

THE MODEL COMPANION OF DIFFERENTIAL FIELDS WITH FREE OPERATORS

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Abstract. A model companion is shown to exist for the theory of partial differential fields of characteristic zero equipped with free operators that commute with the derivations. The free operators here are those introduced in [R. Moosa and T. Scanlon, *Model theory of fields with free operators in characteristic zero*, Journal of Mathematical Logic 14(2), 2014]. The proof relies on a new lifting lemma in differential algebra: a differential version of Hensel's Lemma for local finite algebras over differentially closed fields.

§1. Introduction. A new rather general formalism for the model theory of fields with operators was introduced and developed by the second author and Thomas Scanlon in [8]. That theory treats differential fields and difference fields uniformly, and also allows for a wide variety of other additive operators, or systems of operators, whose multiplicative rules are induced by a ring homomorphism from the field to a finite algebra over the field. These were called *free* operators in [8], and we delay recalling the precise definition until Section 4 below. A model companion was shown to exist in characteristic zero, and both the basic model theory as well as versions of the Canonical Base Property and the Zilber Dichotomy for the model companion were established. An important shortcoming of [8] is that the theories dealt with in that paper cannot impose any commutativity on the operators (this is partly why they are called “free”). Indeed, at the level of generality of [8] it is known that commutativity cannot always be imposed: the theory of a pair of commuting automorphisms has no model companion. Nevertheless, there are several natural examples of theories of commuting operators that have very well behaved and understood model companions. In this paper we extend [8] to deal with some of these commuting situations; namely the context where the fields come already equipped with several commuting derivations and the new operators are assumed to commute with the derivations. That is, we prove that *the theory of partial differential fields of characteristic zero equipped with free operators that commute with the derivations has a model companion*. This is done in Section 6.

A particular case is the theory of partial differential fields equipped with an automorphism. The existence of the model companion for this theory was established by the first author in [6], and served as a model for us.

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There are two main obstacles in passing from fields to differential fields. The first is that in order to extend the geometric characterisation of the existentially closed models that appears in [8] one requires lifting nonsingular solutions of systems of differential polynomial equations from a differentially closed field to a local finite algebra over that field. Such a differential analogue of Hensel’s Lemma appears new and should be of independent interest. It is established here as Theorem 3.1.

The second obstacle is that the geometric characterisation one obtains quantifies over irreducible differential varieties. Because irreducibility of Kolchin closed sets is not known to be a definable property of the parameters, one has to modify the characterisation into a provably first-order axiomatisation of the existentially closed models. Here we follow the method established by the first author in [5] and [6], which uses characteristic sets of prime differential ideals.

A word about the proofs: we have tried to focus on those aspects of the arguments where the differential setting poses a significant challenge. Whenever the results of [8] extend automatically, or by using well known standard methods, we either omit proofs or content ourselves with brief sketches.

§2. Some preliminaries on differential algebra. We will be assuming familiarity with the model theory of differential fields and the differential algebra that it entails. However, we take this opportunity to fix some notation and also to recall some of the more technical aspects of differential algebra that may not be top of mind among model theorists. For further details we suggest Chapters I and IV of [3].

Suppose R is a unital and commutative ring (all of our rings will be so) equipped with m commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$. For $x = (x_1, \dots, x_n)$ a tuple of indeterminates, we denote by $R\{x\}$ the Δ -ring of Δ -polynomials over R . We denote by Θ the free commutative monoid generated by Δ , and consider the set of *algebraic indeterminates*

$$\Theta x = \{\theta x_i : 1 \leq i \leq n, \theta \in \Theta\}.$$

So the underlying R -algebra structure on $R\{x\}$ is that of the polynomial ring $R[\Theta x]$. For $\Lambda \subseteq R\{x\}$ we will use $[\Lambda]$ to denote the Δ -ideal and (Λ) to denote the (usual algebraic) ideal, generated by Λ .

There is a canonical ranking on Θx given by

$$\delta_m^{e_m} \dots \delta_1^{e_1} x_i < \delta_m^{r_m} \dots \delta_1^{r_1} x_j \iff \left(\sum e_k, i, e_m, \dots, e_1 \right) < \left(\sum r_k, j, r_m, \dots, r_1 \right),$$

where the ordering on the right-hand-side is the lexicographic one. This allows us to view Θx as a sequence enumerated with respect to the above linear ordering. Let $f \in R\{x\} \setminus R$. The *leader* of f , v_f , is the highest ranking algebraic indeterminate that appears in f . The *leading degree* of f , d_f , is the degree of v_f in f . The *rank* of f is the pair (v_f, d_f) . The set of ranks is ordered lexicographically. By convention, an element of R has lower rank than all the elements of $R\{x\} \setminus R$. The *separant* of f , S_f , is the formal partial derivative of f with respect to v_f . The *initial* of f , I_f , is the leading coefficient of f when viewed as a polynomial in v_f . Note that both S_f and I_f have lower rank than f . Given a finite subset $\Lambda \subset R\{x\}$, we set $H_\Lambda := \prod_{f \in \Lambda} I_f S_f$.

DEFINITION 2.1. Suppose Λ is a subset of $R\{x\} \setminus R$.

- (1) Λ is said to be *autoreduced* if for each $f \neq g$ in Λ , no proper derivative of v_f appears in g , and if v_f appears at all in g then it does so with strictly smaller degree.
- (2) Λ is said to be *coherent* if the following condition is satisfied: Suppose that for some $f \neq g$ in Λ , there are derivatives θ_f and θ_g such that $\theta_f v_f = \theta_g v_g =: v$. Suppose moreover that v is the least algebraic indeterminate for which this happens. Then for some N ,

$$H_\Lambda^N(S_g \theta_f f - S_f \theta_g g) \in (\Lambda)_v,$$

where $(\Lambda)_v$ is the ideal generated by $\{\theta f : f \in \Lambda, v_{\theta f} < v\}$.

REMARK 2.2. Autoreducedness implies that the elements of Λ have distinct leaders. This implies that any autoreduced set is finite (see [3, Section I.9] for details).

LEMMA 2.3. Let $\phi : R \rightarrow S$ be a homomorphism of differential rings. Suppose Λ is a coherent autoreduced subset of $R\{x\}$, and let $\Lambda^\phi := \{f^\phi : f \in \Lambda\}$, where $f^\phi \in S\{x\}$ is obtained by applying ϕ to the coefficients of f . If $H_\Lambda^\phi \neq 0$, then $H_{\Lambda^\phi} = H_\Lambda^\phi$ and Λ^ϕ is coherent autoreduced.

PROOF. In general, every algebraic indeterminate that appears in f^ϕ appears in f . Moreover,

$$\frac{\partial f^\phi}{\partial v_f} = \left(\frac{\partial f}{\partial v_f} \right)^\phi = S_f^\phi.$$

It follows that if $S_f^\phi \neq 0$, then f and f^ϕ have the same leader and $S_{f^\phi} = S_f^\phi$. In this case, if $I_f^\phi \neq 0$ we get also that $I_{f^\phi} = I_f^\phi$.

Since $H_\Lambda^\phi \neq 0$, it follows from the above that Λ and Λ^ϕ have the same leaders and that $H_{\Lambda^\phi} = H_\Lambda^\phi$. Since the algebraic indeterminates of f^ϕ appear in f with the same degree, we also get that Λ^ϕ is autoreduced. To see that the coherency condition is satisfied we need only observe that if $g \in (\Lambda)_v$ then $g^\phi \in (\Lambda^\phi)_v$. -†

Suppose K is a Δ -field of characteristic zero. While it is not the case that all prime Δ -ideals of $K\{x\}$ are finitely generated as Δ -ideals (though they are finitely generated as radical Δ -ideals), something close is true; they are determined by certain canonical finite subsets. Namely by autoreduced subsets that are minimal with respect to the following standard ranking: Writing finite sets of differential polynomial in nondecreasing order by rank, define $\{g_1, \dots, g_r\} < \{f_1, \dots, f_s\}$ to mean that either there is $i \leq r, s$ such that $rank(g_j) = rank(f_j)$ for $j < i$ and $rank(g_i) < rank(f_i)$, or $r > s$ and $rank(g_j) = rank(f_j)$ for $j \leq s$.

