MONEY

Suppose that in the Diamond (1965) overlapping-generations economy with production there is a stochastic asset bubble, i.e., an unbacked and intrinsically worthless asset such as fiat money that is traded against the good and that is risky in the sense that with some probability $\pi \in (0, 1)$ the money accepted by generation t in exchange for goods will still be legal tender at $t+1$, but with some positive probability $\tilde{\pi} = 1 - \pi$ it will not.[17] In the second case, a financial disaster is assumed to have happened between the moment of which agent t decides to accept intrinsically worthless money in exchange for goods as a means of saving, and the date at which he or she intends to spend the monetary savings in old age consumption. As a result of that event, part of *his claims* over second-period resources, specifically those held in money, have thus been wiped out. Note that it is only the old agent's claims over these resources and not the resources themselves that disappear, so that in equilibrium these resources go to someone else, which in this setup can only be the contemporaneous young agent. An abrupt, sudden redistribution of wealth takes place when this happens, as it is the case when, for instance, bubbles burst, devaluations take place, debt is repudiated, or a currency issued by a toppled government is dismissed during wars, revolutions, on other types of social crises. Because in a stationary environment the previous generation $t - 1$ faced the same risk, generation t may find itself, when young, with its real wage being worth in a newly issued money enough to afford the resources that could have been claimed by the old had their money not become worthless. Effectively, the risk of loss of value of the monetary savings of the old is *as if* the young found themselves with the (real) monetary savings of the previous generation in their hands with some probability $\tilde{\pi}$ as well. Nevertheless, the representative agent chooses the amount and composition of his savings portfolio prior to this uncertainty being resolved. The representative agent's problem therefore becomes in this case

$$\max_{0\leq c_0^t, \tilde{c}_0^t, c_1^t, \tilde{c}_1^t, k^t, m^t} \pi u\left(c_0^t\right) + \tilde{\pi} u\left(\tilde{c}_0^t\right) + \pi v\left(c_1^t\right) + \tilde{\pi} v\left(\tilde{c}_1^t\right), \tag{8}$$

$$\begin{aligned} c_0^t + k^t + m^t &\leq w_t, \\ \tilde{c}_0^t + k^t + m^t &\leq w_t + \frac{1}{1+n}\rho_t m^{t-1}, \\ c_1^t &\leq r_{t+1}k^t + \rho_{t+1}m^t, \\ \tilde{c}_1^t &\leq r_{t+1}k^t, \end{aligned}$$

where $\tilde{c}_i^t, c_i^t$ are agent t's consumption at $t+i$, for $i = 0, 1$, conditional on the money bubble bursting then or not, respectively, m^t is the real savings in risky money by agent t, and ρ_{t+1} is the real return of money if still valued at $t+1$.[18] Note that, given the monotonicity of preferences, the problem of agent t reduces to choosing k^t and m^t *before* the uncertainty about the exchange value of agent $t-1$'s money holdings (and a fortiori of agent t's as well) is resolved.[19]

Note again that, according to (8), the problem faced by the representative agent is *as if* there is in every period a probability $\tilde{\pi}$ that the old agent transfers the returns to his monetary savings to the contemporaneous young agent. This is actually equivalent to the situation in which there is in every period a probability $\tilde{\pi}$ that the monetary price of the good becomes infinity in the old money held entirely by the old agents, whereas in the new money in which the young agents get paid their labor income, that price adjusts to clear markets by allowing the holders of the new money to be able to claim with their labor income the resources $\rho_t m^{t-1}/(1+n)$ that the old agents cannot claim any more.[20]

Under the standard assumptions on u and v, the unique solution to problem (8) is characterized by the first-order conditions

$$\begin{aligned} \frac{\pi u'\left(c_0^t\right) + \tilde{\pi} u'\left(\tilde{c}_0^t\right)}{\pi v'\left(c_1^t\right) + \tilde{\pi} v'\left(\tilde{c}_1^t\right)} &= r_{t+1}, \\ \frac{\pi u'\left(c_0^t\right) + \tilde{\pi} u'\left(\tilde{c}_0^t\right)}{\pi v'\left(c_1^t\right)} &= \rho_{t+1}, \end{aligned} \tag{9}$$

along with the budget constraints in (8).[21]

From the constant returns to scale of the production function, at equilibrium capital and labor are remunerated by their marginal productivities, so that

$$\begin{aligned} r_{t+1} &= F_K\left(\frac{k^t}{1+n}, 1\right), \\ w_t &= F_L\left(\frac{k^{t-1}}{1+n}, 1\right) \end{aligned} \tag{10}$$

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must hold at every period t as well. Moreover, because the population grows at a rate $n > -1$, from the agents' budget constraints it follows that at equilibrium, whether the money bubble bursts or not at any given period t,[22]

$$c_0^t + \frac{c_1^{t-1}}{1+n} + k^t + m^t = F_L\left(\frac{k^{t-1}}{1+n}, 1\right) + F_K\left(\frac{k^{t-1}}{1+n}, 1\right)\frac{k^{t-1}}{1+n} + \frac{1}{1+n}\rho_t m^{t-1}, \tag{11}$$

from which the feasibility condition is equivalent to

$$\rho_t \frac{m^t}{m^{t+1}} = 1+n. \tag{12}$$

In a competitive-equilibrium steady state it then necessarily holds that

$$\rho_t = 1+n \tag{13}$$

for all t, and letting the per-worker steady state demand for real balances be m, the profile of contingent consumptions and monetary and capital savings of a competitive equilibrium steady state $(c_0^e, c_1^e, \tilde{c}_0^e, \tilde{c}_1^e, k^e, m^e)$ is characterized by satisfying the equations[23]

$$\begin{aligned}
\frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)}{\pi v'(c_1) + \tilde{\pi} v'(\tilde{c}_1)} &= F_K\left(\frac{k}{1+n}, 1\right), \\
\frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)}{\pi v'(c_1)} &= 1+n, \\
c_0 + k + m &= F_L\left(\frac{k}{1+n}, 1\right), \\
\tilde{c}_0 + k &= F_L\left(\frac{k}{1+n}, 1\right), \\
c_1 &= F_K\left(\frac{k}{1+n}, 1\right)k + (1+n)m, \\
\tilde{c}_1 &= F_K\left(\frac{k}{1+n}, 1\right)k.
\end{aligned} \tag{14}$$

It follows from the existence of a unique competitive-equilibrium steady state with sure money (i.e., with $\tilde{\pi} = 0$), namely the unique first-best steady state (c_0^*, c_1^*, k^*) (see Proposition 1), that there also exists a unique monetary competitive-equilibrium steady state $(c_0^e, c_1^e, \tilde{c}_0^e, \tilde{c}_1^e, k^e, m^e)$ when money risks becoming worthless, as long as the probability $\tilde{\pi}$ of this event is small enough, as the next proposition establishes.

