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Eigenvalue Ratios of Non-Negatively Curved Graphs

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We derive an optimal eigenvalue ratio estimate for finite weighted graphs satisfying the curvature-dimension inequality $CD(0,\infty)$. This estimate is independent of the size of the graph and provides a general method to obtain higher-order spectral estimates. The operation of taking Cartesian products is shown to be an efficient way for constructing new weighted graphs satisfying $CD(0,\infty)$. We also discuss a higher-order Cheeger constant-ratio estimate and related topics about expanders.

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1. Introduction

1.1. Some historical background

Exploring the influence of eigenvalues on graph structures is one of the central topics in spectral graph theory; see *e.g.* [1, 9, 10, 11, 31]. In this area, the first non-zero Laplacian eigenvalue and the Cheeger constant play a fundamentally important role, and their close relations have found tremendous applications in both theoretical and applied fields, such as the study of expander graphs.

Let $G=(V,E)$ be a non-oriented and connected graph with vertex set V and edge set E. For simplicity, we consider in this subsection the special case of the *normalized* Laplacian $\Delta = D^{-1}A - \mathrm{Id}$ (where D is a diagonal matrix containing the vertex degrees and A is the adjacency matrix). *Cheeger's isoperimetric constant* is defined by

$$h(G) = \inf_{\emptyset \neq S \subset V} \frac{|E(S,V\backslash S)|}{\min\{\mu(S),\mu(V\backslash S)\}}, \tag{1.1}$$

where $E(S_1,S_2)$ is the set of all edges connecting a vertex in S_1 with a vertex in S_2 and $\mu(S) = \sum_{x\in S} d_x$, d_x equals the vertex degree of $x \in V$. The classical *Cheeger inequality* states the following relation between $h(G)$ and the first non-zero eigenvalue $\lambda_2(G) > 0$ of $-\Delta$:

$$\frac{h^2(G)}{4} \leqslant \lambda_2(G) \leqslant 2h(G).$$

Recently there have been two major developments in this area:

- higher-order Cheeger constants $h_k(G)$ and higher-order Cheeger inequalities

$$\frac{Ch_k^2(G)}{k^4} \leqslant \lambda_k(G) \leqslant 2h_k(G) \tag{1.2}$$

  with a universal constant $C > 0$, by Miclo [29] and Lee, Oveis Gharan and Trevisan [23], where $h_2(G)$ agrees with the classical Cheeger constant $h(G)$,
- an improved Cheeger inequality

$$h_2(G) \leqslant Ck \frac{\lambda_2(G)}{\sqrt{\lambda_k(G)}} \tag{1.3}$$

  with a universal constant $C > 0$, by Kwok, Lau, Lee, Oveis Gharan and Trevisan [21].

Remark 1.1. When the gap between λ_2 and λ_k is large, (1.3) gives a lower bound of $\lambda_2(G)$ linear in $h_2(G)$. Another such kind of result is due to Miclo [28], which asserts that

$$\frac{h_2(G)}{\operatorname{diam}(G)} \leqslant \lambda_2(G), \tag{1.4}$$

where $\operatorname{diam}(G)$ denotes the diameter of the graph G.

In the manifold context, another classical spectral result is Buser's inequality [6], providing, under the additional assumption of non-negative Ricci curvature, an estimate for λ_1 from above by h^2, which depends on the dimension of the manifold. Later, a dimension-independent Buser-type estimate was proved by Ledoux [22] in the manifold setting.

To formulate such a result in the graph-theoretical context, a suitable curvature notion for graphs is required. Klartag, Kozma, Ralli and Tetali [20] proved such a Buser-type inequality for finite graphs satisfying the *curvature-dimension condition* $CD(0,\infty)$:

$$\lambda_2(G) \leqslant Cd_G h_2^2(G), \tag{1.5}$$

with a universal constant $C > 0$, where d_G denotes the maximal vertex degree of G.

In this article, we combine (1.3) and (1.5) (in the more general setting of weighted graphs) to derive an eigenvalue ratio result and discuss its optimality. This result has various applications such as higher-order Buser estimates, a higher-order eigenvalue-diameter estimate, and higher-order Cheeger constant-ratio estimates. We also provide a discussion of the underlying curvature notion.

1.2. The curvature-dimension condition $CD(K,n)$

This notion goes back to Bakry and Émery and was studied by Elworthy [14], Schmuckenschläger [32] and Lin and Yau [24] on graphs. Since all results in this paper require such a curvature-dimension condition, we now provide a motivation and brief introduction for this notion.

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In the setting of an n-dimensional Riemannian manifold (M,g), Bochner's formula implies the following inequality relating Ricci curvature and the Laplacian:

$$\frac{1}{2}\Delta\|\nabla f\|^2 - \langle \nabla f, \nabla \Delta f\rangle \geqslant \frac{1}{n}(\Delta f)^2 + \mathrm{Ric}(\nabla f).$$

Assuming $\mathrm{Ric} \geqslant K$, this inequality can be transformed with the Bakry–Émery Γ-calculus, defined by

$$2\Gamma(f,g) = \Delta(fg) - f(\Delta g) - (\Delta f)g = 2\langle \nabla f, \nabla g\rangle \tag{1.6}$$

and

$$2\Gamma_2(f,g) = \Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g), \tag{1.7}$$

into

$$\Gamma_2(f) \geqslant \frac{1}{n}(\Delta f)^2 + K\Gamma(f), \tag{1.8}$$

where $\Gamma(f) = \Gamma(f,f)$ and $\Gamma_2(f) = \Gamma_2(f,f)$. Note that (1.8) involves a curvature parameter K and a dimension parameter n. This inequality also makes sense in the graph-theoretical setting, and if it is satisfied for all functions f, we say that the graph satisfies the *curvature-dimension inequality* $CD(K,n)$. In this paper we are in particular concerned with graphs satisfying the condition $CD(0,\infty)$. This condition holds for all abelian Cayley graphs (see [24, Proposition 1.6 (1)], [20, Theorem 2.3]) but not for trees of degree $\geqslant 3$ (see *e.g.* [19, Remark 16]). As a general guideline, $CD(0,\infty)$ requires that every vertex is contained in sufficiently many short cycles, which can be understood as a kind of local connectivity.

1.3. General setting

Our results are given in the more general setting of weighted graphs (G,μ), where $G = (V,E,w)$ is an undirected weighted finite connected graph and V and E are the sets of vertices and edges, respectively. Edge weights on G are assigned via the symmetric function $w : V \times V \to \mathbb{R}_{\geqslant 0}$ with $w_{xy} = w_{yx} > 0$ if and only if $x \sim y$. We say the graph G is *unweighted* if $w_{xy} = 1$ for any $x \sim y$; for short, $w = \mathbf{1}_E$. Moreover, we assign a positive measure on the vertex set V via the function $\mu : V \to \mathbb{R}_{>0}$. Let $d_x := \sum_{y, y\sim x} w_{xy}$ be the degree of a vertex x and let $d_G := \max_{x\in V} d_x$ be the maximal degree of the graph G. For any function $f : V \to \mathbb{R}$ and any vertex $x \in V$, the associated Laplacian Δ is defined as

$$\Delta f(x) := \frac{1}{\mu(x)} \sum_{y, y\sim x} w_{xy}(f(y) - f(x)).$$

This operator is called μ-Laplacian in [5].

The normalized and non-normalized Laplacians are contained in this general setting as the following special cases.

- Non-normalized Laplacian: if $\mu(x) = 1$ for all $x \in V$ ($\mu = \mathbf{1}_V$ for short).
- Normalized Laplacian: if $\mu(x) = d_x$ for all $x \in V$ ($\mu = \mathbf{d}_V$ for short).

Note that the curvature condition $CD(0,\infty)$ of a graph (G,w,μ) depends on the choice of Laplacian via the formulas (1.6), (1.7) and (1.8).

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The following two quantities, D_G^{non} and D_G^{nor}, appear naturally in our arguments:

$$D_G^{non} := \max_{x\in V} \frac{\sum_{y,y\sim x} w_{xy}}{\mu(x)} \quad \text{and} \quad D_G^{nor} := \max_{x\in V} \max_{y,y\sim x} \frac{\mu(x)}{w_{xy}}.$$

Observe that on an unweighted graph, in either of the cases $\mu = \mathbf{1}_V$ or $\mu = \mathbf{d}_V$ we always have $D_G^{non} D_G^{nor} = d_G$.