Now, if $I \subset K\{x\}$ is a prime Δ -ideal then a *characteristic set* Λ for I is a minimal autoreduced subset of I with respect to this ranking. These always exist, are in addition coherent, and determine the ideal I in the sense that

$$I = \{f \in K\{x\} : \text{for some } \ell \geq 0, H_\Lambda^\ell f \in [\Lambda]\}. \tag{1}$$

Taking zero sets in some differentially closed extension of K we have that

$$V(I) \setminus V(H_\Lambda) = V(\Lambda) \setminus V(H_\Lambda).$$

A consequence of the minimality of Λ is that $H_\Lambda \notin I$, and hence the above equality says that $V(I)$ and $V(\Lambda)$ agree off a proper Kolchin closed subset.

§3. A Differential Lifting Lemma. The following theorem, which we will use later but which may also be of independent interest, is a differential analogue of the artinian case of Hensel’s Lemma.

THEOREM 3.1. *Suppose L is a differentially closed field and R is a local finite L -algebra, equipped with a differential structure extending that of L , and such that the maximal ideal of R is a Δ -ideal. Suppose $\Lambda = \{f_1, \dots, f_s\} \subset R\{x\}$ is a coherent autoreduced set of differential polynomials in one variable over R . Suppose, finally, that $a \in L$ is such that $\text{res}(f)(a) = 0$ for all $f \in \Lambda$ but $\text{res}(H_\Lambda)(a) \neq 0$. Then there exists $b \in R$ such that $\text{res}(b) = a$ and $f(b) = 0$ for all $f \in \Lambda$.*

REMARK 3.2. A proof of this theorem in the special case when $R = L[\epsilon]/(\epsilon^2)$ and ϵ is a Δ -constant can be extracted from Kolchin [4, Chapter 0, Section 4], and served as a model for us. In the ordinary case, when $\Delta = \{\delta\}$, other special cases are also known. For example the case of $R = L[\epsilon]/(\epsilon^{n+1})$ with ϵ a δ -constant follows from Scanlon’s D -Hensel’s Lemma [10, Proposition 6.1]. The main difficulty that our proof below must contend with is the presence of several commuting derivations.

PROOF OF THEOREM 3.1. Let \mathfrak{m} be the maximal ideal of R . Note that as L is algebraically closed, the residue field of R is L .

We will recursively define a sequence b_1, b_2, \dots in R such that

- $\text{res}(b_1) = a$,
- $b_{i+1} \equiv b_i \pmod{\mathfrak{m}^i}$, and
- $f(b_i) \equiv 0 \pmod{\mathfrak{m}^i}$ for each $f \in \Lambda$

As \mathfrak{m} is nilpotent, this will produce, in finitely many steps, the desired root of Λ lifting a . For the base case, we may take any $b_1 \in R$ with $\text{res}(b_1) = a$. Suppose $i \geq 1$ and we have found b_i with the desired properties.

As \mathfrak{m} is a Δ -ideal, and R is a finite dimensional vector space over L , we have that $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite dimensional Δ -module over L . That is, each δ_j induces an additive map on $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, which we also denote by δ_j , and which satisfies $\delta_j(rv) = \delta_j(r)v + r\delta_j(v)$ for all $r \in L$ and $v \in \mathfrak{m}^i/\mathfrak{m}^{i+1}$, and these induced maps commute with each other.

In this context it is well known that $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ will have an L -basis made up of Δ -constants; let us recall why. Let $\{e_1, \dots, e_n\}$ be any L -basis of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. For each $j = 1, \dots, m$, let $A_j \in M_n(L)$ be the matrix whose k -th column consists of the coefficients appearing in the expression of $\delta_j(e_k)$ as an L -linear combination of $\{e_1, \dots, e_n\}$. Under the linear transformation that takes the basis $\{e_1, \dots, e_n\}$ to the standard basis of L^n , the Δ -module $(\mathfrak{m}^i/\mathfrak{m}^{i+1}, \Delta)$ is transformed into the Δ -module $(L^n, \{\delta_1 + A_1, \dots, \delta_m + A_m\})$. Hence, it suffices to find an L -basis made up of constants for this Δ -module. In other words, it suffices to find a nonsingular $n \times n$ matrix M over L such that $\delta_j(M) + A_jM = 0$ for $j = 1, \dots, m$. Let X be a $n \times n$ matrix of variables. Extend Δ from L to derivations on $L(X)$ by letting $\delta_j(X) = -A_jX$ for $j = 1, \dots, m$. The commutativity of the δ_j ’s on $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ yields

$$\delta_j(A_k) - \delta_k(A_j) = [A_k, A_j], \quad j, k = 1, \dots, m.$$

These equalities imply that the derivations Δ on $L(X)$ commute. Hence, $L(X)$ is a differential field extension of L such that $\delta_j(X) + A_j X = 0$. As (L, Δ) is differentially closed we can find a nonsingular matrix M over L with the desired properties.

We can therefore find $\epsilon_1, \dots, \epsilon_n \in \mathfrak{m}^i$ whose image is an L -basis for $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, and such that $\delta(\epsilon_k) \equiv 0 \pmod{\mathfrak{m}^{i+1}}$ for all $\delta \in \Delta$ and all $k = 1, \dots, n$. Now consider the L -linear maps that pick out the coefficients, namely $\lambda_1, \dots, \lambda_k : \mathfrak{m}^i \rightarrow L$ such that for all $v \in \mathfrak{m}^i$,

$$v \equiv \sum_{k=1}^n \lambda_k(v) \epsilon_k \pmod{\mathfrak{m}^{i+1}}. \tag{2}$$

The fact that we have chosen the basis to be made up of Δ -constants modulo \mathfrak{m}^{i+1} implies that the λ_k 's commute with Δ . Indeed, for each $v \in \mathfrak{m}^i$ and $\delta_j \in \Delta$, we have that $\delta_j(v) \in \mathfrak{m}^i$, and so applying (2) with $\delta_j(v)$ instead of v we get

$$\delta_j(v) \equiv \sum_{k=1}^n \lambda_k(\delta_j(v)) \epsilon_k \pmod{\mathfrak{m}^{i+1}}.$$

On the other hand, applying δ_j to both sides of (2) and using the fact that the ϵ_k 's are Δ -constants modulo \mathfrak{m}^{i+1} , we get

$$\delta_j(v) \equiv \sum_{k=1}^n \delta_j(\lambda_k(v)) \epsilon_k \pmod{\mathfrak{m}^{i+1}}.$$

Putting the previous two congruences together yields $\lambda_k(\delta_j(v)) = \delta_j(\lambda_k(v))$.

We are looking for $y_1, \dots, y_n \in L$ such that

$$f(b_i + \sum_{k=1}^n y_k \epsilon_k) \equiv 0 \pmod{\mathfrak{m}^{i+1}} \quad \text{for all } f \in \Lambda. \tag{3}$$

Indeed, we could then set $b_{i+1} := b_i + \sum y_k \epsilon_k$, and the construction would be complete.

To find the y_1, \dots, y_n so that equation (3) holds, we write $f(x) = F(\Theta x)$, where F is an algebraic polynomial in the variables $\Theta x = (\theta x : \theta \in \Theta)$, and compute the Taylor expansion of $F(\Theta b_i + \Theta(\sum_{k=1}^n y_k \epsilon_k))$ around the point Θb_i . To do this, we use standard mult-index notation. Let $\mathbb{N}_*^\Theta = \{\alpha \in \mathbb{N}^\Theta : \alpha \text{ has finite support}\}$. We think of elements of \mathbb{N}^Θ as functions from Θ to \mathbb{N} . For $\alpha \in \mathbb{N}_*^\Theta$, we let $|\alpha| = \sum_{\theta \in \Theta} \alpha(\theta)$ and $\alpha! = \prod_{\theta \in \Theta} \alpha(\theta)!$, since α has finite support both of these expressions have only finitely many terms. We also let $(\Theta x)^\alpha = \prod_{\theta \in \Theta} (\theta x)^{\alpha(\theta)}$ and, finally,

$$D^\alpha F = \frac{\partial^{|\alpha|} F}{\prod_{\theta \in \Theta} \partial(\theta x)^{\alpha(\theta)}}.$$

Now, using Taylor's expansion formula for multivariate polynomials (in this case, in the variables Θx), the Taylor expansion of F around Θb_i is given by

$$F(\Theta x) = \sum_{\alpha \in \mathbb{N}_*^\Theta} \frac{D^\alpha F(\Theta b_i)}{\alpha!} \cdot (\Theta x - \Theta b_i)^\alpha.$$

