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NASH EQUILIBRIUMS IN TWO-PERSON RED-AND-BLACK GAMES

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In a two-person red-and-black game, each player wants to maximize the probability of winning the entire fortune of his opponent by gambling repeatedly with suitably chosen stakes. We find the multiplicativity (including submultiplicative and supermultiplicative) of the win probability function is important for the profiles (bold, timid) or (bold, bold) to be a Nash equilibrium. Surprisingly, a Nash equilibrium condition for the profile (bold, any strategy) is also given in terms of multiplicativity. Finally, we search for some suitable conditions such that the profile (timid, timid) is also a Nash equilibrium.

1. INTRODUCTION

The red-and-black gambling problem has taken its name from the game of roulette. The discrete version of this gambling problem can be described as follows. A player beginning with a positive integral fortune of x units can stake any positive integer amount $a \leq x$. His fortune becomes $x + a$ if he wins with a fixed probability w $(0 < w < 1)$ and $x - a$ if he loses with probability $1 - w$. The player seeks to maximize the probability of reaching a prespecified goal M by gambling repeatedly with suitably chosen stakes. Dubins and Savage [5] showed that in the subfair case (i.e., $w \leq 1/2$) an optimal strategy is bold play, which corresponds to always betting the entire fortune or just enough to reach the goal, whichever is smaller. This seems intuitively reasonable in that a shorter game seems to give a better chance to the subfair player since he will surely lose in the long run. In the superfair case (i.e., $w \geq 1/2$), Ross [9] proved that it is optimal for the player to bet timidly, that is, always to stake 1 unit of his current fortune at each stage. Intuitively, if the player is superfair, to prolong the game is better for him.

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The discrete version of red-and-black game has been extended in several ways. One of the extensions is a two-person red-and-black game which was introduced by Secchi [10]. Later, Pontiggia [7] proposed two different formulations: a weighted two-person red-and-black game and a proportional two-person red-and-black game. She showed that in each model, it is a Nash equilibrium for subfair player to play boldly and for superfair player to play timidly. Chen and Hsiau [3] proposed the bet-dependent win probability functions to generalize Pontiggia's results. They showed that if the subfair player's win probability function $f : [0, 1] \to [0, 1]$ is convex and satisfies that $f(0) = 0, f(s) \leq s$ and $f(s)f(t) \leq f(st)$ for all $0 \leq t \leq s \leq 1$, then it is a Nash equilibrium for the subfair player to play boldly and for the superfair player to play timidly. Later, two new models of two-person red-and-black game were proposed by Chen and Hsiau [4]. One is called a bet-exchangeable game in which at each stage there is a positive probability that two players exchange their bets. The other one is called a stage-dependent game in which the win probability functions are stage-dependent. In each model, they searched for some suitable conditions such that the subfair player playing boldly and the superfair player playing timidly is a Nash equilibrium. Recently, Chen [2] considered the two-person red-and-black game with lower limit and proposed a modified timid strategy. For her model, she showed that if the subfair player's win probability function satisfies some suitable conditions, then the subfair player using a bold strategy and the superfair one using the modified timid strategy forms a Nash equilibrium.

Pontiggia [7] also introduced an N-person model, called proportional N-person red-and-black game, and proposed a conjecture about it. For this Chen and Hsiau [3] gave a counterexample of the conjecture and Chen [1] showed that the conjecture is true in proportional three-person red-and-black game with suitable weights on each player. In [8], Pontiggia proposed an N-person nonconstant sum game, for which she gave some suitable conditions on the winning probability function to ensure that it is a Nash equilibrium for each player to play boldly.

In this paper, we propose different Nash equilibrium conditions for two-person red-and-black games which can be described as follows. There are two people gambling in stages with positive integer initial fortune. Denote the two players by I and II. Assume during the game the total amount of fortune $M, M \geq 2$, is fixed. At each stage, assume each player chooses his stakes without any knowledge of the stakes chosen by the other. Let f be a nonzero and nondecreasing function from $[0, 1]$ to $[0, 1]$ with $f(0) = 0$. Suppose at stage m, player I has x_m chips and bets $a_m \in \{1, \ldots, x_m\}$ chips, while player II bets $b_m \in \{1, \ldots, M - x_m\}$ chips. The law of motion for player I is defined by

$$x_{m+1} = \begin{cases} x_m + b_m & \text{with probability } f\left(\dfrac{a_m}{a_m + b_m}\right), \\ x_m - a_m & \text{with probability } 1 - f\left(\dfrac{a_m}{a_m + b_m}\right), \end{cases}$$

for $1 \leq x_m \leq M - 1$, and by $x_{m+1} = x_m$ with probability 1 for $x_m = 0$ or $x_m = M$, which means that once one of the players reaches M, the state of neither player can change. Here, f is called player I's win probability function. The goal of each player is to maximize his probability of winning the entire fortune of his opponent, i.e., reaching M, by gambling repeatedly with suitably chosen stakes.