We order the eigenvalues of Δ with multiplicities by

$$0 = \lambda_1(G,\mu) < \lambda_2(G,\mu) \leqslant \cdots \leqslant \lambda_{|V|}(G,\mu) \leqslant 2D_G^{non},$$

where $\lambda \geqslant 0$ is an eigenvalue if there exists a non-zero solution of $\Delta f + \lambda f = 0$.

1.4. Results

Combining the improved Cheeger inequality and Buser's inequality leads to the following eigenvalue ratio.

Theorem 1.2. *For any finite graph (G,μ) satisfying $CD(0,\infty)$ and any natural number $k \geqslant 2$, we have*

$$\lambda_k(G,\mu) \leqslant \left(\frac{20\sqrt{2}e}{e-1}\right)^2 D_G^{non} D_G^{nor} k^2 \lambda_2(G,\mu). \tag{1.9}$$

It is natural to ask about the optimality of this result: Are the curvature condition and the dependence on the $D_G^{non} D_G^{nor}$ necessary and can the quadratic term k^2 in (1.9) be improved?

- The unweighted dumbbell graph in Example 3.6 provides a counterexample to (1.9) if we drop the curvature condition $CD(0,\infty)$.
- Weighted triangles and tetrahedra in Examples 3.4 and 3.5 show that the factor $D_G^{non} D_G^{nor}$ cannot be dropped.
- Unweighted cycles in Example 3.3 show that the quadratic exponent in (1.9) is optimal.

Another natural question is: How restrictive is the $CD(0,\infty)$ condition? It is possible to produce many new examples from given graphs satisfying $CD(0,\infty)$ by taking Cartesian products due to the following fundamental result.

Theorem 1.3. *If $(G_1, \mathbf{1}_{V_1})$ and $(G_2, \mathbf{1}_{V_2})$ satisfy $CD(K_1,n_1)$ and $CD(K_2,n_2)$ respectively, then $(G_1 \times G_2, \mathbf{1}_{V_1\times V_2})$ satisfies $CD(K_1 \wedge K_2, n_1 + n_2)$.*

Here we used the notion $K_1 \wedge K_2 := \min\{K_1, K_2\}$. The above estimate is optimal at least for the Cartesian product of a graph G with itself (Remark 2.8). Theorem 1.3 can be extended to include the case of regular graphs with normalized Laplacian operators (Remark 2.7). In particular, the property of satisfying $CD(0,\infty)$ is preserved when taking Cartesian products in many cases.

Theorem 1.2 can be used as a general source to derive various interesting higher-order estimates between geometric invariants and spectra. Of particular interest are the higher-order Cheeger constants $h_k(G,\mu)$ defined as follows. For a given (G,μ), the expansion $\phi_{w,\mu}(S)$ of a non-empty

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subset S of V is given by

$$\phi_{w,\mu}(S) := \frac{|E(S, V \setminus S)|_w}{\mu(S)},$$

where $|E(S,V\setminus S)|_w := \sum_{x\sim y, x\in S, y\notin S} w_{xy}$ and $\mu(S) := \sum_{x\in S}\mu(x)$.

Definition 1.4 (higher-order Cheeger constants [23, 29]). For a natural number k, the kth Cheeger constant of (G,μ) is defined as

$$h_k(G,\mu) := \min_{S_1,\dots,S_k} \max_{1\leqslant i\leqslant k} \phi_{w,\mu}(S_i),$$

where the minimum is taken over all collections of k non-empty, mutually disjoint subsets $S_1,\dots,S_k$, that is, all k-subpartitions of V.

Note that $h_2(G,\mu)$ coincides with the classical Cheeger constant, and $h_k(G,\mu) \leqslant h_{k+1}(G,\mu)$. We use Theorem 1.2 to derive the following higher-order Buser inequality.

Corollary 1.5. *For any graph (G,μ) satisfying $CD(0,\infty)$ and any natural number k, we have*

$$h_k(G,\mu) \geqslant h_2(G,\mu) \geqslant \frac{(e-1)^2}{40\sqrt{2}e^2}\frac{1}{D_G^{nor}\sqrt{D_G^{non}}}\frac{\sqrt{\lambda_k(G,\mu)}}{k}.$$

Combining the inequalities of Alon and Milman [1] and Theorem 1.2 leads to the following higher-order eigenvalue-diameter estimate.

Corollary 1.6. *Let $(G,\mathbf{1}_V)$ be an unweighted finite graph satisfying $CD(0,\infty)$. Then, for any $k\geqslant 2$ we have*

$$\operatorname{diam}(G) \leqslant \frac{80e}{e-1} d_G \log_2|V| \frac{k}{\sqrt{\lambda_k(G,\mu)}}.$$

This result compares nicely with the celebrated Cheng estimate ([8, Corollary 2.2])

$$\operatorname{diam}(M) \leqslant \sqrt{2n(n+4)}\frac{k}{\sqrt{\lambda_k(M,g)}} \tag{1.10}$$

for compact Riemannian manifolds (M,g) with non-negative Ricci curvature.

In combination with the higher-order Cheeger inequalities in [23], Theorem 1.2 implies the following higher-order Cheeger constant-ratio estimate.

Corollary 1.7. *There exists a universal constant C such that for any graph (G,μ) satisfying $CD(0,\infty)$ and any natural number $k\geqslant 2$, we have*

$$h_k(G,\mu) \leqslant C D_G^{non} D_G^{nor} k\sqrt{\log k}\, h_2(G,\mu). \tag{1.11}$$

Higher-order Cheeger constants lead naturally to the notion of *k-way expanders* introduced by Tanaka [33] and Mimura [30] (where 2-way expander families coincide with the classical

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families of expanders). The condition of being a k-way expander family is strictly weaker than the property of being a classical expander family (see [30, p. 2525]). A consequence of Corollary 1.7 is the fact that the concepts of k-way expanders for all $k \geqslant 2$ are equivalent in the class of all graphs satisfying $CD(0,\infty)$. This can be viewed as an analogue (for the $CD(0,\infty)$-class) to Mimura's result [30, Corollary 1.5] for the class of all vertex-transitive graphs.

1.5. Organization of the paper

In Section 2, we discuss in detail the curvature-dimension inequality in the graph setting, introduce two interesting examples for later use concerning the optimality of Theorem 1.2, and provide a proof of Theorem 1.3. In Section 3, we derive the eigenvalue ratio estimate, discuss its optimality with the help of examples, and present applications. In Section 4, we discuss a higher-order Cheeger constant-ratio estimate and related topics about multi-way expanders. Finally, in the Appendix, we give more details about the curvature-dimension inequality calculations in some examples and also a self-contained proof of Buser's inequality for graphs satisfying $CD(0,\infty)$.

2. Information for a better understanding of curvature

The curvature-dimension inequality (CD-inequality for short) was introduced by Bakry and Émery [4] as a substitute for the lower Ricci curvature bound of the underlying space. It was studied for graphs by Elworthy [14], Schmuckenschläger [32] and Lin and Yau [24]; see also [12, 19]. The operators Γ and Γ_2 are defined iteratively as follows.