In particular, the Taylor expansion of $F(\Theta b_i + \Theta(\sum_{k=1}^n y_k \epsilon_k))$ around the point Θb_i is given by

$$F(\Theta b_i + \Theta(\sum_{k=1}^n y_k \epsilon_k)) = \sum_{\alpha \in \mathbb{N}_*^{\Theta}} \frac{D^\alpha F(\Theta b_i)}{\alpha!} \cdot (\Theta(\sum_{k=1}^n y_k \epsilon_k))^\alpha.$$

Note that as the ϵ_k 's are Δ -constants modulo \mathfrak{m}^{i+1} , $\Theta(\sum_{k=1}^n y_k \epsilon_k) \equiv \sum_{k=1}^n \Theta(y_k) \epsilon_k \pmod{\mathfrak{m}^{i+1}}$. Hence, for those $\alpha \in \mathbb{N}_*^{\Theta}$ such that $|\alpha| \geq 2$, the terms $(\Theta(\sum_{k=1}^n y_k \epsilon_k))^\alpha$ vanish modulo \mathfrak{m}^{i+1} . We thus get that

$$F(\Theta b_i + \Theta(\sum_{k=1}^n y_k \epsilon_k)) \equiv F(\Theta b_i) + \sum_{\theta \in \Theta} \frac{\partial F}{\partial(\theta X)}(\Theta b_i) \cdot (\sum_{k=1}^n \theta(y_k) \epsilon_k) \pmod{\mathfrak{m}^{i+1}}.$$

Since $f(b_i) = F(\Theta b_i)$ and $f(b_i + \sum_{k=1}^n y_k \epsilon_k) = F(\Theta b_i + \Theta(\sum_{k=1}^n y_k \epsilon_k))$, what one obtains, therefore, is

$$\begin{aligned} f(b_i + \sum_{k=1}^n y_k \epsilon_k) &\equiv f(b_i) + \sum_{\theta \in \Theta} \frac{\partial F}{\partial(\theta X)}(\Theta b_i) \cdot (\sum_{k=1}^n \theta(y_k) \epsilon_k) \pmod{\mathfrak{m}^{i+1}} \\ &\equiv \sum_{k=1}^n \lambda_k(f(b_i)) \epsilon_k + \sum_{k=1}^n \left(\sum_{\theta \in \Theta} \frac{\partial F}{\partial(\theta X)}(\Theta b_i) \cdot \theta(y_k) \right) \epsilon_k \pmod{\mathfrak{m}^{i+1}} \\ &\equiv \sum_{k=1}^n \left[\lambda_k(f(b_i)) + \sum_{\theta \in \Theta} \text{res} \left(\frac{\partial F}{\partial(\theta X)}(\Theta b_i) \right) \theta(y_k) \right] \epsilon_k \pmod{\mathfrak{m}^{i+1}} \end{aligned}$$

where in the last step we use the fact that for all $r \in R$, $r \epsilon_k \equiv \text{res}(r) \epsilon_k \pmod{\mathfrak{m}^{i+1}}$. Hence, solving the system of congruences (3) is thereby reduced to solving, for each fixed $k = 1, \dots, n$, the following inhomogeneous system of linear differential equations in y_k over L :

$$\lambda_k(f(b_i)) + \sum_{\theta \in \Theta} \text{res} \left(\frac{\partial F}{\partial(\theta X)}(\Theta b_i) \right) \theta(y_k) = 0 \quad \text{for all } f \in \Lambda. \tag{4}$$

Note that i, b_i , and k are all fixed.

Let us denote the linear differential polynomial over L that is the left-hand-side of (4) by ℓ_f . So

$$\ell_f(y_k) := \lambda_k(f(b_i)) + \sum_{\theta \in \Theta} \text{res} \left(\frac{\partial F}{\partial(\theta X)}(\Theta b_i) \right) \theta(y_k).$$

To complete the proof of the theorem, therefore, we need to show that the linear differential system $\{\ell_f(y_k) = 0 : f \in \Lambda\}$ has a solution in L . This would follow if we knew that $\ell\Lambda := \{\ell_f : f \in \Lambda\}$ was a coherent autoreduced set in $L\{y_k\}$. Indeed, any coherent autoreduced set of linear differential polynomials generates a prime differential ideal (see for example statement (2) on page 399 of [9]), and hence the differentially closed L would contain a root of $\ell\Lambda$. Actually the situation is a little more complicated as $\ell\Lambda$ need not be autoreduced. But we will find a coherent autoreduced set of linear differential polynomials that generate the same differential ideal as $\ell\Lambda$, and that will of course suffice for the above argument to go through.

Note that ℓ_f is defined for any $f \in R\{x\}$ with the property that $f(b_i) \in \mathfrak{m}^i$. Consider the Δ -ideal $V := \{f \in R\{x\} : f(b_i) \in \mathfrak{m}^i\}$ of $R\{x\}$, and the map $\ell : V \rightarrow L\{y_k\}$ given by $f \mapsto \ell_f$. By the induction hypothesis, $\Lambda \subset V$.

CLAIM 3.3. *The map $\ell : V \rightarrow L\{y_k\}$ satisfies the following properties:*

- (a) *it is a homomorphism of Δ -modules over L ,*
- (b) *if $f \in V$ then, for any $g \in R\{x\}$, $\ell_{gf} = \text{res}(g)(a)\ell_f$,*
- (c) *for $f \in V$, f and ℓ_f have the same leaders (though in x and y_k , respectively), and moreover, the separant of ℓ_f is $\text{res}(S_f)(a)$.*

PROOF OF CLAIM 3.3. That ℓ is L -linear is clear, so for part (a) we need to show that for all $\delta \in \Delta$, $\delta(\ell_f) = \ell_{\delta f}$. This can be directly verified from the definition of ℓ , using the fact that λ_k and res commute with δ . The computation is tedious but routine, and we leave it to the reader.

For part (b), suppose $f \in V$. By construction $\ell_{gf}(y_k)$ is the coefficient of ϵ_k in the expansion of

$$gf \left(b_i + \sum_{j=1}^n y_j \epsilon_j \right) = g \left(b_i + \sum_{j=1}^n y_j \epsilon_j \right) f \left(b_i + \sum_{j=1}^n y_j \epsilon_j \right)$$

modulo \mathfrak{m}^{i+1} . We have already seen that for any $y_1, \dots, y_n \in L$, $f \left(b_i + \sum_{j=1}^n y_j \epsilon_j \right)$ is an L -linear combination of $\{\epsilon_1, \dots, \epsilon_n\}$ and hence lands in \mathfrak{m}^i . It follows that, working modulo \mathfrak{m}^{i+1} , the only contribution of $g \left(b_i + \sum_{j=1}^n y_j \epsilon_j \right)$ to the above product will be $\text{res} \left(g \left(b_i + \sum_{j=1}^n y_j \epsilon_j \right) \right) = \text{res}(g)(a)$. This proves part (b).

Finally, for part (c), note that if an algebraic indeterminate $\theta(y_k)$ appears in $\ell_f \in V$, then $\text{res} \left(\frac{\partial F}{\partial(\theta x)}(\Theta b_i) \right) \neq 0$, and so θx must appear in f . On the other hand, if $v_f = \theta x$ is the leader of f , then the coefficient of $\theta(y_k)$ in ℓ_f is $\text{res} \left(\frac{\partial F}{\partial v_f}(\Theta b_i) \right) = \text{res}(S_f(b_i)) = \text{res}(S_f)(a)$, which is nonzero because by assumption $\text{res}(H_\Lambda)(a) \neq 0$. It follows that f and ℓ_f have the same leaders (though in x and y_k , respectively). ⊖

We use the claim to prove that $\ell\Lambda := \{\ell_f : f \in \Lambda\} \subset L\{y_k\}$ satisfies the coherency condition, that is, part (2) of Definition 2.1. Suppose $f \neq g$ in Λ and there are derivatives θ_f and θ_g such that $\theta_f v_{\ell_f} = \theta_g v_{\ell_g} =: v$, and v is the least such occurrence. We show that $S_{\ell_g} \theta_f \ell_f - S_{\ell_f} \theta_g \ell_g \in (\ell\Lambda)_v$. Indeed, by 3.3(c), we have that $v = \theta_f v_f = \theta_g v_g$, and this is the least such occurrence for Λ . Since Λ is coherent, there is N such that $h := H_\Lambda^N(S_g \theta_f f - S_f \theta_g g) \in (\Lambda)_v$. It follows by Claim 3.3 parts (a) and (b) that $\ell_h \in (\ell\Lambda)_v$. But, also by 3.3,

$$\begin{aligned} \ell_h &= \text{res}(H_\Lambda)(a) (\text{res}(S_g)(a)\theta_f \ell_f - \text{res}(S_f)(a)\theta_g \ell_g) \\ &= \text{res}(H_\Lambda)(a) (S_{\ell_g} \theta_f \ell_f - S_{\ell_f} \theta_g \ell_g). \end{aligned}$$

Since $\text{res}(H_\Lambda)(a) \neq 0$, we get that $S_{\ell_g} \theta_f \ell_f - S_{\ell_f} \theta_g \ell_g \in (\ell\Lambda)_v$, as desired.

