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Abstract

We show that the anti-canonical volume of an n-dimensional Kähler–Einstein $\mathbb{Q}$-Fano variety is bounded from above by certain invariants of the local singularities, namely $\mathrm{lct}^n \cdot \mathrm{mult}$ for ideals and the normalized volume function for real valuations. This refines a recent result by Fujita. As an application, we get sharp volume upper bounds for Kähler–Einstein Fano varieties with quotient singularities. Based on very recent results by Li and the author, we show that a Fano manifold is K-semistable if and only if a de Fernex–Ein–Mustaţă type inequality holds on its affine cone.

1. Introduction

An n-dimensional complex projective variety X is said to be a $\mathbb{Q}$-*Fano variety* if X has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. When $n \geqslant 2$, the (anti-canonical) volume $((-K_X)^n)$ of a $\mathbb{Q}$-Fano variety X can be arbitrarily large, such as volumes of weighted projective spaces (see [Dol82]). In the smooth case, Kollár *et al.* [KMM92] and Campana [Cam92] showed that there exists a uniform volume upper bound C_n for n-dimensional Fano manifolds. If $n \leqslant 3$, then $\mathbb{P}^n$ has the largest volume among all n-dimensional Fano manifolds. This fails immediately when $n \geqslant 4$ by examples of Batyrev [Bat81]. However, if a $\mathbb{Q}$-Fano variety X admits a Kähler–Einstein metric (in the sense of [BBEGZ11]), then it is expected that $\mathbb{P}^n$ has the largest volume (see [Tia90, OSS16] for $n = 2$, and [BB11, BB17] for smooth cases). Recently, Fujita [Fuj15] showed that if X is a n-dimensional $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric, then the anti-canonical volume $((-K_X)^n)$ is less than or equal to $(n+1)^n$ (= the volume of $\mathbb{P}^n$).

In this paper, we refine the result [Fuj15] by looking at invariants of the local singularities. Our first main result is as follows.

Theorem 1. *Let X be a Kähler–Einstein $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let Z be a closed subscheme of X with* $\mathrm{Supp}\, Z = \{p\}$. *Then we have*

$$((-K_X)^n) \leqslant \left(1 + \frac{1}{n}\right)^n \mathrm{lct}(X; I_Z)^n \, \mathrm{mult}_Z X, \tag{1.1}$$

where $\mathrm{lct}(X; I_Z)$ is the log canonical threshold of the ideal sheaf I_Z, and $\mathrm{mult}_Z X$ is the Hilbert–Samuel multiplicity of X along Z.

Note that the invariant $\mathrm{lct}^n \cdot \mathrm{mult}$ has been studied by de Fernex, Ein and Mustaţă in [Mus02, dFEM03, dFEM04, dFM15].

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There are three major aspects of Theorem 1:

– If we take $Z = p$ to be a smooth point on X, then $\mathrm{lct}(X;\mathfrak{m}_p) = n$ and $\mathrm{mult}_p X = 1$. In particular, we recover Fujita's result [Fuj15, Corollary 1.3].
– By definition, $\mathrm{lct}(X; I_Z)$ is the infimum of $(1 + \mathrm{ord}_E(K_{Y/X}))/\mathrm{ord}_E(I_Z)$ among all prime divisors E on a resolution Y of X. Although $\mathrm{lct}(X; I_Z)$ is not easy to compute in general, we can always get a volume upper bound by considering one divisor E, e.g. Theorems 3 and 25.
– On the other hand, if we fix a singularity p on a Kähler–Einstein $\mathbb{Q}$-Fano variety X, we get a lower bound of $\mathrm{lct}^n \cdot \mathrm{mult}$ among all thickenings of p. For example, if $X = \mathbb{P}^n$ then the inequality (1.1) recovers the main result in [dFEM04] (see Theorem 22). More generally, we get de Fernex–Ein–Mustaţă type inequalities for all cone singularities with K-semistable base using a logarithmic version of Theorem 1 (see Theorem 7).

For a real valuation v on $\mathbb{C}(X)$ centered at a closed point p, the normalized volume function $\widehat{\mathrm{vol}}$, introduced by C. Li in [Li15], is defined as $\widehat{\mathrm{vol}}(v) := A_X(v)^n \cdot \mathrm{vol}(v)$, where $A_X(v)$ is the log discrepancy of v and $\mathrm{vol}(v)$ is the volume of v. The following theorem gives an upper bound for the anti-canonical volume of X in terms of the normalized volume of a real valuation.

Theorem 2. *Let X be a Kähler–Einstein $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let v be a real valuation on $\mathbb{C}(X)$ centered at p. Then we have*

$$((-K_X)^n) \leqslant \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v). \tag{1.2}$$

Minimizing the normalized volume function, $\widehat{\mathrm{vol}}$ has been studied by Li, Xu, Blum and the author in [Li15, Li17, LL16, LX16, Blu18]. In [LL16], a logarithmic version of Theorem 2 was developed to obtain the sharp lower bound of the normalized volume function for cone singularities over Kähler–Einstein $\mathbb{Q}$-Fano varieties.

To compare Theorems 1 and 2, we know from [Mus02, Li15] that the infimum of $\mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X$ among all thickenings Z of p is no bigger than the infimum of $\widehat{\mathrm{vol}}(v)$ among all real valuations centered at p. In fact, we show that these two infimums are the same (see Theorem 27). In particular, this means that Theorems 1 and 2 are actually equivalent.

As an application of Theorems 1 and 2, we get sharp volume upper bounds for Kähler–Einstein $\mathbb{Q}$-Fano varieties with quotient singularities.

Theorem 3. *Let X be a Kähler–Einstein $\mathbb{Q}$-Fano variety of dimension n. Let $p \in X$ be a closed point. Suppose (X, p) is a quotient singularity with local analytic model $\mathbb{C}^n/G$ where $G \subset \mathrm{GL}(n, \mathbb{C})$ acts freely in codimension 1. Then*

$$((-K_X)^n) \leqslant \frac{(n+1)^n}{|G|},$$

with equality if and only if $|G \cap \mathbb{G}_m| = 1$ and $X \cong \mathbb{P}^n/G$, where $\mathbb{G}_m \subset \mathrm{GL}(n, \mathbb{C})$ is the subgroup consisting of non-zero scalar matrices.

It is well known that for algebraic surfaces, klt singularities are the same as quotient singularities. In [Tia90], Tian showed that the anti-canonical volume of a Kähler–Einstein log Del Pezzo surface is bounded from above by $48/|G|$ where $G \subset \mathrm{GL}(2, \mathbb{C})$ is the orbifold group at a closed point. In [OSS16], Odaka, Spotti and Sun improved the volume upper bounds to $12/|G|$.

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As a direct consequence of Theorem 3, we get the sharp volume upper bounds $9/|G|$ for Kähler–Einstein log Del Pezzo surfaces.

COROLLARY 4. *Let X be a Kähler–Einstein log Del Pezzo surface. Let $p \in X$ be a closed point with local analytic model $\mathbb{C}^2/G$, where $G \subset \mathrm{GL}(2,\mathbb{C})$ acts freely in codimension 1. Then*

$$((-K_X)^2) \leqslant \frac{9}{|G|},$$

with equality if and only if $|G \cap \mathbb{G}_m| = 1$ and $X \cong \mathbb{P}^2/G$.

Remark 5. The author was informed by Kento Fujita that he independently obtained cases of Corollary 4 when (X,p) is Du Val.

It was conjectured by Cheltsov and Kosta in [CK14, Conjecture 1.18] that a Gorenstein Del Pezzo surface admits Kähler–Einstein metrics if and only if the singularities are of certain types depending on the anti-canonical volume. This conjecture was proved by Odaka *et al.* in [OSS16, p. 165] where $\mathbb{Q}$-Gorenstein smoothable Kähler–Einstein log Del Pezzo surfaces are classified. As a quick application of Corollary 4, we give an alternative proof of the 'only if' part of this conjecture. We remark that partial results are known before [OSS16], e.g. [DT92, Jef97, Won13] on the 'only if' part and [MM93, GK07, Che08, Shi10, CK14] on the 'if' part.

COROLLARY 6. *Let X be a Kähler–Einstein log Del Pezzo surface with at most Du Val singularities.*

(i) *If $((-K_X)^2) = 1$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$ or $\mathbb{D}_4$.*
(ii) *If $((-K_X)^2) = 2$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$.*
(iii) *If $((-K_X)^2) = 3$, then X has at most singularities of type $\mathbb{A}_1$ or $\mathbb{A}_2$.*
(iv) *If $((-K_X)^2) = 4$, then X has at most singularities of type $\mathbb{A}_1$.*
(v) *If $((-K_X)^2) \geqslant 5$, then X is smooth.*

The comparison between Theorems 1 and 2 also allows us to rephrase the results from [Li17, LL16] in terms of $\mathrm{lct}^n \cdot \mathrm{mult}$. The following theorem gives a necessary and sufficient condition for a Fano manifold to be K-semistable.

THEOREM 7. *Let V be a Fano manifold of dimension $n-1$. Assume $H = -rK_V$ is an ample Cartier divisor for some $r \in \mathbb{Q}_{>0}$. Let $X := C(V,H) = \mathbf{Spec} \bigoplus_{k=0}^{\infty} H^0(V, mH)$ be the affine cone with cone vertex o. Then V is K-semistable if and only if for any closed subscheme Z of X supported at o, the following inequality holds:*

$$\mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X \geqslant \frac{1}{r}((-K_V)^{n-1}). \tag{1.3}$$

In particular, from the 'only if' part of Theorem 7 we get de Fernex–Ein–Mustaţă type inequalities (1.3) on cone singularities with K-semistable bases (see [dFEM04] or Theorem 22 for the smooth case). Note that the 'if' part is a straight-forward consequence after [Li17] using the relations between $\mathrm{lct}^n \cdot \mathrm{mult}$ and $\widehat{\mathrm{vol}}$ from [Mus02, Li15].

The proofs of Theorems 1 and 2 rely on the recent result [Fuj15] where Fujita settled the optimal volume upper bounds for Ding-semistable $\mathbb{Q}$-Fano varieties. Recall that a $\mathbb{Q}$-Fano variety X is said to be *Ding-semistable* if the Ding invariant $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ is non-negative for any

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normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$. The results in [Ber16, § 3] show that, if a $\mathbb{Q}$-Fano variety X admits a Kähler–Einstein metric, then X is Ding-semistable (see also [Fuj15, Theorem 3.2]). This allows us to prove Theorems 1 and 2 under the weaker assumption that X is Ding-semistable (see Theorems 16 and 21). To prove the equality case of Theorem 3, we blow up a flag ideal associated to the $\widehat{\mathrm{vol}}$-minimizing valuation to obtain a test configuration with zero CM weight. Then K-polystability is enough to characterize the $\mathbb{Q}$-Fano variety with maximal volume (see Lemma 33 for details).

The paper is organized as follows. In § 2, we recall the notions of Seshadri constants, Ding-semistability, filtrations and normalized volume of real valuations. As a generalization of [Fuj15, Theorem 2.3], we characterize the Seshadri constant of an arbitrary thickening of a close point in terms of the volume function in Lemma 10. In § 3.1, we prove Theorems 1 and 2. The proof of Theorem 1 is a quick application of [Fuj15, Theorem 1.2] and Lemma 10. To prove Theorem 2, we apply [Fuj15, Theorem 4.9] to the filtration associated to any real valuation. In § 3.2, we give some applications of Theorems 1 and 2. We recover the main result from [dFEM04]. We prove the inequality part of Theorem 3. We also get a volume upper bound for a Ding-semistable $\mathbb{Q}$-Fano variety with an isolated singularity that is not terminal (see Theorem 25). Section 4 is devoted to comparing the two invariants occurred in the volume upper bounds, i.e. $\mathrm{lct}^n \cdot \mathrm{mult}$ and $\widehat{\mathrm{vol}}$. In Theorem 27, we show that these two invariants have the same infimums among ideals supported (respectively valuations centered) at a klt point. This implies that Theorems 1 and 2 are equivalent to each other. We prove Theorem 7 using Theorem 27 and results from [Li17, LL16]. In § 4.2, we study the existence of minimizers of $\mathrm{lct}^n \cdot \mathrm{mult}$ and $\widehat{\mathrm{vol}}$. In § 5, we study the equality case of Theorems 1 and 2. Lemma 33 characterizes the equality case of Theorem 2 when X is K-polystable with extra conditions on the real valuation. Applying this lemma, we prove the equality case of Theorem 3 and Corollary 6. We also prove that $\mathbb{P}^n$ is the only Ding-semistable $\mathbb{Q}$-Fano variety of volume at least $(n+1)^n$ (see Theorem 36). This theorem improves the equality case of [Fuj15, Theorem 1.1] where Fujita proved for Ding-semistable Fano manifolds.

Notation

Throughout this paper, we work over the complex numbers $\mathbb{C}$. Let X be an n-dimensional normal variety. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is a klt pair. For a closed subscheme Z of X, the *log canonical threshold* of its ideal sheaf I_Z with respect to (X, Δ) is defined as

$$\mathrm{lct}(X, \Delta; I_Z) := \inf_E \frac{1 + \mathrm{ord}_E(K_Y - f^*(K_X + \Delta))}{\mathrm{ord}_E(I_Z)},$$

where the infimum is taken over all prime divisors E on a log resolution $f : Y \to (X, \Delta)$. We also denote by $\mathrm{lct}(X; I_Z) = \mathrm{lct}(X, 0; I_Z)$ when $\Delta = 0$. If $\dim Z = 0$, the *Hilbert–Samuel multiplicity* of X along Z is defined as

$$\mathrm{mult}_Z(X) := \lim_{k\to\infty} \frac{\ell(\mathcal{O}_X/I_Z^k)}{k^n/n!}.$$

We will also use the notation $\mathrm{lct}(\mathfrak{a})$ (and $\mathrm{mult}(\mathfrak{a})$) as abbreviation of $\mathrm{lct}(X; \mathfrak{a})$ (and $\mathrm{mult}_{V(\mathfrak{a})} X$) for a coherent ideal sheaf $\mathfrak{a}$ (of dimension 0) once the variety X is specified.