Hereafter, a bold strategy is that a player always bets his entire fortune at each stage of the game, and a timid strategy is that a player always bets one unit at each stage of the game. For convenience, the profile (strategy α, strategy β) will denote that player I plays strategy α and II plays strategy β. A profile (strategy α, strategy β) is said to be a Nash equilibrium if strategy α is optimal for player I while player II plays strategy β and strategy β is optimal for player II while player I plays strategy α.

A function f is said to be *multiplicative* if

$$f(st) = f(s)f(t) \quad \text{for all } s, t \in (0, 1).$$

A function f is said to be *submultiplicative* if

$$f(st) \le f(s)f(t) \quad \text{for all } s, t \in (0, 1).$$

Moreover, if the above strict inequality holds, then we say that f is strictly submultiplicative. A function f is said to be *supermultiplicative* if

$$f(st) \ge f(s)f(t) \quad \text{for all } s, t \in (0, 1).$$

Moreover, if the above strict inequality holds, then we say that f is strictly submultiplicative. Notice that if f_1 and f_2 are two submultiplicative win probability functions, then $f_1 \circ f_2$ is nonzero and nondecreasing, $f_1 \circ f_2(0) = f_1(0) = 0$ and

$$f_1(f_2(st)) \le f_1(f_2(s)f_2(t)) \le (f_1 \circ f_2(s))(f_1 \circ f_2(t)) \quad \text{for all } s, t \in (0, 1),$$

which imply that $f_1 \circ f_2$ is also a submultiplicative win probability function. Similarly, the composition of two supermultiplicative win probability functions is also a supermultiplicative win probability function. This simple property makes it easy to produce more submultiplicative (or supermultiplicative) win probability functions.

The organization of this paper is as follows. In Section 2, we first show that if player I's win probability function is supermultiplicative and player II's win probability function is submultiplicative, then the profile (bold, timid) is a Nash equilibrium. Next, we prove that if the two players' win probability functions are submultiplicative, then the profile (bold, bold) is a Nash equilibrium. These two results infer that if player I's win probability function is multiplicative and player II's win probability function is submultiplicative, then both the profiles (bold, timid) and (bold, bold) are Nash equilibrium. In fact, in Section 3, we show that the profile (bold, any strategy) is a Nash equilibrium if player I's win probability function is multiplicative and player II's win probability function is submultiplicative. Finally, in Section 4, we search for some suitable conditions such that the profile (timid, timid) is a Nash equilibrium.

2. NASH EQUILIBRIUMS FOR THE PROFILES (BOLD, TIMID) AND (BOLD, BOLD)

In this section, we find that the win probability function being submultiplicative or supermultiplicative is important for the profile (bold, timid) or (bold, bold) to be a Nash equilibrium.

THEOREM 2.1: *If f is supermultiplicative and g is submultiplicative, then the profile (bold, timid) is a Nash equilibrium. If, in addition, f is strictly supermultiplicative and g is strictly submultiplicative, then the profile (bold, timid) is the unique Nash equilibrium.*

THEOREM 2.2: *If f and g are submultiplicative, then the profile (bold, bold) is a Nash equilibrium for this game. Moreover, if f and g are strictly submultiplicative, then the profile (bold, bold) is the unique Nash equilibrium.*

Before proving Theorems 2.1 and 2.2, three lemmas about the submultiplicative win probability functions and supermultiplicative win probability functions are investigated first. For convenience, denote, hereafter, player I's win probability function by f and player II's win probability by g. Note that $g(s) = 1 - f(1-s)$ for all $s \in [0, 1]$.