It may not be the case, however, that $\ell\Lambda$ is autoreduced. The leaders of distinct members are distinct, and it is *partially* autoreduced in the sense that no proper derivative of the leader of ℓ_f appears in ℓ_g for $f \neq g$ in Λ . Indeed, this follows from the fact that Λ is autoreduced, that ℓ_f and f have the same leader, and that

every algebraic indeterminate appearing in ℓ_g also appears in g . The obstacle to autoreducedness is that the leader of ℓ_f may still appear in ℓ_g , just not as its leader. But now a standard reduction process (see the second paragraph of the proof of Proposition 5 of [4, Chapter 0, Section 4], for example) can be used to transform $\ell\Lambda$ into an autoreduced set. The kinds of transformations we have in mind are as follows: if v_{ℓ_f} appears with coefficient c in ℓ_f and coefficient d in ℓ_g , then we can replace ℓ_g by $\ell_g - \frac{d}{c}\ell_f$. This process will produce a coherent autoreduced set of linear differential polynomials $\Gamma \subset L\{y_k\}$ such that $[\Gamma] = [\ell\Lambda]$. It follows, as explained earlier, that $[\ell\Lambda]$ is a prime differential ideal and so the system (4) has a solution in L . This completes the proof of Theorem 3.1. \dashv

REMARK 3.4. The above theorem remains true even if we remove the assumption that R is local. In this case one replaces the maximal ideal by the Jacobson radical \mathfrak{J} of R , the residue map by the quotient map $\text{pr} : R \rightarrow R/\mathfrak{J}$, and the condition $\text{res}(H_\Lambda)(a) \neq 0$ by $\text{pr}(H_\Lambda)(a)$ being a unit.

PROOF. The general case reduces to the local case of Theorem 3.1 by the following claim: *Suppose $(K, \Delta) \subseteq (R, \Delta)$ is an extension of differential rings where K is a field of characteristic zero and R is a finite K -algebra. If $R = \prod_{i=0}^t R_i$ is the decomposition of R into local K -algebras, then Δ induces a differential structure on each R_i . Moreover, if the Jacobson radical of R is a Δ -ideal, then so is the maximal ideal of each R_i .* To prove the above claim, for each $i = 0, \dots, t$, and each $r \in R_i$, let \widehat{r} denote the element of R whose R_i -th co-ordinate is r and whose R_j -th co-ordinate is 0_{R_j} for all $j \neq i$. Let $\widehat{R}_i = \{\widehat{r} : r \in R_i\}$. Then for all $\delta \in \Delta$ and $r \in R_i$ we have

$$\delta(\widehat{r}) = \delta(\widehat{r}\widehat{1}_{R_i}) = \widehat{r}\delta(\widehat{1}_{R_i}) + \delta(\widehat{r})\widehat{1}_{R_i} \in \widehat{R}_i,$$

which shows that $\delta(\widehat{R}_i) \subseteq \widehat{R}_i$. So we get an induced Δ -structure on each R_i . For the “moreover” clause, note that if \mathfrak{m}_i is the maximal ideal of R_i , then $\prod_{i=0}^t \mathfrak{m}_i$ is the Jacobson radical of R .

Note that the reduction to the local case also uses the fact that the co-ordinate projection $R \rightarrow R_i$ of a coherent autoreduced set is again coherent autoreduced. This follows from Lemma 2.3; one uses the assumption that $\text{pr}(H_\Lambda)(a)$ is a unit to see that the co-ordinate projection of H_Λ is nonzero and so Lemma 2.3 applies. \dashv

§4. \mathcal{D} -differential fields. We wish to study differential fields (K, Δ) equipped with additional linear operators, or stacks of operators, that commute with the derivations in Δ . The behaviour of these operators with respect to multiplication in the field should be governed by a fixed finite K -algebra, in a way that we now make precise.

Fix throughout the rest of the paper the following data:

- a base field k of characteristic zero,
- a finite k -algebra B with the property that for each of the (finitely many) maximal ideals \mathfrak{m} of B , $B/\mathfrak{m} = k$,
- a k -algebra homomorphism $\pi : B \rightarrow k$, and
- a k -basis $(\epsilon_0, \dots, \epsilon_\ell)$ of B such that $\pi(\epsilon_0) = 1$ and $\pi(\epsilon_i) = 0$ if $i \neq 0$.

Note that the condition on the residue fields above is automatically satisfied if k is algebraically closed.

Given any k -algebra R we let

$$\mathcal{D}(R) := B \otimes_k R$$

denote the base extension, which will be a finite and free R -algebra whose corresponding basis we will also denote by $(\epsilon_0, \dots, \epsilon_\ell)$. We can also lift $\pi : B \rightarrow k$ to a surjective R -algebra homomorphism $\pi^R : \mathcal{D}(R) \rightarrow R$, by base extension.

Now, if R is equipped with k -linear derivations $\Delta = \{\delta_1, \dots, \delta_m\}$, then we can canonically make $\mathcal{D}(R)$ a Δ -ring by defining δ_j on $\mathcal{D}(R)$ to be the unique B -linear derivation that extends δ_j on R ; that is, by defining $\delta_j(b \otimes r) := b \otimes \delta_j(r)$.

DEFINITION 4.1. A \mathcal{D} -differential ring is a differential k -algebra (R, Δ) equipped with a sequence of operators $\partial = (\partial_1, \dots, \partial_\ell)$ such that $e : R \rightarrow \mathcal{D}(R)$ given by

$$r \mapsto r\epsilon_0 + \partial_1(r)\epsilon_1 + \dots + \partial_\ell(r)\epsilon_\ell$$

is a homomorphism of differential k -algebras.

An equivalent formulation of (R, Δ, ∂) being a \mathcal{D} -differential ring is that:

- (a) R is a k -algebra,
- (b) $\Delta = (\delta_1, \dots, \delta_m)$ are commuting k -linear derivations on R ,
- (c) (R, ∂) is a \mathcal{D} -ring in the sense of [8]; that is, the map $e : R \rightarrow \mathcal{D}(R)$ defined above is a homomorphism of k -algebras, and
- (d) the ∂ commute with the Δ .

Indeed, it is an easy computation to see that e will preserve the differential structure if and only if each operator in ∂ commutes with each operator in Δ .

Thus, all the examples of free operators described in Section 3 of [8] have \mathcal{D} -differential analogues by imposing the further condition that they commute with Δ . In particular, difference-differential rings are \mathcal{D} -differential where \mathcal{D} is given by setting $B = k \times k$ and taking the standard basis.

REMARK 4.2. As in [8], we often identify a \mathcal{D} -differential structure on (R, Δ) with a differential k -algebra homomorphism $e : R \rightarrow \mathcal{D}(R)$ having the property that $\pi^R \circ e = \text{id}$. Indeed, given such an e , if we let ∂_i be the operator that takes r to the ϵ_i -coefficient of $e(r)$ in $\mathcal{D}(R)$, then (R, Δ, ∂) is a \mathcal{D} -differential ring.