Following [BBEGZ11, Ber16], on an n-dimensional $\mathbb{Q}$-Fano variety X, a Kähler metric ω on the smooth locus X_{reg} is said to be a *Kähler–Einstein metric on* X if it has constant Ricci curvature, i.e. $\mathrm{Ric}(\omega) = \omega$ on X_{reg}, and $\int_{X_{\mathrm{reg}}} \omega^n = ((-K_X)^n)$. By [BBEGZ11, Definition 3.5 and Proposition 3.8], this is equivalent to saying that ω is a closed positive $(1,1)$-current with full Monge–Ampère mass on X satisfying $V^{-1}\omega^n = \mu_\omega$ (see [BBEGZ11, § 3] for details).

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Given a normal variety X and a (not necessarily effective) $\mathbb{Q}$-divisor Δ, we say that (X,Δ) is a *sub pair* if $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let $f: Y\to (X,\Delta)$ be a log resolution, then we have the following equality

$$K_Y = f^*(K_X+\Delta)+\sum_i a_i E_i,$$

where E_i are distinct prime divisors on Y. We say that (X,Δ) is *sub log canonical* if $a_i \geqslant -1$ for all i. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then the *log canonical threshold* of D with respect to a sub pair (X,Δ) is defined as

$$\operatorname{lct}(X,\Delta;D) := \sup\{c\in\mathbb{R} : (X,\Delta+cD) \text{ is sub log canonical}\}. \tag{1.4}$$

2. Preliminaries

2.1 Seshadri constants

Definition 8 [Laz04]. Let X be a projective variety of dimension n. For a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X, we define the *volume* of L on X to be

$$\operatorname{vol}_X(L) := \limsup_{k\to\infty, kL \text{ Cartier}} \frac{h^0(X,\mathcal{O}_X(kL))}{k^n/n!}.$$

We know that the lim sup computing $\operatorname{vol}_X(L)$ is actually a limit. If $L\equiv L'$, then $\operatorname{vol}_X(L) = \operatorname{vol}_X(L')$. Moreover, we can extend it uniquely to a continuous function

$$\operatorname{vol}_X : N^1(X)_{\mathbb{R}} \to \mathbb{R}_{\geqslant 0}.$$

Definition 9. Let X be a projective variety, L be an ample $\mathbb{Q}$-Cartier divisor on X. Let Z be a non-empty proper closed subscheme of X with ideal sheaf I_Z. Denote by $\sigma: \hat{X}\to X$ the blow up of X along Z. Let F be the Cartier divisor on $\hat{X}$ given by $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1}(I_Z)\cdot\mathcal{O}_{\hat{X}}$. The *Seshadri constant* of L along Z, denoted by $\epsilon_Z(L)$, is defined as

$$\epsilon_Z(L) := \sup\{x\in\mathbb{R}_{>0} \mid \sigma^*L - xF \text{ is ample}\}.$$

Lemma 10. *Let X be an n-dimensional normal projective variety with $n\geqslant 2$, L be an ample $\mathbb{Q}$-divisor on X, $p\in X$ be a closed point. Let Z be a closed subscheme of X with* $\operatorname{Supp} Z = \{p\}$. *Let $\sigma: \hat{X}\to X$ be the blow up along Z, and $F\subset\hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1}I_Z\cdot\mathcal{O}_{\hat{X}}$.*

(i) *For any $x\in\mathbb{R}_{\geqslant 0}$, we have*

$$\operatorname{vol}_{\hat{X}}(\sigma^*L - xF) \geqslant ((\sigma^*L-xF)^n) = (L^n) - \operatorname{mult}_Z X\cdot x^n.$$

(ii) *Set $\Lambda_Z(L) := \{x\in\mathbb{R}_{\geqslant 0} \mid \operatorname{vol}_{\hat{X}}(\sigma^*L-xF) = ((\sigma^*L-xF)^n)\}$. Then we have*

$$\epsilon_Z(L) = \max\{x\in\mathbb{R}_{\geqslant 0} \mid y\in\Lambda_Z(L) \textit{ for all } y\in[0,x]\}.$$

Proof. The proof goes along the same line as [Fuj15, Theorem 2.3].

(i) The equality $((\sigma^*L-xF)^n) = (L^n) - \operatorname{mult}_Z X\cdot x^n$ follows from the fact that $\operatorname{mult}_Z X = (-1)^{n-1}(F^n)$ (see [Ram73]).

For the inequality part, we can assume that $x\in\mathbb{Q}_{>0}$ since the function $\operatorname{vol}_{\hat{X}}(\sigma^*L-xF)$ is continuous. Take any sufficiently large $k\in\mathbb{Z}_{>0}$ with $kx\in\mathbb{Z}_{>0}$ and kL Cartier. Notice that we have the long exact sequence

$$0\to H^0(\hat{X},\sigma^*(kL)-kxF)\to H^0(\hat{X},\sigma^*(kL))\to H^0(kxF,\sigma^*(kL)|_{kxF})\to\cdots \tag{2.1}$$

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Thus

$$\begin{aligned} h^0(\hat{X},\sigma^*(kL)-kxF) &\geqslant h^0(\hat{X},\sigma^*(kL))-h^0(kxF,\sigma^*(kL)|_{kxF}) \\ &= h^0(X,kL)-h^0(kxF,\mathcal{O}_{kxF}). \end{aligned}$$

Consequently,

$$\mathrm{vol}_{\hat{X}}(\sigma^*L-xF)\geqslant \mathrm{vol}_X(L)-\lim_{k\to\infty}\frac{h^0(kxF,\mathcal{O}_{kxF})}{(kx)^n/n!}x^n.$$

Hence it suffices to show that

$$\lim_{j\to\infty}\frac{h^0(jF,\mathcal{O}_{jF})}{j^n/n!}=\mathrm{mult}_Z X.$$

We look at the exact sequence

$$0\to\mathcal{O}_{\hat{X}}(-jF)|_F\to\mathcal{O}_{(j+1)F}\to\mathcal{O}_{jF}\to 0. \tag{2.2}$$

The isomorphism $F\cong\mathbf{Proj}\bigoplus_{m\geqslant 0}I_Z^m/I_Z^{m+1}$ yields $\mathcal{O}_{\hat{X}}(-jF)|_F\cong\mathcal{O}_F(j)$. Since $H^1(F,\mathcal{O}_F(j))=0$ for $j\gg 0$, we have an exact sequence

$$0\to H^0(F,\mathcal{O}_F(j))\to H^0((j+1)F,\mathcal{O}_{(j+1)F})\to H^0(jF,\mathcal{O}_{jF})\to 0.$$

Hence $h^0((j+1)F,\mathcal{O}_{(j+1)F})-h^0(jF,\mathcal{O}_{jF})=h^0(F,\mathcal{O}_F(j))=\ell(I_Z^j/I_Z^{j+1})$ for $j\gg 0$. As a result,

$$h^0(jF,\mathcal{O}_{jF})=\ell(\mathcal{O}_X/I_Z^j)+\mathrm{const}.$$

for $j\gg 0$. Hence we prove (i).

(ii) Assume $\mathrm{vol}_{\hat{X}}(\sigma^*L-xF)=(L^n)-\mathrm{mult}_Z X\cdot x^n$ for some $x\in\mathbb{Q}_{>0}$. We will show that

$$h^i(\hat{X},\sigma^*(kL)-kxF)=o(k^n)\quad\text{for } i\geqslant 1, k\gg 0 \text{ with } kx\in\mathbb{Z}_{>0}. \tag{2.3}$$

For $i\geqslant 2$, the long exact sequence (2.1) yields

$$h^i(\hat{X},\sigma^*(kL)-kxF)\leqslant h^{i-1}(kxF,\mathcal{O}_{kxF})+h^i(\hat{X},\sigma^*(kL)).$$

Since $\sigma^*(kL)$ is nef, we know that $h^i(\hat{X},\sigma^*(kL))=o(k^n)$. Serre vanishing theorem implies that $H^{i-1}(F,\mathcal{O}_F(j))=0$ for $i\geqslant 2$, $j\gg 0$. Then the exact sequence (2.2) implies that

$$H^{i-1}((j+1)F,\mathcal{O}_{(j+1)F})\cong H^{i-1}(jF,\mathcal{O}_{jF})$$

for $i\geqslant 2$, $j\gg 0$. As a result, $h^{i-1}(kxF,\mathcal{O}_{kxF})$ is constant for $k\gg 0$. Therefore, $h^i(\hat{X},\sigma^*(kL)-kxF)=o(k^n)$ for any $i\geqslant 2$.

By the asymptotic Riemann–Roch theorem we know that

$$\begin{aligned} \lim_{k\to\infty}\frac{\chi(\hat{X},\sigma^*(kL)-kxF)}{k^n/n!} &= ((\sigma^*L-xF)^n) \\ &= (L^n)-\mathrm{mult}_Z X\cdot x^n \\ &= \mathrm{vol}_{\hat{X}}(\sigma^*L-xF). \end{aligned}$$

Hence we have

$$\lim_{k\to\infty}\frac{1}{k^n}\sum_{i=1}^n(-1)^i h^i(\hat{X},\sigma^*(kL)-kxF)=0.$$

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Since $h^i(\hat{X}, \sigma^*(kL) - kxF) = o(k^n)$ for $i \geqslant 2$, we conclude that $h^1(\hat{X}, \sigma^*(kL) - kxF) = o(k^n)$ as well.

Now let us prove (ii). Denote by a the right-hand side of the equation in (ii). For any nef divisor, its volume is equal to its self intersection number. Hence it is obvious that $\epsilon_Z(L) \leqslant a$. Then it suffices to show that $a \leqslant \epsilon_Z(L)$. Equivalently, we only need to show that for $\epsilon > 0$ sufficiently small with $a - \epsilon \in \mathbb{Q}_{>0}$, $\sigma^*L - (a-\epsilon)F$ is ample. Fix $\delta \in \mathbb{Q}_{>0}$ such that $\delta < \epsilon_Z(L)$, that is, $\sigma^*L - \delta F$ is ample. By [dFKL07, Theorem A], we only need to show that

$$h^i(\hat{X}, m((\sigma^*L - (a-\epsilon)F) - t(\sigma^*L - \delta F))) = o(m^n) \tag{2.4}$$

for any $i > 0$, any sufficiently small $t \in \mathbb{Q}_{>0}$ and suitably divisible m. Denote $b := (a - \epsilon + t\delta)/(1-t)$. Notice that

$$\frac{1}{1-t}((\sigma^*L - (a-\epsilon)F) - t(\sigma^*L - \delta F)) = \sigma^*L - \frac{a-\epsilon+t\delta}{1-t}F = \sigma^*L - bF.$$

It is clear that $b < a$ for $t \in \mathbb{Q}_{>0}$ sufficiently small. Hence from the description of a we get:

$$\mathrm{vol}_{\hat{X}}(\sigma^*L - bF) = (L^n) - \mathrm{mult}_Z X \cdot b^n.$$

Then applying (2.3) yields that $h^i(\hat{X}, \sigma^*(kL) - kbF) = o(k^n)$ for $i > 0$, sufficiently small $t \in \mathbb{Q}_{>0}$ and suitably divisible k. In other words, (2.4) is true under the same condition for i, t and $m = (1-t)k$. Hence we prove the lemma. □

2.2 K-semistability and Ding-semistability

In this section we recall the definition of K-semistability, and its recent equivalence Ding-semistability.

DEFINITION 11 ([Tia97, Don02], see also [LX14]). Let X be an n-dimensional $\mathbb{Q}$-Fano variety.

(a) Let $r \in \mathbb{Q}_{>0}$ such that $L := -rK_X$ is Cartier. A *test configuration* (respectively a *semi test configuration*) of (X, L) consists of the following data:

- a variety $\mathcal{X}$ admitting a $\mathbb{G}_m$-action and a $\mathbb{G}_m$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{A}^1$, where the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ is given by the standard multiplication;
- a $\mathbb{G}_m$-equivariant π-ample (respectively π-semiample) line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{L})|_{\pi^{-1}(\mathbb{A}^1\setminus\{0\})}$ is equivariantly isomorphic to $(X, L) \times (\mathbb{A}^1\setminus\{0\})$ with the natural $\mathbb{G}_m$-action.

Moreover, if $\mathcal{X}$ is normal in addition, then we say that $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is a *normal test configuration* (respectively a *normal semi test configuration*) of (X, L).

(b) Assume that $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a normal test configuration. Let $\bar{\pi} : (\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ be the natural equivariant compactification of $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$. The *CM weight* (equivalently, *Donaldson–Futaki invariant*) of $(\mathcal{X}, \mathcal{L})$ is defined by the intersection formula:

$$\mathrm{CM}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n+1)((-K_X)^n)}\left(\frac{n}{r^{n+1}}(\bar{\mathcal{L}}^{n+1}) + \frac{n+1}{r^n}(\bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1})\right).$$

(c)
- The pair $(X, -K_X)$ is called *K-semistable* if $\mathrm{CM}(\mathcal{X}, \mathcal{L}) \geqslant 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$.
- The pair $(X, -K_X)$ is called *K-polystable* if $\mathrm{CM}(\mathcal{X}, \mathcal{L}) \geqslant 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$, and the equality holds if and only if $\mathcal{X} \cong X \times \mathbb{A}^1$.