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PROPOSITION 2. *The Diamond* (1965) *overlapping-generations economy with production has a unique competitive-equilibrium steady state if the probability* $\tilde{\pi} \in [0, 1)$ *of money becoming worthless is small enough.*

Proof. The system of equations characterizing a competitive-equilibrium steady state, for any $\tilde{\pi}$, is

$$\begin{aligned}
\pi\left[u'(c_0) - F_K\left(\frac{k}{1+n}, 1\right)v'(c_1)\right] + \tilde{\pi}\left[u'(\tilde{c}_0) - F_K\left(\frac{k}{1+n}, 1\right)v'(\tilde{c}_1)\right] &= 0,\\
\pi[u'(c_0) - (1+n)v'(c_1)] + \tilde{\pi}u'(\tilde{c}_0) &= 0,\\
c_0 + k + m - F_L\left(\frac{k}{1+n}, 1\right) &= 0,\\
\tilde{c}_0 + k - F_L\left(\frac{k}{1+n}, 1\right) &= 0,\\
c_1 - F_K\left(\frac{k}{1+n}, 1\right)k - (1+n)m &= 0,\\
\tilde{c}_1 - F_K\left(\frac{k}{1+n}, 1\right)k &= 0.
\end{aligned} \qquad (15)$$

When $\tilde{\pi} = 0$, the system (15) has the unique first-best steady state (c_0^*, c_1^*, k^*) solution to (2) as solution, and in fact, by Proposition 1, as the only solution, m being the level of monetary savings $m^* = \frac{c_1^*}{1+n} - k^*$ implementing the first-best steady state as a competitive equilibrium when $\frac{c_1^*}{1+n} - k^* \geq 0$,[24] and $\tilde{c}_0$ and $\tilde{c}_1$ being variables determined by k^* simultaneously, but irrelevant in this case.[25]

The Jacobian of the system in (15) is, with obvious notation,[26]

$$\begin{pmatrix}
\pi u'' & \tilde{\pi}\tilde{u}'' & -\pi F_K v'' & -\tilde{\pi}F_K\tilde{v}'' & 0 & -[\pi v' + \tilde{\pi}\tilde{v}']F_{KK}\frac{1}{1+n}\\
\pi u'' & \tilde{\pi}\tilde{u}'' & -\pi(1+n)v'' & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 1 - F_{LK}\frac{1}{1+n}\\
0 & 1 & 0 & 0 & 0 & 1 - F_{LK}\frac{1}{1+n}\\
0 & 0 & 1 & 0 & -(1+n) & -F_K - F_{KK}\frac{1}{1+n}\\
0 & 0 & 0 & 1 & 0 & -F_K - F_{KK}\frac{1}{1+n}
\end{pmatrix}, \qquad (16)$$

and it is regular for $\tilde{\pi} = 0$. In effect, the last four rows are clearly linearly independent, and the first two rows can be combined linearly with the last four rows in order to turn them into (i) a block of zeros in their first four columns, and

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(ii) in their last two columns the block, for $\tilde{\pi} = 0$,

$$\begin{pmatrix} -u'' - F_K(1+n)v'' & -\left(F_K + F_{KK}\frac{1}{1+n}\right)\left(F_K v'' + u''\frac{1}{1+n}\right) - v' F_{KK}\frac{1}{1+n} \\ -u'' - (1+n)^2 v'' & -\left(F_K + F_{KK}\frac{1}{1+n}\right)\left((1+n)v'' + u''\frac{1}{1+n}\right) \end{pmatrix}. \quad \mathbf{(17)}$$

Because for $\tilde{\pi} = 0$, k is the only k^* such that $F_K\left(\frac{k}{1+n}, 1\right) = 1+n$, this block is regular, and therefore so is the entire Jacobian. By the continuity of the determinant of the Jacobian in (16)[27] with respect to $\tilde{\pi}$, it is still regular for any $\tilde{\pi} < \varepsilon$ and some $\varepsilon > 0$ small enough. As a consequence, the existence and uniqueness of the solution to the system with $\tilde{\pi} = 0$ (i.e., the existence and uniqueness of the first-best steady state) imply, by the implicit function theorem, the existence and local uniqueness of the competitive-equilibrium steady state for all $\tilde{\pi} \in [0, \varepsilon)$. Moreover, the local uniqueness is global for all $\tilde{\pi} \in [0, \varepsilon)$ because otherwise either the correspondence from $\tilde{\pi}$ to the set solutions to (15) is not locally a function at $\tilde{\pi} = 0$, which we just proved it is, or it is not upper hemicontinuous, which it is as well.[28] ■

It is worth noting that the existence result provided in Proposition 2 is a general property that does not depend on the uniqueness of the moneyless steady state $(\bar{c}_0, \bar{c}_1, \bar{k})$ solution to the first, fourth, and sixth equations in (14) with $\tilde{\pi} = 1$, as opposed to the condition for existence provided in Proposition 3 in Weil (1987). In effect, it is established in Weil (1987) that, *if there is a unique steady state for the moneyless economy*, then there exists a competitive-equilibrium steady state with a stochastic asset bubble[29] if, and only if,

$$\pi > \frac{F_K\left(\frac{\bar{k}}{1+n}, l\right)}{1+n}; \quad \mathbf{(18)}$$

that is to say, if, and only if, the probability of money losing completely its value is low enough. As a consequence, *if there is a unique steady state for the moneyless economy*, there cannot be a competitive-equilibrium steady state with a stochastic asset bubble if

$$F_K\left(\frac{\bar{k}}{1+n}, l\right) \geq 1+n; \quad \mathbf{(19)}$$

that is to say, in the case where the *moneyless* competitive-equilibrium steady state is dynamically efficient. It turns out this leaves open the question of whether there exist stochastic asset bubbles when there are several steady states (possibly dynamically efficient) of the moneyless economy.