Associated to a \mathcal{D} -differential ring (R, Δ, ∂) we have canonical endomorphisms of (R, Δ) . Indeed, these are precisely the endomorphisms of R associated to (R, ∂) in [8, Section 4]. To describe them, let $B = \prod_{i=0}^t B_i$ be the decomposition of B into local k -algebras. By assumption the residue field of each B_i is k , and so we get surjective k -algebra homomorphisms $\pi_i : B \rightarrow k$ obtained by precomposing the residue map $B_i \rightarrow k$ with the co-ordinate projection $B \rightarrow B_i$. Now, given a k -algebra R , we have the corresponding decomposition $\mathcal{D}(R) = \prod_{i=0}^t B_i \otimes_k R$, and surjective R -algebra homomorphisms $\pi_i^R : \mathcal{D}(R) \rightarrow R$. Note that because the maximal ideal of B_i is nilpotent, if L is any field extension of k then $B_i \otimes_k L$ is again a local L -algebra

with residue field L . Hence in this case π_i^L is the composition of the co-ordinate projection $\mathcal{D}(L) \rightarrow B_i \otimes_k L$ with the residue map $B_i \otimes_k L \rightarrow L$.

If (R, ∂) is a \mathcal{D} -ring then the associated endomorphisms of R introduced in [8] are the maps $\sigma_i := \pi_i^R \circ e$. If, moreover, (R, Δ, ∂) is a \mathcal{D} -differential ring, then π_i^R and $e : R \rightarrow \mathcal{D}(R)$ both preserve the differential structure, and so the σ_i are endomorphisms of (R, Δ) . We may, and do, assume that $\pi_0 = \pi$, so that $\sigma_0 = \text{id}$.

Extending \mathcal{D} -differential structure will require the associated endomorphisms to be injective:

LEMMA 4.3. *Suppose R is a Δ -subring of a Δ -field (L, Δ) , extending k . Suppose $e : R \rightarrow \mathcal{D}(L)$ is a k -linear homomorphism of Δ -rings, such that for all $i = 0, 1, \dots, t$, the maps $\pi_i^L \circ e : R \rightarrow L$ are injective. Then there is a unique extension of e to $(\text{Frac}(R), \Delta)$.*

PROOF. Forgetting about Δ for the moment, by Lemma 4.9 of [8], we can extend e to a k -algebra homomorphism on $\text{Frac}(R)$. But such an extension is unique and automatically preserves the differential structure. \dashv

The following proposition reduces the problem of extending \mathcal{D} -differential structure to the much easier problem of extending difference-differential structure. It is here that our differential lifting lemma (Theorem 3.1) is used.

PROPOSITION 4.4. *Suppose (R, Δ) is a Δ -subring of a differentially closed field (L, Δ) extending k , and $e : R \rightarrow \mathcal{D}(L)$ is a homomorphism of differential k -algebras such that $\pi^L \circ e : R \rightarrow L$ is the inclusion map. Suppose moreover that we have extensions of $\pi_i^L \circ e, \dots, \pi_t^L \circ e$ to endomorphisms of (L, Δ) , say $\sigma_1, \dots, \sigma_t$. Then there exists an extension of e to a \mathcal{D} -differential field structure on L whose associated endomorphisms are $(\sigma_1, \dots, \sigma_t)$.*

PROOF. The assumptions imply that the $\pi_i^L \circ e$ are all injective on R . We can therefore extend e canonically to $K := \text{Frac}(R)$ by Lemma 4.3.

Suppose that for arbitrary $a \in L$ we can extend e to a homomorphism on the Δ -ring generated by a over K , denoted by $K\{a\}$, in such a way that $\pi_i^L \circ e(a) = \sigma_i(a)$ for all $i = 0, \dots, t$. It would then follow that each $\pi_i^L \circ e$ agrees with σ_i on all of $K\{a\}$, and hence is in particular injective. By Lemma 4.3 again, e would extend to the Δ -field generated by a over K , denoted $K\langle a \rangle$, and $\pi_i^L \circ e$ would agree with σ_i on all of $K\langle a \rangle$. As a was arbitrary, a standard argument using Zorn’s Lemma would then complete the proof of the Proposition.

It suffices, therefore, to prove that for arbitrary $a \in L$, e extends to $K\{a\}$ in such a way that $\pi_i^L \circ e(a) = \sigma_i(a)$ for all $i = 0, \dots, t$. Suppose first that a is Δ -transcendental over K . Let $b_i \in B_i \otimes_k L$ be any element with residue $\sigma_i(a)$. Now, by Δ -transcendentality, there is a unique extension of e to a homomorphism on $(K\{a\}, \Delta)$ which takes a to (b_0, b_1, \dots, b_t) . By construction, $\pi_i^L \circ e(a) = \sigma_i(a)$.

It remains to consider the case when a is Δ -algebraic over K . In order to extend e to a homomorphism on $(K\{a\}, \Delta)$ we need to find $b \in \mathcal{D}(L)$ such that $g^e(b) = 0$ for all $g \in I_\Delta(a/K) := \{f \in K\{x\} : f(a) = 0\}$. (Recall that g^e denotes the differential polynomial over $\mathcal{D}(L)$ obtained from g by applying e to the coefficients.) Moreover, we require that $\pi_i^L(b) = \sigma_i(a)$ for all $i = 0, \dots, t$. The desired extension

would then send a to b . Using the decomposition $\mathcal{D}(L) = \prod_{i=0}^t (B_i \otimes_k L)$, it suffices to show:

- (*) Fix an arbitrary $i \in \{0, \dots, t\}$, let $R_i := B_i \otimes_k L$ equipped with the unique B_i -linear differential structure extending Δ on L , and let $e_i : K \rightarrow R_i$ be the composition of e with the projection onto the i th factor. Then there exists $b_i \in R_i$ such that $g^{e_i}(b_i) = 0$ for all $g \in I_\Delta(a/K)$, and such that $\text{res}_i^{L_i}(b_i) = \sigma_i(a)$.

Indeed, $b = (b_0, \dots, b_t)$ would then work.

We show (*) by first lifting $\sigma_i(a)$ to a root of Λ^{e_i} in R_i , where Λ is a characteristic set for $I_\Delta(a/K)$. To do this we need to verify the hypotheses of Theorem 3.1. Note that L is differentially closed by assumption (and this is the only place we use that assumption), and R_i is a local finite differential L -algebra whose maximal ideal is a Δ -ideal as it is generated by the maximal ideal of B_i which are all Δ -constants. If $f \in \Lambda$ then $f(a) = 0$, so that applying σ_i shows that $\text{res}_i^L(f^{e_i})$ vanishes on $\sigma_i(a)$. On the other hand, since $H_\Lambda(a) \neq 0$, applying σ_i shows that $\text{res}_i^L(H_\Lambda^{e_i})$ does not vanish on $\sigma_i(a)$. In particular, $H_\Lambda^{e_i}$ is not zero in $R_i\{x\}$ and so, by Lemma 2.3, $H_{\Lambda^{e_i}} = H_\Lambda^{e_i}$ and Λ^{e_i} is coherent autoreduced. Theorem 3.1 therefore applies, and we have $b_i \in R_i$ with residue $\sigma_i(a)$ and such that $g^{e_i}(b_i) = 0$ for all $g \in \Lambda$. It follows that $g^{e_i}(b_i) = 0$ for all $g \in [\Lambda]$. If $g \in I_\Delta(a/K)$ then, by (1) of §2, we have that $H_\Lambda^N g \in [\Lambda]$ for some N , and so $(H_{\Lambda^{e_i}}(b_i))^N g^{e_i}(b_i) = 0$. But we know that $H_{\Lambda^{e_i}}(b_i) \neq 0$, since its residue is $\sigma_i(H_\Lambda(a)) \neq 0$. It follows that $g^{e_i}(b_i) = 0$ for all $g \in I_\Delta(a/K)$. This completes the proof of (*), and hence of Proposition 4.4. ⊖

§5. \mathcal{D} -prolongations of Δ -varieties. Given a \mathcal{D} -differential field (K, Δ, ∂) and an affine Δ -variety $V \subseteq \mathbb{A}^n$ over K we wish to define the \mathcal{D} -prolongation of V . Taking our cue from the general theory of abstract prolongations of algebraic varieties developed in [7], the right way to construct the prolongations is as follows: base change V from K to $\mathcal{D}(K)$ via the homomorphism $e : K \rightarrow \mathcal{D}(K)$ and then take the Weil restriction back down to K using the standard K -algebra structure on $\mathcal{D}(K) = K \otimes_k B$. The problem is that the theory of Weil restrictions has not, to our knowledge, been developed in the differential-algebraic context. Instead of taking on that project here, we will simply construct the prolongations explicitly in co-ordinates.