In [Din88], Ding introduced the so-called *Ding functional* on the space of Kähler metrics on Fano manifolds and showed that Kähler–Einstein metrics are critical points of the Ding functional. Recently, Berman [Ber16] established a formula expressing CM-weights in terms of the slope of Ding functional along a geodesic ray in the space of all bounded positively curved metrics on the anti-canonical $\mathbb{Q}$-line bundle of a $\mathbb{Q}$-Fano variety. Later on, this slope was called *Ding invariant* by Fujita [Fuj15] in order to define the concept of *Ding-semistability*. Next we will recall the recent work by Berman [Ber16] and Fujita [Fuj15].

Definition 12 [Ber16, Fuj15]. Let X be an n-dimensional $\mathbb{Q}$-Fano variety.

(a) Let $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ be a normal semi test configuration of $(X,-rK_X)$ and $(\bar{\mathcal{X}},\bar{\mathcal{L}})/\mathbb{P}^1$ be its natural compactification. Let $D_{(\mathcal{X},\mathcal{L})}$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ satisfying the following conditions:

- the support $\mathrm{Supp} D_{(\mathcal{X},\mathcal{L})}$ is contained in $\mathcal{X}_0$;
- the divisor $-rD_{(\mathcal{X},\mathcal{L})}$ is a $\mathbb{Z}$-divisor corresponding to the divisorial sheaf $\bar{\mathcal{L}}(rK_{\bar{\mathcal{X}}/\mathbb{P}^1})$.

(b) The *Ding invariant* $\mathrm{Ding}(\mathcal{X},\mathcal{L})$ of $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ is defined as

$$\mathrm{Ding}(\mathcal{X},\mathcal{L}) := \frac{-(\bar{\mathcal{L}}^{n+1})}{(n+1)r^{n+1}((-K_X)^n)} - (1-\mathrm{lct}(\mathcal{X},D_{(\mathcal{X},\mathcal{L})};\mathcal{X}_0)).$$

Here $\mathrm{lct}(\mathcal{X},D_{(\mathcal{X},\mathcal{L})};\mathcal{X}_0)$ is defined in the sense of (1.4).

(c) X is called *Ding-semistable* if $\mathrm{Ding}(\mathcal{X},\mathcal{L}) \geqslant 0$ for any normal test configuration $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ of $(X,-rK_X)$.

The following theorem illustrates relations among the existence of Kähler–Einstein metrics, K-semistability and Ding-semistability of $\mathbb{Q}$-Fano varieties.

Theorem 13. (i) *If a $\mathbb{Q}$-Fano variety X admits a Kähler–Einstein metric, then X is Ding-semistable.*

(ii) *A $\mathbb{Q}$-Fano variety is K-semistable if and only if it is Ding-semistable.*

Proof. Part (i) is proved by Berman in [Ber16] (see also [Fuj15, Theorem 3.2]).

In (ii), the direction that Ding-semistability implies K-semistability is proved by Berman in [Ber16] using the inequality that $\mathrm{Ding}(\mathcal{X},\mathcal{L}) \geqslant \mathrm{CM}(\mathcal{X},\mathcal{L})$ for any normal test configuration $(\mathcal{X},\mathcal{L})$ (see also [Fuj15, Theorem 3.2.2]). The direction that K-semistability implies Ding-semistability is proved in [BBJ15] based on [LX14] (see also [Fuj16, § 3]). □

2.3 Filtrations

The use of filtrations to study stability notions was initiated by Witt Nyström in [WN12]. We recall the relevant definitions about filtrations after [BC11, WN12] (see also [BHJ17] and [Fuj15, § 4.1]).

Definition 14. A *good filtration* of a graded $\mathbb{C}$-algebra $S = \bigoplus_{m=0}^{\infty} S_m$ is a decreasing, left continuous, multiplicative and linearly bounded $\mathbb{R}$-filtrations of S. In other words, for each $m \geqslant 0 \in \mathbb{Z}$, there is a family of subspaces $\{\mathcal{F}^x S_m\}_{x\in\mathbb{R}}$ of S_m such that:

- $\mathcal{F}^x S_m \subseteq \mathcal{F}^{x'} S_m$, if $x \geqslant x'$;
- $\mathcal{F}^x S_m = \bigcap_{x'<x} \mathcal{F}^{x'} S_m$;
- $\mathcal{F}^x S_m \cdot \mathcal{F}^{x'} S_{m'} \subseteq \mathcal{F}^{x+x'} S_{m+m'}$, for any $x, x' \in \mathbb{R}$ and $m, m' \in \mathbb{Z}_{\geqslant 0}$;

– $e_{\min}(\mathcal{F}) > -\infty$ and $e_{\max}(\mathcal{F}) < +\infty$, where $e_{\min}(\mathcal{F})$ and $e_{\max}(\mathcal{F})$ are defined by the following operations:

$$\begin{aligned} e_{\min}(S_m, \mathcal{F}) &:= \inf\{t \in \mathbb{R} : \mathcal{F}^t S_m \neq S_m\}; \\ e_{\max}(S_m, \mathcal{F}) &:= \sup\{t \in \mathbb{R} : \mathcal{F}^t S_m \neq 0\}; \\ e_{\min}(\mathcal{F}) = e_{\min}(S_\bullet, \mathcal{F}) &:= \liminf_{m\to\infty} \frac{e_{\min}(S_m, \mathcal{F})}{m}; \\ e_{\max}(\mathcal{F}) = e_{\max}(S_\bullet, \mathcal{F}) &:= \limsup_{i\to\infty} \frac{e_{\max}(S_m, \mathcal{F})}{m}. \end{aligned} \tag{2.5}$$

Define $S^{(t)} := \bigoplus_{k=0}^{\infty} \mathcal{F}^{kt} S_k$. When we want to emphasize the dependence of $S^{(t)}$ on the filtration $\mathcal{F}$, we also denote $S^{(t)}$ by $\mathcal{F}S^{(t)}$. The following concept of volume will be important for us:

$$\mathrm{vol}(S^{(t)}) = \mathrm{vol}(\mathcal{F}S^{(t)}) := \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} S_m}{m^n/n!}. \tag{2.6}$$

Now assume L is an ample line bundle over X and $S = \bigoplus_{m=0}^{\infty} H^0(X, L^m) = \bigoplus_{m=0}^{\infty} S_m$ is the section ring of (X, L). Then following [Fuj15], we can define a sequence of ideal sheaves on X:

$$I^{\mathcal{F}}_{(m,x)} = \mathrm{Image}(\mathcal{F}^x S_m \otimes L^{-m} \to \mathcal{O}_X), \tag{2.7}$$

and define $\overline{\mathcal{F}}^x S_m := H^0(V, L^m \cdot I^{\mathcal{F}}_{(m,x)})$ to be the saturation of $\mathcal{F}^x S_m$ such that $\mathcal{F}^x S^m \subseteq \overline{\mathcal{F}}^x S_m$. $\mathcal{F}$ is called *saturated* if $\overline{\mathcal{F}}^x S_m = \mathcal{F}^x S_m$ for any $x \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geqslant 0}$. Notice that with the notation above we have:

$$\mathrm{vol}(\overline{\mathcal{F}}S^{(t)}) := \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} \overline{\mathcal{F}}^{kt} H^0(X, kL)}{k^n/n!}.$$

2.4 Normalized volume of real valuations

Let (X, p) be a normal $\mathbb{Q}$-Gorenstein singularity with $\dim X = n$. A *real valuation* v on the function field $\mathbb{C}(X)$ is a map $v : \mathbb{C}(X) \to \mathbb{R}$, satisfying:

- $v(fg) = v(f) + v(g)$;
- $v(f + g) \geqslant \min\{v(f), v(g)\}$;
- $v(\mathbb{C}^*) = 0$.

Denote by $\mathcal{O}_v := \{f \in \mathbb{C}(X) : v(f) \geqslant 0\}$ the *valuation ring* of v. Denote by $(\mathcal{O}_p, \mathfrak{m}_p)$ the local ring of X at p. We say that a real valuation v on $\mathbb{C}(X)$ is *centered at* p if the local ring $\mathcal{O}_p$ is dominated by $\mathcal{O}_v$. In other words, v is non-negative on $\mathcal{O}_p$ and strictly positive on $\mathfrak{m}_p$. Denote by $\mathrm{Val}_{X,p}$ the set of real valuations centered at p. Given a valuation $v \in \mathrm{Val}_{X,p}$, define the valuation ideals for any $x \in \mathbb{R}$ as follows:

$$\mathfrak{a}_x(v) := \{f \in \mathcal{O}_X : v(f) \geqslant x\}.$$

The *volume* of the valuation v is defined as

$$\mathrm{vol}(v) := \lim_{x\to+\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}_x(v))}{x^n/n!}.$$

By [ELS03, Mus02, Cut13], the limit on the right-hand side exists and is equal to the multiplicity of the graded family of ideals $\mathfrak{a}_\bullet(v)$:

$$\mathrm{vol}(v) = \lim_{m\to\infty} \frac{\mathrm{mult}(\mathfrak{a}_m(v))}{m^n} =: \mathrm{mult}(\mathfrak{a}_\bullet(v)).$$

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Following [JM12, BdFFU15], we can define the *log discrepancy* for any valuation $v \in \mathrm{Val}_X$. This is achieved in three steps in [JM12, BdFFU15]. Firstly, for a divisorial valuation ord_E associated to a prime divisor E over X, define $A_X(E) := 1 + \mathrm{ord}_E(K_{Y/X})$, where $\pi : Y \to X$ is a resolution of X containing E. Next we define the log discrepancy of a quasi-monomial valuation (also called Abhyankar valuation). For a resolution $\pi : Y \to X$, let $\underline{y} := (y_1, \dots, y_r)$ be a system of algebraic coordinate of a point $\eta \in Y$. By [JM12, Proposition 3.1], for every $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^n_{\geqslant 0}$ one can associate a unique valuation $\mathrm{val}_\alpha = \mathrm{val}_{\underline{y},\alpha} \in \mathrm{Val}_X$ with the following property: whenever $f \in \mathcal{O}_{Y,\eta}$ is written in $\widehat{\mathcal{O}_{Y,\eta}}$ as $f = \sum_{\beta \in \mathbb{Z}^r_{\geqslant 0}} c_\beta y^\beta$, with each $c_\beta \in \widehat{\mathcal{O}_{Y,\eta}}$ either zero or a unit, we have

$$\mathrm{val}_\alpha(f) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}.$$

Such valuations are called *quasi-monomial valuations* (or equivalently, *Abhyankar valuations* by [ELS03]). Let $(Y, D = \sum_{k=1}^N D_k)$ be a log smooth model of X, i.e. $\pi : Y \to X$ is an isomorphism outside of the support of D. We denote by $\mathrm{QM}_\eta(Y, D)$ the set of all quasi-monomial valuations v that can be described at the point $\eta \in Y$ with respect to coordinates $(y_1, \dots, y_r)$ such that each y_i defines at η an irreducible component of D (hence η is the generic point of a connected component of the intersection of some of the divisors D_i). We put $\mathrm{QM}(Y, D) := \bigcup_\eta \mathrm{QM}(Y, D)$. Suppose η is a generic point of a connected component of $D_{i_1} \cap \cdots \cap D_{i_r}$, then the log discrepancy of val_α is defined as

$$A_X(\mathrm{val}_\alpha) := \sum_{j=1}^r \alpha_j \cdot A_X(\mathrm{ord}_{D_{i_j}}) = \sum_{j=1}^r \alpha_j \cdot (1 + \mathrm{ord}_{D_{i_j}}(K_{Y/X})).$$

Finally, in [JM12] Jonsson and Mustaţă showed that there exists a retraction map $r_{Y,D} : \mathrm{Val}_X \to \mathrm{QM}(Y, D)$ for any log smooth model (Y, D) over X, such that it induces a homeomorphism $\mathrm{Val}_X \to \varprojlim_{(Y,D)} \mathrm{QM}(Y, D)$. For any real valuation $v \in \mathrm{Val}_X$, we define

$$A_X(v) = \sup_{(Y,D)} A_X(r_{(Y,D)}(v)),$$

where (Y, D) ranges over all log smooth models over X. For details, see [JM12] and [BdFFU15, Theorem 3.1]. It is possible that $A_X(v) = +\infty$ for some $v \in \mathrm{Val}_X$, see e.g. [JM12, Remark 5.12].

Following [Li15], the *normalized volume function* $\widehat{\mathrm{vol}}(v)$ for any $v \in \mathrm{Val}_{X,p}$ is defined as

$$\widehat{\mathrm{vol}}(v) := \begin{cases} A_X(v)^n \cdot \mathrm{vol}(v) & \text{if } A_X(v) < +\infty, \\ +\infty & \text{if } A_X(v) = +\infty. \end{cases}$$

Notice that $\widehat{\mathrm{vol}}(v)$ is rescaling invariant: $\widehat{\mathrm{vol}}(\lambda v) = \widehat{\mathrm{vol}}(v)$ for any $\lambda > 0$. By Izumi's theorem (see [Izu85, Ree89, ELS03, JM12, BFJ14, Li15]), one can show that $\widehat{\mathrm{vol}}$ is uniformly bounded from below by a positive number on $\mathrm{Val}_{X,p}$.