4. THE BEST STEADY STATE THAT "RISKY" MONEY CAN BUY

Let us consider now the best monetary steady state—i.e., the steady state maximizing the utility of the representative agent under the constraints of using the risky money for intergenerational transfers and remunerating factors by their marginal productivities. It would be characterized as a solution to[30]

$$\max_{0\leq c_0, c_1, \tilde{c}_0, \tilde{c}_1, k, m} \pi u(c_0) + \tilde{\pi} u(\tilde{c}_0) + \pi v(c_1) + \tilde{\pi} v(\tilde{c}_1), \tag{20}$$

$$c_0 + k + m \leq F_L\left(\frac{k}{1+n}, 1\right),$$

$$\tilde{c}_0 + k \leq F_L\left(\frac{k}{1+n}, 1\right),$$

$$c_1 \leq F_K\left(\frac{k}{1+n}, 1\right) k + (1+n)m,$$

$$\tilde{c}_1 \leq F_K\left(\frac{k}{1+n}, 1\right) k.$$

Thus, the best monetary steady state is a profile $(c_0, c_1, \tilde{c}_0, \tilde{c}_1, k, m)$ satisfying the set of first-order conditions and budget constraints[31]

$$\frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)}{\pi v'(c_1) + \tilde{\pi} v'(\tilde{c}_1)} = \frac{F_K\left(\frac{k}{1+n}, 1\right) + F_{KK}\left(\frac{k}{1+n}, 1\right)\frac{k}{1+n}}{1 - F_{LK}\left(\frac{k}{1+n}, 1\right)\frac{1}{1+n}}, \tag{21}$$

$$\frac{u'(c_0)}{v'(c_1)} = 1+n,$$

$$c_0 + k + m = F_L\left(\frac{k}{1+n}, 1\right),$$

$$\tilde{c}_0 + k = F_L\left(\frac{k}{1+n}, 1\right),$$

$$c_1 = F_K\left(\frac{k}{1+n}, 1\right) k + (1+n)m,$$

$$\tilde{c}_1 = F_K\left(\frac{k}{1+n}, 1\right) k.$$

Note that, as opposed to what happens in the competitive-equilibrium steady state, the impact of savings in terms of capital on the real wage and the return to capital is now taken into account in the first equation in (21) through the changes in the marginal productivities of capital and labor that an increase in savings in terms of capital induces.

It follows from the existence of a unique first-best steady state that there exists a unique steady state implementable with the risky money that maximizes the representative agent's utility as long as the probability $\tilde{\pi}$ of money becoming worthless is small enough, as the next proposition establishes.

PROPOSITION 3. *The Diamond* (1965) *overlapping-generations economy with production has a unique best monetary steady state*[32] *if the probability* $\tilde{\pi} \in (0, 1)$ *of money becoming worthless is small enough.*

Proof. The system of equations characterizing the steady state that maximizes the representative agent's utility while remunerating factors by their marginal productivities and saving in risky money to implement intergenerational transfers is (using notation previously introduced)[33]

$$\begin{aligned}
&\left(1+n+F_{KK}\frac{k}{1+n}\right)[\pi u' + \tilde{\pi}\tilde{u}'] \\
&\quad - \left(F_K + F_{KK}\frac{k}{1+n}\right)[\pi v' + \tilde{\pi}\tilde{v}'](1+n) = 0, \\
&u'(c_0) - (1+n)v'(c_1) = 0, \\
&c_0 + k + m - F_L\left(\frac{k}{1+n}, 1\right) = 0, \\
&\tilde{c}_0 + k - F_L\left(\frac{k}{1+n}, 1\right) = 0, \\
&c_1 - F_K\left(\frac{k}{1+n}, 1\right)k - (1+n)m = 0, \\
&\tilde{c}_1 - F_K\left(\frac{k}{1+n}, 1\right)k = 0.
\end{aligned} \tag{22}$$

When $\tilde{\pi} = 0$, the system has as its only solution the unique first-best steady state (c_0^*, c_1^*, k^*) (m being again the level of per-worker monetary savings[34] implementing the first-best steady state as a competitive equilibrium, and $\tilde{c}_0$ and $\tilde{c}_1$ being variables once more determined simultaneously, but irrelevant in this case).[35]

The Jacobian of the system is

$$\begin{pmatrix}
\pi A u'' & \tilde{\pi} A \tilde{u}'' & -\pi B v'' & -\tilde{\pi} B \tilde{v}'' & 0 & C \\
u'' & 0 & -(1+n)v'' & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 - F_{LK}\frac{1}{1+n} \\
0 & 1 & 0 & 0 & 0 & 1 - F_{LK}\frac{1}{1+n} \\
0 & 0 & 1 & 0 & -(1+n) & -F_K - F_{KK}\frac{1}{1+n} \\
0 & 0 & 0 & 1 & 0 & -F_K - F_{KK}\frac{1}{1+n}
\end{pmatrix} \tag{23}$$

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with the notation introduced previously and with $A = 1 + n + F_{KK}\frac{k}{1+n}$, $B = (F_K + F_{KK}\frac{k}{1+n})(1+n)$, and

$$C = \left(F_{KKK}\frac{k}{(1+n)^2} + F_{KK}\frac{1}{1+n}\right)[\pi u' + \tilde{\pi}\tilde{u}'] - \left(F_{KKK}\frac{k}{(1+n)^2} + F_{KK}\frac{2}{1+n}\right)[\pi v' + \tilde{\pi}\tilde{v}'](1+n). \quad (24)$$