Definition 2.1. For any two functions $f,g: V\to\mathbb{R}$, we define

$$\Gamma(f,g) := \frac{1}{2}\{\Delta(fg) - f\Delta g - g\Delta f\}, \tag{2.1}$$

and

$$\Gamma_2(f,g) := \frac{1}{2}\{\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f)\}. \tag{2.2}$$

We also write $\Gamma(f) := \Gamma(f,f)$ and $\Gamma_2(f) := \Gamma_2(f,f)$ for short. In particular, by the definition above we have for any $x\in V$ and any f,g

$$\Gamma(f,g)(x) = \frac{1}{2\mu(x)}\sum_{y,y\sim x} w_{xy}(f(y)-f(x))(g(y)-g(x)). \tag{2.3}$$

A useful fact is the summation by parts formula

$$\sum_{x\in V}\mu(x)\Gamma(f,g)(x) = -\sum_{x\in V}\mu(x)f(x)\Delta g(x), \tag{2.4}$$

and also

$$\Gamma(f,g) \leqslant \sqrt{\Gamma(f)}\sqrt{\Gamma(g)}. \tag{2.5}$$

Rewriting (2.1) provides the chain rule

$$\Delta(f^2) = 2\Gamma(f) + 2f\Delta(f). \tag{2.6}$$

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Definition 2.2. Let $K \in \mathbb{R}$ and $n \in \mathbb{R}_+$. We say that (G,μ) satisfies the CD-inequality $CD(K,n)$ if, for any functions f and any vertex x, the following inequality holds:

$$\Gamma_2(f)(x) \geqslant \frac{1}{n}(\Delta f(x))^2 + K\Gamma(f)(x). \tag{2.7}$$

In particular, we say that (G,μ) satisfies $CD(0,\infty)$ if for any functions f we have $\Gamma_2(f) \geqslant 0$.

In the following subsection we present some illustrative examples of weighted graphs and their curvature. Section 2.2 describes a method to construct many more examples satisfying $CD(0,\infty)$ from given ones via Cartesian products and provides a proof of Theorem 1.3.

2.1. Examples of graphs satisfying $CD(0,\infty)$

We will be mainly concerned with the class of graphs satisfying $CD(0,\infty)$. For the purpose of illustration and for later use concerning the optimality of Theorem 1.2, we present some simple examples. Whereas explicit curvature calculations are given in Appendix A, we first mention some basic principles used in our curvature calculations.

From (2.2), we see that Γ_2 is a symmetric bilinear form. At every vertex $x \in V$, we can write $\Gamma_2(f,g)(x) = f^\top \Gamma_2(x) g$, where $f,g \in \mathbb{R}^V$ on the right-hand side denotes the (column) vector representation of the functions f and g. Let $B_2(x) := \{y \in V : \mathrm{dist}(y,x) \leqslant 2\}$, where dist stands for the usual shortest-path metric on V. Then $\Gamma_2(x)$ is a symmetric matrix, which is non-trivial only on a submatrix of size $|B_2(x)| \times |B_2(x)|$, which we again denote by $\Gamma_2(x)$, for simplicity. A graph (G,μ) satisfies $CD(0,\infty)$ if and only if $\Gamma_2(x)$ is positive semidefinite at every vertex $x \in V$. Observe that the entries of each row of $\Gamma_2(x)$ sum up to zero since $\Gamma_2(x)\mathbf{c} = 0$ for any constant vector $\mathbf{c}$. In particular, if all the off-diagonal entries are non-positive, then the matrix $\Gamma_2(x)$ is diagonally dominant and hence positive semidefinite.

Example 2.3. Consider the triangle graph $\triangle_{xyz}$ with positive edge weights a,b,c, as shown in Figure 1(a). Assign a measure μ to the vertices such that $\mu(x) := C, \mu(y) := B, \mu(z) := A$.

- *Normalized case*: $A = b+c, B = a+c, C = a+b$. Then $(\triangle_{xyz},\mu)$ satisfies $CD(0,\infty)$.
- *Non-normalized case*: $A = B = C = 1$. Then $(\triangle_{xyz},\mu)$ does not always satisfy $CD(0,\infty)$. If in particular $a = c$, it satisfies $CD(0,\infty)$. But when $a = 1, c = 1/b$, it does not satisfy $CD(0,\infty)$ if b is large/small enough. In fact, when $b \geqslant 5.01$ or $b \leqslant 0.12$, the symmetric curvature matrix $\Gamma_2(x)$ has a negative eigenvalue.

This example illustrates the general observation that positivity of the non-normalized Bakry–Émery curvature at a vertex is more sensitive to large differences in the weights of the adjacent edges than normalized Bakry–Émery curvature.

Example 2.4. Consider the tetrahedron graph T_4 with positive edge weights a,b,c, as shown in Figure 1(b). Observe that this graph is regular, that is, $d_{x_i} = a+b+c$ is a constant for every i. Assign a measure μ on the vertices such that $\mu(x_i) = A$ for all i, where A is a positive constant. (Note that this includes both the cases of normalized and non-normalized Laplacians.) Then (T_4,μ) always satisfies $CD(0,\infty)$.

Details about the curvature matrix Γ_2 of the triangle graph and the tetrahedron graph are given in Appendix A. For the normalized case, the curvature of unweighted triangle graphs was

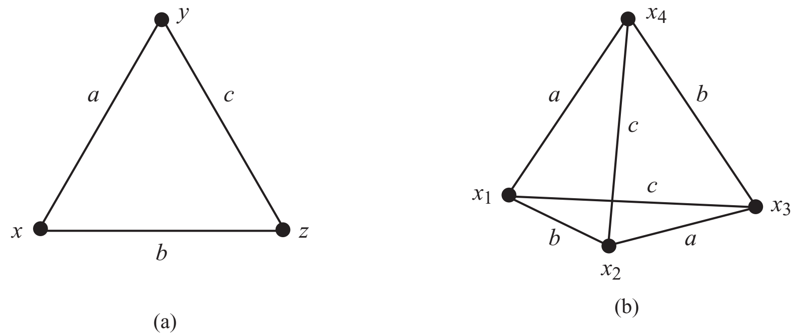

Figure 1. (a) Triangle. (b) Tetrahedron.

calculated in [24, Proposition 1.6], and the curvature of general unweighted complete graphs was calculated in [19, Proposition 3].

In fact, the tetrahedron graph in Figure 1(b) belongs to a large class of graphs called Ricci flat graphs with consistent edge weights. The concept of a Ricci flat graph was introduced by Chung and Yau [13] and that of consistent edge weights was further introduced in Bauer, Horn, Lin, Lippner, Mangoubi and Yau [5]. We refer the reader to [5, 13] for the precise definitions. Every graph in this class is a regular graph (in fact both its unweighted and weighted degree are constant) and satisfies $CD(0,\infty)$ if we assign a measure μ such that $\mu(x) = A$ for all vertices x (see [13, 24] for the unweighted case; the weighted case follows from the same calculations). In particular, every abelian Cayley graph is Ricci flat and hence satisfies $CD(0,\infty)$.

2.2. CD-inequalities of Cartesian product graphs

In this subsection we discuss a method for constructing new graphs satisfying certain CD-inequalities from known examples, that is, taking the Cartesian product.

Given two (possibly infinite) graphs $G_1 = (V_1, E_1, w)$ and $G_2 = (V_2, E_2, \overline{w})$, their Cartesian product $G_1 \times G_2 = (V_1 \times V_2, E_{12}, w^{12})$ is a weighted graph with vertex set $V_1 \times V_2$ and edge set E_{12} given by the following rule. Two vertices $(x_1, y_1), (x_2, y_2) \in V_1 \times V_2$ are connected by an edge in E_{12} if

$$x_1 = x_2, y_1 \sim y_2 \text{ in } E_2 \quad \text{or} \quad x_1 \sim x_2 \text{ in } E_1, y_1 = y_2.$$

In the first case above we chose the edge weight to be $\overline{w}_{y_1 y_2}$ and in the second case $w_{x_1 x_2}$.

Recall the following result from the Introduction, which we will prove in this subsection.

Theorem 1.3. *If* $(G_1, \mathbf{1}_{V_1})$ *and* $(G_2, \mathbf{1}_{V_2})$ *satisfy* $CD(K_1, n_1)$ *and* $CD(K_2, n_2)$ *respectively, then* $(G_1 \times G_2, \mathbf{1}_{V_1 \times V_2})$ *satisfies* $CD(K_1 \wedge K_2, n_1 + n_2)$.

Let $f : V_1 \times V_2 \to \mathbb{R}$ be a function on the product graph. For fixed $y \in V_2$, we will write $f_y(\cdot) := f(\cdot, y)$ as a function on V_1. Similarly, $f^x(\cdot) := f(x, \cdot)$. The following lemma is crucial for the proof.

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Lemma 2.5. *For any function $f : V_1 \times V_2 \to \mathbb{R}$ and any $(x,y) \in V_1 \times V_2$, we have*

$$\Gamma_2(f)(x,y) \geqslant \Gamma_2(f_y)(x) + \Gamma_2(f^x)(y), \tag{2.8}$$

where the operators Γ_2 are understood to be on different graphs according to the functions they are acting on.