3. Proofs of main theorems

3.1 Proofs

It was proved by Berman [Ber16] that the existence of a Kähler–Einstein metric on a $\mathbb{Q}$-Fano variety implies Ding-semistability (see Theorem 13). In this section, we will prove Theorem 1 and 2 with the weaker assumption that X is Ding-semistable.

The following result by Fujita is crucial for proving Theorem 1.

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THEOREM 15 [Fuj15, Theorem 1.2]. *Let X be a $\mathbb{Q}$-Fano variety. Assume that X is Ding-semistable. Take any non-empty proper closed subscheme $\emptyset \neq Z \subsetneq X$ corresponds to an ideal sheaf $0 \neq I_Z \subsetneq \mathcal{O}_X$. Let $\sigma : \hat{X} \to X$ be the blow up along Z, let $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1} I_Z \cdot \mathcal{O}_{\hat{X}}$. Then we have $\beta(Z) \geqslant 0$, where*

$$\beta(Z) := \operatorname{lct}(X; I_Z) \cdot \operatorname{vol}_X(-K_X) - \int_0^{+\infty} \operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)\, dx.$$

We will prove the following theorem, a stronger result that implies Theorem 1.

THEOREM 16. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let Z be a closed subscheme of X with* $\operatorname{Supp} Z = \{p\}$. *Then we have*

$$((-K_X)^n) \leqslant \left(1 + \frac{1}{n}\right)^n \operatorname{lct}(X; I_Z)^n \operatorname{mult}_Z X. \tag{3.1}$$

Proof. Let $\sigma : \hat{X} \to X$ be the blow up along Z, and let $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1} I_Z \cdot \mathcal{O}_{\hat{X}}$. By Theorem 15, we have

$$\operatorname{lct}(X; I_Z) \cdot ((-K_X)^n) \geqslant \int_0^{+\infty} \operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)\, dx.$$

On the other hand, by Lemma 10, we have

$$\begin{aligned}\int_0^{+\infty} \operatorname{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)\, dx &\geqslant \int_0^{\sqrt[n]{((-K_X)^n)/\operatorname{mult}_Z X}} (((-K_X)^n) - \operatorname{mult}_Z X \cdot x^n)\, dx \\ &= \frac{n}{n+1} ((-K_X))^n \cdot \sqrt[n]{((-K_X)^n)/\operatorname{mult}_Z X}.\end{aligned}$$

Hence we get the desired inequality

$$((-K_X)^n) \leqslant \left(1 + \frac{1}{n}\right)^n \operatorname{lct}(X; I_Z)^n \operatorname{mult}_Z X. \qquad \square$$

Remark 17. As we see in the proof above, if the equality of (3.1) holds, then

$$\operatorname{vol}_{\hat{X}}(\sigma_*(-K_X) - xF) = \max\{((-K_X)^n) - \operatorname{mult}_Z X \cdot x^n, 0\}$$

for any $x \in \mathbb{R}_{\geqslant 0}$. Then by Lemma 10 (ii) we have that $\epsilon_Z(-K_X) = (1 + 1/n) \operatorname{lct}(X; I_Z)$.

The following result by Fujita is the main tool to prove Theorem 2.

THEOREM 18 [Fuj15, Theorem 4.9]. *Let X be a $\mathbb{Q}$-Fano variety of dimension n. Assume that X is Ding-semistable. Let $\mathcal{F}$ be a good graded filtration of $S = \bigoplus_{m=0}^{+\infty} H^0(X, mL)$ where $L = -rK_X$ is Cartier. Then the pair $(X \times \mathbb{A}^1, \mathcal{I}_\bullet^{\cdot(1/r)} \cdot (t)^{\cdot d_\infty})$ is sub log canonical in the sense of [Fuj15, Definition 2.4], where*

$$\begin{aligned}\mathcal{I}_m &= I^{\mathcal{F}}_{(m, me_+)} + I^{\mathcal{F}}_{(m, me_+ - 1)} t^1 + \cdots + I^{\mathcal{F}}_{(m, me_- + 1)} t^{m(e_+ - e_-) - 1} + (t^{m(e_+ - e_-)}), \\ d_\infty &= 1 - \frac{e_+ - e_-}{r} + \frac{1}{r^{n+1}((-K_X)^n)} \int_{e_-}^{e_+} \operatorname{vol}(\overline{\mathcal{F}} S^{(t)})\, dt,\end{aligned}$$

and $e_+, e_- \in \mathbb{Z}$ with $e_+ \geqslant e_{\max}(S_\bullet, \mathcal{F})$ and $e_- \leqslant e_{\min}(S_\bullet, \mathcal{F})$.

We will follow the notation of Theorem 18. Let $v \in \mathrm{Val}_{X,p}$ be a real valuation centered at p. Define an $\mathbb{R}$-filtration $\mathcal{F}_v$ of S as

$$\mathcal{F}_v^x S_m := H^0(X, L^m \cdot \mathfrak{a}_x(v)).$$

LEMMA 19. *The $\mathbb{R}$-filtration $\mathcal{F}_v$ of S is decreasing, left-continuous, multiplicative and saturated. Moreover, if $A_X(v) < +\infty$ then $\mathcal{F}_v$ is also linearly bounded.*

Proof. The decreasing, left-continuous and multiplicative properties of $\mathcal{F}_v$ follows from the corresponding properties of $\{\mathfrak{a}_x(v)\}_{x\in\mathbb{R}}$. To prove that $\mathcal{F}_v$ is saturated, notice that the homomorphism

$$\mathcal{F}_v^x S_m \otimes_{\mathbb{C}} L^{-m} \twoheadrightarrow I_{(m,x)}^{\mathcal{F}_v}$$

induces the inclusion $I_{(m,x)}^{\mathcal{F}_v} \subset \mathfrak{a}_x(v)$ for any $x \in \mathbb{R}$. Thus $\overline{\mathcal{F}}^x S_m = H^0(X, L^m \cdot I_{(m,x)}^{\mathcal{F}_v}) \subset \mathcal{F}^x S_m$, which implies that $\mathcal{F}_v$ is saturated.

If $A_X(v) < +\infty$, then by [Li15, Theorem 3.1] there exists a constant $c = c(X,p)$ such that $v \leqslant cA_X(v)\mathrm{ord}_p$. So it is easy to see that

$$e_{\max}(S_\bullet, \mathcal{F}_v) \leqslant cA_X(v) \cdot e_{\max}(S_\bullet, \mathcal{F}_{\mathrm{ord}_p}) < +\infty.$$

The second inequality follows from the work in [BKMS15]. Next, it is clear that $\mathfrak{a}_x(v) = \mathcal{O}_X$ for $x \leqslant 0$. Hence $\mathcal{F}_v^x S_m = S_m$ for $x \leqslant 0$, which implies $e_{\min}(S_\bullet, \mathcal{F}_v) \geqslant 0$. Hence $\mathcal{F}_v$ is linearly bounded. □

It is clear that $\mathcal{F}_v^x S_m = S_m$ for $x \leqslant 0$. Hence we may choose $e_- = 0$. The graded sequence of ideal sheaves $\mathcal{I}_\bullet$ on $X \times \mathbb{A}^1$ becomes:

$$\mathcal{I}_m = I_{(m,me_+)}^{\mathcal{F}_v} + I_{(m,me_+-1)}^{\mathcal{F}_v} t^1 + \cdots + I_{(m,1)}^{\mathcal{F}_v} t^{me_+-1} + (t^{me_+}).$$

We also have that

$$d_\infty = 1 - \frac{e_+}{r} + \frac{1}{r^{n+1}((-K_X)^n)} \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_v S^{(t)})\, dt.$$

The valuation v extends to a $\mathbb{G}_m$-invariant valuation $\bar{v}$ on $\mathbb{C}(X \times \mathbb{A}^1) = \mathbb{C}(X)(t)$, such that for any $f = \sum_j f_j t^j$ we have

$$\bar{v}(f) = \min_j \{v(f_j) + j\}. \tag{3.2}$$

PROPOSITION 20. *Let X be a $\mathbb{Q}$-Fano variety of dimension n. Assume that X is Ding-semistable. Let $p \in X$ be a closed point. Let $v \in \mathrm{Val}_{X,p}$ be a real valuation centered at p. Then we have*

$$A_X(v) \geqslant \frac{1}{r^{n+1}((-K_X)^n)} \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_v S^{(t)})\, dt. \tag{3.3}$$

Proof. We may assume that $A_X(v) < +\infty$, since otherwise the inequality holds automatically. Hence Lemma 19 implies that $\mathcal{F}_v$ is good and saturated.

By Theorem 18, we know that $(X \times \mathbb{A}^1, \mathcal{I}_\bullet^{\cdot(1/r)} \cdot (t)^{\cdot d_\infty})$ is sub log canonical. Since $X \times \mathbb{A}^1$ has klt singularities, for any $0 < \epsilon \ll 1$ there exists $m = m(\epsilon)$ such that

$$\mathcal{O}_{X\times\mathbb{A}^1} \subset \mathcal{J}(X \times \mathbb{A}^1, \mathcal{I}_m^{\cdot(1-\epsilon)/(rm)} \cdot (t)^{\cdot(1-\epsilon)d_\infty}),$$

where the right-hand side is the multiplier ideal sheaf. By [BdFFU15, Theorem 1.2] we know that the following inequality holds for any real valuation u on $X \times \mathbb{A}^1$:

$$A_{X\times\mathbb{A}^1}(u) > \frac{(1-\epsilon)}{rm} u(\mathcal{I}_m) + (1-\epsilon) d_\infty u(t). \tag{3.4}$$

Let us choose a sequence of quasi-monomial real valuations $\{v_n\}$ on X such that $v_n \to v$ and $A_X(v_n) \to A_X(v)$ when $n \to \infty$. It is easy to see that $\{\bar{v}_n\}$ is a sequence of quasi-monomial valuations on $X \times \mathbb{A}^1$ satisfying $\bar{v}_n(t) = 1$ and

$$A_{X\times\mathbb{A}^1}(\bar{v}_n) = A_X(v_n) + 1.$$

Hence we have

$$\begin{aligned} A_X(v) + 1 &= \lim_{n\to\infty} A_X(\bar{v}_n) \\ &\geqslant \frac{(1-\epsilon)}{rm} \lim_{n\to\infty} \bar{v}_n(\mathcal{I}_m) + (1-\epsilon) d_\infty. \end{aligned}$$

From the definition of $\mathcal{F}_v^x S_m$ we get:

$$v(I_{(m,x)}^{\mathcal{F}_v}) \geqslant x.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \bar{v}_n(\mathcal{I}_m) &= \lim_{n\to\infty} \min_{0\leqslant j\leqslant re_+} \left\{ v_n(I_{(m,j)}^{\mathcal{F}_v}) + me_+ - j \right\} \\ &= \min_{0\leqslant j\leqslant re_+} \left\{ \lim_{n\to\infty} v_n(I_{(m,j)}^{\mathcal{F}_v}) + me_+ - j \right\} \\ &= \min_{0\leqslant j\leqslant re_+} \{ v(I_{(m,j)}^{\mathcal{F}_v}) + me_+ - j \} \\ &\geqslant me_+. \end{aligned}$$

Hence when $\epsilon \to 0+$ we get:

$$A_X(v) + 1 \geqslant \frac{e_+}{r} + d_\infty. \tag{3.5}$$

Therefore,

$$A_X(v) \geqslant -1 + \frac{e_+}{r} + d_\infty = \frac{1}{r^{n+1}((-K_X)^n)} \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_v S^{(t)})\, dt.$$

Hence we get the desired inequality. □

The following theorem is a stronger result that implies Theorem 2.

THEOREM 21. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let $v \in \mathrm{Val}_{X,p}$ be a real valuation centered at p. Then we have*

$$((-K_X)^n) \leqslant \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v).$$

Proof. We may assume $A_X(v) < +\infty$, since otherwise the inequality holds automatically. Since $\mathcal{F}_v^{mx} S_m = H^0(X, L^m \cdot \mathfrak{a}_{mx}(v))$, we have the exact sequence

$$0 \to \mathcal{F}_v^{mx} S_m \to H^0(X, L^m) \to H^0(X, L^m \otimes (\mathcal{O}_X/\mathfrak{a}_{mx}(v))).$$

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Hence we have

$$\dim \mathcal{F}_v^{mx} S_m \geqslant h^0(X, L^m) - \ell(\mathcal{O}_X/\mathfrak{a}_{mx}(v)).$$

This implies

$$\operatorname{vol}(\mathcal{F}_v S^{(x)}) \geqslant (L^n) - \operatorname{vol}(v) x^n. \tag{3.6}$$

So we have the estimate:

$$\begin{aligned}\int_0^{+\infty} \operatorname{vol}(\mathcal{F}_v S^{(x)})\, dx &\geqslant \int_0^{\sqrt[n]{(L^n)/\operatorname{vol}(v)}} ((L^n) - \operatorname{vol}(v) x^n)\, dx \\ &= \frac{n}{n+1}(L^n) \cdot \sqrt[n]{(L^n)/\operatorname{vol}(v)}.\end{aligned}$$

Applying (3.3) we get the inequality:

$$\begin{aligned}A_X(v) &\geqslant \frac{1}{r(L^n)} \int_0^{+\infty} \operatorname{vol}(\mathcal{F}_v S^{(t)})\, dt \\ &\geqslant \frac{1}{r(L^n)} \frac{n}{n+1} (L^n) \sqrt[n]{(L^n)/\operatorname{vol}(v)} = \frac{n}{n+1} \sqrt[n]{((-K_X)^n)/\operatorname{vol}(v)}.\end{aligned}$$

This is exactly equivalent to

$$((-K_X)^n) \leqslant \left(1 + \frac{1}{n}\right)^n A_X(v)^n \operatorname{vol}(v) = \left(1 + \frac{1}{n}\right)^n \widehat{\operatorname{vol}}(v). \qquad \square$$

3.2 Applications

The following result is an easy consequence of Theorem 1.