This Jacobian is regular for $\tilde{\pi} = 0$. In effect, the last four rows are clearly linearly independent, and the first two rows can be combined linearly with the last four rows to turn them into (1) a block of zeros in their first four columns, and (2) in their last two columns the block, when $\tilde{\pi} = 0$,

$$\begin{pmatrix} -Au'' - (1+n)Bv'' & C - \left(\frac{1}{1+n}Au'' + Bv''\right)\left(F_K + F_{KK}\frac{1}{1+n}\right) \\ -u'' - (1+n)^2 v'' & -\left(\frac{1}{1+n}u'' + (1+n)v''\right)\left(F_K + F_{KK}\frac{1}{1+n}\right) \end{pmatrix}, \quad (25)$$

because for $\tilde{\pi} = 0$, $F_K = 1+n$. But because for $\tilde{\pi} = 0$ it holds that $(1+n)A = B$ and $u' = (1+n)v'$, so that

$$C = \left(F_{KKK}\frac{k}{(1+n)^2} + F_{KK}\frac{1}{1+n}\right)\left(u' - (1+n)v'\right) - F_{KK}\frac{1}{1+n}v'(1+n) = -F_{KK}v' > 0, \quad (26)$$

it follows that the block (25) is regular, and therefore so is the entire Jacobian. By the continuity of the determinant of the Jacobian in (23)[36] with respect to $\tilde{\pi}$, it is still regular for any $\tilde{\pi} < \varepsilon$ and some $\varepsilon > 0$ small enough. As a consequence, the existence and uniqueness of the solution to the system with $\tilde{\pi} = 0$ (i.e., the existence and uniqueness of the first-best steady state) imply, by the implicit function theorem, the existence and local uniqueness of the best monetary steady state for all $\pi \in [0, \varepsilon)$. Moreover, the local uniqueness is global for all $\tilde{\pi} \in [0, \varepsilon)$ because otherwise either the correspondence from $\tilde{\pi}$ to the set solutions to (22) is not locally a function at $\tilde{\pi} = 0$ (which we just proved it is), or it is not upper hemicontinuous, which it is as well.[37] ■

Comparing equations (14) characterizing the laissez-faire competitive-equilibrium steady state with risky money with equations (21) characterizing the best steady state with risky money, it becomes apparent that the two steady states do not coincide, as the next proposition establishes.

PROPOSITION 4. *In the Diamond* (1965) *overlapping-generations economy with production, the best monetary steady state is not a competitive-equilibrium outcome under laissez-faire if the probability $\tilde{\pi}$ of money becoming worthless is positive.*

Proof. In effect, should the best monetary steady state $(c_0, \tilde{c}_0, c_1, \tilde{c}_1, k, m)$ solution to (21) coincide with a competitive-equilibrium steady state $(c_0^e, \tilde{c}_0^e, c_1^e, \tilde{c}_1^e, k^e, m^e)$ solution to (14), then from the second equation in both (14) and (21) it would hold that

$$\frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)}{\pi v'(c_1)} = 1 + n = \frac{u'(c_0)}{v'(c_1)}, \qquad (27)$$

from which $\tilde{\pi} u'(\tilde{c}_0)/\pi v'(c_1) = 0$ would follow, which under the assumptions made on u and v cannot be.[38] ■

As a consequence of Proposition 4, the laissez-faire competitive-equilibrium steady state is not the best steady state in which the economy can be when the agents can save in terms of the risky money besides capital. There is nonetheless an active fiscal policy making it possible to decentralize the best monetary steady state when money is risky as a competitive equilibrium, as the next section establishes.

5. IMPLEMENTING THE BEST MONETARY STEADY STATE THROUGH TAXES AND TRANSFERS

Assume the government announces at each period t that it will

(1) tax linearly the capital returns of generation t at $t+1$ at a rate

$$\tau^t = 1 - \frac{1+n}{F_K\left(\frac{k^{t-1}}{1+n}, 1\right)} \cdot \frac{F_K\left(\frac{k^{t-1}}{1+n}, 1\right) + F_{KK}\left(\frac{k^{t-1}}{1+n}, 1\right)\frac{k^{t-1}}{1+n}}{(1+n) + F_{KK}\left(\frac{k^{t-1}}{1+n}, 1\right)\frac{k^{t-1}}{1+n}}; \qquad (28)$$

(2) subsidize returns from monetary savings of generation t at $t+1$ at a rate[39]

$$\mu^t = \frac{\tilde{\pi} u'\left(F_L\left(\frac{k^{t-2}}{1+n}, 1\right) - k^{t-1}\right)}{\pi u'\left(F_L\left(\frac{k^{t-2}}{1+n}, 1\right) - k^{t-1} - m^{t-1}\right)}; \qquad (29)$$

and

(3) transfer to agent t at $t+1$ the following lump sum, depending on whether agent t's monetary savings have lost or not completely lost their value:

$$\tilde{T}^t = \tau^t F_K\left(\frac{k^{t-1}}{1+n}, 1\right) k^{t-1}, \qquad (30)$$

$$T^t = \tau^t F_K\left(\frac{k^{t-1}}{1+n}, 1\right) k^{t-1} - \mu^t (1+n) m^{t-1}.$$

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Then the representative agent's problem becomes

$$
\begin{aligned}
\max_{0\le c_0^t,\tilde c_0^t,c_1^t,\tilde c_1^t,k^t,m^t} \; & \pi u\left(c_0^t\right)+\tilde\pi u\left(\tilde c_0^t\right)+\pi v\left(c_1^t\right)+\tilde\pi v\left(\tilde c_1^t\right), \\
& c_0^t+k^t+m^t \le w_t, \\
& \tilde c_0^t+k^t+m^t \le w_t+\frac{1}{1+n}\rho_t m^{t-1}, \\
& c_1^t \le (1-\tau^t) r_{t+1} k^t+(1+\mu^t)\rho_{t+1} m^t+T^t, \\
& \tilde c_1^t \le (1-\tau^t) r_{t+1} k^t+\tilde T^t
\end{aligned} \tag{31}
$$