Let $x := (x_1, \dots, x_n)$ be co-ordinates for \mathbb{A}^n , and $\bar{x} := (x = x^{(0)}, x^{(1)}, \dots, x^{(\ell)})$ co-ordinates for $\mathbb{A}^{(\ell+1)n}$, where recall ℓ is such that $\dim_k B = \ell + 1$. Given $f \in K\{x\}$, let $f^{(0)}, \dots, f^{(\ell)} \in K\{\bar{x}\}$ be such that

$$f^e \left(\sum_{j=0}^{\ell} x^{(j)} \epsilon_j \right) = \sum_{j=0}^{\ell} f^{(j)}(\bar{x}) \epsilon_j \tag{5}$$

in $\mathcal{D}(K)\{\bar{x}\}$. Note that $\mathcal{D}(K)\{\bar{x}\} = \mathcal{D}(K\{\bar{x}\})$ is a free $K\{\bar{x}\}$ -module with basis $\{\epsilon_0, \dots, \epsilon_\ell\}$, so that these $f^{(j)}$ are uniquely determined.

Fix a differentially closed field (L, Δ) extending (K, Δ) .

DEFINITION 5.1. Suppose $V \subseteq L^n$ is a Δ -closed set defined over K . The \mathcal{D} -prolongation of V , denoted by $\tau(V, \mathcal{D}, e)$, or just τV for short, is the Δ -closed subset of $L^{(\ell+1)n}$ defined by the equations $f^{(j)}(\bar{x}) = 0$ for all $f \in I_\Delta(V/K)$ and all $j = 0, \dots, \ell$.

REMARK 5.2.

- (a) A straightforward computation shows that if $I_\Delta(V/K) = [\Lambda]$ then in defining τV it suffices to consider $f \in \Lambda$. It follows that τ commutes with base change, and so preserves \mathcal{D} -differential fields of definition.
- (b) The association $\bar{a} \mapsto \sum_{j=0}^\ell a^{(j)} \epsilon_j$ gives us a natural identification of τV with the solutions in $\mathcal{D}(L)^n$ of the set of equations $\{f^e(x) = 0 : f \in I_\Delta(V/K)\}$.
- (c) In the pure \mathcal{D} -fields case when $\Delta = \emptyset$, the prolongations introduced here coincide with the \mathcal{D} -prolongations introduced in [7] and used in [8].
- (d) When $\mathcal{D}(R) = R \times R$, the prolongation introduced here is the difference-differential prolongation $V \times V^\sigma$ that was used in [6].

Recall that our basis elements were chosen so that the projection onto the ϵ_0 -co-ordinate is the ring homomorphism $\pi : B \rightarrow k$. It follows that the coefficient of ϵ_0 in $\epsilon_j \epsilon_k$ is 0 unless $j = k = 0$ in which case it is 1. This, together with the fact that the ϵ_j are Δ -constants, implies that under the identification of $x^{(0)} = x$, $f^{(0)}(\bar{x}) = f(x)$. Hence, the $x^{(0)}$ -coordinate projection maps τV to V . We denote this morphism of Δ -closed sets by $\hat{\pi} : \tau V \rightarrow V$.

On K -points, $\hat{\pi}$ has a section denoted by $\nabla : V(K) \rightarrow \tau V(K)$, and given by $a \mapsto (a, \partial_1 a, \dots, \partial_\ell a)$. Indeed, if $a \in V(K)$ and $f \in I_\Delta(V/K)$, then computing in $\mathcal{D}(K)$

$$0 = e(f(a)) = f^e(a + \partial_1 a + \dots + \partial_\ell a) = \sum_{j=0}^\ell f^{(j)}(a, \partial_1 a, \dots, \partial_\ell a) \epsilon_j$$

so that each $f^{(j)}(\nabla a) = 0$, as desired.

Finally, for each $i = 0, \dots, t$, we also have natural morphisms $\hat{\pi}_i : \tau V \rightarrow V^{\sigma_i}$, where V^{σ_i} refers to the transform of V obtained by applying the associated endomorphism σ_i to the coefficients of the defining equations of V . This map is the

one given on L -points by $\bar{a} \mapsto \sum_{j=0}^\ell a^{(j)} \pi_i(\epsilon_j)$, where recall that π_i is the composi-

tion of the k -algebra projection $B \rightarrow B_i$ in the decomposition $B = \prod_{i=0}^t B_i$, with the residue map $B_i \rightarrow k$. That this works is a consequence of the following easy computation:

LEMMA 5.3. Suppose $f \in K\{x\}$ and $\bar{a} \in L^{(\ell+1)n}$ is such that $f^{(j)}(\bar{a}) = 0$ for all $j = 0, \dots, \ell$. Then for each $i = 0, \dots, t$, $\sum_{j=0}^\ell a^{(j)} \pi_i(\epsilon_j)$ is a root of f^{σ_i} .

PROOF. We compute in L ,

$$0 = \pi_i^L \left(\sum_{j=0}^\ell f^{(j)}(\bar{a}) \epsilon_j \right) = \pi_i^L \left(f^e \left(\sum_{j=0}^\ell a^{(j)} \epsilon_j \right) \right) = f^{\sigma_i} \left(\sum_{j=0}^\ell a^{(j)} \pi_i(\epsilon_j) \right)$$

where the second equality is by (5) and the third is because $\pi_i^K \circ e = \sigma_i$. ←

One checks easily that $\hat{\pi}_0 = \hat{\pi}$. Note also that $\hat{\pi}_i \circ \nabla(a) = \sigma_i(a)$, for all i .

§6. \mathcal{D} -differentially closed fields. The language in which we study \mathcal{D} -differential rings is that of k -algebras equipped with function symbols for the derivations in Δ and the operators in ∂ . Note that the associated endomorphisms are 0-definable as they are in fact k -linear combinations of the operators in ∂ , as was pointed out in [8, Section 4.1].

Let \mathcal{K} denote the class of \mathcal{D} -differential rings that are integral domains and for which the associated endomorphisms are injective. By Lemma 4.3, \mathcal{K} is precisely the set of models of the universal part of the theory of \mathcal{D} -differential fields. Here is a geometric characterisation of existentially closed \mathcal{D} -differential fields that precisely extends Theorem 4.6 of [8] to the differential setting.

THEOREM 6.1. *A \mathcal{D} -differential ring (K, Δ, ∂) is existentially closed in the class \mathcal{K} if and only if*

- I (K, Δ) is a differentially closed field,
- II the associated endomorphisms are automorphisms, and
- III if $V \subseteq K^n$ is an irreducible Δ -closed set and $W \subseteq \tau V$ is an irreducible Δ -closed subset of the \mathcal{D} -prolongation such that $\widehat{\pi}_i(W)$ is Δ -dense in V^{σ_i} for all $i = 0, \dots, t$, then there exists $a \in V$ with $\nabla(a) \in W$.

PROOF. The only aspect of the proof of [8, Theorem 4.6] that does not readily extend to the differential context is the reliance on the extension theorems which in turn rely on a Hensel’s Lemma for local finite algebras over algebraically closed fields. So replacing those arguments in [8] with our extension theorem (Proposition 4.4 above), which followed from our differential Hensel’s Lemma (Theorem 3.1 above), we get a proof Theorem 6.1 quite easily. Let us give a few details.

Suppose (K, Δ, ∂) is existentially closed in \mathcal{K} . Now, as the associated endomorphisms are isomorphisms between differential integral domains, and since $\text{DCF}_{0,m}$ is the model completion of the theory of differential integral domains, the $\sigma_1, \dots, \sigma_t$ can be extended to automorphisms of some differentially closed field (L, Δ) , which we also denote by $\sigma_1, \dots, \sigma_t$. By Proposition 4.4, we can then extend the \mathcal{D} -structure on K to L so that (L, Δ, ∂) is a \mathcal{D} -differential field whose associated endomorphisms are the automorphisms $\sigma_1, \dots, \sigma_t$. As (K, Δ, ∂) is existentially closed, it follows that (K, Δ) is already a differentially closed field and the associated endomorphisms are already surjective, thus establishing I and II.