Proof. For simplicity, we will let x_i denote a neighbour of $x \in V_1$, and write $w_i := w_{xx_i}$ for short. Similar notions are used for $y \in V_2$ and $\overline{w}$.

Recall that $2\Gamma_2(f)(x,y) = \Delta\Gamma(f)(x,y) - 2\Gamma(f,\Delta f)(x,y)$. By definition, we have

$$\Delta\Gamma(f)(x,y) = \sum_{x_i \sim x} w_i(\Gamma(f)(x_i,y) - \Gamma(f)(x,y)) + \sum_{y_k \sim y} \overline{w}_k(\Gamma(f)(x,y_k) - \Gamma(f)(x,y)) := L_1 + L_2.$$

For the first term L_1, we calculate

$$\begin{aligned} L_1 &= \sum_{x_i \sim x} w_i[\Gamma(f_y)(x_i) + \Gamma(f^{x_i})(y) - \Gamma(f_y)(x) - \Gamma(f^x)(y)] \\ &= \Delta\Gamma(f_y)(x) + \frac{1}{2}\sum_{x_i \sim x}\sum_{y_k \sim y} w_i\overline{w}_k[(f(x_i,y_k) - f(x_i,y))^2 - (f(x,y_k) - f(x,y))^2]. \end{aligned}$$

Similarly, we obtain

$$L_2 = \Delta\Gamma(f^x)(y) + \frac{1}{2}\sum_{y_k \sim y}\sum_{x_i \sim x} \overline{w}_k w_i[(f(x_i,y_k) - f(x,y_k))^2 - (f(x_i,y) - f(x,y))^2].$$

Furthermore, we have

$$\begin{aligned} 2\Gamma(f,\Delta f)(x,y) &= \sum_{x_i \sim x} w_i(f(x_i,y) - f(x,y))(\Delta f(x_i,y) - \Delta f(x,y)) \\ &\quad + \sum_{y_k \sim y} \overline{w}_k(f(x,y_k) - f(x,y))(\Delta f(x,y_k) - \Delta f(x,y)) \\ &:= T_1 + T_2. \end{aligned}$$

Then, for the term T_1 we have

$$\begin{aligned} T_1 &= \sum_{x_i \sim x} w_i(f(x_i,y) - f(x,y))(\Delta f_y(x_i) + \Delta f^{x_i}(y) - \Delta f_y(x) - \Delta f^x(y)) \\ &= 2\Gamma(f_y,\Delta f_y)(x) + \sum_{x_i \sim x}\sum_{y_k \sim y} w_i\overline{w}_k(f(x_i,y) - f(x,y)) \\ &\quad \times (f(x_i,y_k) - f(x_i,y) - f(x,y_k) + f(x,y)). \end{aligned}$$

Similarly, we also have

$$\begin{aligned} T_2 &= 2\Gamma(f^x,\Delta f^x)(y) + \sum_{y_k \sim y}\sum_{x_i \sim x} \overline{w}_k w_i(f(x,y_k) - f(x,y)) \\ &\quad \times (f(x_i,y_k) - f(x,y_k) - f(x_i,y) + f(x,y)). \end{aligned}$$

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Observing the fact that

$$\begin{aligned}&(f(x_i,y_k)-f(x,y_k))^2-(f(x_i,y)-f(x,y))^2\\&\quad=(f(x_i,y_k)-f(x,y_k)-f(x_i,y)+f(x,y))^2\\&\qquad+2(f(x_i,y_k)-f(x,y_k)-f(x_i,y)+f(x,y))(f(x_i,y)-f(x,y)),\end{aligned}$$

we arrive at

$$L_2-\Delta\Gamma(f^x)(y)-(T_1-2\Gamma(f_y,\Delta f_y)(x))\geqslant 0 \tag{2.9}$$

and

$$L_1-\Delta\Gamma(f_y)(x)-(T_2-2\Gamma(f^x,\Delta f^x)(y))\geqslant 0. \tag{2.10}$$

This completes the proof. □

Remark 2.6. The intuition of the above calculation is that the mixed terms are 'flat'. In fact, Lemma 2.5 still holds if we replace $\Gamma_2(f)$ with

$$\widetilde{\Gamma}_2(f):=\frac{1}{2}\Delta\Gamma(f)-\Gamma\left(f,\frac{\Delta(f^2)}{2f}\right).$$

Explicitly, for any positive function $f:V_1\times V_2\to\mathbb{R}$ and any $(x,y)\in V_1\times V_2$, we have

$$\widetilde{\Gamma}_2(f)(x,y)\geqslant\widetilde{\Gamma}_2(f_y)(x)+\widetilde{\Gamma}_2(f^x)(y). \tag{2.11}$$

The proof is done in a similar way. The operator $\widetilde{\Gamma}_2$ was introduced in [5] to define a modification of the CD-inequality, called the exponential curvature-dimension inequality $CDE(K,n)$ (see Definition 3.9 in [5]). Under the assumption of their new notion of curvature lower bound, they prove Li–Yau-type gradient estimates (dimension-dependent) for the heat kernels on graphs.

Proof of Theorem 1.3. By Lemma 2.5, we have for any function $f:V_1\times V_2\to\mathbb{R}$ and any $(x,y)\in V_1\times V_2$,

$$\begin{aligned}\Gamma_2(f)(x,y)&\geqslant\Gamma_2(f_y)(x)+\Gamma_2(f^x)(y)\\&\geqslant\frac{1}{n_1}(\Delta f_y(x))^2+\frac{1}{n_2}(\Delta f^x(y))^2+K_1\Gamma(f_y)(x)+K_2\Gamma(f^x)(y)\\&\geqslant\frac{1}{n_1+n_2}(\Delta f_y(x)+\Delta f^x(y))^2+K_1\wedge K_2(\Gamma(f_y)(x)+\Gamma(f^x)(y)).\end{aligned} \tag{2.12}$$

In the last inequality above we used Young's inequality. Recalling the facts $\Delta f_y(x)+\Delta f^x(y)=\Delta f(x,y)$ and $\Gamma(f_y)(x)+\Gamma(f^x)(y)=\Gamma(f)(x,y)$, we complete the proof. □

Remark 2.7. We can have more flexibility concerning the measures assigned to vertices. Suppose the vertex measures assigned to G_1,G_2 and $G_1\times G_2$ take the constant values μ_1,μ_2 and μ_{12} on each vertex, respectively, then the modified conclusion of Theorem 1.3 is that $(G_1\times G_2,\mu_{12})$ satisfies

$$CD\left(\frac{1}{\mu_{12}}(\mu_1K_1\wedge\mu_2K_2),n_1+n_2\right). \tag{2.13}$$

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This modification covers the case of normalized Laplacians on regular graphs. In particular, if both (G_1, μ_1) and (G_2, μ_2) satisfy $CD(0,\infty)$, then $(G_1 \times G_2, \mu_{12})$ also satisfies $CD(0,\infty)$.

Remark 2.8. The estimates of the CD-inequality in Theorem 1.3 (in fact also (2.13)) are tight at least for the Cartesian product of a graph G with itself. That is, if G satisfies $CD(K,n)$ precisely (*i.e.* for given dimension n, K is chosen largest possible), then the CD-inequality in Theorem 1.3 (or in (2.13)) is optimal for $G \times G$. This can be seen as follows. First note that this tightness depends on that of (2.9), (2.10) and (2.12). By assumption, there exists a function f on the graph G and a vertex x of the graph such that

$$\Gamma_2(f)(x) = \frac{1}{n}(\Delta f(x))^2 + K\Gamma(f)(x),$$

with $\Gamma(f)(x) \neq 0$. We can then choose a particular function F on $G \times G$ such that:

(i) $F(x,x) = f(x)$,
(ii) $F(x_i,x) = f(x_i)$ for all neighbours x_i of x in G,
(iii) $F(x,x_k) = f(x_k)$ for all neighbours x_k of x in G,
(iv) $F(x_i,x_k) = F(x,x_k) + F(x_i,x) - F(x,x)$.