(where τ^t, μ^t, T^t and $\tilde T^t$ are not affected by the agent t's choice variables and hence are taken as given by him) and the new equilibrium conditions are the first-order conditions

$$
\begin{aligned}
\frac{\pi u'\left(c_0^t\right)+\tilde\pi u'\left(\tilde c_0^t\right)}{\pi v'\left(c_1^t\right)+\tilde\pi v'\left(\tilde c_1^t\right)} &= (1-\tau^t) r_{t+1}, \\
\frac{\pi u'\left(c_0^t\right)+\tilde\pi u'\left(\tilde c_0^t\right)}{\pi v'\left(c_1^t\right)} &= (1+\mu^t)\rho_{t+1}
\end{aligned} \tag{32}
$$

along with the budget constraints in (31) and the remunerations to factors by their marginal productivities. Thus the competitive-equilibrium steady state is now characterized by

$$
\begin{aligned}
\frac{\pi u'(c_0)+\tilde\pi u'(\tilde c_0)}{\pi v'(c_1)+\tilde\pi v'(\tilde c_1)} &= (1-\tau) F_K\left(\frac{k}{1+n},1\right), \\
\frac{\pi u'(c_0)+\tilde\pi u'(\tilde c_0)}{\pi v'(c_1)} &= (1+\mu)(1+n), \\
c_0+k+m &= F_L\left(\frac{k}{1+n},1\right), \\
\tilde c_0+k &= F_L\left(\frac{k}{1+n},1\right), \\
c_1 &= (1-\tau) F_K\left(\frac{k}{1+n},1\right) k+(1+\mu)(1+n) m+T, \\
\tilde c_1 &= (1-\tau) F_K\left(\frac{k}{1+n},1\right) k+\tilde T.
\end{aligned} \tag{33}
$$

Because it follows from this policy that, in a steady state,

$$(1-\tau)F_K\left(\frac{k}{1+n},1\right)=(1+n)\frac{F_K\left(\frac{k}{1+n},1\right)+F_{KK}\left(\frac{k}{1+n},1\right)\frac{k}{1+n}}{(1+n)+F_{KK}\left(\frac{k}{1+n},1\right)\frac{k}{1+n}}, \quad \textbf{(34)}$$

$$(1+\mu)(1+n)=\frac{\pi u'(c_0)+\tilde{\pi}u'(\tilde{c}_0)}{\pi u'(c_0)}(1+n) \quad \textbf{(35)}$$

and

$$\tilde{T}=\tau F_K\left(\frac{k}{1+n},1\right)k, \quad \textbf{(36)}$$

$$T=\tau F_K\left(\frac{k}{1+n},1\right)k-\mu(1+n)m,$$

it is straightforward to check that the systems (33) and (21) are the same one,[40] and therefore the competitive-equilibrium steady state under this policy provides the representative agent with the same contingent consumptions profile, savings, and portfolio as the best steady state implementable through the risky money. This result is summarized in the following proposition.

PROPOSITION 5. *In the Diamond* (1965) *overlapping-generations economy with production, if money risks becoming worthless with a positive but small enough probability* $\tilde{\pi}$*, then the best monetary steady state is the unique competitive-equilibrium steady state under the following policy:*

(1) *tax capital returns at* $t+1$ *at a rate*

$$\tau^t=1-\frac{1+n}{F_K\left(\frac{k^{t-1}}{1+n},1\right)}\cdot\frac{F_K\left(\frac{k^{t-1}}{1+n},1\right)+F_{KK}\left(\frac{k^{t-1}}{1+n},1\right)\frac{k^{t-1}}{1+n}}{(1+n)+F_{KK}\left(\frac{k^{t-1}}{1+n},1\right)\frac{k^{t-1}}{1+n}}; \quad \textbf{(37)}$$

(2) *subsidize monetary savings returns at* $t+1$ *at a rate*

$$\mu^t=\frac{\tilde{\pi}u'\left(F_L\left(\frac{k^{t-2}}{1+n},1\right)-k^{t-1}\right)}{\pi u'\left(F_L\left(\frac{k^{t-2}}{1+n},1\right)-k^{t-1}-m^{t-1}\right)}; \quad \textbf{(38)}$$

(3) *transfer to each agent of generation* t *at* $t+1$ *the lump sum* $\tilde{T}_t$ *or* T_t*, defined as*

$$\tilde{T}^t=\tau^tF_K\left(\frac{k^{t-1}}{1+n},1\right)k^{t-1} \quad \textbf{(39)}$$

$$T^t=\tau^tF_K\left(\frac{k^{t-1}}{1+n},1\right)k^{t-1}-\mu^t(1+n)m^{t-1},$$

depending on whether agent t*'s monetary savings have lost or not completely lost their value at* $t+1$.

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A few remarks are in order at this point. First note that the tax and transfers policy announced in any period t is defined as a function of the capital savings decided by the generation born at $t-1$. Therefore, the policy is defined in terms of information that is both known at the time of its announcement and not manipulable by the agents to which it applies. Second, by construction, the government does not incur any deficit or surplus at the steady state, because the amount raised by the tax in a distortionary way is given back as a lump sum to the same agents in the same period.

Note finally that whether the returns to capital need to be taxed (as opposed to subsidized), i.e., whether $\tau > 0$, hinges on the following inequality holding true,

$$\frac{F_K\left(\frac{k}{1+n}, 1\right)}{1+n} > \frac{F_K\left(\frac{k}{1+n}, 1\right) + F_{KK}\left(\frac{k}{1+n}, 1\right)\frac{k}{1+n}}{(1+n) + F_{KK}\left(\frac{k}{1+n}, 1\right)\frac{k}{1+n}} \tag{40}$$

for the best monetary steady state allocation $(c_0, \tilde{c}_0, c_1, \tilde{c}_1, k, m)$, which—given that $(1+n) + F_{KK}(\frac{k}{1+n}, 1)\frac{k}{1+n} > 0$ for any Cobb–Douglas or CES technology with an elasticity of substitution smaller than 1[41]—holds if, and only if,

$$1 > \frac{F_K\left(\frac{k}{1+n}, 1\right)}{1+n} \tag{41}$$

or equivalently if, and only if, $k^* < k$, where k^* is the first-best level of per-worker capital savings satisfying $F_K(\frac{k^*}{1+n}, 1) = 1+n$, and k is the level of capital for the best monetary steady state solving (21). In other words, if for the best monetary steady state the level of capital is higher than for the first-best steady state, then its implementation as a competitive equilibrium requires taxing the return of capital savings. Otherwise the returns to capital need to be subsidized as well. Note, however, that according to Proposition 3 in Weil (1987) this case can only happen if in the absence of money the economy has more than one steady state.