LEMMA 2.3: *If f is submultiplicative, then a bold strategy is optimal for player II while player I plays boldly. Moreover, if f is strictly submultiplicative, then a bold strategy is the unique optimal strategy for player II while player I plays boldly.*

PROOF: Assume that player I plays boldly. If player II also adopts a bold strategy, then the corresponding law of motion at stage m for player I having x_m units and playing boldly is given by

$$x_{m+1} = \begin{cases} M & \text{with probability } f\left(\dfrac{x_m}{M}\right) \\ 0 & \text{with probability } 1 - f\left(\dfrac{x_m}{M}\right) \end{cases}$$

for $1 \leq x_m \leq M-1$ and $x_{m+1} = x_m$ with probability 1 for $x_m = 0$ or M. Set

$$T(x) = P(\text{player II reaches } M \text{ with an initial fortune } M - x).$$

From the above, it is clear that $T(0) = 1$, $T(M) = 0$, and it is not difficult to derive the identity:

$$T(x) = 1 - f\left(\frac{x}{M}\right) \quad \text{if } 1 \leq x \leq M-1. \tag{2.1}$$

To prove a bold strategy is optimal for player II while player I plays boldly, it suffices to show that $T(\cdot)$ is excessive (see Theorem 3.3.10 of [6]) or, equivalently, to

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show that for every $x \in \{1, \ldots, M-1\}$ and every $b \in \{1, \ldots, M-x\}$:

$$T(x) \geq f\left(\frac{x}{x+b}\right) T(x+b) + \left[1 - f\left(\frac{x}{x+b}\right)\right] T(0). \qquad \textbf{(2.2)}$$

Substituting (2.1) into (2.2), we see that the inequality (2.2) equals to

$$f\left(\frac{x}{M}\right) \leq f\left(\frac{x}{x+b}\right) f\left(\frac{x+b}{M}\right) \qquad \textbf{(2.3)}$$

which holds since f is submultiplicative. Hence, the bold strategy is optimal for player II.

Moreover, if f is strictly submultiplicative, then (2.3) is actually an equality only at $b = M - x$. Thus, the bold strategy is the unique optimal strategy for player II when player I plays boldly. Hence the proof is complete. ■

Remark 2.1: Note that the inequality (2.3) means that if player I plays boldly, then the win probability of player I reaching M while player II always plays boldly is not greater than that while player II first bets b units and then plays boldly. That is, if player I plays boldly, then the probability of player II going broke by always playing boldly is not greater than that by first betting b units and then playing boldly. Thus, from the above observation, we see that if f is submultiplicative, then to bet less times makes the probability of player II going broke smaller. Hence, Lemma 2.3 seems intuitively reasonable, in that a shorter game seems to give a better chance to player II.

LEMMA 2.4: *If f is supermultiplicative, then a timid strategy is optimal for player II while player I plays boldly. Moreover, if f is strictly supermultiplicative, then a timid strategy is the unique optimal strategy for player II while player I plays boldly.*

The proof of Lemma 2.4 is similar to the proof of Theorem 2.2 in [3]. Here we omit it. Note that as in Remark 2.1, we see that if f is supermultiplicative, then to bet more times makes the probability of player II going broke smaller. Hence, Lemma 2.4 seems intuitively reasonable, in that to prolong the game is better for player II.

LEMMA 2.5: *If f is supermultiplicative and g is submultiplicative, then a bold strategy is optimal for player I while player II plays timidly. Moreover, if g is strictly submultiplicative, then a bold strategy is the unique optimal strategy for player I while player II plays timidly.*

PROOF: Assume player II plays a timid strategy. If player I adopts a bold strategy, set

$$Q(x) = P(\text{player I reaches } M \text{ with an initial fortune } x).$$

Notice that when player I plays boldly, he reaches $y, x < y \le M$, if and only if he wins $y-x$ successive games, his capital increasing by 1 at the end of each game, and so

$$Q(x) = \left[\prod_{i=0}^{y-x-1} f\left(\frac{x+i}{x+i+1}\right)\right] Q(y). \tag{2.4}$$

As in the proof of Lemma 2.3, to prove that a bold strategy is optimal for player I, it suffices to show that for all $x \in \{1, \ldots, M-1\}$ and $a \in \{1, \ldots, x\}$,

$$\begin{aligned} Q(x) &\ge f\left(\frac{a}{a+1}\right) Q(x+1) + \left[1 - f\left(\frac{a}{a+1}\right)\right] Q(x-a) \\ &= f\left(\frac{a}{a+1}\right) Q(x+1) + g\left(\frac{1}{a+1}\right) Q(x-a). \end{aligned} \tag{2.5}$$

From (2.4), we have that $Q(x) = f(x/(x+1))Q(x+1)$ and $Q(x-a) = \left[\prod_{i=0}^{a} f((x-i)/(x-i+1))\right] Q(x+1)$ and so the inequality (2.5) is equivalent to

$$f\left(\frac{x}{x+1}\right) \ge f\left(\frac{a}{a+1}\right) + g\left(\frac{1}{a+1}\right) \prod_{i=0}^{a} f\left(\frac{x-i}{x-i+1}\right) \tag{2.6}$$

since $Q(x+1) \ge 0$. Since f is supermultiplicative, the inequality (2.6) is satisfied if

$$\begin{aligned} f\left(\frac{x}{x+1}\right) &\ge f\left(\frac{a}{a+1}\right) + g\left(\frac{1}{a+1}\right) f\left(\frac{x-a}{x+1}\right) \\ &= f\left(\frac{a}{a+1}\right) + g\left(\frac{1}{a+1}\right)\left[1 - g\left(\frac{a+1}{x+1}\right)\right]. \end{aligned}$$