For such an F the equalities in (2.9) and (2.10) are attained at (x,x) and $\Delta F_x(x) = \Delta F^x(x)$ and (by consequence) $\Gamma(F_x)(x) = \Gamma(F^x)(x)$, hence the equality in (2.12) is also attained. Therefore we obtain

$$\Gamma_2(F)(x,x) = \frac{1}{2n}(\Delta F(x,x))^2 + K\Gamma(F)(x,x),$$

which confirms the postulated tightness.

In the specific example $(G, \mathbf{1}_V)$, where G is the unweighted graph consisting of just one edge with end-points x,y, and writing $\Gamma(f,g)(x) = f^\top \Gamma(x) g$ and $\Delta f(x) = \Delta(x) f$, an easy calculation leads to

$$\Gamma_2(x) = 2\Gamma(x) = \Delta(x)^\top \Delta(x) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and the curvature-dimension condition $CD(K,n)$ translates into $K \leqslant 2 - 2/n$. This means that G satisfies $CD(2-2/n, n)$ precisely and $G \times G$ satisfies $CD(2-2/n, 2n)$ precisely, as well.

3. Eigenvalue ratios and higher-order spectral bounds

The first subsection is concerned with the eigenvalue ratio estimate under the $CD(0,\infty)$ condition and its optimality properties. Sections 3.2 and 3.3 discuss applications: lower estimates for higher-order Cheeger constants and upper diameter estimates in term of eigenvalues.

3.1. Eigenvalue ratio

As in [25] for the Riemannian manifold case, we need to combine the improved Cheeger inequality with the following Buser-type inequality.

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Theorem 3.1. *Let (G,μ) satisfy $CD(0,\infty)$. Then we have*

$$h_2(G,\mu) \geqslant \frac{e-1}{2e} \frac{1}{\sqrt{D_G^{nor}}} \sqrt{\lambda_2(G,\mu)}. \tag{3.1}$$

This is an adaptation of the Buser inequality in [20] to our setting of weighted graphs with a slightly better constant in (3.1). We refer to the arXiv version of this paper [27, Appendix I.2] for a detailed proof for (3.1). The dependence on D_G^{nor} comes from [27, Lemma 1.5]; see also [5].

We also need the following improved Cheeger inequality in [21] to obtain the eigenvalue ratio estimate. Their context was the weighted normalized setting.

Theorem 3.2 (Kwok, Lau, Lee, Oveis Gharan and Trevisan). *On (G,μ) we have, for any natural number $k \geqslant 2$,*

$$h_2(G,\mu) \leqslant 10\sqrt{2D_G^{non}}\,k \frac{\lambda_2(G,\mu)}{\sqrt{\lambda_k(G,\mu)}}. \tag{3.2}$$

Here the setting is slightly more general than that in [21]. To obtain (3.2), one needs to be careful about the final calculations in the proof of Proposition 3.2 in [21] (pp. 16 in the full version of [21]) and the fact that $\lambda_k \leqslant 2D_G^{non}$.

Combining (3.1) and (3.2), we get the following eigenvalue ratio estimate stated in the Introduction.

Theorem 1.2. *For any finite graph (G,μ) satisfying $CD(0,\infty)$ and any natural number $k \geqslant 2$, we have*

$$\lambda_k(G,\mu) \leqslant \left(\frac{20\sqrt{2}e}{e-1}\right)^2 D_G^{non} D_G^{nor} k^2 \lambda_2(G,\mu). \tag{3.3}$$

We remark that this estimate does not depend on the size of the graph. The following examples are concerned with the optimality of this result.

The first example shows that the order of k in the above estimate is optimal.

Example 3.3. Consider an unweighted cycle $\mathcal{C}_N$ with $N \geqslant 3$ vertices. Note that $\mathcal{C}_N$ can be considered as an abelian Cayley graph and hence satisfies $CD(0,\infty)$. Assign to it a measure μ which takes the constant value 2 on every vertex. Then the eigenvalues of the associated Laplacian are given by (see *e.g.* Example 1.5 in [10] or Section 7 in [26]),

$$\lambda_k(\mathcal{C}_N) = 1 - \cos\left(\frac{2\pi}{N}\left\lfloor\frac{k}{2}\right\rfloor\right), \quad k = 1,2,\ldots,N.$$

Observe that we have

$$\lim_{N\to\infty} \frac{\lambda_k(\mathcal{C}_N)}{\lambda_2(\mathcal{C}_N)} = \left\lfloor\frac{k}{2}\right\rfloor^2.$$

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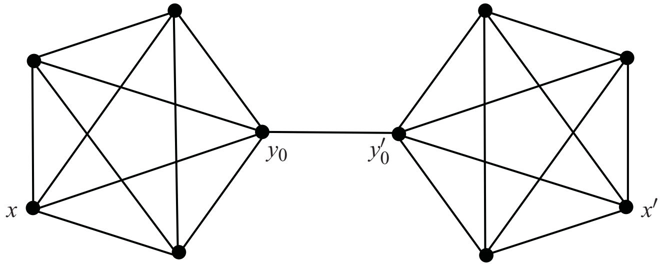

Figure 2. The dumbbell graph G_5.

The dependence on the term $D_G^{non}D_G^{nor}$ is also necessary in the estimate (3.3). This can be concluded from the following examples.

Example 3.4. Let us revisit the triangle graph $(\triangle_{xyz},\mu)$ in Example 2.3. Consider the special case that $A=B=C=1$ and $a=c$. Suppose $b\geqslant a$. Then this graph satisfies $CD(0,\infty)$. The eigenvalues of the non-normalized Laplacian are

$$\lambda_1=0<\lambda_2=3a\leqslant\lambda_3=a+2b.$$

Note further that $D_G^{non}D_G^{nor}=(a+b)/a$. Therefore, we have

$$\frac{1}{3}D_G^{non}D_G^{nor}\leqslant\frac{\lambda_3(\triangle_{xyz})}{\lambda_2(\triangle_{xyz})}\leqslant\frac{2}{3}D_G^{non}D_G^{nor}. \tag{3.4}$$

We give another example which works for the eigenvalue ratios of both non-normalized and normalized Laplacians.

Example 3.5. Consider the tetrahedron graph (T_4,μ) in Example 2.4 with the assumption that $b\geqslant a=c$. Recall that $\mu=A$ is a constant measure. Then the eigenvalues of the μ-Laplacian are

$$\lambda_1=0<\lambda_2=\frac{4a}{A}\leqslant\lambda_3=\lambda_4=\frac{2a+2b}{A}.$$

Moreover, we have $D_G^{non}D_G^{nor}=(2a+b)/a$. Hence we obtain

$$\frac{1}{4}D_G^{non}D_G^{nor}\leqslant\frac{\lambda_3(T_4)}{\lambda_2(T_4)}\leqslant\frac{1}{2}D_G^{non}D_G^{nor}. \tag{3.5}$$

The following example shows that we cannot expect that the eigenvalue ratio estimate (3.3) remains valid if a graph (G,μ) possesses a small portion of vertices not satisfying $CD(0,\infty)$.

Example 3.6. Consider a sequence of dumbbell graphs $\{G_N\}_{N=3}^{\infty}$. Given two copies of complete graphs over N vertices, $\mathcal{K}_N$ and $\mathcal{K}'_N$, G_N is the graph obtained via connecting them by a new edge $e=(y_0,y'_0)$, as shown in Figure 2. It was shown in [19] that the complete graph $\mathcal{K}_N$ with normalized Laplacian satisfies

$$CD\left(\frac{N+2}{2(N-1)},\infty\right).$$

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Modifying the calculation in the proof of this fact in [19], we obtain the following results.

- With the normalized Laplacian, G_N satisfies $CD(1/2,\infty)$ at every vertex which is not y_0, y_0'. At y_0, y_0', $CD(0,\infty)$ does not hold when $N \geqslant 3$.
- With the non-normalized Laplacian, G_N satisfies $CD(N/2,\infty)$ at every vertex which is not y_0, y_0'. At y_0, y_0', $CD(0,\infty)$ does not hold when $N \geqslant 3$.