6. CONCLUDING REMARKS

To conclude, and for the sake of completeness, some comments on the usual argument about the implementability of the first-best steady state as a monetary equilibrium with riskless money are in order. Clearly, the implementation of the first-best steady state as a competitive equilibrium typically requires holding strictly positive amounts of both money (or public debt)[42] and capital. Therefore, in the absence of uncertainty, both assets must have the same return in the steady state, so that the agents are indifferent about the composition of their savings portfolio. In other words, the agents' (although not the planner's) choice of the *composition* of their savings portfolio is completely undetermined in the first-best steady state (even if the *level* is not). Thus, although there indeed exists a way

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to support the first-best steady state using money to place some of the agents' savings, nothing in the model explains why the agents would actually *choose* to place their savings precisely the way that makes it possible to do so. As a matter of fact, nothing *within the model* leads the agents to choose the composition of the portfolio that implements the first-best.[43]

Note finally that this indeterminacy is not of the same nature as, say, that of the production plan at equilibrium of a firm with a constant–returns to scale technology. In effect, in that case it is widely assumed that production just adjusts to a demand that is well determined by prices. Nevertheless, in the case of the choice savings portfolio in the first-best steady state, on the two sides of the money market sits the same representative agent, and *both* sides then face the same indeterminacy. As a result, there is no well-determined side of the market in this case that is able to anchor an indeterminate side. This points to the existence of an element, missing from the model, that would explain why the agents would choose to save exactly the right amounts of capital and money that make it possible to put the economy into the best possible steady state. Interestingly enough, the introduction of the risk of money completely losing value, even if this risk is minimal, helps to pin down completely both the level and the composition of the agents' savings portfolio in the steady state.

A number of issues clearly remain to be addressed in this setup, such as the dynamics out of the steady state, the cost of moving to such a steady state, and the endogenization of the probability of breakdown of the intergenerational transfers mechanism. These and other issues are left for future research.

NOTES

1. That is to say, without resorting to redistribution across generations as, for instance, a pay-as-you-go pension scheme does.

2. Maybe because of lack of consensus on a social security system or because of the weakness of the state to implement it.

3. A fiat money in Samuelson (1958), or public debt that is rolled over every period in Diamond (1965).

4. Which, incidentally, increases the marginal productivity of capital and hence the return to their own savings to an extent that offsets their lower level of savings.

5. Although not a completely unbacked asset, the bonds issued by the Confederate States of America to finance the war effort during the American Civil War were pledged not to be honored by the Union, and the paper money issued by the Confederacy was just paper at the end of the conflict (although it had already almost zero exchange value by then, because of both massive printing by the Confederacy and deliberate flooding of the South with counterfeit CSA dollars by the Union).

6. The analysis presented later actually generalizes, in ways that will be made precise, to the case in which a money suffers a sharp and sudden drop in its real value following a monetary reform or an outburst of hyperinflation. The choice of a complete loss of value of a money is made just for expositional purposes.

7. One can of course always debate what a government can and cannot do and why. To try to circumvent that, I assume here the least possible intrusive government and consider how a government can improve upon the competitive-equilibrium steady state *without resorting to redistribution* (implicitly assuming that agents do not oppose fiscal policies that turn out to be nonredistributive in

the steady state, which seems a sensible assumption, especially when such policies help to implement a unanimously preferred steady state).

8. That is to say, intergenerational transfers that divert saved resources toward someone else's consumption, instead of toward production, actually.

9. In the final section I will further assume, in order to establish a sign for the tax rate on capital returns, that the technology is described by a constant–returns to scale Cobb–Douglas production function or a CES production function with elasticity of substitution smaller than 1. For all the other results constant returns to scale suffice.

10. Although the choice of notation is always debatable, I will choose to write the model in terms of the *choice variables* of the agents and hence keep k for the *per-worker savings in capital*, instead of (as it is traditional) the level of *capital per old agent*. In the same vein, I will prefer explicit marginal productivities to so-called "intensive form" expressions that may obscure relations that are otherwise pretty clear (this is, at any rate, particularly true for the arguments and proofs presented later). Needless to say, the two choices are equivalent.

11. In the case in which $\frac{c_1^*}{1+n} - k^* < 0$, if every period the young agents receive a transfer from the old agents of an amount m^* of the good, then the first-best steady state (c_0^*, c_1^*, k^*) obtains as well, but for this transfers to be implemented in a competitive equilibrium there needs to be an additional infinitely lived agent, a bank, willing to buy from the young agent t an IOU bearing interest of $1+n$ in exchange for an amount $-m^* > 0$ of the good paid back to the bank by agent $t-1$ to cancel his own outstanding IOU to the bank. Note that it is implicitly assumed that the agents pay back their IOUs when old *with probability* 1. Note also that the intermediary bank would thus make neither gains nor losses.

12. Similarly for the assumption that the agents repay with probability 1 their IOUs when old in the case $\frac{c_1^*}{1+n} - k^* < 0$.

13. As in Weil (1987), except for the fact that in that paper the "steady state" equilibrium switches to the moneyless steady state once money becomes worthless. Here, in contrast, a new money replaces the dismissed one when that happens and, as a consequence, the equilibrium will be truly stationary.