Since g is submultiplicative, the above inequality is satisfied if

$$f\left(\frac{x}{x+1}\right) \ge f\left(\frac{a}{a+1}\right) + g\left(\frac{1}{a+1}\right) - g\left(\frac{1}{x+1}\right)$$

which is true with equality, since $f(s) + g(1-s) = 1$. Hence, a bold strategy is optimal for player I while player II plays timidly.

Moreover, if g is strictly submultiplicative, it can be proved that the inequality (2.5) is actually an equality if and only if $a = x$. This means that in this case, the bold strategy is the unique strategy for player I when player II plays timidly. ■

From the definition of Nash equilibriums, Lemmas 2.4 and 2.5 combined with Lemma A.1 of [7] imply Theorem 2.1. Notice that Theorem 2.3 in [3] gave a different conditions on win probability function such that the profile (bold, timid) is a Nash equilibrium. Next, Lemma 2.3 combined with Lemma A.1 of [7] implies Theorem 2.2.

Remark 2.2: Recall that the composition of two submultiplicative (or supermultiplicative) win probability functions is also a submultiplicative (or supermultiplicative, respectively) win probability function. Moreover, if $g_i(s) = 1 - f_i(1-s)$, $i = 1, 2$, then $1 - f_1 \circ f_2(1-s) = 1 - f_1(1 - g_2(s)) = g_1 \circ g_2(s)$. It follows that $g_1 \circ g_2$ is also player II's win probability function if $f_1 \circ f_2$ is player I's win probability function. Therefore, if for each $i = 1, 2$, f_i is supermultiplicative and g_i is submultiplicative, the profile (bold, timid), by Theorem 2.1, is a Nash equilibrium for a two-person red-and-black game with player I's win probability function $f_1 \circ f_2$.

On the other hand, if for each $i = 1, 2$, f_i and g_i are submultiplicative, then $f_1 \circ f_2$ and $g_1 \circ g_2$ are submultiplicative and so by Theorem 2.2, the profile (bold, bold) is a Nash equilibrium for a two-person red-and-black game with player I's win probability function $f_1 \circ f_2$.

The following two examples apply Theorem 2.1.

Example 2.1: Recall the proportional two-person red-and-black game, proposed by Pontiggia, where $f(s) = sw/(sw + \bar{s}\bar{w})$, $0 < w \leq 1/2$, $\bar{w} = 1 - w$ and $\bar{s} = 1 - s$. Then $g(s) = 1 - f(1-s) = s\bar{w}/(s\bar{w} + \bar{s}w)$. Note that for all $s, t \in (0, 1)$,

$$\begin{aligned} f(st) - f(s)f(t) &= \left(\frac{stw}{stw + (1-st)\bar{w}}\right) - \left(\frac{sw}{sw + \bar{s}\bar{w}}\right)\left(\frac{tw}{tw + \bar{t}\bar{w}}\right) \\ &= \frac{s\bar{s}t\bar{t}w\bar{w}(1-2w)}{[stw + (1-st)\bar{w}]\,(sw + \bar{s}\bar{w})(tw + \bar{t}\bar{w})} > 0. \end{aligned}$$

Clearly, $g(st) - g(s)g(t)$ is given by the same expression, but with w and $\bar{w}$ interchanged, and so it is negative. Hence f strictly supermultiplicative and g is strictly submultiplicative, so by Theorem 2.1, the profile (bold, timid) is the unique Nash equilibrium.

Example 2.2: Here, we give player I's win probability function f, which is not convex, so Theorem 2.3 in [3] cannot be applied. But by applying Theorem 2.1, we can show that the profile (bold, timid) is the unique Nash equilibrium.