For condition III, let (L, Δ) be a differentially closed field extending (K, Δ) such that $W(L)$ has a point \bar{a} that is Kolchin generic over K . So $\bar{a} \in \tau V(L)$ and $a^{(0)} \in V(L)$. Letting $a' = \sum_{j=0}^{\ell} a^{(j)}\epsilon_j$, we see that a' is a root in $\mathcal{D}(L)^n$ of $f^e(x)$ for all $f \in I_{\Delta}(V/K)$; see Remark 5.2(b). By the Δ -density of $\widehat{\pi}(W)$ in V , we have that $I_{\Delta}(a^{(0)}/K) = I_{\Delta}(V/K)$, and so we can extend $e : K \rightarrow \mathcal{D}(L)$ to $K\{a^{(0)}\}$ by sending $a^{(0)} \mapsto a'$. Just as in [8, Theorem 4.6], the Δ -density of $\widehat{\pi}_i(W)$ in V^{σ_i} , for $i = 1, \dots, t$, will imply that the endomorphisms associated to $e : K\{a^{(0)}\} \rightarrow \mathcal{D}(L)$ are injective. Now, as in the previous paragraph, extending these endomorphisms to automorphisms of some $(L', \Delta) \supseteq (L, \Delta)$ and then using Proposition 4.4, we obtain a \mathcal{D} -differential structure on L' extending e on $K\{a^{(0)}\}$. So $(L', \Delta, \partial) \supseteq (K, \Delta, \partial)$ and $\nabla(a^{(0)}) = \bar{a}$. As (K, Δ, ∂) is existentially closed, it follows that there must exist $a \in V(K)$ with $\nabla(a) \in W(K)$, as desired.

As the readers can verify for themselves, the converse direction of Theorem 6.1 is proved exactly as the corresponding direction of [8, Theorem 4.6], with algebraically

closed fields replaced by differentially closed fields and the Zariski topology replaced by the Kolchin topology. –1

REMARK 6.2. Inspecting the proof of property III for (K, Δ, ∂) existentially closed reveals a stronger conclusion: given any nonempty Δ -open set $O \subseteq W$ the $a \in V$ could be found so that $\nabla(a) \in O$. The point is that $\bar{a} \in W(L)$, being generic over K , must land in $O(L)$.

As in the case of existentially closed difference-differential fields considered in [6], we do not know if the above characterisation amounts to a first-order axiomatisation because in differentially closed fields the property of being Kolchin irreducible is not known to be a definable property of the parameters of a Δ -closed set. Our strategy to avoid this difficulty is precisely the same as that of the first author in [6].

Recall that for Λ a finite subset of a Δ -polynomial ring over a Δ -field, H_Λ is the product of the initials and separants of the members of Λ . As was explained in §2, when Λ is a characteristic set for a prime Δ -ideal I , the zero sets of Λ and I agree off the proper Δ -closed hypersurface defined by H_Λ . It will be useful to introduce notation for this Δ -open set, we denote it by $V^*(\Lambda) := V(\Lambda) \setminus V(H_\Lambda)$.

PROPOSITION 6.3. *Suppose (K, Δ, ∂) is a \mathcal{D} -differential field where (K, Δ) is differentially closed. Then condition III of Theorem 6.1 is equivalent to:*

- III' *Let $x = (x_1, \dots, x_n)$ be co-ordinates for K^n and $\bar{x} = (x^{(0)}, x^{(1)}, \dots, x^{(\ell)})$ co-ordinates for $\tau(K^n) = K^{(\ell+1)n}$. Suppose $\Lambda \subset K\{x\}$ and $\Gamma \subset K\{\bar{x}\}$ are characteristic sets for prime ideals such that*
 - (i) *in $\tau(K^n)$, $V^*(\Gamma) \subseteq V(\{f^{(j)} : j = 0, \dots, \ell, f \in \Lambda\})$, and*
 - (ii) *in K^n , $\hat{\pi}_i(V^*(\Gamma))$ contains a nonempty Δ -open subset of $V^*(\Lambda^{\sigma_i})$, for all $i = 0, \dots, t$.*

Then there exists $a \in V^(\Lambda)$ with $\nabla(a) \in V^*(\Gamma)$.*

PROOF. Note that the $\hat{\pi}_i$ referred to in (ii) is $\hat{\pi}_i : \tau(K^n) \rightarrow (K^n)^{\sigma_i} = K^n$.

Assume that III' holds, and suppose V and W are as in III of Theorem 6.1. Let Λ be a characteristic set for $I_\Delta(V/K)$ and Γ a characteristic set for $I_\Delta(W/K)$. The fact that $W \subseteq \tau(V)$ and $V^*(\Gamma)$ is contained in W implies assumption (i). Since Λ^{σ_i} is a characteristic set for V^{σ_i} , $V^*(\Lambda^{\sigma_i})$ is Δ -open in V^{σ_i} . Hence the fact that $\hat{\pi}_i(W)$ is Δ -dense in V^{σ_i} implies, using quantifier elimination for differentially closed fields, the truth of (ii). Therefore there exists $a \in V^*(\Lambda) \subseteq V$ with $\nabla(a) \in V^*(\Gamma) \subseteq W$, as desired.

For the converse we assume III and prove III'. Suppose Λ and Γ are given. Let $V \subset K^n$ denote the irreducible Δ -closed set defined by the prime Δ -ideal that has Λ as a characteristic set, and let $W \subseteq \tau(K^n)$ be the irreducible Δ -closed set defined by the prime Δ -ideal that has Γ as a characteristic set. Since $W \setminus V(H_\Gamma) = V^*(\Gamma)$, assumption (i) implies that $W \subseteq V(\{f^{(j)} : j = 0, \dots, \ell, f \in \Lambda\})$. But because $I_\Delta(V/K)$ need not be $[\Lambda]$, this does not imply that $W \subseteq \tau V$. Let us assume for now, however, that $W \subseteq \tau V$. Since $V^*(\Lambda^{\sigma_i})$ is Δ -open in V^{σ_i} , assumption (ii) does imply that $\hat{\pi}_i(W)$ is Δ -dense in V^{σ_i} . We can apply III of Theorem 6.1, or rather the refinement mentioned in Remark 6.2, to get $a \in V^*(\Lambda)$ with $\nabla(a) \in V^*(\Gamma)$.

It therefore remains only to prove that $W \subseteq \tau V$. Since we know that $W \subseteq V(\{f^{(j)} : j = 0, \dots, \ell, f \in \Lambda\})$, Lemma 5.3 implies that $\hat{\pi}_i(W) \subseteq V(\Lambda^{\sigma_i})$ for all $i = 0, \dots, t$. If we therefore let O_i be the nonempty Δ -open subset of $V(\Lambda^{\sigma_i})$ whose

existence is asserted by (ii), then $\widehat{\pi}_i^{-1}(O_i) \cap W$ is a nonempty Δ -open subset of W .

Hence $O := W \cap \bigcap_{i=0}^t \widehat{\pi}_i^{-1}(O_i)$ is a nonempty Δ -open subset of W . So to prove that

$W \subseteq \tau V$ it suffices to prove that $O \subseteq \tau V$. Let $\bar{b} \in O$. To show that $\bar{b} \in \tau V$ we need to show that $f^{(j)}(\bar{b}) = 0$ for all $f \in I_\Delta(V/K)$ and $j = 0, \dots, \ell$. We know this is true for all $f \in \Lambda$ since $\bar{b} \in W$, and it is easy to see that the same must hold for all $f \in [\Lambda]$. But as Λ is characteristic of $I_\Delta(V/K)$, if $f \in I_\Delta(V/K)$ then $H_\Lambda^N f \in [\Lambda]$ for some N , see equality (1) of Section 2. Hence, using identity (5) of Section 5, we get that for all $f \in I_\Delta(V/K)$

$$\begin{aligned} 0 &= \sum_{j=0}^{\ell} (H_\Lambda^N f)^{(j)}(\bar{b}) \epsilon_j \\ &= (H_\Lambda^N f)^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right) \\ &= (H_\Lambda^N)^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right) \cdot f^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right) \\ &= (H_\Lambda^N)^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right) \cdot \left(\sum_{j=0}^{\ell} f^{(j)}(\bar{b}) \epsilon_j \right). \end{aligned}$$