14. For a small enough probability of money becoming worthless (see Proposition 2).

15. Unique as well for a small enough probability of money becoming worthless (see Proposition 3).

16. In the absence of money or any other intergenerational transfer mechanism, the competitive-equilibrium steady state differs typically also not only from the first-best steady state, but also from the best steady state that can be implemented through the existing markets for capital and labor. An adequate policy of taxes and transfers nevertheless makes it possible to implement this constrained-best steady state [see Dávila (2008)].

17. More generally, $\tilde{\pi}$ is the probability with which a fraction θ of the agent monetary savings are lost, following a monetary reform—possibly as a result of a hyperinflationary period—in which the old money is exchanged for the new one at a rate $1-\theta$ (the case of a complete loss of value corresponding to $\theta = 1$).

18. In the case $\theta \in (0, 1)$ in which money loses only a fraction of its value, the second and fourth constraints in (8) are respectively

$$\tilde{c}_0^t + k^t + m^t \le w_t + \theta \frac{1}{1+n} \rho_t m^{t-1},$$

$$\tilde{c}_1^t \le r_{t+1} k^t + (1-\theta) \rho_{t+1} m^t.$$

19. As in note 11, a negative $m^t < 0$ stands for the resources bought by agent t by issuing an IOU to the bank. There is the risk in every period that the IOU issued by agent $t-1$ will not be repaid at t, thus reducing to zero the real value of agent t's issuance of his or her IOU decided prior to this uncertainty being resolved and hence still owed at $t+1$. Note that the constrained set is compact, and hence the agent's problem is well defined, as long as $r_{t+1} < \rho_{t+1}$, i.e., if the risky asset (the money or IOU) bears a higher return than the safe asset (capital).

20. In the case $m_t < 0$, with probability $\tilde{\pi}$ the price of the good in terms of agent t's IOU to the bank becomes infinity (because agent $t-1$ does not sell any amount of the good to cancel a debt that he or she is not paying back anymore), whereas in terms of the young agents' labor income it adjusts to clear markets, given that the old will not put the resources $\rho_t m^{t-1}/(1+n)$ into the market any more.

21. More generally, if $\theta \in (0, 1)$,

$$\frac{\pi u'(c_0^t) + \tilde{\pi} u'(\tilde{c}_0^t)}{\pi v'(c_1^t) + \tilde{\pi} v'(\tilde{c}_1^t)} = r_{t+1},$$

$$\frac{\pi u'(c_0^t) + \tilde{\pi} u'(\tilde{c}_0^t)}{\pi v'(c_1^t) + \tilde{\pi} v'(\tilde{c}_1^t)(1-\theta)} = \rho_{t+1}.$$

Note also that the agent's optimal choice determines now not only the overall level of savings $k^t + m^t$ chosen by the agent, given w_t, r_{t+1}, and ρ_{t+1}, but also the very composition of the savings portfolio, i.e., k^t and m^t (the system of equilibrium equations (9) along with the budget constraints in (8) reduces in fact to a system of two equations in k^t and m^t). This is in sharp contrast to what happens in the absence of risk, when only the market clearing condition for capital pins down the individual's savings portfolio. In effect, for both assets to be held simultaneously they must earn the same return, making the agent indifferent to the composition of his or her portfolio. Interestingly, it follows from the conditions (9) that, at any competitive equilibrium, the return to money (unproductive savings) has to be necessarily larger than the return to capital (productive savings), that is to say, $r_{t+1} < \rho_{t+1}$. The higher real return for monetary savings is clearly a consequence of the fact that money is a riskier asset than capital in this setup, so that it needs to bear a higher return for the agents to be willing to accept it at equilibrium. It may seem surprising at first, because the only productive investments here are, at least directly, those made in terms of capital. It is worth stressing, at any rate, that money (or by the same token public debt, a pay-as-you-go pension system, or any other intergenerational transfer mechanism) is an unproductive investment only in a strictly direct technological and physical sense, because by diverting excess capital towards consumption and hence making it possible to support higher levels of net output at equilibrium, it cannot be deemed socially unproductive, if only because it implements a better steady state. Social arrangements or institutions thus certainly matter.

22. In the event that the bubble bursts, condition (11) holds for $\tilde{c}_0^t$ and $\tilde{c}_1^{t-1}$, and also for $\theta \in (0, 1)$.

23. The modifications needed for the case $\theta \in (0, 1)$ are straightforward.

24. In the case $\frac{c_1^*}{1+n} - k^* < 0$, this is the amount young agents borrow from the infinitely lived intermediary in the credit market, to be repaid with an interest n when they are old.

25. When $\tilde{\pi} = 1$ the system has as solution for $(\tilde{c}_0, \tilde{c}_1, k)$ the competitive-equilibrium steady state in the absence of a money solution $(\bar{c}_0, \bar{c}_1, \bar{k})$ to the first, fourth, and sixth equations in (14), or equivalently with money known to be worthless next period and hence worthless today (with c_0, c_1, m required to satify the third and fifth equations in (15) at indeterminate but irrelevant levels).

26. That is, to say $u'' = u''(c_0)$, $\tilde{u}'' = u''(\tilde{c}_0)$, $v'' = v''(c_1)$, $\tilde{v}'' = v''(\tilde{c}_1)$, $F_K = F_K(\frac{k}{1+n}, 1)$, $F_{KK} = F_{KK}(\frac{k}{1+n}, 1)$, $F_{LK} = F_{LK}(\frac{k}{1+n}, 1)$ at a solution $c_0, \tilde{c}_0, c_1, \tilde{c}_1, k, m$ to the system.