Assume player I's win probability function

$$f(s) = \begin{cases} s^2 & \text{if } 0 \leq s < 1/2, \\ \dfrac{s/4}{s/4 + 3(1-s)/4} = \dfrac{s}{3-2s} & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Note that $f(1/2) - [f(2/5) + f(3/5)]/2 = 1/300 > 0$ and so f is not convex. Since $g(s) = 1 - f(1-s)$, it follows that

$$g(s) = \begin{cases} s(2-s) & \text{if } 0 \leq s < 1/2, \\ \dfrac{3s}{1+2s} & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

If $s, t \in (0, 1/2]$, $f(st) = s^2t^2 = f(s)f(t)$ and $g(st) = st(2 - st) < st[2 - st + 2(1 - s)(1 - t)] = st(2 - s)(2 - t) = g(s)g(t)$. If $0 < t < 1/2 < s < 1$, then

$$f(st) - f(s)f(t) = s^2t^2 - \frac{st^2}{3 - 2s} = \frac{st^2(1 - s)(2s - 1)}{3 - 2s} > 0,$$
$$g(st) - g(s)g(t) = st(2 - st) - \frac{t(2 - t)3s}{1 + 2s} = \frac{st(1 - s)[(2st - 1) + 3(t - 1)]}{1 + 2s} < 0.$$

Similarly, it can be proved that $f(st) > f(s)f(t)$ and $g(st) < g(s)g(t)$ for $s, t \in (1/2, 1)$. Hence f strictly supermultiplicative and g is strictly submultiplicative.

The following two examples apply Theorem 2.2.

Example 2.3: Assume player I's win probability function $f(s) = s^\gamma/[s^\gamma + (1 - s)^\gamma]$, where $\gamma \geq 1$. Then

$$g(s) = 1 - \frac{(1 - s)^\gamma}{(1 - s)^\gamma + s^\gamma} = f(s).$$

To apply Theorem 2.2, it remains to prove that f is submultiplicative. Note that for $s, t \in (0, 1)$,

$$f(st) - f(s)f(t) = \left(\frac{s^\gamma t^\gamma}{s^\gamma t^\gamma + (1 - st)^\gamma}\right) - \left(\frac{s^\gamma}{s^\gamma + (1 - s)^\gamma}\right)\left(\frac{t^\gamma}{t^\gamma + (1 - t)^\gamma}\right)$$
$$= -\frac{s^\gamma t^\gamma[(1 - st)^\gamma - (s - st)^\gamma - (t - st)^\gamma - (1 + st - s - t)^\gamma]}{[s^\gamma + (1 - s)^\gamma][t^\gamma + (1 - t)^\gamma][s^\gamma t^\gamma + (1 - st)^\gamma]}.$$

Since $1 - st = (s - st) + (t - st) + (1 + st - s - t)$ and $\gamma \geq 1$, it follows that $(1 - st)^\gamma \geq (s - st)^\gamma + (t - st)^\gamma + (1 + st - s - t)^\gamma$ and so f is submultiplicative. Hence the profile (bold, bold) is a Nash equilibrium for this game.

Moreover, if $\gamma > 1$, then f is strictly submultiplicative. Hence, the profile (bold, bold) is the unique Nash equilibrium for the game with $\gamma > 1$.

In Example 2.3, we see that $g(s) = f(s)$ for all $s \in [0, 1]$. Intuitively, an question thus arises: if the profile (bold, bold) is a Nash equilibrium, is it a necessary condition that $g(s) = f(s)$ for all $s \in [0, 1]$? The following example shows that even if $g \neq f$, the profile (bold, bold) is a Nash equilibrium.

Example 2.4: Assume player I's win probability function $f(s) = s^3(4 - 3s)$. Then $g(s) = s^2(3s^2 - 8s + 6)$. Note that for all $s, t \in (0, 1)$,

$$f(st) - f(s)f(t) = (st)^3(4 - 3st) - s^3(4 - 3s)t^3(4 - 3s)$$
$$= -12s^3t^3(1 - s)(1 - t) < 0$$

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and

$$\begin{aligned} g(st) - g(s)g(t) &= (st)^2[3(st)^2 - 8st + 6] - s^2(3s^2 - 8s + 6)t^2(3t^2 - 8t + 6) \\ &= 6s^2t^2(1-s)(1-t)[4 - (3-s)(3-t)] < 0. \end{aligned}$$

Then f and g are strictly submultiplicative. By Theorem 2.2, the profile (bold, bold) is the unique Nash equilibrium for this game.

3. NASH EQUILIBRIUM CONDITIONS FOR (BOLD, ANY STRATEGY)

From Theorems 2.1 and 2.2, we have known that if f is multiplicative and g is submultiplicative, then both the profiles (bold, timid) and (bold, bold) are Nash equilibriums. In fact, we can prove a stronger result if f is multiplicative and g is submultiplicative.