We present the calculations in Appendix B. With only 2 of $2N$ vertices violating the curvature condition, the eigenvalue ratio estimate (3.3) no longer holds. Indeed, for the normalized Laplacian, we observe by Cheeger's inequality that

$$\lambda_2(G_N) \leqslant 2h(G) \leqslant 2\frac{|E(S,V\backslash S)|}{\mu(S)} = \frac{2}{N(N-1)+1},$$

choosing $S = \mathcal{K}_N$ to estimate $h(G)$, given in (1.1).

Recall that the spectrum of a complete graph $\mathcal{K}_N$ is the simple eigenvalue 0 and the eigenvalue $N/(N-1)$ with multiplicity $N-1$. Deleting the edge $\{y_0, y_0'\}$ from G_N, we obtain two disjoint copies of $\mathcal{K}_N$ with combined spectrum $\lambda_1 = \lambda_2 = 0 < \lambda_3 = \lambda_4 = \cdots = N/(N-1)$. By an interlacing theorem for edge-deleting in [7], we conclude that $\lambda_4(G_N) \geqslant N/(N-1)$. (Note that the Laplacian $\mathcal{L}$ there is slightly different but unitarily equivalent to our normalized Laplacian, since $\mathcal{L} = D^{1/2}\Delta D^{-1/2}$.) Therefore we have

$$\frac{\lambda_4(G_N)}{\lambda_2(G_N)} \geqslant \frac{1}{2}N^2.$$

Since in this case $D_G^{non}D_G^{nor} = N$, (3.3) does not hold when N is large. Similar arguments show also for the non-normalized Laplacian that (3.3) is no longer true for all N. (The interlacing theorem for non-normalized Laplacian is well-known; see *e.g.* [17]).

Remark 3.7. Replacing the sequence of complete graphs $\mathcal{K}_\mathcal{N}$ above with a sequence of expanders, we obtain graphs *of bounded degree* violating the curvature condition and for which (3.3) does not hold.

3.2. Higher-order Buser inequalities

Higher-order Buser inequalities were first established by Funano [16] in the Riemannian setting and then improved in [25]. The following result from the Introduction seems to be the first higher-order Buser-type inequality in the graph setting.

Corollary 1.5. *For any graph (G,μ) satisfying $CD(0,\infty)$ and any natural number k, we have*

$$h_k(G,\mu) \geqslant h_2(G,\mu) \geqslant \frac{(e-1)^2}{40\sqrt{2}e^2}\frac{1}{D_G^{nor}\sqrt{D_G^{non}}}\frac{\sqrt{\lambda_k(G,\mu)}}{k}. \tag{3.6}$$

Proof. The first inequality is given by the monotonicity of the higher-order Cheeger constants $h_k(G,\mu)$ (as functions in k). The second inequality follows from Buser's inequality (3.1) and Theorem 1.2. □

Remark 3.8. Inequalities in the other direction complementing (3.6) (without any curvature condition) are given by the higher-order Cheeger inequalities (1.2) by Lee, Oveis Gharan and

Trevisan [23] from the Introduction. In our setting of weighted graphs, they read as

$$h_k(G,\mu) \leqslant C\sqrt{D_G^{non}}k^2\sqrt{\lambda_k(G,\mu)}, \tag{3.7}$$

where C is a universal constant. (For the generalization to our setting, one needs to slightly modify the calculation for $\mathbb{E}(\sum_{i=1}^m w(E(\hat{S}_i, \bar{\hat{S}}_i)))$ in Lemma 4.7 of [23].) Hence, for a graph (G,μ) satisfying $CD(0,\infty)$ and with bounded degree, $h_k(G,\mu)$ and $\sqrt{\lambda_k(G,\mu)}$ are equivalent up to polynomials of k of degree smaller than or equal to 2.

Remark 3.9. In [5], Bauer, Horn, Lin, Lippner, Mangoubi and Yau proved for a graph (G,μ) satisfying another, related curvature condition, namely, the *exponential curvature-dimension inequality* $CDE(0,n)$ (see [5, Definition 3.9]) and for a fixed $0<\alpha<1$ that there exists a constant $C(\alpha)$, depending only on α, such that

$$\lambda_2(G,\mu) \leqslant C(\alpha)D_G^{nor}nh_2(G,\mu)^2. \tag{3.8}$$

That is, they obtain a dimension-dependent Buser inequality. Our approach also applies to their setting. In particular, we obtain the following eigenvalue ratio estimate and higher-order Buser inequalities under the condition $CDE(0,n)$,

$$\lambda_k(G,\mu) \leqslant C_1(\alpha)D_G^{nor}D_G^{non}nk^2\lambda_2(G,\mu), \tag{3.9}$$

$$h_k(G,\mu) \geqslant h_2(G,\mu) \geqslant C_2(\alpha)\frac{1}{D_G^{nor}\sqrt{D_G^{non}}}\frac{1}{nk}\sqrt{\lambda_k(G,\mu)}, \tag{3.10}$$

where $C_1(\alpha)$, $C_2(\alpha)$ are constants depending only on α.

3.3. A discrete analogue of Cheng's theorem

In the manifold setting, Cheng's theorem [8] provides a relation between the diameter and the k-eigenvalue of the Laplacian under the non-negative Ricci curvature assumption, presented in (1.10) in the Introduction. In this subsection, we derive a graph-theoretical analogue. To do so, we restrict our considerations to Alon and Milman's setting [1] of unweighted non-normalized graphs (G,μ) with $\mu = \mathbf{1}_V$. We recall the following eigenvalue-diameter estimate from [1, Theorem 2.7].

Theorem 3.10 (Alon and Milman). *Let $G=(V,E)$ be a finite connected graph with maximal degree d_G and let Δ be the non-normalized Laplacian. Then we have*

$$\operatorname{diam}(G) \leqslant 2\sqrt{\frac{2d_G}{\lambda_2(G)}}\log_2|V|. \tag{3.11}$$

Combining Theorem 3.10 with Theorem 1.2, we obtain the following result from the Introduction.

Corollary 1.6. *Let $(G,\mathbf{1}_V)$ be an unweighted finite graph satisfying $CD(0,\infty)$. Then, for any $k\geqslant 2$ we have*

$$\operatorname{diam}(G) \leqslant \frac{80e}{e-1}d_G\log_2|V|\frac{k}{\sqrt{\lambda_k(G,\mu)}}. \tag{3.12}$$

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Remark 3.11. Note that there are various further developments in connection with Alon and Milman's estimate (3.11); see for example the work of Chung [9], Mohar [31], Chung, Grigor'yan and Yau [11] and Houdré and Tetali [18]. In principle, the estimate (3.12) can be improved accordingly.

4. Ratios of higher-order Cheeger constants and multi-way expanders

In this section we derive the following result from the Introduction and discuss applications in the topic of multi-way expanders.

Corollary 1.7. *There exists a universal constant C such that for any graph (G,μ) satisfying $CD(0,\infty)$ and any natural number $k \geqslant 2$, we have*

$$h_k(G,\mu) \leqslant C D_G^{non} D_G^{nor} k\sqrt{\log k}\, h_2(G,\mu). \tag{4.1}$$

First, we recall the following results of Lee, Oveis Gharan and Trevisan [23, Theorems 1.2, 3.9 and Corollary 4.2] in our general setting.

Theorem 4.1 (Lee, Oveis Gharan and Trevisan). *Let (G,μ) be a weighted graph with vertex measure μ. Then we have*

$$h_k(G,\mu) \leqslant C\sqrt{D_G^{non} \log k \lambda_{2k}}, \tag{4.2}$$

with a universal constant $C > 0$. Moreover, if the graph G has genus at most $g \geqslant 1$ (i.e. G can be embedded into an orientable surface of genus at most g without edge crossings), we have

$$h_k(G,\mu) \leqslant C' \log(g+1)\sqrt{D_G^{non}\lambda_{2k}}, \tag{4.3}$$

with another universal constant $C' > 0$.