THEOREM 22 [dFEM04]. *Let X be a variety of dimension n. Let Z be a closed subscheme of X supported at a single smooth closed point. Then*

$$\left(\frac{\operatorname{lct}(X; I_Z)}{n}\right)^n \operatorname{mult}_Z X \geqslant 1.$$

Proof. It is clear that the multiplicity is preserved under completion. Next, [dFEM11, Proposition 2.11] implies that lct is also preserved under completion. Hence it suffices to prove the theorem for $X = \mathbb{P}^n$. This follows directly by applying Theorem 1 to the case $X = \mathbb{P}^n$. $\square$

Now we will illustrate some volume bounds for singular X.

THEOREM 23. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Suppose (X, p) is a quotient singularity with local analytic model $\mathbb{C}^2/G$ where $G \subset \mathrm{GL}(n, \mathbb{C})$ acts freely in codimension 1. Then*

$$((-K_X)^n) \leqslant \frac{(n+1)^n}{|G|}.$$

Proof. Let $(Y, o) := (\mathbb{C}^n/G, 0)$ be the local analytic model of the quotient singularity (X, p). Let $H := G \cap \mathbb{G}_m$ with $d := |H|$. Define the real valuation u_0 on Y to be the pushforward of the valuation ord_0 on $\mathbb{C}^n$ under the quotient map $\mathbb{C}^n \to Y$.

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We first show that $\widehat{\mathrm{vol}}(u_0) = n^n/|G|$. Let $\widehat{\mathbb{C}^n}$ be the blow up of $\mathbb{C}^n$ at the origin 0 with exceptional divisor E. Denote by $\pi : \mathbb{C}^n \to Y$ the quotient map. Then π lifts to $\widehat{\mathbb{C}^n}$ as $\hat{\pi} : \widehat{\mathbb{C}^n} \to \hat{Y}$, where $\hat{Y} := \widehat{\mathbb{C}^n}/G$. We have the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathbb{C}^n} & \xrightarrow{\hat{\pi}} & \hat{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{C}^n & \xrightarrow{\pi} & Y \end{array}$$

Let $F \subset \hat{Y}$ be the exceptional divisor of h. For a general point on F, its stabilizer under the G-action is exactly H. So $\hat{\pi}^* F = dE$, which implies that $u_0 = \pi_*(\mathrm{ord}_E) = d\,\mathrm{ord}_F$. It is clear that

$$K_{\widehat{\mathbb{C}^n}} = \hat{\pi}^*\left(K_{\hat{Y}} + \left(1 - \frac{1}{d}\right)F\right).$$

Then combining these equalities with $K_{\widehat{\mathbb{C}^n}} = g^* K_{\mathbb{C}^n} + (n-1)E = g^*(\pi^* K_Y) + (n-1)E$, we get

$$K_{\hat{Y}} = h^* K_Y + \left(\frac{n}{d} - 1\right)F.$$

Hence $A_X(\mathrm{ord}_F) = n/d$ and $A_X(u_0) = d \cdot A_X(\mathrm{ord}_F) = n$. Lemma 24 implies that $\mathrm{vol}(u_0) = 1/|G|$. Therefore, $\widehat{\mathrm{vol}}(u_0) = n^n/|G|$.

Since $A_Y(u_0) < +\infty$, a singular version of [JM12, Corollary 5.11] for klt singularities (such a statement is true thanks to the Izumi inequality for klt singularities [Li15, Proposition 3.1]) implies that u_0 has a unique extension, say $\hat{u}_0$, as a valuation on $\mathbf{Spec}\,\widehat{\mathcal{O}_{Y,o}}$ that centered at the closed point. By assumption we have an isomorphism $\phi : \widehat{\mathcal{O}_{X,p}} \to \widehat{\mathcal{O}_{Y,o}}$. Let $v_0 := (\phi^* \hat{u}_0)|_{\mathcal{O}_{X,p}}$, then $v_0 \in \mathrm{Val}_{X,p}$. By a singular version of [JM12, Proposition 5.13], we know that $A_X(v_0) = A_Y(u_0)$. On the other hand, since the colength of any $\mathfrak{m}$-primary ideal does not change under completion, we have $\mathrm{vol}(v_0) = \mathrm{vol}(\hat{u}_0) = \mathrm{vol}(u_0)$. As a result, we have

$$\widehat{\mathrm{vol}}(v_0) = \widehat{\mathrm{vol}}(u_0) = \frac{n^n}{|G|}.$$

Hence the theorem follows as a direct application of Theorem 2 to $v = v_0$. □

Lemma 24. *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite group acting on $\mathbb{C}^n = \mathbf{Spec}\,\mathbb{C}[x_1, \ldots, x_n]$. Then we have*

$$\lim_{m\to\infty} \frac{\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]^G_{<m}}{m^n/n!} = \frac{1}{|G|}.$$

Proof. Denote $W := \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. Then we have that

$$\mathbb{C}[x_1, \ldots, x_n] \cong \bigoplus_{m\geqslant 0} \mathrm{Sym}^m W.$$

Denote by $\rho_m : G \to \mathrm{GL}(\mathrm{Sym}^m W)$ the induced representation of G on $\mathrm{Sym}^m W$. Since G is finite, $\rho_m(g)$ is diagonalizable for any $m \geqslant 0$ and $g \in G$. Denote the n eigenvalues of $\rho_1(g)$ by $\lambda_{g,1}, \ldots, \lambda_{g,n}$. Then the eigenvalues of $\rho_m(g)$ are exactly all monomials in $\lambda_{g,i}$ of degree m. Let $c_m := \dim_{\mathbb{C}}(\mathrm{Sym}^m W)^G$ and $d_m := \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]^G_{\leqslant m}$. From representation theory we know that

$$c_m = \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(\rho_m(g)).$$

Hence

$$c(t) := \sum_{m=0}^{\infty} c_m t^m = \frac{1}{|G|} \sum_{g\in G} \frac{1}{(1-\lambda_{g,1}t)\cdots(1-\lambda_{g,n}t)}.$$

This is also known as Molien's theorem [Mol97]. Since $d_m = \sum_{i=0}^{m} c_i$, we have

$$\begin{aligned} d(t) &:= \sum_{m=0}^{\infty} d_m t^m = \frac{c(t)}{1-t} \\ &= \frac{1}{|G|} \sum_{g\in G} \frac{1}{(1-t)(1-\lambda_{g,1}t)\cdots(1-\lambda_{g,n}t)}. \end{aligned}$$

Here all eigenvalues $\lambda_{g,i}$ are roots of unity. Using partial fraction decomposition, for any $g \neq id$ we have

$$\frac{1}{(1-t)(1-\lambda_{g,1}t)\cdots(1-\lambda_{g,n}t)} = \sum_{m=0}^{\infty} o(m^n)t^m.$$

Hence

$$\begin{aligned} d(t) &= \frac{1}{|G|}\cdot\frac{1}{(1-t)^{n+1}} + \sum_{m=0}^{\infty} o(m^n)t^m \\ &= \sum_{m=0}^{\infty}\left(\frac{1}{|G|}\binom{n+m}{n} + o(m^n)\right)t^m. \end{aligned}$$

As a result,

$$\lim_{m\to\infty}\frac{d_{m-1}}{m^n/n!} = \lim_{m\to\infty}\frac{1/|G|\binom{n+m-1}{n}}{m^n/n!} = \frac{1}{|G|}.$$ □

THEOREM 25. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be an isolated singularity. If X is not terminal at p, then*

$$((-K_X)^n) \leqslant \left(1+\frac{1}{n}\right)^n \operatorname{mult}_p X < e\ \operatorname{mult}_p X.$$

Proof. Denote by $\mathfrak{m}_p$ the maximal ideal at p. We first show that $\operatorname{lct}(X;\mathfrak{m}_p) \leqslant 1$. Since p is an isolated singularity, we may take a log resolution of $(X, \mathfrak{m}_p)$, namely $\pi : Y \to X$ such that π is an isomorphism away from p and $\pi^{-1}\mathfrak{m}_p \cdot \mathcal{O}_Y$ is an invertible ideal sheaf on Y. Let E_i be the exceptional divisors of π. We define the numbers a_i and b_i by

$$K_Y = \pi^* K_X + \sum_i a_i E_i \quad \text{and} \quad \pi^{-1}\mathfrak{m}_p \cdot \mathcal{O}_Y = \mathcal{O}_Y\Big(-\sum_i b_i E_i\Big).$$

It is clear that $\operatorname{lct}(X;\mathfrak{m}_p) = \min_i((1+a_i)/b_i)$. Since π is an isomorphism away from p, we have $b_i \geqslant 1$ for any i. Since X is not terminal at p, there exists an index i_0 such that $a_{i_0} \leqslant 0$. Hence

$$\operatorname{lct}(X;\mathfrak{m}_p) \leqslant \frac{1+a_{i_0}}{b_{i_0}} \leqslant 1.$$

So we finish the proof by applying Theorem 1 to $Z = p$. □

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4. Comparing invariants for ideals and valuations

In this section, we will always assume that (X, p) is a normal $\mathbb{Q}$-Gorenstein klt singularity of dimension n with $p \in X$ a closed point.

4.1 The infimums of invariants

Lemma 26. *Let $\mathfrak{a}$ be an ideal sheaf on X supported at p. Let $v_0 \in \mathrm{Val}_X$ be a divisorial valuation that computes $\mathrm{lct}(\mathfrak{a})$. Then v_0 is centered at p, and we have*

$$\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) \geqslant \widehat{\mathrm{vol}}(v_0).$$

Proof. Since v_0 computes $\mathrm{lct}(\mathfrak{a})$, we have that $\mathrm{lct}(\mathfrak{a}) = A_X(v_0)/v_0(\mathfrak{a})$. Denote $\alpha := v_0(\mathfrak{a})$. Then $\mathfrak{a}^m \subset \mathfrak{a}_{m\alpha}(v_0)$ since $v_0(\mathfrak{a}^m) = m\alpha$. Therefore,

$$\begin{aligned} \mathrm{mult}(\mathfrak{a}) &= \lim_{m\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}^m)}{m^n/n!} \\ &\geqslant \lim_{m\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}_{m\alpha}(v_0))}{m^n/n!} \\ &= \alpha \ \mathrm{vol}(v_0). \end{aligned}$$

Hence we prove the lemma. □

Following [ELS03], for a graded sequence of ideals $\mathfrak{a}_\bullet$ supported at p, the *volume* of $\mathfrak{a}_\bullet$ is defined as

$$\mathrm{vol}(\mathfrak{a}_\bullet) := \limsup_{m\to\infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!},$$

while the *multiplicity* of $\mathfrak{a}_\bullet$ is defined as

$$\mathrm{mult}(\mathfrak{a}_\bullet) := \lim_{m\to\infty} \frac{\mathrm{mult}(\mathfrak{a}_m)}{m^n}.$$

By [LM09, Cut13], we know that $\mathrm{mult}(\mathfrak{a}_\bullet) = \mathrm{vol}(\mathfrak{a}_\bullet)$ when X is normal.

Theorem 27. *We have*

$$\inf_{\mathfrak{a}} (\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})) = \inf_{\mathfrak{a}_\bullet} (\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet)) = \inf_v \widehat{\mathrm{vol}}(v), \tag{4.1}$$

where the infimums are taken over all ideals $\mathfrak{a}$ supported at p, all graded sequences of ideals $\mathfrak{a}_\bullet$ supported at p and all real valuations $v \in \mathrm{Val}_{X,p}$, respectively. We also set $\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet) = +\infty$ if $\mathrm{lct}(\mathfrak{a}_\bullet) = +\infty$.

Proof. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideal sheaves supported at p. By [JM12, BdFFU15], we have

$$\lim_{m\to\infty} m \ \mathrm{lct}(\mathfrak{a}_m) = \mathrm{lct}(\mathfrak{a}_\bullet).$$

Since $\mathrm{mult}(\mathfrak{a}_\bullet) = \mathrm{vol}(\mathfrak{a}_\bullet)$, if $\mathrm{lct}(\mathfrak{a}_\bullet) < +\infty$ then

$$\lim_{m\to\infty} \mathrm{lct}(\mathfrak{a}_m)^n \cdot \mathrm{mult}(\mathfrak{a}_m) = \mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet).$$

Thus we have

$$\inf_{\mathfrak{a}} (\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})) \leqslant \inf_{\mathfrak{a}_\bullet} (\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet)). \tag{4.2}$$

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Next, Lemma 26 implies that

$$\inf_{\mathfrak{a}}(\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})) \geqslant \inf_v \widehat{\mathrm{vol}}(v). \tag{4.3}$$

Finally, for any real valuation $v \in \mathrm{Val}_{X,p}$ with $A_X(v) < +\infty$, by [BdFFU15, Theorem 1.2] we have

$$\mathrm{lct}(\mathfrak{a}_\bullet(v)) \leqslant \frac{A_X(v)}{v(\mathfrak{a}_\bullet(v))} = A_X(v) < +\infty.$$

By definition we have $\mathrm{vol}(\mathfrak{a}_\bullet(v)) = \mathrm{vol}(v)$. Therefore,

$$\mathrm{lct}(\mathfrak{a}_\bullet(v))^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet(v)) \leqslant A_X(v)^n \cdot \mathrm{vol}(v) = \widehat{\mathrm{vol}}(v). \tag{4.4}$$

Hence we have

$$\inf_{\mathfrak{a}_\bullet}(\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet)) \leqslant \inf_v \widehat{\mathrm{vol}}(v). \tag{4.5}$$

Combining (4.2), (4.3) and (4.5) together, we finish the proof. □

Remark 28. (a) The first equality in (4.1) was observed by Mustaţă in [Mus02] where he showed that $\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet) \geqslant n^n$ for any graded sequence of $\mathfrak{m}$-primary ideals $\mathfrak{a}_\bullet$ of a regular local ring of dimension n. The inequality (4.4) was essentially realized by Li in [Li15, Remark 2.8] where he considered the smooth case.