THEOREM 3.1: *If f is multiplicative and g is submultiplicative, then for any strategy β, the profile (bold, strategy β) is a Nash equilibrium.*

PROOF: To prove the profile (bold, strategy β) is a Nash equilibrium, it needs to prove (i) a bold strategy is optimal for player I while player II uses strategy β; (ii) while player I plays boldly, the probability of player II reaching M is independent on strategy β. For convenience, at each stage, denote player II's bet by $\beta(x)$ which is a function of his fortune $M - x$ and $\beta(x) \in \{1, \ldots, M - x\}$.

To prove (i), assume player II adopts strategy β. If player I uses a bold strategy, set

$$Q(x) = P(\text{ player I reaches } M \text{ with a fortune } x).$$

The corresponding law of motion at stage m for player I having x_m chips and playing boldly is given by

$$x_{m+1} = \begin{cases} x_m + \beta(x_m) & \text{with probability } f\left(\dfrac{x_m}{x_m + \beta(x_m)}\right), \\ 0 & \text{with probability } 1 - f\left(\dfrac{x_m}{x_m + \beta(x_m)}\right) \end{cases}$$

for $1 \le x_m \le M - 1$ and by $x_{m+1} = x_m$ with probability 1 for $x_m = 0$ or $x_m = M$. From this, it is not difficult to derive the recurrence relation:

$$Q(x) = f\left(\frac{x}{x + \beta(x)}\right) Q(x + \beta(x)), \tag{3.1}$$

where $1 \le x \le M - 1$. Note that $Q(0) = 0$ and $Q(M) = 1$ for all $m \in \mathbb{N}$.

As the proof of Lemma 2.3, to prove that a bold strategy is optimal for player I, it suffices to show that for every $x \in \{1, \ldots, M - 1\}$ and $a \in \{1, \ldots, x\}$:

$$Q(x) \ge f\left(\frac{a}{a + \beta(x)}\right) Q(x + \beta(x)) + \left[1 - f\left(\frac{a}{a + \beta(x)}\right)\right] Q(x - a). \tag{3.2}$$

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Repeatedly using (3.1) yields

$$Q(x) = f\left(\frac{x}{b_1}\right) Q(b_1) = \cdots = \left[\prod_{i=1}^{k} f\left(\frac{b_{k-1}}{b_k}\right)\right] Q(M) = f\left(\frac{x}{M}\right), \tag{3.3}$$

where $b_1 = x + \beta(x)$, $b_{i+1} = b_i + \beta(b_i)$, $i \geq 1$, and $b_k = M$ for some k. The last identity holds since f is multiplicative and $Q(M) = 1$.

Substituting (3.3) into (3.2), we see that the inequality (3.2) becomes

$$f\left(\frac{x}{M}\right) \geq f\left(\frac{a}{a+\beta(x)}\right) f\left(\frac{x+\beta(x)}{M}\right) + \left[1 - f\left(\frac{a}{a+\beta(x)}\right)\right] f\left(\frac{x-a}{M}\right),$$

which is equivalent to

$$f\left(\frac{x}{M}\right) - f\left(\frac{x-a}{M}\right) \geq f\left(\frac{a}{a+\beta(x)}\right) \left[f\left(\frac{x+\beta(x)}{M}\right) - f\left(\frac{x-a}{M}\right)\right]. \tag{3.4}$$

If $f((x+\beta(x))/M) = 0$, then the inequality (3.4) holds. If $f((x+\beta(x))/M) > 0$, we have for all $1 \leq b \leq x + \beta(x)$,

$$\frac{f(b/M)}{f((x+\beta(x))/M)} = f\left(\frac{b}{x+\beta(x)}\right)$$

since f is multiplicative and so the inequality (3.4) becomes

$$f\left(\frac{x}{x+\beta(x)}\right) - f\left(\frac{x-a}{x+\beta(x)}\right) \geq f\left(\frac{a}{a+\beta(x)}\right) - f\left(\frac{a(x-a)}{(a+\beta(x))(x+\beta(x))}\right). \tag{3.5}$$

Let $s = a/(a+\beta(x))$ and $t = (x-a)/(x+\beta(x))$. Then the inequality (3.5) becomes $f(s+t-st) - f(t) \geq f(s) - f(st)$, which is equivalent to $g((1-s)(1-t)) \leq g(1-s)g(1-t)$. Since g is submultiplicative, the inequality (3.5) holds and hence (i) holds.