Proof of Corollary 1.7. Using (4.2) and Theorems 1.2 and 3.1, we obtain

$$\begin{aligned} h_k(G,\mu) &\leqslant C\sqrt{D_G^{non}\log k\lambda_{2k}} \\ &\leqslant C'\sqrt{D_G^{non}\log k}\sqrt{D_G^{non}D_G^{nor}}(2k)\sqrt{\lambda_2(G,\mu)} \\ &\leqslant C''\sqrt{D_G^{non}\log k}\sqrt{D_G^{non}D_G^{nor}}(2k)\sqrt{D_G^{nor}}h_2(G,\mu) \\ &= 2C'' D_G^{non}D_G^{nor}k\sqrt{\log k}\, h_2(G,\mu), \end{aligned}$$

with various universal constants C, C', C''. □

Moreover, if we replace (4.2) with (4.3) in the above proof, we obtain the following result.

Corollary 4.2. *There exists a universal constant C such that if (G,μ) satisfies $CD(0,\infty)$, then for any $k \geqslant 2$,*

$$h_k(G,\mu) \leqslant C D_G^{non} D_G^{nor} \log(g_G+1) k h_2(G,\mu), \tag{4.4}$$

where $g_G \geqslant 1$ is an upper bound of the genus of G.

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Remark 4.3. The order of k in (4.4) is optimal. This follows from the example of unweighted cycles $\mathcal{C}_N$ (which are planar) with the same measure μ as in Example 3.3, since we have (see *e.g.* [26, Proposition 7.3])

$$h_k(\mathcal{C}_N) = \frac{1}{\lfloor N/k \rfloor}, \quad \text{for } 2 \leqslant k \leqslant N.$$

The dependence on $D_G^{non} D_G^{nor}$ of the ratio estimate is also necessary. This follows from the following example analysed in Mimura [30].

Example 4.4. Consider the Cartesian product graph $G_{N,2}$ of the unweighted complete graphs $\mathcal{K}_N$ and $\mathcal{K}_2$. Assign the measure $\mu = \mathbf{1}$ to it. Since complete graphs satisfy $CD(0,\infty)$ (in fact the complete graph $\mathcal{K}_N$ is the Cayley graph of $\mathbb{Z}/N\mathbb{Z}$ when all its elements are taken as generators), we know by Theorem 1.3 that $G_{N,2}$ satisfies $CD(0,\infty)$. It is straightforward to see that $h_2(G_{N,2}) \leqslant 1$. Observe that we can partition $G_{N,2}$ into two induced subgraphs $\mathcal{K}_N$ and $\mathcal{K}'_N$. By Lemma 1 of Tanaka [33] (see also [30]), we have

$$h_3(G_{N,2}) \geqslant h_2(\mathcal{K}_N) = \frac{N}{2}.$$

(Note that Tanaka's lemma was stated for the constants $\{\mathfrak{h}_k(G)\}$ defined below. One can check that it also works for $\{h_k(G)\}$ here.) Therefore, we obtain

$$\frac{h_3(G_{N,2})}{h_2(G_{N,2})} \geqslant \frac{N}{2} = \frac{1}{2} d_G. \tag{4.5}$$

This shows the necessity of the dependence on the term $D_G^{non} D_G^{nor} = d_G$. Note that (4.5) also holds for the normalized measure μ. We comment that one can also analyse the eigenvalues of this example to show the necessity of the dependence on the degree in (3.3) (see also [30]) using an interlacing theorem or Lemma 6 of [33].

Now we restrict our considerations to the setting $w = \mathbf{1}_E$ and $\mu = \mathbf{1}_V$, that is, $G = (V,E)$ is now an unweighted graph with non-normalized Laplacian. Recently, the concept of multi-way expanders was defined and studied in Tanaka [33] and Mimura [30]. We denote $\mathfrak{h}_k(G)$ to be the following larger k-way isoperimetric constant (compare with Definition 1.4)

$$\mathfrak{h}_k(G) := \min_{S_1,\ldots,S_k} \max_{1 \leqslant i \leqslant k} \phi_{1,\mathbf{1}}(S_i), \tag{4.6}$$

where the minimum is taken over all partitions of V, that is, $V = \bigsqcup_{i=1}^k S_i$, $S_i \neq \emptyset$ for all i.

Definition 4.5 (multi-way expanders [30, 33]). Let $k \geqslant 2$ be a natural number. A sequence of finite graphs $\{G_m = (V_m, E_m)\}_{m \in \mathbb{N}}$ is called a sequence of k-way expanders if (i) $\sup_m d_{G_m} < \infty$, (ii) $\lim_{m\to\infty} |V_m| = \infty$, and (iii) $\inf_m \mathfrak{h}_k(G_m) > 0$.

Observe that 2-way expander families coincide with classical families of expanders. In general, the property of being $(k+1)$-way expanders is strictly weaker than being k-way expanders (see [30]). However, Mimura [30] proved that the concepts of k-way expanders for all $k \geqslant 2$ are equivalent within the class of finite, connected, vertex-transitive graphs.

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As a consequence of Corollary 1.7, we have the following result.

Corollary 4.6. *For the class of finite connected graphs satisfying $CD(0,\infty)$, the concepts of k-way expanders for all $k \geqslant 2$ are equivalent.*

Proof. Using the relation (see [23, Theorem 3.8], and [30])

$$h_k(G) \leqslant \mathfrak{h}_k(G) \leqslant k h_k(G) \tag{4.7}$$

and employing Corollary 1.7 yields

$$\mathfrak{h}_k(G) \leqslant C d_G k^2 \sqrt{\log k}\, \mathfrak{h}_2(G). \tag{4.8}$$

Hence, when $d_G < \infty$, $\inf_m \mathfrak{h}_k(G_m) > 0$ implies $\inf_m \mathfrak{h}_2(G_m) > 0$. This completes the proof. □

Abelian Cayley graphs lie in the intersection of the class of vertex-transitive graphs and the class of graphs satisfying $CD(0,\infty)$. It is well known that there are no expanders in the class of abelian Cayley graphs (see Alon and Roichman [2]). Moreover, Friedman, Murty and Tillich [15] proved an explicit upper estimate for λ_2 which implies this fact. Therefore, we also obtain the following explicit upper estimate for λ_k implying the non-existence of sequences of multi-way expanders in this class of abelian Cayley graphs.

Corollary 4.7. *For any abelian Cayley graph $G = (V,E)$ of degree d of size $N = |V|$, there exists a universal constant C such that, for any $k \geqslant 2$,*

$$\lambda_k(G) \leqslant C k^2 d^2 N^{-4/d}. \tag{4.9}$$

This is a direct consequence of Theorem 1.2 and the estimate $\lambda_2 \leqslant C d N^{-4/d}$ in [15]. Therefore, it is natural to ask the following question.

Question 4.8. Does there exist a sequence of expanders satisfying $CD(0,\infty)$?

We are inclined to a negative answer. For example, the non-existence of expander families satisfying $CD(0,\infty)$ would follow if one could prove that every graph of vertex degree at most d and satisfying $CD(0,\infty)$ possesses polynomial volume growth with degree depending only on d. In fact, a sequence of expanders, in contrast, have exponential volume growth as their Cheeger constant has uniformly positive lower bound.

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Appendix A: Curvature matrix of the triangle and tetrahedron graphs

The curvature matrix $\Gamma_2(x)$ for the graph $(\triangle_{xyz},\mu)$ in Figure 1(a) is

$$\frac{1}{4C}\begin{pmatrix} \frac{3a^2}{B}+\frac{3b^2}{A}+\frac{(a+b)^2}{C} & \frac{bc}{A}-\frac{a(3a+c)}{B}-\frac{a(a+b)}{C} & \frac{ac}{B}-\frac{b(3b+c)}{A}-\frac{b(a+b)}{C} \\ \frac{bc}{A}-\frac{a(3a+c)}{B}-\frac{a(a+b)}{C} & \frac{bc}{A}+\frac{3a(a+c)}{B}+\frac{a(a-b)}{C} & \frac{2ab}{C}-\frac{2ac}{B}-\frac{2bc}{A} \\ \frac{ac}{B}-\frac{b(3b+c)}{A}-\frac{b(a+b)}{C} & \frac{2ab}{C}-\frac{2ac}{B}-\frac{2bc}{A} & \frac{3b(b+c)}{A}+\frac{b(b-a)}{C}+\frac{ac}{B} \end{pmatrix}.$$

Let us have a closer look at the special case that $A=B=C=1$ and $a=c$. Then the matrix $4\Gamma_2(x)=4\Gamma_2(z)$ reduces to

$$\begin{pmatrix} 4a^2+2ab+4b^2 & -5a^2 & a^2-2ab-4b^2 \\ -5a^2 & 7a^2 & -2a^2 \\ a^2-2ab-4b^2 & -2a^2 & a^2+2ab+4b^2 \end{pmatrix},$$

and the matrix $4\Gamma_2(y)$ is

$$\begin{pmatrix} 10a^2 & -5a^2 & -5a^2 \\ -5a^2 & 3a^2+4ab & 2a^2-4ab \\ -5a^2 & 2a^2-4ab & 3a^2+4ab \end{pmatrix}.$$

Observe that when $b \geqslant a/2$, the above two matrices are both diagonally dominant and hence positive semidefinite. In fact, they are always positive semidefinite for any $a,b \geqslant 0$. (We checked this via Maple.)