(b) By Izumi's theorem, the infimums in Theorem 27 are always positive (see § 2.4).

We will use Theorem 27 to prove Theorem 7 based on [Li17, LL16].

Theorem 29 (= Theorem 7). *Let V be a Fano manifold of dimension $n-1$. Assume $H = -rK_V$ is an ample Cartier divisor for some $r \in \mathbb{Q}_{>0}$. Let $X := C(V, H) = \mathbf{Spec} \bigoplus_{k=0}^{\infty} H^0(V, mH)$ be the affine cone with cone vertex o. Then V is K-semistable if and only if for any closed subscheme Z of X supported at o, the following inequality holds:*

$$\mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X \geqslant \frac{1}{r}((-K_V)^{n-1}). \tag{4.6}$$

Proof. It is clear that $A_X(\mathrm{ord}_V) = r^{-1}$ and $\mathrm{vol}(\mathrm{ord}_V) = r^{n-1}((-K_V)^{n-1})$. Hence the right-hand side of (4.6) is equal to $\widehat{\mathrm{vol}}(\mathrm{ord}_V)$. From Theorem 27 we know that (4.6) is equivalent to saying that the normalized volume $\widehat{\mathrm{vol}}$ is minimized at ord_V over (X, o). This is equivalent to V being K-semistable by [LL16, Corollary 1.5]. □

4.2 Finding minimizers

Assume that the infimum of $\mathrm{lct}^n(\mathfrak{a}) \cdot \mathrm{mult}(\mathfrak{a})$ is attained by some ideal $\mathfrak{a} = \mathfrak{a}_0$. Then Lemma 26 and Theorem 27 together imply that the divisorial valuation v_0 that computes $\mathrm{lct}(\mathfrak{a}_0)$ is a minimizer of $\widehat{\mathrm{vol}}$. In this subsection, we will study the converse problem, i.e. assume there exists a minimizer v_* of $\widehat{\mathrm{vol}}$ that is divisorial, can the infimum of $\mathrm{lct}^n \cdot \mathrm{mult}$ for ideals be achieved? In Proposition 31, we give an affirmative answer to this problem assuming an extra condition on v_*.

Definition 30. Let R be a Noetherian local domain. For a real valuation v of $K(R)$ dominating R, we define the *associate graded algebra* of v as

$$\mathrm{gr}_v R := \bigoplus_{m \in \Phi} \mathfrak{a}_m(v)/\mathfrak{a}_{>m}(v),$$

where Φ is the valuation semigroup of v.

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PROPOSITION 31. *Assume that* $\widehat{\mathrm{vol}}(\cdot)$ *is minimized at a divisorial valuation* v_*. *If the graded algebra* $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ *is a finitely generated* $\mathbb{C}$*-algebra, then for any sufficiently divisible* $k \in \mathbb{Z}_{>0}$ *we have*

$$\mathrm{lct}(\mathfrak{a}_k(v_*))^n \cdot \mathrm{mult}(\mathfrak{a}_k(v_*)) = \widehat{\mathrm{vol}}(v_*).$$

In particular, the function $(\mathrm{lct}(\cdot)^n \cdot \mathrm{mult}(\cdot))$ *attains its minimum at* $\mathfrak{a}_k(v_*)$.

Proof. By Izumi's theorem we know that the filtration $\mathfrak{a}_1(v_*) \supset \mathfrak{a}_2(v_*) \supset \cdots$ defines the same topology on $\mathcal{O}_{X,p}$ as the $\mathfrak{m}_p$-adic topology on $\mathcal{O}_{X,p}$. Then Lemma 32 implies $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(v_*)$ is a finitely generated $\mathcal{O}_{X,p}$-algebra. Therefore, the degree k Veronese subalgebra $\bigoplus_{m\geqslant 0}\mathfrak{a}_{km}(v_*)$ is generated in degree 1 for $k\in\mathbb{Z}_{>0}$ sufficiently divisible. This means $\mathfrak{a}_{km} = \mathfrak{a}_k^m$ for any $m \geqslant 0$. From inequality (4.4) we have

$$\begin{aligned}\widehat{\mathrm{vol}}(v_*) &\geqslant \mathrm{lct}(\mathfrak{a}_\bullet(v_*))^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet(v_*))\\ &= \mathrm{lct}(\mathfrak{a}_{k\bullet}(v_*))^n \cdot \mathrm{vol}(\mathfrak{a}_{k\bullet}(v_*))\\ &= \mathrm{lct}(\mathfrak{a}_k(v_*))^n \cdot \mathrm{mult}(\mathfrak{a}_k(v_*)).\end{aligned}$$

On the other hand, Theorem 27 also implies that $\mathrm{lct}(\mathfrak{a}_k(v_*))^n \cdot \mathrm{mult}(\mathfrak{a}_k(v_*)) \geqslant \widehat{\mathrm{vol}}(v_*)$ because v_* is a minimizer of $\widehat{\mathrm{vol}}(\cdot)$. So we prove the proposition. □

LEMMA 32. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring. Let* $\mathfrak{a}_\bullet$ *be a graded sequence of* $\mathfrak{m}$*-primary ideals. Assume that the linear topology on* R *defined by the filtration* $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ *is the same as the* $\mathfrak{m}$*-adic topology on* R. *Assume that* $\bigoplus_{m\geqslant 0}\mathfrak{a}_m/\mathfrak{a}_{m+1}$ *is a finitely generated* $R/\mathfrak{a}_1$*-algebra. Then* $\bigoplus_{m\geqslant 0}\mathfrak{a}_m$ *is a finitely generated* R*-algebra.*

Proof. We take a set of homogeneous generators of $\bigoplus_{m\geqslant 0}\mathfrak{a}_m/\mathfrak{a}_{m+1}$, say $x_1,\ldots,x_r$. Let d_i be the degree of x_i, i.e. $x_i \in \mathfrak{a}_{d_i}/\mathfrak{a}_{d_i+1}$. Take $y_i \in \mathfrak{a}_{d_i}$ such that $x_i = y_i + \mathfrak{a}_{d_i+1}$. Denote by $\bigoplus_{m\geqslant 0}\mathfrak{b}_m$ the graded R-subalgebra of $\bigoplus_{m\geqslant 0}\mathfrak{a}_m$ generated by $y_1,\ldots,y_r$.

Since $\{x_i\}$ generates $\bigoplus_{m\geqslant 0}\mathfrak{a}_m/\mathfrak{a}_{m+1}$, we have $\mathfrak{a}_m = \mathfrak{b}_m + \mathfrak{a}_{m+1}$. Hence for any $l>0$ the induction yields

$$\mathfrak{a}_m = (\mathfrak{b}_m + \mathfrak{b}_{m+1} + \cdots + \mathfrak{b}_{m+l-1}) + \mathfrak{a}_{m+l}.$$

Because R is Noetherian, the ideals $(\mathfrak{b}_m + \mathfrak{b}_{m+1} + \cdots + \mathfrak{b}_{m+l-1})$ stabilize for l sufficiently large. Denote the stabilized ideal by $\mathfrak{c}_m$, then we have $\mathfrak{a}_m = \mathfrak{c}_m + \mathfrak{a}_{m+l}$ for l sufficiently large. Since the filtration $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ defines the same topology as the $\mathfrak{m}$-adic topology, we may take l sufficiently large such that $\mathfrak{a}_{m+l} \subset \mathfrak{m}\mathfrak{a}_m$. As a result, we have $\mathfrak{a}_m = \mathfrak{c}_m + \mathfrak{m}\mathfrak{a}_m$ which implies that $\mathfrak{a}_m = \mathfrak{c}_m$ by Nakayama's lemma.

Now it suffices to show that $\bigoplus_{m\geqslant 0}\mathfrak{c}_m$ is a finitely generated graded R-algebra. By definition, $\mathfrak{c}_m$ is generated by monomials in $\{y_i\}$ of weighted degree at least m. For any $0 \leqslant j \leqslant d_i$, denote by $y_{i,j}$ the homogeneous element $y_i \in \mathfrak{a}_j$ of degree j. Then any monomial in $\{y_i\}$ of weighted degree at least m is equal to a monomial in $\{y_{i,j}\}$ of weighted degree m. Hence $\bigoplus_{m\geqslant 0}\mathfrak{c}_m$ is generated by $\{y_{i,j}\}$ as a graded R-algebra, which finishes the proof. □

Postscript remark. Since the first version of this article was posted on the arXiv, there has been a lot of progresses studying properties of the minimizer of $\widehat{\mathrm{vol}}$. Here we mention two of them which are useful in our presentation.

- Blum [Blu18] proved that there always exists a minimizer of $\widehat{\mathrm{vol}}$ for any klt singularity $x \in X$;
- If a divisorial valuation $v_* = \mathrm{ord}_S$ minimizes $\widehat{\mathrm{vol}}$ in $\mathrm{Val}_{X,x}$, then S is necessarily a *Kollár component* (see [LX16] for a definition). In particular, $\mathrm{gr}_{v_*}\mathcal{O}_{X,x}$ is finitely generated. This was proved by Blum [Blu18] and Li and Xu [LX16] independently.

5. Maximal volume cases

5.1 K-polystable varieties of maximal volume

The following lemma is the main technical tool to prove the equality case of Theorem 3 and Corollary 6.

LEMMA 33. *Let X be a K-polystable $\mathbb{Q}$-Fano variety of dimension n. Let $p \in X$ be a closed point. Assume that there exists a divisorial valuation $v_* \in \mathrm{Val}_{X,p}$ satisfying $((-K_X)^n) = (1+1/n)^n \widehat{\mathrm{vol}}(v_*)$. Then we have:*

- (i) *v_* is a minimizer of the normalized volume function $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{X,p}$;*
- (ii) *$X \cong \mathbf{Proj}\,(\mathrm{gr}_{v_*}\mathcal{O}_{X,p})[x]$, where x is a homogeneous element of degree 1 (we may view X as an orbifold projective cone over $F := \mathbf{Proj}\,\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$).*

Proof. Part (i) is a direct consequence of Theorem 2.

We need some preparation to prove part (ii). Since v_* is a divisorial minimizer of $\widehat{\mathrm{vol}}$ by (i), it is induced by a Kollár component by [Blu18, Proposition 4.9] and [LX16, Theorem 1.2]. In particular, $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ is a finitely generated $\mathbb{C}$-algebra. Let $\bar{v}_*$ be the corresponding divisorial valuation on $X \times \mathbb{A}^1$ (see (3.2)). Let t be the parameter of $\mathbb{A}^1$. Let $\mathbf{Spec}\,R$ be an affine Zariski open subset of X. By Lemma 32, $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(v_*)$ is a finitely generated R-algebra. Then

$$\mathfrak{a}_m(\bar{v}_*) = \mathfrak{a}_m(v_*) + \mathfrak{a}_{m-1}(v_*)t + \cdots + \mathfrak{a}_1(v_*)t^{m-1} + (t^m).$$

In order to keep track of the degree of the graded ring $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(\bar{v}_*)$, denote by s the element t in $\mathfrak{a}_1(\bar{v}_*) = \mathfrak{a}_1(v_*) + (t)$. Hence $\deg s = 1$, $\deg t = 0$ and we have

$$\mathfrak{a}_m(\bar{v}_*) = \mathfrak{a}_m(v_*) + \mathfrak{a}_{m-1}(v_*)s + \cdots + \mathfrak{a}_1(v_*)s^{m-1} + s^m R[t],$$

where $\deg \mathfrak{a}_i(v_*) = i$. As an $R[t]$-algebra, $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(\bar{v}_*)$ is generated by $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(v_*)$ and s. Since $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(v_*)$ is a finitely generated R-algebra, we have that $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(\bar{v}_*)$ is of finite type over $\mathcal{O}_{X\times\mathbb{A}^1}$.