For (ii), assume that player I adopts a bold strategy and player II adopts strategy β. From (3.3), we see that the win probability that player I reaches M is $f(x/M)$, which is independent on strategy β. Hence (ii) holds. ■

Remark 3.1: If a continuous win probability function f is multiplicative, then $f(s) = s^\delta$ for all $s \in [0, 1]$. Notice that if $f(s) = s^\delta$, then

$$\begin{aligned} g(st) - g(s)g(t) &= 1 - (1-st)^\delta - [1-(1-s)^\delta][1-(1-t)^\delta] \\ &= (1-s)^\delta + (1-t)^\delta - (1-s)^\delta(1-t)^\delta - (1-st)^\delta \\ &= u^\delta + v^\delta - u^\delta v^\delta - (u+v-uv)^\delta, \end{aligned}$$

where $u = 1-s$ and $v = 1-t$. It follows that $g(st) \leq g(s)g(t)$ if and only if $\delta \geq 1$. Hence, if a continuous win probability function f is multiplicative and g is submultiplicative, then $f(s) = s^\delta$, where $\delta \geq 1$.

4. NASH EQUILIBRIUM CONDITIONS FOR (TIMID, TIMID)

In this section, we find suitable conditions such that the profile (timid, timid) is a Nash equilibrium. Before doing this, we first give a necessary and sufficient condition on player I's win probability function such that a timid strategy is optimal for player I while player II plays timidly.

LEMMA 4.1: *Assume player II plays timidly and* $0 < f(1/2) < 1$. *Denote* $p = f(1/2)$ *and* $q = 1 - p$. *A timid strategy is optimal for player I if and only if*

$$\left[1 - f\left(\frac{i}{i+1}\right)\right]\left[1 + \frac{p}{q} + \cdots + \left(\frac{p}{q}\right)^i\right] \geq 1 \tag{4.1}$$

for all $i \in \{1, 2, \cdots, M-1\}$. *Moreover, if the strict inequality holds for the inequality* (4.1) *holds except at* $i = 1$, *then a timid strategy is the unique optimal strategy for player I.*

PROOF: Assume that player II plays a timid strategy. If player I also uses a timid strategy, set

$$Q(x) = P(\text{player I reaches } M \text{ with an initial fortune } x).$$

The corresponding law of motion at stage m for player I having x_m units and playing timidly is given by

$$x_{m+1} = \begin{cases} x_m + 1 & \text{with probability } f\left(\frac{1}{2}\right) = p, \\ x_m - 1 & \text{with probability } 1 - f\left(\frac{1}{2}\right) = q \end{cases}$$

for $1 \leq x_m \leq M - 1$ and by $x_{m+1} = x_m$ with probability 1 for $x_m = 0$ or M. Following a well-known result from the theory of random walks, we have that

$$Q(x) = \begin{cases} x/M & \text{if } p = q = 1/2, \\ \dfrac{1 - (q/p)^x}{1 - (q/p)^M} & \text{if } p \neq q \end{cases} \tag{4.2}$$

for every $0 \leq x \leq M$.

As the proof of Lemma 2.3, a timid strategy is optimal for player I if and only if the following inequality holds for every $x \in \{1, \cdots, M-1\}$ and every $a \in A_I(x)$:

$$f\left(\frac{a}{a+1}\right) Q(x+1) + \left[1 - f\left(\frac{a}{a+1}\right)\right] Q(x-a) \leq Q(x),$$

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which is equivalent to

$$f\left(\frac{a}{a+1}\right)[Q(x+1)-Q(x)] \leq \left[1-f\left(\frac{a}{a+1}\right)\right][Q(x)-Q(x-a)].$$

By (4.2), we have $Q(x+1)-Q(x)=(q/p)^x Q(1)$, $Q(x)-Q(x-a)=(q/p)^x$ $[\sum_{i=1}^{a}(p/q)^i]Q(1)$, and $Q(1)>0$. Then the above inequality is equivalent to for $1 \leq a \leq M-1$,

$$f\left(\frac{a}{a+1}\right)\left(\frac{q}{p}\right)^x \leq \left[1-f\left(\frac{a}{a+1}\right)\right]\left[\left(\frac{q}{p}\right)^{x-1}+\left(\frac{q}{p}\right)^{x-2}+\cdots+\left(\frac{q}{p}\right)^{x-a}\right],$$

which is equivalent to (4.1). Hence a timid strategy is optimal for player I if and only if (4.1) holds.

Moreover, if the strict inequality holds for (4.1) except at $a=1$, then (4.1) is actually an equality only at $a=1$. Thus, the timid strategy is the unique optimal strategy for player I while player II plays timidly. Hence the proof is complete. ■

THEOREM 4.2: *In a two-person red-and-black game with* $f(1/2)=1/2$, *the profile (timid, timid) is a Nash equilibrium if*

$$\begin{aligned} f(s) &\geq s \quad \text{if } s \leq 1/3, \\ f(s) &\leq s \quad \text{if } s \geq 2/3. \end{aligned} \tag{4.3}$$

Moreover, for $0<s<1$, *if* $f(s) \neq s$ *except at* $s=1/2$, *then the profile (timid, timid) is the unique Nash equilibrium.*

PROOF: From the definition of Nash equilibriums and Lemma 4.1, to prove that the profile (timid, timid) is a Nash equilibrium is to prove that the inequality (4.1) holds for f and for g, respectively.