It therefore suffices to verify that $(H_\Lambda^N)^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right)$ is a unit in $\mathcal{D}(K)$, or equivalently that for each $i = 0, \dots, t$, $\pi_i^K (H_\Lambda^N)^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right) \neq 0$ in K . But

$$\pi_i^K (H_\Lambda^N)^e \left(\sum_{j=0}^{\ell} b^{(j)} \epsilon_j \right) = (H_\Lambda^N)^{\sigma_i} \left(\sum_{j=0}^{\ell} b^{(j)} \pi_i \epsilon_j \right) = (H_{\Lambda^{\sigma_i}}^N)(\widehat{\pi}_i \bar{b})$$

and the latter is not zero because $\widehat{\pi}_i(\bar{b}) \in O_i$ and O_i is disjoint of $V(H_{\Lambda^{\sigma_i}})$ by assumption. This completes the proof that $O \subseteq \tau V$, and hence that $W \subseteq \tau V$. \dashv

COROLLARY 6.4. *The theory of \mathcal{D} -differential fields has a model companion.*

PROOF. This is because conditions I and II of Theorem 6.1, and III' of Proposition 6.3, are first-order axiomatisable. For the latter the main point is that, by a criterion of Rosenfeld's, being the characteristic set of a prime Δ -ideal is definable in parameters. See [6] for a more detailed discussion. \dashv

We call this model companion the theory of *\mathcal{D} -differentially closed fields* and denote it by $\mathcal{D}\text{-DCF}_{0,m}$. When $m = 0$ (so $\Delta = \emptyset$) we recover the theory of existentially closed \mathcal{D} -fields, $\mathcal{D}\text{-CF}_0$, introduced in [8]. On the other hand, when $\mathcal{D}(R) = R \times R$ we recover the theory of existentially closed difference-differential fields $\text{DCF}_{0,m}$ whose existence was established in [6].

§7. Properties of $\mathcal{D}\text{-DCF}_{0,m}$. Given that the model companion exists, it is by now routine to show that this model companion is tame in the various ways one expects. We list below five basic model-theoretic properties of $\mathcal{D}\text{-DCF}_{0,m}$ that are readily verified using the results and methods of [8, Section 5], which were themselves based

on the work of Chatzidakis and Hrushovski on difference fields [2]. As no serious complications arise in the differential setting, we omit proofs entirely. (See also [6] for the difference-differential case when $\mathcal{D}(R) = R \times R$.)

We will need a bit more notation. A \mathcal{D} -differential field is said to be *inversive* if the associated endomorphisms are surjective. Given a subset B of a model of \mathcal{D} -DCF $_{0,m}$ we denote by $\langle B \rangle$ the smallest inversive \mathcal{D} -differential subfield that contains B .

1. *Completions.* The completions of \mathcal{D} -DCF $_{0,m}$ are determined by the action of the associated endomorphisms on k^{alg} . In fact, if (K, Δ, ∂) and (L, Δ', ∂') are models of \mathcal{D} -DCF $_{0,m}$ with a common algebraically closed inversive \mathcal{D} -differential subfield F , then $(K, \Delta, \partial) \equiv_F (L, \Delta', \partial')$.
2. *acl.* If B is a subset of a model of \mathcal{D} -DCF $_{0,m}$ then $\text{acl}(B) = \langle B \rangle^{\text{alg}}$. In the case when $t = 0$ (so B is a local k -algebra), $\text{dcl}(B) = \langle B \rangle$.
3. *Quantifier reduction.* Suppose $(L, \Delta, \partial) \models \mathcal{D}$ -DCF $_{0,m}$, K is an inversive \mathcal{D} -differential subfield, and $a, b \in L^n$. Then $\text{tp}(a/K) = \text{tp}(b/K)$ if and only if there is an isomorphism between $(K\langle a \rangle, \Delta, \partial)$ and $(K\langle b \rangle, \Delta, \partial)$ that fixes K , takes a to b , and extends to an isomorphism from $(K\langle a \rangle^{\text{alg}}, \sigma)$ to $(K\langle b \rangle^{\text{alg}}, \sigma)$. In particular, if $t = 0$ then \mathcal{D} -DCF $_{0,m}$ admits quantifier elimination.
4. *Simplicity.* Every completion of \mathcal{D} -DCF $_{0,m}$ is simple with the independence theorem holding over (model-theoretically) algebraically closed sets. Non-forking independence is given by $A \downarrow_C B$ if and only if $\langle A \rangle$ and $\langle B \rangle$ are algebraically disjoint over $\langle C \rangle$. In the case when $t = 0$, \mathcal{D} -DCF $_{0,m}$ is a complete stable theory.
5. *Imaginaries.* Every completion of \mathcal{D} -DCF $_{0,m}$ eliminates imaginaries.

It is a little less straightforward to extend the methods of [8] to obtain the appropriate Canonical Base Property and Zilber Dichotomy for finite dimensional types. These results were shown to hold in the special case of an automorphism, that is for the model companion of differential fields of characteristic zero equipped with an automorphism, first in the ordinary case [1] and then generalised to the partial setting [6, Section 4]. In fact, we expect the Zilber Dichotomy to hold in any setting of fields equipped with commuting free operators for which a model companion exists.

We conclude by discussing some of the natural reducts of \mathcal{D} -DCF $_{0,m}$.

PROPOSITION 7.1. *Suppose $(K, \Delta, \partial) \models \mathcal{D}$ -DCF $_{0,m}$.*

- (i) *For each associated automorphism σ , $(K, \Delta, \sigma) \models \text{DCF}_{0,m} \text{ A}$.*
- (ii) *For any $r \leq m$ and $\Delta_1 \subseteq \Delta$ a subset of r -many derivations, $(K, \Delta_1, \partial) \models \mathcal{D}$ -DCF $_{0,r}$. In particular, $(K, \partial) \models \mathcal{D}$ -CF $_0$.*
- (iii) *For Δ_1 as above, and setting $\Delta_2 := \Delta \setminus \Delta_1$, we have that $(K^{\Delta_2}, \Delta_1, \partial) \models \mathcal{D}$ -DCF $_{0,r}$. Here K^{Δ_2} denotes the field of Δ_2 -constants.*

PROOF. The following are adaptations of the arguments of Proposition 4.12 of [8] and Proposition 3.4 of [6].

(i) By the geometric characterization of the models of DCF $_{0,m} \text{ A}$ given in [6], it suffices to show that if V and W are irreducible Δ -closed sets such that $W \subset V \times V^\sigma$ projects Δ -dominantly onto each factor, then there is $a \in V$ with $(a, \sigma a) \in W$. To prove this, consider the pull-back W' of W under $\tau V \rightarrow V \times V^\sigma$, and apply condition (III) of Theorem 6.1 to an irreducible component of W' that projects Δ -dominantly onto W .

(ii) Checking the axioms for \mathcal{D} -DCF $_{0,r}$, but working with (III') of 6.3 rather than (III) of 6.1, we see that it suffices to show that if Λ is a characteristic set of a prime Δ_1 -ideal of the Δ_1 -polynomial ring over K , then Λ is also characteristic for a prime Δ -ideal of $K\{x\}$. That this is indeed the case is part of the proof of Kolchin's Irreducibility Theorem [4, Chapter 0, Section 6].

(iii) We check that $(K^{\Delta_2}, \Delta_1, \partial)$ is an existentially closed \mathcal{D} - Δ_1 -field. Let $\phi(x)$ be a quantifier free formula in the language of \mathcal{D} - Δ_1 -rings with parameters from K^{Δ_2} and with a realisation in some \mathcal{D} - Δ_1 -field extension (F, Δ_1, ∂) . We can expand (F, Δ_1, ∂) to a \mathcal{D} - Δ -field (F, Δ, ∂) by interpreting all the derivation in Δ_2 to be trivial on F . So (F, Δ, ∂) extends $(K^{\Delta_2}, \Delta, \partial)$, and itself extends to a model $(L, \Delta, \partial) \models \mathcal{D}$ -DCF $_{0,m}$. Since K^{Δ_2} is a common algebraically closed inversive \mathcal{D} -differential subfield of (K, Δ, ∂) and (L, Δ, ∂) , property (1) above implies that there is $b \in K$ such that

$$(K, \Delta, \partial) \models \phi(b) \wedge \bigwedge_{\delta \in \Delta_2} \delta(b) = 0.$$

Since ϕ is quantifier free and refers only to Δ_1 , we have $(K^{\Delta_2}, \Delta_1, \partial) \models \phi(b)$, as desired. \dashv

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