Let $\mathcal{X} := \mathbf{Proj}_{X\times\mathbb{A}^1}\oplus_{m\geqslant 0}\mathfrak{a}_m(\bar{v}_*)$, hence $\mathcal{X}$ is normal by Lemma 34. Denote by $g : \mathcal{X} \to X\times\mathbb{A}^1$ the projection. Denote the composite map by $\pi : \mathcal{X} \to \mathbb{A}^1$. The $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ is defined as

$$\mathcal{L} := g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) + \left(1 + \frac{1}{n}\right)A_X(v_*)\mathcal{O}_{\mathcal{X}}(1),$$

where we treat $\mathcal{O}_{\mathcal{X}}(1) = (1/k)\mathcal{O}_{\mathcal{X}}(k)$ as a $\mathbb{Q}$-line bundle for sufficiently divisible $k \in \mathbb{Z}_{>0}$. Let $\hat{X} := g_*^{-1}(X \times \{0\})$ and $\sigma = g|_{\hat{X}} : \hat{X} \to X$. Let E be the exceptional divisor of g given by $I_E := \mathcal{O}_{\mathcal{X}}(1)$. Hence

$$E \cong \mathbf{Proj}\bigoplus_{m\geqslant 0}\mathfrak{a}_m(\bar{v}_*)/\mathfrak{a}_{m+1}(\bar{v}_*) = \mathbf{Proj}(\mathrm{gr}_{v_*}\mathcal{O}_{X,p})[s],$$

where s is a homogeneous element of degree 1. Since $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ is an integral domain, E is irreducible and reduced.

We will show the following statements:

- (a) $\mathcal{L}$ is π-nef with $\mathcal{L}^{\perp} = \hat{X}$;
- (b) $\mathcal{L}$ is π-semiample, hence $(\mathcal{X}, \mathcal{L})$ is a semi $\mathbb{Q}$-test configuration of $(X, -K_X)$;
- (c) $\mathrm{CM}(\mathcal{X}, \mathcal{L}) = 0$.

For (a), we see that $\mathcal{L}_t \cong -K_{\mathcal{X}_t}$ is ample whenever $t \neq 0$. From the definition we know that $\mathcal{L}$ is g-ample, hence $\mathcal{L}|_E$ is ample. On $\hat{X}$, we have that

$$\mathcal{L}|_{\hat{X}} \cong \sigma^*(-K_X) + \left(1+\frac{1}{n}\right)A_X(v_*)\mathcal{O}_{\hat{X}}(1).$$

For $k \in \mathbb{Z}_{>0}$ sufficiently divisible, let Z be the thickening of p in X such that $I_Z = \mathfrak{a}_k(v_*)$. From (i) we know that v_* minimizes $\widehat{\mathrm{vol}}$, hence $\mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X = \widehat{\mathrm{vol}}(v_*)$ by Proposition 31. Since

$$((-K_X)^n) = \left(1+\frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v_*) = \left(1+\frac{1}{n}\right)^n \mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X,$$

by Lemma 10 and Remark 17 we have that $\mathcal{L}|_{\hat{X}}$ is nef with zero top self intersection number. In addition, $\epsilon_Z(-K_X) = A_X(v_*)/m$. As a result, $\mathcal{L}^{\perp} = \hat{X}$.

For (b), we first show that $\mathcal{X}$ is normal $\mathbb{Q}$-Gorenstein with klt singularities. By Lemma 34 we have that $E = \mathrm{Ex}(g)$ is a $\mathbb{Q}$-Cartier prime divisor on $\mathcal{X}$. Hence

$$K_{\mathcal{X}} = g^*K_{X\times\mathbb{A}^1} + (A_{\mathcal{X}}(\mathrm{ord}_E) - 1)E$$

is $\mathbb{Q}$-Cartier. To show that $\mathcal{X}$ has klt singularities, it suffices to show that $(\mathcal{X}, \hat{X})$ is plt. By Lemma 34, we have $\bar{v}_* = \mathrm{ord}_E$. Hence $\mathcal{X}_0 = \hat{X} + E$ as Weil divisors since $\bar{v}_*(t) = 1$. In particular, $K_{\mathcal{X}} + \hat{X}$ is $\mathbb{Q}$-Cartier.

It is clear that

$$I_E/(I_E \cdot I_{\hat{X}}) = I_E|_{\hat{X}} = \mathcal{O}_{\mathcal{X}}(1)|_{\hat{X}} = \mathcal{O}_{\hat{X}}(1) \subset \mathcal{O}_{\hat{X}}.$$

Since the kernel of $I_E \to \mathcal{O}_{\hat{X}}$ is $I_E \cap I_{\hat{X}}$, we have that $I_E \cdot I_{\hat{X}} = I_E \cap I_{\hat{X}}$. On the other hand, by computing graded ideals we know that $I_E + I_{\hat{X}} = I_F$. Denote by η the generic point of F, then $\mathcal{O}_{\hat{X},\eta} = \mathcal{O}_{\mathcal{X},\eta}/I_{\hat{X},\eta}$ is a DVR since $\hat{X}$ is normal. Applying Lemma 35 to $(R, \mathfrak{p}, \mathfrak{q}) = (\mathcal{O}_{\mathcal{X},\eta}, I_{E,\eta}, I_{\hat{X},\eta})$ yields that E is Cartier at η. Since $\mathcal{X}_0 = \hat{X} + E$ is Cartier and $\hat{X} \cap E = F$, we have that $\hat{X} = \mathcal{X}_0 - E$ is Cartier in codimension 2. Next we notice that $\hat{X} \to X$ produces a Kollár component, hence $\hat{X}$ has klt singularities. Thus $(\mathcal{X}, \hat{X})$ is plt by inverse of adjunction [KM98, Theorem 5.50].

By Shokurov's base-point-free theorem, to show $\mathcal{L}$ is π-semiample we only need to show that $\mathcal{L} - K_{\mathcal{X}/\mathbb{A}^1}$ is π-ample. It follows from Lemma 34 that $\bar{v}_* = \mathrm{ord}_E$ and $E \sim_{\mathbb{Q}} \mathcal{O}_{\mathcal{X}}(-1)$. Hence we have

$$\begin{aligned} -K_{\mathcal{X}/\mathbb{A}^1} &= g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) - (A_{X\times\mathbb{A}^1}(\bar{v}_*) - 1)E \\ &= g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) + A_X(v_*)\mathcal{O}_{\mathcal{X}}(1). \end{aligned}$$

Since $0 < A_X(v_*) < (1 + 1/n)A_X(v_*)$, we see that $-K_{\mathcal{X}/\mathbb{A}^1}$ is π-ample. Hence $\mathcal{L} - K_{\mathcal{X}/\mathbb{A}^1}$ is π-ample.

For (c), we know that

$$\mathrm{CM}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n+1)((-K_X)^n)}(n(\bar{\mathcal{L}}^{n+1}) + (n+1)(\bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1})).$$

By definition of $\mathcal{L}$ we know that

$$\begin{aligned} \bar{\mathcal{L}} &= \bar{g}^*(-K_{X\times\mathbb{P}^1/\mathbb{P}^1}) + \left(1+\frac{1}{n}\right)A_X(v_*)\mathcal{O}_{\bar{\mathcal{X}}}(1), \\ K_{\bar{\mathcal{X}}/\mathbb{P}^1} &= \bar{g}^*K_{X\times\mathbb{P}^1/\mathbb{P}^1} - A_X(v_*)\mathcal{O}_{\bar{\mathcal{X}}}(1). \end{aligned}$$

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Since $\mathcal{O}_{\bar{\mathcal{X}}}(1)$ is supported in $\bar{g}^{-1}((p,0))$, we have $(\bar{g}^*K_{X\times\mathbb{P}^1/\mathbb{P}^1}\cdot\mathcal{O}_{\bar{\mathcal{X}}}(1))=0$ as a cycle. Next, it is clear that $((\bar{g}^*K_{X\times\mathbb{P}^1/\mathbb{P}^1})^{n+1})=(K^{n+1}_{X\times\mathbb{P}^1/\mathbb{P}^1})=0$. Hence we have

$$(\bar{\mathcal{L}}^{n+1})=\left(1+\frac{1}{n}\right)^{n+1}A_X(v_*)^{n+1}(\mathcal{O}_{\bar{\mathcal{X}}}(1)^{n+1}),$$
$$(\bar{\mathcal{L}}^{n}\cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1})=-\left(1+\frac{1}{n}\right)^{n}A_X(v_*)^{n+1}(\mathcal{O}_{\bar{\mathcal{X}}}(1)^{n+1}).$$

Thus $\mathrm{CM}(\mathcal{X},\mathcal{L})=0$.

Now we are ready to prove part (ii). By (b) we know that $\mathcal{L}$ is semiample. Denote the ample model of $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ by $(\mathcal{Y},\mathcal{M})$, with $h:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{L}=h^*\mathcal{M}$. Then $(\mathcal{Y},\mathcal{M})$ is a normal $\mathbb{Q}$-test configuration of $(X,-K_X)$. Since $\bar{h}_*K_{\bar{\mathcal{X}}/\mathbb{P}^1}=K_{\bar{\mathcal{Y}}/\mathbb{P}^1}$, we have $\mathrm{CM}(\mathcal{Y},\mathcal{M})=\mathrm{CM}(\mathcal{X},\mathcal{L})=0$. From (a) we know that $\mathcal{L}^\perp=\hat{X}$, so $\mathrm{Ex}(h)=\hat{X}$. Hence $\mathcal{Y}_0=h_*\mathcal{X}_0=h_*(\hat{X}+E)=h_*E$. In particular, $\mathcal{Y}_0$ is a prime divisor on $\mathcal{Y}$. It is clear that $\mathcal{L}|_E=(1+1/n)A_X(v_*)\mathcal{O}_E(1)$ is ample, hence $h|_E:E\to\mathcal{Y}_0$ is a finite birational morphism. Since X is K-polystable by assumption, hence $\mathrm{CM}(\mathcal{Y},\mathcal{M})=0$ yields that $X\cong\mathcal{Y}_0$. Then $h|_E:E\to\mathcal{Y}_0$ has to be an isomorphism because $\mathcal{Y}_0\cong X$ is normal. Hence $X\cong E\cong \mathbf{Proj}\,(\mathrm{gr}_{v_*}\mathcal{O}_{X,p})[s]$. □

Lemma 34. *Let (X,p) be a normal singularity. Let v_* be a divisorial valuation on X centered at p. Assume that $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ is a finitely generated $\mathbb{C}$-algebra. Let $F:=\mathbf{Proj}\,\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$. Define $\hat{X}:=\mathbf{Proj}_X\bigoplus_{m\geqslant 0}\mathfrak{a}_m(v_*)$ with the projection $\sigma:\hat{X}\to X$. Then $\hat{X}$ is normal, $F=\mathrm{Ex}(\sigma)$ is a $\mathbb{Q}$-Cartier prime divisor on $\hat{X}$, and $v_*=\mathrm{ord}_F$. (Note that σ is called a prime blow-up in [Ish04].)*

Proof. By Lemma 32, we know that $\bigoplus_{m\geqslant 0}\mathfrak{a}_m(v_*)$ is a finitely generated $\mathcal{O}_X$-algebra. Since the valuation ideals $\mathfrak{a}_m(v_*)$ are always integrally closed, $\hat{X}$ is normal. By definition of F we know that the ideal sheaf I_F is the same as the coherent sheaf $\mathcal{O}_{\hat{X}}(1)$. Let $k\in\mathbb{Z}_{>0}$ be sufficiently divisible so that $\mathfrak{a}_{mk}(v_*)=\mathfrak{a}_k(v_*)^m$ for any positive integer m. Thus $\hat{X}$ is naturally isomorphic to the blow up of X along the ideal sheaf $\mathfrak{a}_k(v_*)$. In particular, $\mathcal{O}_{\hat{X}}(k)=\sigma^{-1}\mathfrak{a}_k(v_*)\cdot\mathcal{O}_{\hat{X}}$ is an invertible ideal sheaf. Next, $\mathcal{O}_{\hat{X}}/\mathcal{O}_{\hat{X}}(k)$ is supported at the exceptional locus of σ. As ideal sheaves on $\hat{X}$, we know that $\mathcal{O}_{\hat{X}}(1)^k\subset\mathcal{O}_{\hat{X}}(k)\subset\mathcal{O}_{\hat{X}}(1)$. Hence $\mathcal{O}_{\hat{X}}(1)$ and $\mathcal{O}_{\hat{X}}(k)$ have the same nilradical as ideal sheaves, and

$$F_{\mathrm{red}}=\mathrm{Supp}(\mathcal{O}_{\hat{X}}/\mathcal{O}_{\hat{X}}(1))=\mathrm{Supp}(\mathcal{O}_{\hat{X}}/\mathcal{O}_{\hat{X}}(k))$$

which is the reduced exceptional locus of σ. On the other hand, it is easy to see that $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ is an integral domain, so F is an integral scheme.