Notice that $g(1/2)=1-f(1/2)=1/2=f(1/2)$. Thus, the inequality (4.1) can be simplified and so it remains to prove that

$$f\left(\frac{i}{i+1}\right) \leq \frac{i}{i+1} \quad \text{and} \quad g\left(\frac{i}{i+1}\right) \leq \frac{i}{i+1} \tag{4.4}$$

for all $i \in \{1,2,\cdots,M-1\}$. From $g(s)=1-f(1-s)$ and (4.3), it follows that $f(s) \leq s$ and $g(s) \leq s$ for all $s \geq 2/3$ and so the inequalities (4.4) hold.

Moreover, for $0<s<1$, if $f(s) \neq s$ except at $s=1/2$, then it can be proved that the (4.4) is strict inequalities except at $i=1$. Hence, if a player plays timidly, then the timid play is the unique optimal strategy for his opponent. From Lemma A.1 of [7], it follows that the profile (timid, timid) is the unique Nash equilibrium for this game. Hence the proof is complete. ■

The following examples can be analyzed by applying Theorem 4.2. In Example 4.1, the win probability functions of player I and II are the same, but in Example 4.2, the two players' win probability functions are different.

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Example 4.1: Let player I's win probability function be $f(s) = s^\delta/[s^\delta + (1-s)^\delta]$, where $0 < \delta \leq 1$. Then $f(1/2) = 1/2$ and $g(s) = 1 - f(1-s) = f(s)$. Thus if $f(s) \leq s$ on $[2/3, 1]$, then for $s \leq 1/3$, $f(s) = 1 - f(1-s) \geq 1 - (1-s) = s$ holds. Therefore, by Theorem 4.2, to prove that the profile (timid, timid) is a Nash equilibrium, it remains to prove that $f(s) \leq s$ on $[2/3, 1]$.

Note that for $s \geq 2/3$ and $0 < \delta \leq 1$, $(1-s)^{1-\delta} \leq s^{1-\delta}$, which is equivalent to

$$s^\delta(1-s) \leq s(1-s)^\delta$$

or

$$s^\delta \leq s[s^\delta + (1-s)^\delta].$$

Hence $f(s) \leq s$ while $s \geq 2/3$. Moreover, for $0 < s < 1$, if $0 < \delta < 1$, then $f(s) \neq s$ except at $s = 1/2$ and so (timid, timid) is the unique Nash equilibrium for this game.

Example 4.2: Let player I's win probability function be $f(s) = 2s(1-s)(1-2s)^2(1-3s) + s$. Then $f(1/2) = 1/2$, $f(s) \leq s$ while $s \geq 2/3$ and $f(s) \geq s$ while $s \leq 1/3$. Hence by Theorem 4.2, the profile (timid, timid) is a Nash equilibrium.

Recall that if f_1 and f_2 are two win probability functions, then $f_1 \circ f_2$ is also a win probability function. The following is about how to produce more win probability functions such that the profile (timid, timid) is a Nash equilibrium.

THEOREM 4.3: *Let f_1 and f_2 be two win probability functions with $f_1(1/2) = 1/2 = f_2(1/2)$. Suppose that for each $i = 1, 2$, f_i satisfies that*

$$f_i(s) \geq s \quad \text{if } s \leq 1/3,$$
$$f_i(s) \leq s \quad \text{if } s \geq 2/3.$$

Then the profile (timid, timid) is a Nash equilibrium for a two-person red-and-black game with player I's win probability function $f_1 \circ f_2$.

PROOF: Since $f_i(1/2) = 1/2$ for all $i = 1, 2$, we have that $f_1 \circ f_2(1/2) = f_1(1/2) = 1/2$. Since f is increasing and for $i = 1, 2$

$$f_i(s) \geq s \quad \text{if } s \leq 1/3,$$
$$f_i(s) \leq s \quad \text{if } s \geq 2/3,$$

we have

$$f_1 \circ f_2(s) \geq f_1(s) \geq s \quad \text{if } s \leq 1/3,$$
$$f_1 \circ f_2(s) \leq f_1(s) \leq s \quad \text{if } s \geq 2/3.$$

Applying Theorem 4.2, the proof of this is complete. ■

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