27. Which is clearly equal to the determinant of the block (17); i.e.,

$$-\left[(1+n)^2 v'' + u''\right] v' F_{KK} \frac{1}{1+n} < 0.$$

28. In effect, because at every point of the graph of the correspondence from $\tilde{\pi}$ to the set of solutions to (15) the correspondence is locally a C^1 function of $\tilde{\pi}$ for every $\tilde{\pi} \in [0, \varepsilon)$, for every sequence $\{\tilde{\pi}_n, (c_0^n, \tilde{c}_0^n, c_1^n, \tilde{c}_1^n, k^n, m^n)\}_{n \in \mathbf{N}}$ within the graph of that correspondence such that $\{\tilde{\pi}_n\}_{n \in \mathbf{N}}$ converges to 0 there exists a convergent subsequence. Because moreover, all the left-hand sides in (15) are continuous with respect to all $c_0, \tilde{c}_0, c_1, \tilde{c}_1, k, m$ and $\tilde{\pi}$, (15) still holds true in the limit of the subsequence when $\tilde{\pi} \to 0^+$, which establishes upper hemicontinuity at $\tilde{\pi} = 0$. Incidentally, it can be easily checked that for $\tilde{\pi} = 1$ the Jacobian is singular. When money is worthless, the system (15) has a trivial indeterminacy in c_0, c_1, m.

29. That reverts to the moneyless steady state once it bursts. The equilibrium conditions in Proposition 3 in Weil (1987) are nonetheless equivalent to those characterizing the recurrent stochastic asset bubbles being considered here, i.e., a profile $(c_0^e, c_1^e, \tilde{c}_0^e, \tilde{c}_1^e, k^e, m^e)$ solution to (14).

30. In the case $\theta \in (0, 1)$ in which money loses only a fraction of its value, the second and fourth constraints in (20) are, respectively,

$$\tilde{c}_0 + k + m \leq F_L\left(\frac{k}{1+n}, 1\right) + \theta m,$$

$$\tilde{c}_1 \leq F_K\left(\frac{k}{1+n}, 1\right) k + (1-\theta)m.$$

31. In the case $\theta \in (0, 1)$, the second equation becomes

$$\frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)(1-\theta)}{\pi v'(c_1) + \tilde{\pi} v'(\tilde{c}_1)(1-\theta)} = 1 + n,$$

whereas the binding budget constraints get modified as mentioned in note 30.

32. That is to say, a unique steady state that maximizes the representative agent's utility *among those implementable through savings in risky money* (on top of savings in capital).

33. That is to say, $u'' = u''(c_0)$, $\tilde{u}'' = u''(\tilde{c}_0)$, $v'' = v''(c_1)$, $\tilde{v}'' = v''(\tilde{c}_1)$, $F_K = F_K(\frac{k}{1+n}, 1)$, $F_{KK} = F_{KK}(\frac{k}{1+n}, 1)$, $F_{LK} = F_{LK}(\frac{k}{1+n}, 1)$, at a solution $c_0, \tilde{c}_0, c_1, \tilde{c}_1, k, m$ to the system.

34. Or IOUs, if negative.

35. When $\tilde{\pi} = 1$, the system determines the choice $(\tilde{c}_0, \tilde{c}_1, k)$ maximizing the representative agent's utility with no intergenerational transfers but remunerating factors by their productivities (with c_0, c_1, m required to satisfy the second, third, and fifth equations in (20) given k).

36. Which is clearly equal to the determinant of the block (25), i.e.,

$$-\left[(1+n)^2 v'' + u''\right] F_{KK} v' < 0.$$

37. For the same reasons as in Proposition 2. Incidentally, it can easily be checked that for $\tilde{\pi} = 1$ the Jacobian is singular.

38. In the case where money loses value partially, i.e., $\theta \in (0, 1)$, the competitive-equilibrium steady state is the first-best one only if

$$\frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)(1-\theta)}{\pi v'(c_1) + \tilde{\pi} v'(\tilde{c}_1)(1-\theta)} = \frac{\pi u'(c_0) + \tilde{\pi} u'(\tilde{c}_0)}{\pi v'(c_1) + \tilde{\pi} v'(\tilde{c}_1)(1-\theta)},$$

which requires

$$\tilde{\pi} u'(\tilde{c}_0)\theta = 0!!$$

39. In the case $\theta \in (0, 1)$, the rate μ^t is

$$\mu^t = \frac{\pi u'\left(F_L\left(\frac{k^{t-2}}{1+n}, 1\right) - k^{t-1} - m^{t-1}\right) + \tilde{\pi} u'\left(F_L\left(\frac{k^{t-2}}{1+n}, 1\right) - k^{t-1}\right)}{\pi u'\left(F_L\left(\frac{k^{t-2}}{1+n}, 1\right) - k^{t-1} - m^{t-1}\right) + \tilde{\pi} u'\left(F_L\left(\frac{k^{t-2}}{1+n}, 1\right) - k^{t-1}\right)(1-\theta)} - 1.$$

40. The same is true in the general case $\theta \in (0, 1)$ for the rate μ^t in note 39, for which Proposition 5 applies as well.

41. In effect, the right-hand side of the first equation must be positive in (21), i.e.,

$$\frac{F_K\left(\frac{k}{1+n}, 1\right) + F_{KK}\left(\frac{k}{1+n}, 1\right)\frac{k}{1+n}}{(1+n) + F_{KK}\left(\frac{k}{1+n}, 1\right)\frac{k}{1+n}} > 0,$$

but the numerator is positive for any constant–returns to scale Cobb–Douglas technology $F(K, L) = AK^{\alpha}L^{1-\alpha}$ and any CES technology $F(K, L) = A[aK^r + (1-a)L^r]^{1/r}$ with $r < 0$ and hence elasticity of substitution $s = \frac{1}{1-r} \in [0, 1)$, so that the denominator is positive as well.

42. Or the issuance of IOUs by the agents when young if they want to transfer income from their old age to their young age.

43. That the modeler knows this to be the right thing to do does not seem to be a very compelling argument.

REFERENCES

Dávila, Julio (2008) The Taxation of Capital Returns in Overlapping Generations Economies without Financial Assets. CORE Discussion Paper 2008/75.

Diamond, Peter A. (1965) National debt in a neoclassical growth model. *American Economic Review* 55, 1126–1150.

Samuelson, Paul A. (1958) An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy* 66, 467–482.

Tirole, Jean (1985) Asset bubbles and overlapping generations economies. *Econometrica* 53, 1499–1528

Weil, Philipppe (1987) Confidence and the real value of money in an overlapping generations economy. *Quarterly Journal of Economics* 107, 29–42.

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