The matrix $4A^2\Gamma_2(x)$ for the tetrahedron graph (T_4,μ) in Figure 1(b) is given by

$$\begin{pmatrix} 2ab+2ac+2bc+4a^2+4b^2+4c^2 & -2ab+2ac-2bc-4b^2 & 2ab-2ac-2bc-4c^2 & -2ab-2ac+2bc-4a^2 \\ -2ab+2ac-2bc-4b^2 & 2ac+2ab+2bc+4b^2 & -2ab-2ac+2bc & -2bc-2ac+2ab \\ 2ab-2ac-2bc-4c^2 & -2ab-2ac+2bc & 2ab+2ac+2bc+4c^2 & -2ab+2ac-2bc \\ -2ab-2ac+2bc-4a^2 & -2bc-2ac+2ab & -2ab+2ac-2bc & 2ab+2ac+2bc+4a^2 \end{pmatrix}.$$

This is a positive semidefinite matrix.

Appendix B: CD-inequalities of dumbbell graphs

In this section we present the calculations for the CD-inequalities of dumbbell graphs G_N claimed in Example 3.6. They are modified from that of [19, Proposition 3].

A general formula representing $\Gamma_2(f)$ is given by

$$\begin{aligned}\Gamma_2(f)(x) &= Hf(x)+\frac{1}{2}(\Delta f(x))^2-\frac{1}{2}\frac{\sum_{y,y\sim x} w_{xy}}{\mu(x)}\Gamma(f)(x) \\ &\quad -\frac{1}{4}\frac{1}{\mu(x)}\sum_{y,y\sim x} w_{xy}(f(y)-f(x))^2\frac{\sum_{z,z\sim y} w_{yz}}{\mu(y)},\end{aligned} \tag{B.1}$$

where

$$Hf(x) := \frac{1}{4}\frac{1}{\mu(x)}\sum_{y,y\sim x}\frac{w_{xy}}{\mu(y)}\sum_{z,z\sim y} w_{yz}(f(x)-2f(y)+f(z))^2.$$

This is an extension of [19, (2.9)] to our general setting (G,μ).

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Let us first consider the case of the unweighted normalized Laplacian. Let x be a vertex of G_N which is different from y_0 or y_0' (see Figure 2). First observe that

$$Hf(x) \geqslant \frac{1}{4N(N-1)} \sum_{y,y\sim x} \sum_{\substack{z,z\sim y\\ z\neq y_0'}} (f(x)-2f(y)+f(z))^2.$$

Now our calculations reduce to the complete graph $\mathcal{K}_N$ itself. Note that when $y,z\neq x$,

$$\begin{aligned}&(f(x)-2f(y)+f(z))^2+(f(x)-2f(z)+f(y))^2\\ &\quad=(f(x)-f(y))^2+(f(x)-f(z))^2+4(f(y)-f(z))^2.\end{aligned}$$

Then we have

$$Hf(x) \geqslant \frac{N+2}{2N}\Gamma(f)(x)+\frac{1}{N(N-1)}\sum_{\{y,z\}}(f(y)-f(z))^2,$$

where the second summation is over all unordered pair of neighbours of x. By (B.1), we arrive at

$$\Gamma_2(f)(x) \geqslant \frac{2-N}{2N}\Gamma(f)(x)+\frac{1}{2}(\Delta f(x))^2+\frac{1}{N(N-1)}\sum_{\{y,z\}}(f(y)-f(z))^2.$$

The last two terms above can be further manipulated as follows:

$$\begin{aligned}&\frac{1}{2(N-1)^2}\left(\sum_{y,y\sim x}(f(y)-f(x))\right)^2+\frac{1}{N(N-1)}\sum_{\{y,z\}}(f(y)-f(z))^2\\
&\geqslant \frac{1}{N(N-1)}\Bigg[\frac{1}{2}\sum_{y,y\sim x}(f(y)-f(x))^2-\sum_{\{y,z\}}(f(y)-f(x))(f(z)-f(x))\\
&\qquad+\sum_{\{y,z\}}((f(y)-f(x))^2+(f(z)-f(x))^2)\Bigg]\\
&=\frac{1}{N(N-1)}\left[\left(\frac{1}{2}+\frac{N-2}{2}\right)\sum_{y,y\sim x}(f(y)-f(x))^2+\frac{1}{2}\sum_{\{y,z\}}(f(y)-f(z))^2\right]\\
&\geqslant \frac{N-1}{N}\Gamma(f)(x).\end{aligned}$$

In the equality above, we use the facts that

$$\begin{aligned}&\frac{1}{2}\sum_{\{y,z\}}((f(y)-f(x))^2+(f(z)-f(x))^2)-\sum_{\{y,z\}}(f(y)-f(x))(f(z)-f(x))\\
&\quad=\frac{1}{2}\sum_{\{y,z\}}(f(y)-f(z))^2\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\sum_{\{y,z\}}((f(y)-f(x))^2+(f(z)-f(x))^2)\\
&\qquad=\frac{1}{4}\sum_{y,y\sim x}\sum_{z,z\sim x,z\neq y}(f(y)-f(x))^2+\frac{1}{4}\sum_{z,z\sim x}\sum_{y,y\sim x,y\neq z}(f(z)-f(x))^2\\
&\qquad=\frac{N-2}{2}\sum_{y,y\sim x}(f(y)-f(x))^2.\end{aligned}$$

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Therefore we have

$$\Gamma_2(f)(x) \geqslant \frac{1}{2}\Gamma(f)(x).$$

That is, G_N satisfies $CD(1/2,\infty)$ at any vertex $x \neq y_0, y_0'$.

Remark. We note that this CD-inequality at vertex x still holds even if we attach different graphs to every vertex in $\mathcal{K}_N$ other than x via single edges.

At y_0, $CD(0,\infty)$ does not hold. Let f_0 be the function taking the value 1 at y_0', 2 at all other vertices in $\mathcal{K}_N'$, and 0 at all vertices in $\mathcal{K}_N$. Then one can check by (B.1) that

$$\Gamma_2(f_0)(y_0) = \frac{3-N}{2N^2} < 0, \quad \text{if } N \geqslant 4.$$

In the case $N=3$, we can use another function g_0 taking the value 1 at y_0, -1 at all other vertices in $\mathcal{K}_3$, 4 at y_0', and 7 at other vertices in $\mathcal{K}_3'$. One can then check directly that $\Gamma_2(g_0)(y_0) = -1/9 < 0$.

For the case of the unweighted non-normalized Laplacian, the calculations are similar. Note in this case at $x \neq y_0, y_0'$, we have

$$\begin{aligned}\Gamma_2(f)(x) &= Hf(x) + \frac{1}{2}(\Delta f(x))^2 - \frac{d_x}{2}\Gamma(f)(x) - \frac{1}{4}\sum_{y,y\sim x}(f(y)-f(x))^2 d_y \\ &\geqslant Hf(x) + \frac{1}{2}(\Delta f(x))^2 - N\Gamma(f)(x).\end{aligned}$$

Carrying out the calculation in the same way as in the normalized case, we finally conclude that

$$\Gamma_2(f)(x) \geqslant \frac{N}{2}\Gamma(f)(x).$$

The arguments for CD-inequalities at y_0, y_0' can be done with the same special functions as in the normalized case.

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