For k sufficiently divisible, we have that $\mathcal{O}_{\hat{X}}(k)$ is invertible and $\sigma_*\mathcal{O}_{\hat{X}}(mk)=\mathfrak{a}_{mk}(v_*)$ for any $m\in\mathbb{Z}_{\geqslant 0}$. Thus $\mathcal{O}_{\hat{X}}(k)=\mathcal{O}_{\hat{X}}(-lF)$ for some positive integer l (so F is $\mathbb{Q}$-Cartier). Then for any $m\in\mathbb{Z}_{\geqslant 0}$ we have the following equivalences:

$$v_*(f)\geqslant m\Leftrightarrow v_*(f^k)\geqslant mk\Leftrightarrow f^k\in\mathfrak{a}_{mk}(v_*)\Leftrightarrow\sigma^*f^k\in\mathcal{O}_{\hat{X}}(mk)$$
$$\Leftrightarrow\mathrm{ord}_F(\sigma^*f^k)\geqslant ml\Leftrightarrow\mathrm{ord}_F(f)\geqslant\frac{ml}{k}.$$

Assume $v_*(f)=m$. If $\mathrm{ord}_F(f)>ml/k$, then we have $\mathrm{ord}_F(f)\geqslant(ml+1)/k$. Hence $\mathrm{ord}_F(f^l)\geqslant(ml+1)l/k$ which implies that $v_*(f^l)\geqslant ml+1$, a contradiction! Hence we have $\mathrm{ord}_F=(l/k)v_*$. Therefore, $v_*=\mathrm{ord}_F$ since both v_* and ord_F are divisorial. □

Lemma 35. *Let $(R,\mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{p},\mathfrak{q}$ be two prime ideals in R satisfying that $\mathfrak{p}+\mathfrak{q}=\mathfrak{m}$, $\mathfrak{p}\cap\mathfrak{q}=\mathfrak{p}\mathfrak{q}$ and $R/\mathfrak{q}$ is a DVR. Then $\mathfrak{p}$ is principal.*

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Proof. Let $x+\mathfrak{q}$ be a uniformizer of $R/\mathfrak{q}$. Since $\mathfrak{m}=\mathfrak{p}+\mathfrak{q}$, we may choose x so that $x\in\mathfrak{p}$. Hence we have $(x)+\mathfrak{q}=\mathfrak{m}$. As a result,

$$\mathfrak{p}=(x)+\mathfrak{p}\cap\mathfrak{q}=(x)+\mathfrak{p}\mathfrak{q}.$$

So $\mathfrak{p}=(x)$ by Nakayama lemma. □

5.2 Applications

The following theorem improves Fujita's result on the equality case in [Fuj15, Theorem 5.1].

Theorem 36. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety of dimension n. If $((-K_X)^n)\geqslant(n+1)^n$, then $X\cong\mathbb{P}^n$.*

Proof 1. Notice that $((-K_X)^n)\leqslant(n+1)^n$ by [Fuj15, Corollary 1.3]. Thus we have $((-K_X)^n)=(n+1)^n$. Let $p\in X$ be a smooth point. From [Fuj15, Proof of Theorem 5.1] or Remark 17, we see that $\epsilon_p(-K_X)=n+1$. Then the theorem is a consequence of the forthcoming paper [LZ16] (joint with Ziquan Zhuang), where we show that if an n-dimensional $\mathbb{Q}$-Fano variety X satisfies $\epsilon_p(-K_X)>n$ for some smooth point $p\in X$, then $X\cong\mathbb{P}^n$. □

Proof 2. We follow the strategy and notation of the proof of Lemma 33. Let $v_*:=\mathrm{ord}_p$ for a smooth point $p\in X$. As argued in the first proof, the assumptions of Lemma 33 are fulfilled except that X is only Ding-semistable rather than K-polystable. Nevertheless, we still have a semi $\mathbb{Q}$-test configuration $(\mathcal{X},\mathcal{L})$ of $(X,-K_X)$ such that $\mathcal{L}^\perp=\hat{X}$ and $\mathrm{CM}(\mathcal{X},\mathcal{L})=0$. Let $(\mathcal{Y},\mathcal{M})$ be the ample model of $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$, with $h:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{L}=h^*\mathcal{M}$. We will show that $\mathcal{Y}_0\cong\mathbb{P}^n$.

Following the proof of Lemma 33, we know that $h|_E:E\to\mathcal{Y}_0$ is a finite birational morphism. Since $v_*=\mathrm{ord}_p$ and $p\in X$ is a smooth point, we have that $E\cong\mathbb{P}^n$. Therefore, E is the normalization of $\mathcal{Y}_0$. Consider the short exact sequence

$$0\to\mathcal{O}_{\mathcal{X}}(-E)\to\mathcal{O}_{\mathcal{X}}\to\mathcal{O}_E\to 0.$$

By taking h_*, we get a long exact sequence

$$0\to h_*\mathcal{O}_{\mathcal{X}}(-E)\to\mathcal{O}_{\mathcal{Y}}\to h_*\mathcal{O}_E\to R^1h_*\mathcal{O}_{\mathcal{X}}(-E)\to\cdots. \tag{5.1}$$

Since $\mathcal{X}_0=\hat{X}+E$, we have that $h_*\mathcal{O}_{\mathcal{X}}(-E)=\mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_0)\otimes h_*\mathcal{O}_{\mathcal{X}}(\hat{X})$. Notice that $\hat{X}$ is h-exceptional, hence $h_*\mathcal{O}_{\mathcal{X}}(\hat{X})=\mathcal{O}_{\mathcal{Y}}$. As a result, $h_*\mathcal{O}_{\mathcal{X}}(-E)\cong\mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_0)$. On the other hand, we have

$$h^*\mathcal{M}=\mathcal{L}=g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1})-(n+1)E=-K_{\mathcal{X}/\mathbb{A}^1}-E.$$

So $-E=K_{\mathcal{X}/\mathbb{A}^1}+h^*\mathcal{M}\sim_{\mathbb{Q},h}K_{\mathcal{X}}$. Since $\mathcal{X}$ has klt singularities, we have that $R^1h_*\mathcal{O}_{\mathcal{X}}(-E)=0$ by the generalized Kodaira vanishing theorem [Kol95, Theorem 10.19.4]. Thus the exact sequence (5.1) yields that $h_*\mathcal{O}_E=\mathcal{O}_{\mathcal{Y}_0}$, which implies $\mathcal{Y}_0\cong E\cong\mathbb{P}^n$.

Since $\mathbb{P}^n$ is rigid under smooth deformation (cf. [Kol96, Exercise V.1.11.12.2]), we conclude that $\mathcal{Y}_t\cong\mathbb{P}^n$ for general $t\in\mathbb{A}^1\setminus\{0\}$, hence $X\cong\mathcal{X}_t\cong\mathcal{Y}_t\cong\mathbb{P}^n$. □

In the following theorem, we show the equality case of Theorem 3.

Theorem 37. *Let X be a Kähler–Einstein $\mathbb{Q}$-Fano variety. Let $p\in X$ be a closed point. Suppose (X,p) is a quotient singularity with local analytic model $\mathbb{C}^n/G$, where $G\subset\mathrm{GL}(n,\mathbb{C})$ acts freely in codimension 1. Then $((-K_X)^n)=(n+1)^n/|G|$ if and only if $|G\cap\mathbb{G}_m|=1$ and $X\cong\mathbb{P}^n/G$.*

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Proof. For the 'if' part, we may assume that $G \subset U(n)$. Hence G preserves the Fubini–Study metric ω_{FS} on $\mathbb{P}^n$. Since $|G \cap \mathbb{G}_m| = 1$, the G-action on $\mathbb{P}^n$ is free in codimension 1. This implies that the quotient metric of ω_{FS} on $\mathbb{P}^n/G$ is Kähler–Einstein and $((-K_{\mathbb{P}^n/G})^n) = ((-K_{\mathbb{P}^n})^n)/|G| = (n+1)^n/|G|$.

For the 'only if' part, let $(Y, o) := (\mathbb{C}^n/G, 0)$ be the local analytic model of (X, p). Let $d := |G \cap \mathbb{G}_m|$. Following the proof of Theorem 23, we have a partial resolution $h : \hat{Y} \to Y$ of Y that is the quotient of the blow up $g : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$. Since $\hat{Y}$ is the quotient of $\widehat{\mathbb{C}^n}$, it has klt singularities. Denote by F the exceptional divisor of h, then we have $F \cong \mathbb{P}^{n-1}/G$. Let $u_* := \mathrm{ord}_F$ be the divisorial on Y centered at o, then we have $\mathrm{gr}_{u_*}\mathcal{O}_{Y,o} \cong \mathbb{C}[x_1, \ldots, x_n]^G$ is a finitely generated $\mathbb{C}$-algebra. (Here we assume $\deg x_i = 1/d$ so that the grading is preserved under the isomorphism.) Theorem 23 also implies $\widehat{\mathrm{vol}}(u_*) = n^n/|G|$.

Since $\widehat{\mathcal{O}_{X,p}} \cong \widehat{\mathcal{O}_{Y,o}}$, the divisorial valuation u_* induces a divisorial valuation v_* on X centered at p as in the proof of Theorem 23. Thus $\widehat{\mathrm{vol}}(v_*) = \widehat{\mathrm{vol}}(u_*)$, $\mathrm{gr}_{v_*}(\mathcal{O}_{X,p}) \cong \mathrm{gr}_{u_*}(\mathcal{O}_{Y,o})$ is a finitely generated $\mathbb{C}$-algebra, $\hat{X} := \mathbf{Proj}_X \bigoplus_{m \geqslant 0} \mathfrak{a}_m(v_*)$ has klt singularities, and

$$((-K_X)^n) = \frac{(n+1)^n}{|G|} = \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v_*).$$

By [Ber16] we have that X is K-polystable. Hence applying Lemma 33 yields

$$X \cong \mathbf{Proj}\ \mathrm{gr}_{v_*}\mathcal{O}_{X,p}[x] \cong \mathbf{Proj}\ \mathbb{C}[x_1, \ldots, x_n]^G[x],$$

where $\deg x_i = 1/d$ and $\deg x = 1$. Denote $y := x^{1/d}$, then $\mathbb{C}[x_1, \ldots, x_n]^G[x]$ is the Veronese subalgebra of $\mathbb{C}[x_1, \ldots, x_n]^G[y]$ where $\deg x_i = \deg y = 1/d$. As a result,

$$X \cong \mathbf{Proj}\ \mathbb{C}[x_1, \ldots, x_n]^G[y] = \mathbb{P}^n/G.$$

Denote by H_∞ the hyperplane at infinity in $\mathbb{P}^n$. Let D be the prime divisor in X corresponding to H_∞/G in $\mathbb{P}^n/G$. It is clear that $(X, (1 - 1/d)D)$ is an orbifold as a global quotient of $\mathbb{P}^n$. Denote by $\pi : \mathbb{P}^n \to X$ the quotient map, then we have

$$\pi^*\left(K_X + \left(1 - \frac{1}{d}\right)D\right) = K_{\mathbb{P}^n}, \quad \pi^*D = dH_\infty.$$

Hence $\pi^*(-K_X) = -K_{\mathbb{P}^n} + (d-1)H_\infty = \mathcal{O}_{\mathbb{P}^n}(n+d)$. This implies $((-K_X)^n) = (n+d)^n/|G|$, so $d = 1$. □

Remark 38. (i) The restriction on G in Theorem 37 can be dropped in its logarithmic version as follows: suppose (X, D) is a conical Kähler–Einstein log Fano pair, $p \notin \mathrm{Supp}(D)$, (X, p) is analytically isomorphic to $(\mathbb{C}^n/G, 0)$ and $((-K_X - D)^n) = (n+1)^n/|G|$, then $(X, D) \cong (\mathbb{P}^n/G, (1 - 1/d)H_\infty/G)$ where $d = |G \cap \mathbb{G}_m|$.

(ii) The K-polystable condition in Lemma 33 and Kähler–Einstein condition in Theorem 3 cannot be dropped since any cubic surface in $\mathbb{P}^3$ with only one or two $\mathbb{A}_2$ singularities is K-semistable but not K-polystable (hence not Kähler–Einstein) according to [OSS16, § 4.2], but all global quotients of $\mathbb{P}^2$ are Kähler–Einstein (hence K-polystable).

Corollary 39 (= Corollary 6). *Let X be a Kähler–Einstein log Del Pezzo surface with at most Du Val singularities.*

(i) *If $((-K_X)^2) = 1$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$ or $\mathbb{D}_4$.*

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(ii) *If* $((-K_X)^2) = 2$, *then* X *has at most singularities of type* $\mathbb{A}_1$, $\mathbb{A}_2$ *or* $\mathbb{A}_3$.
(iii) *If* $((-K_X)^2) = 3$, *then* X *has at most singularities of type* $\mathbb{A}_1$ *or* $\mathbb{A}_2$.
(iv) *If* $((-K_X)^2) = 4$, *then* X *has at most singularities of type* $\mathbb{A}_1$.
(v) *If* $((-K_X)^2) \geqslant 5$, *then* X *is smooth.*

Proof. Let $\mathbb{C}^2/G$ be the local analytic model of (X, p) for any closed point $p \in X$. By Corollary 4, we know that $|G| \leqslant 9/((-K_X)^2)$.

Recall that the order of orbifold group at an $\mathbb{A}_k$ singularity is $k+1$, a $\mathbb{D}_k$ singularity is $4(k-2)$, an $\mathbb{E}_6$ singularity is 24, an $\mathbb{E}_7$ singularity is 48, and an $\mathbb{E}_8$ singularity is 120. Hence the inequality $|G| \leqslant 9/((-K_X)^2)$ implies (ii)–(v). For part (i), the same argument yields that X has at most singularities type of $\mathbb{A}_k$ with $k \leqslant 8$, or of type $\mathbb{D}_4$. Hence we only need to rule out $\mathbb{A}_8$ cases.

Assume to the contrary that (X, p) is of type $\mathbb{A}_8$. Since $((-K_X)^2) = 1 = 9/|G|$, Theorem 37 implies that $X \cong \mathbb{P}^2/G$, where $G = \mathbb{Z}/9\mathbb{Z}$ acts on $\mathbb{P}^2$ as follows:

$$[x, y, z] \mapsto [\zeta^i x, \zeta^{-i} y, z] \quad \text{for } i \in \mathbb{Z}/9\mathbb{Z},$$

where $\zeta := e^{2\pi i/9}$ is the ninth root of unity. The point p is the quotient of $[0, 0, 1]$ under this action, which is of type $\frac{1}{9}(1, -1)$ as a cyclic quotient singularity. However, the quotient singularities at $[1, 0, 0]$ and $[0, 1, 0]$ are both of type $\frac{1}{9}(1, 2)$ which are not Du Val, contradiction! □

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